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On the 3-parameter generalized quaternions with generalized tribonacci numbers components

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Abstract. In this paper, we aim to combine 3-parameter generalized quaternions (shortly 3PGQs), which are a general form of the quaternion algebra according to 3-parameters, and generalized Tribonacci number (shortly GTNs), which are also quite a big special number family for third-order recurrence sequences and most general form of all of the third-order recurrence sequences. Namely, we investigate a special new number system called 3-parameter generalized quaternions with generalized Tribonacci numbers components (shortly 3PGQs with GTN components) with both nonnegative and negative subscripts and examine some special cases of them. Then, we construct a Maple code of this special number family. Moreover, we obtain some new and classical well-known equations such as; Binet formulas, generating function, exponential generating function, Poisson generating function, summation formulas, polar representation, and matrix equation. In addition to these, we give also determinant, characteristic polynomial, characteristic equation, eigenvalues, and eigenvectors concerning the matrix representation of 3PGQs with GTN components.

1. Introduction

Number theory is becoming more popular since it has several applications in lots of work-frames including some beneficial applications. Every facet of number theory is receiving more notice and taking new materials and methods. As one of the most important and basic parts of number theory, numbers and number systems are well-established concepts for many researchers in various fields, as they have a wide range of applications in a variety of pure and applied working areas such as; robotics, computer graphics, engineering, etc. Quaternions, one of these well-known number systems, as an extension of the complex numbers were invented by W. R. Hamilton in 1843, [28–30]. Quaternion algebra is noncommutative, associative, and 4-dimensional Clifford algebra. The set of quaternions (real or Hamilton quaternion) is denoted by $\mathbb{H}$ and defined as:

$$\mathbb{H} := \{q | q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \ q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

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where e_1, e_2, e_3 are quaternionic units hold the rules $e_1^2 = -1$, $e_2^2 = -1$, $e_3^2 = -1$, $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = e_1$, $e_3e_1 = -e_1e_3 = e_2$. The literature includes many works with respect to the quaternion and its other types. Split quaternions were found by J. Cockle with the property of quaternionic units: $e_1^2 = -1$, $e_2^2 = e_3^2 = 1$, $e_1e_2e_3 = 1$ [15]. Additionally, the generalized quaternions (or 2-parameter generalized quaternions), have been analyzed in several studies (cf. [17, 23, 36–38, 42, 45, 47, 48, 79]). For all $\lambda_1, \lambda_2 \in \mathbb{R}$, the set of the generalized quaternions is represented by $\mathbb{H}_{\lambda_1\lambda_2}$ and defined as follows:

$$\mathbb{H}_{\lambda_1\lambda_2} := \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3,\ q_0, q_1, q_2, q_3, \lambda_1, \lambda_2 \in \mathbb{R}\},$$

where $e_1^2 = -\lambda_1$, $e_2^2 = -\lambda_2$, $e_3^2 = -\lambda_1\lambda_2$, $e_1e_2 = -e_2e_1 = e_3$, $e_2e_3 = -e_3e_2 = \lambda_2e_1$, $e_3e_1 = -e_1e_3 = \lambda_1e_2$. For $\lambda_1 = 1, \lambda_2 = 1$; q is a real quaternion, for $\lambda_1 = 1, \lambda_2 = -1$; q is a split quaternion, for $\lambda_1 = 1, \lambda_2 = 0$; q is a semi-quaternion, for $\lambda_1 = -1, \lambda_2 = 0$; q is a split semi-quaternion, for $\lambda_1 = 0, \lambda_2 = 0$; q is a 1/4-quaternion [15, 17, 23, 30, 36–38, 42, 43, 45, 47, 48, 79].

Additionally, T. D. Şentürk and Z. Ünal [75, 76] investigated a new quite comprehensive quaternion type called 3-parameter generalized quaternions (shortly; 3PGQs). The authors constructed a new and general aspect for the quaternion algebra depending on the 3-parameters to get a generalization of real, split, and 2-parameter generalized quaternions (shortly 2PGQs). The set of 3PGQs is denoted by $\mathbb{H}_{\lambda_1\lambda_2\lambda_3}$ and defined by:

$$\mathbb{H}_{\lambda_1\lambda_2\lambda_3} := \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3,\ q_0, q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\},$$

where quaternionic units hold the rules given in Table 1.

Table 1: Multiplication Rules of 3PGQs [75, 76]

.	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\lambda_1\lambda_2$	λ_1e_3	$-\lambda_2e_2$
e_2	e_2	$-\lambda_1e_3$	$-\lambda_1\lambda_3$	λ_3e_1
e_3	e_3	λ_2e_2	$-\lambda_3e_1$	$-\lambda_2\lambda_3$

According to the values $\lambda_1, \lambda_2, \lambda_3$, some special cases of 3PGQs are listed in the following Table 2, and also it can be written that the other special types with respect to the $\lambda_{i\in\{1,2,3\}}$ can be studied [75, 76].

Table 2: Classification of 3PGQs [75, 76]

For	*Type of 3PGQs*
$\lambda_1 = 1, \lambda_2 = \lambda, \lambda_3 = \mu$	2PGQs [17, 23, 36–38, 42, 45, 47, 48, 79]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$	Split quaternions [15]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$	Hamilton quaternions [28–30]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0$	Semi-quaternions [43, 47]
$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$	Split semi-quaternions [47]
$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$	1/4-quaternions [30, 47]

When the existing studies are examined, there are many special recurrence sequences with different orders such as; second, third, and higher than three-order. In this paper, we are interested in third-order recurrence sequence which is the most general form of all third-order recurrence sequences and called a generalized Tribonacci number sequence (or shortly GTNs). Some particular cases of this family can be seen in Table 3 and Table 4 [1, 6–8, 10, 13, 20, 22, 39, 44, 46, 50–55, 58, 62, 65–70, 72–74, 80–82]. For all $n \geq 3$, the generalized Tribonacci sequence ($\{T_n(T_0, T_1, T_2; r, s, t)\}_{n\geq 0}$ or shortly $\{T_n\}_{n\geq 0}$) satisfy the recurrence relation

$$T_n = rT_{n-1} + sT_{n-2} + tT_{n-3} \tag{1}$$

with the initial values $T_0 = a, T_1 = b, T_2 = c$. Here a, b, c are arbitrary integers and $r, s, t\,(t \neq 0)$ are real numbers [6]. For all $n \in \mathbb{Z}^+$, the GTNs with negative subscripts hold the following recurrence relation [64]:

$$T_{-n} = \frac{1}{t}T_{-(n-3)} - \frac{s}{t}T_{-(n-1)} - \frac{r}{t}T_{-(n-2)}. \tag{2}$$

Indeed, the recurrence relation (1) holds for all $n \in \mathbb{Z}$.

The Tribonacci numbers have some applications and some researchers focus on this part. For example, Basu and Das investigated a new coding theory such as Tribonacci coding theory [2] by using the Tribonacci numbers and based on the Tribonacci matrices. Additionally, Gupta and Sanghi give an innovative and effective digital signature scheme with respect to Tribonacci matrices and factoring in [24]. Moreover, Bezuszka and D'Angelo gave an interesting application for Tribonacci numbers [3]. Also, Demirci and Cangül examined the Tribonacci graphs as an application to graph theory [16].

One can observe that merging the quaternions and special recurrence sequences has been a concentrated work frame, and also several researchers attracted due to their applications and various use areas. In the literature, 2PGQs with some third-order recurrence sequences were studied intensively. In [12], Cerda-Morales studied the third-order Jacobsthal 2PGQs. Padovan and Perrin 2PGQs [32], Pell-Padovan 2PGQs [33], 2PGQs with generalized Jacobsthal numbers components [27], 2PGQs with generalized 3-primes and generalized reverse 3-primes numbers components [34] are investigated by İşbilir and Gürses. Kızılateş et al. introduced the bicomplex generalized Tribonacci quaternions in [41]. Also, Flaut and Shpakivskyi examined the generalized Fibonacci quaternions and Fibonacci-Narayana quaternions in [21]. As mentioned earlier, 2PGQs also 3PGQs have some special cases concerning λ_1 and λ_2 (e.g., real, split, semi, split semi, 1/4-quaternions), Taşcı introduced the Padovan and Pell-Padovan real quaternions in [78]. Günay and Taşkara [26] examined some properties of Padovan real quaternions. Then, Günay studied the real quaternions with some generalized third-order recurrence numbers components [25]. Also, Dişkaya and Menken investigated the (s, t)-Padovan and (s, t)-Perrin real quaternions, and split (s, t)-Padovan and (s, t)-Perrin quaternions in [18, 19]. Cerda-Morales studied the real quaternions with generalized Tribonacci numbers components [6], third-order Jacobsthal quaternions [9], third-order $\bar{h}$-Jacobsthal and third-order $\bar{h}$-Jacobsthal-Lucas sequences and related quaternions [11], and third-order Jacobsthal generalized quaternions [12]. Recently, Bilgici introduced the Fibonacci and Lucas 3PGQs in [4] and 3PGQs with Jacobsthal and Jacobsthal-Lucas numbers components in [5]. Chaker and Boua examined some properties of generalized quaternions algebra with generalized Fibonacci quaternions [14]. Also, Horadam 3PGQs were determined by İşbilir and Gürses in [35].

In this paper, we intend to bring together quite interesting and popular number systems; 3PGQs, which are a general form of the quaternion algebra according to 3-parameters, and GTNs which are quite a big special number family for third-order special recurrence sequences. That is, we construct the 3-parameter generalized quaternions with generalized Tribonacci numbers components (shortly 3PGQs with GTN components) which is quite a general and big number system. Our aim in doing this study is to create the most general form of different type of quaternions (2PGQ, split, real, and the others, cf. Table 2) with third-order special number sequences that have not been done in the literature. As can be seen from special cases, it covers many studies. This paper is organized into four sections as follows. In Section 2, we give the general notions and notations about both 3PGQs and GTNs. In Section 3, we define and examine the 3PGQs with GTN components. Then, we give recurrence relation, Binet formula, generating function, exponential generating function, summing formulas, matrix formulas, determinant equalities which have the role for finding nth and $-(n+1)$th element of the sequence, and some special equalities, as well. Also, we give a Maple code as an application to improve the paper. In addition, we give determinants, characteristic polynomials, characteristic equations, eigenvalues, and eigenvectors concerning the matrix representation of 3PGQs with GTN components. Finally, in Section 4, we give the conclusions.

2. Preliminaries

First of all, we recall some fundamental and necessary general terminology concerning the 3PGQs and GTNs needed throughout this study.

Let $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3,\ \ p = p_0 + p_1e_1 + p_2e_2 + p_3e_3 \in \mathbb{H}_{\lambda_1\lambda_2\lambda_3}$, some basic algebraic properties are listed as follows ([75, 76]):

- ✻ *Equality:* $q = p \Leftrightarrow q_0 = p_0, \quad q_1 = p_1, \quad q_2 = p_2, \quad q_3 = p_3.$
- ✻ *Addition and subtraction:* $q \pm p = q_0 \pm p_0 + (q_1 \pm p_1)\,e_1 + (q_2 \pm p_2)\,e_2 + (q_3 \pm p_3)\,e_3.$
- ✻ *Multiplication by a scalar:* $\dot{c}q = \dot{c}q_0 + \dot{c}q_1e_1 + \dot{c}q_2e_2 + \dot{c}q_3e_3, \quad \dot{c} \in \mathbb{R}.$
- ✻ *Multiplication:* $qp = S_qS_p - f(V_q, V_p) + S_qV_q + S_pV_p + V_q \wedge V_p,$
  where $f\left(V_q, V_p\right) = \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$ and

$$V_q \wedge V_p = \begin{vmatrix} \lambda_3e_1 & \lambda_2e_2 & \lambda_1e_3 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix} = \lambda_3(q_2p_3 - q_3p_2)e_1 + \lambda_2(q_3p_1 - q_1p_3)e_2 + \lambda_1(q_1p_2 - q_2p_1)e_3.$$

- ✻ *Conjugation:* $\overline{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_3.$
- ✻ *Inner product:* $\langle q, p\rangle = q_0p_0 + \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3.$
- ✻ *Norm:* $N_q = q\overline{q} = \overline{q}q = q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2.$
- ✻ *Inverse:* $q^{-1} = \dfrac{\overline{q}}{N_q} = \dfrac{q_0 - q_1e_1 - q_2e_2 - q_3e_3}{q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2}$, where $N_q \neq 0$.

One can see that $S_{q\pm p} = q_0 \pm p_0 = S_q \pm S_p$, $V_{q\pm p} = V_q \pm V_p$, and $\overline{q} = S_q - V_q$. If $N_q = 1$, then q is a 3-parameter generalized unit quaternion. For detailed information for 3PGQs, we can refer to the studies [75, 76].

Additionally, for $N_q > 0$ and $\lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2 \neq 0$, q can be expressed in a polar form as follows:

$$q = \sqrt{N_q}\,(\cos\theta + \hat{q}\sin\theta)\,,$$

where $\hat{q} = \dfrac{1}{\sqrt{\lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2}}(q_1, q_2, q_3)$ · Here $\cos\theta = \dfrac{q_0}{\sqrt{N_q}}$, $\sin\theta = \sqrt{\dfrac{\lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2}{N_q}}$, and $\hat{q}$ is called 3-parameter generalized unit vector. Furthermore, for q, the following fundamental matrix[1)] M_q is obtained:

$$M_q = \begin{pmatrix} q_0 & -\lambda_1\lambda_2q_1 & -\lambda_1\lambda_3q_2 & -\lambda_2\lambda_3q_3 \\ q_1 & q_0 & -\lambda_3q_3 & \lambda_3q_2 \\ q_2 & \lambda_2q_3 & q_0 & -\lambda_2q_1 \\ q_3 & -\lambda_1q_2 & \lambda_1q_1 & q_0 \end{pmatrix}.$$

According to the values of $\lambda_{i\in\{1,2,3\}}$, the matrix M_q can be classified. For $\lambda_1 = 1$, λ_2, $\lambda_3 \in \mathbb{R}$, M_q for 2PGQ is obtained. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$, then M_q for split quaternions is written. Moreover, for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then M_q for Hamilton quaternions is constructed.

Then, one can write some algebraic calculations for M_q:

- ✻ The determinant of M_q is $\det\left(M_q\right) = N_q^2$.
- ✻ The characteristic polynomial of M_q is: $P_{M_q}(\mathrm{u}) = \left(\mathrm{u}^2 - 2\mathrm{u}q_0 + q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2\right)^2.$

[1)] Throughout the paper, this representation, which is actually the left matrix representation, will be used due to the noncommutativity of 3PGQs. Similarly, the right matrix representation can also be considered.

✻ The characteristic equation of M_q is:

$$\det\left(M_q - \mathrm{u}I_4\right) = 0 \Leftrightarrow P_{M_q}(\mathrm{u}) = \left(\mathrm{u}^2 - 2\mathrm{u}q_0 + q_0^2 + \lambda_1\lambda_2 q_1^2 + \lambda_1\lambda_3 q_2^2 + \lambda_2\lambda_3 q_3^2\right)^2 = 0.$$

✻ The characteristic equation enables to calculate the eigenvalues as follows:

$$\mathfrak{T}_{1,2} = q_0 + \sqrt{-\lambda_1\lambda_2 q_1^2 - \lambda_1\lambda_3 q_2^2 - \lambda_2\lambda_3 q_3^2}, \quad \mathfrak{T}_{3,4} = q_0 - \sqrt{-\lambda_1\lambda_2 q_1^2 - \lambda_1\lambda_3 q_2^2 - \lambda_2\lambda_3 q_3^2}.$$

✻ The relation occurs: $\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = q_0^2 + \lambda_1\lambda_2 q_1^2 + \lambda_1\lambda_3 q_2^2 + \lambda_2\lambda_3 q_3^2 = N_q$.

✻ The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ are written as:

$$\begin{pmatrix} \frac{\lambda_1 q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_1\lambda_2 q_1 q_3}{\lambda_1 q_2^2+\lambda_2 q_3^2} & \frac{q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_1 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{\lambda_2 q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_1\lambda_2 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & -\frac{q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_2 q_1 q_3}{\lambda_1 q_2^2+\lambda_2 q_3^2} & 0 & 1 \end{pmatrix}^t.$$

✻ The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ are written as follows:

$$\begin{pmatrix} \frac{\lambda_1 q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_1\lambda_2 q_1 q_3}{\lambda_1 q_2^2+\lambda_2 q_3^2} & -\frac{q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_1 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} -\frac{\lambda_2 q_3\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}-\lambda_1\lambda_2 q_1 q_2}{\lambda_1 q_2^2+\lambda_2 q_3^2} & \frac{q_2\sqrt{-\lambda_1\lambda_2 q_1^2-\lambda_1\lambda_3 q_2^2-\lambda_2\lambda_3 q_3^2}+\lambda_2 q_1 q_3}{\lambda_1 q_2^2\lambda_2 q_3^2} & 0 & 1 \end{pmatrix}^t,$$

where the notation "t" represents the transpose of a matrix.

As for the GTNs, the characteristic equation is as follows (see [6]):

$$x^3 - rx^2 - sx - t = 0. \tag{3}$$

The roots of above equation (3):

$$x_1 = \frac{r}{3} + \alpha + \beta, \quad x_2 = \frac{r}{3} + \kappa\alpha + \kappa^2\beta, \quad x_3 = \frac{r}{3} + \kappa^2\alpha + \kappa\beta, \tag{4}$$

where $\alpha = \sqrt[3]{\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\zeta}}$, $\beta = \sqrt[3]{\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\zeta}}$, $\kappa = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$, $\zeta = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}$, and $x_1 + x_2 + x_3 = r$, $x_1x_2 + x_1x_3 + x_2x_3 = -s$, $x_1x_2x_3 = t$. Equation (1) has one real and two nonreal solutions the latter being conjugate complex on condition that $\zeta > 0$ [6]. The Binet formula for GTNs is written as [6]:

$$T_n = \frac{\widehat{P}x_1^n}{(x_1 - x_2)(x_1 - x_3)} + \frac{\widehat{R}x_2^n}{(x_2 - x_1)(x_2 - x_3)} + \frac{\widehat{S}x_3^n}{(x_3 - x_1)(x_3 - x_2)}, \tag{5}$$

where

$$\widehat{P} = c - b(x_2 + x_3) + ax_2x_3, \quad \widehat{R} = c - b(x_1 + x_3) + ax_1x_3, \quad \widehat{S} = c - b(x_1 + x_2) + ax_1x_2. \tag{6}$$

Howard and Saidak [31] show that the Binet formula for GTNs in Equation (5) holds for all $n \in \mathbb{Z}$ [64]. In addition to these, an efficient method to generate T_n is applying the S-matrix, which is a generalization of the R-matrix, and S-matrix is determined as (cf. [40, 51, 80, 81]): $S = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

The GTNs are the largest family for the third-order recurrence numbers. Due to the received values depending on the r, s, t and T_0, T_1, T_2, this family contains different special cases. According to these, special cases and some special subfamilies of this sequence can be examined in Table 3 and Table 4. In Table 3, this sequence is classified both r, s, t and T_0, T_1, T_2 in Table 4, the generalized Tribonacci sequence is grouped dealing with the r, s, t values. For detailed information see [6, 7, 10, 20, 44, 49, 51, 55–64, 69, 71, 72].

Table 3: Several Special Cases of Generalized Tribonacci Numbers

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	**Recurrence Relation**
Tribonacci	$\{a_n\} = \{T_n(0,1,1;1,1,1)\}$	$a_n = a_{n-1} + a_{n-2} + a_{n-3}$
Tribonacci-Lucas	$\{b_n\} = \{T_n(3,1,3;1,1,1)\}$	$b_n = b_{n-1} + b_{n-2} + b_{n-3}$
Tribonacci-Perrin	$\{c_n\} = \{T_n(3,0,2;1,1,1)\}$	$c_n = c_{n-1} + c_{n-2} + c_{n-3}$
M. Tribonacci	$\{d_n\} = \{T_n(1,1,1;1,1,1)\}$	$d_n = d_{n-1} + d_{n-2} + d_{n-3}$
M. Tribonacci-Lucas	$\{f_n\} = \{T_n(4,4,10;1,1,1)\}$	$f_n = f_{n-1} + f_{n-2} + f_{n-3}$
A. Tribonacci-Lucas	$\{g_n\} = \{T_n(4,2,0;1,1,1)\}$	$g_n = g_{n-1} + g_{n-2} + g_{n-3}$
Padovan (Cordonnier)	$\{h_n\} = \{T_n(1,1,1;0,1,1)\}$	$h_n = h_{n-2} + h_{n-3}$
Perrin	$\{\iota_n\} = \{T_n(3,0,2;0,1,1)\}$	$\iota_n = \iota_{n-2} + \iota_{n-3}$
Van der Laan	$\{i_n\} = \{T_n(1,0,1;0,1,1)\}$	$i_n = i_{n-2} + i_{n-3}$
Padovan-Perrin	$\{j_n\} = \{T_n(0,0,1;0,1,1)\}$	$j_n = j_{n-2} + j_{n-3}$
M. Padovan	$\{k_n\} = \{T_n(3,1,3;0,1,1)\}$	$k_n = k_{n-2} + k_{n-3}$
A. Padovan	$\{l_n\} = \{T_n(0,1,0;0,1,1)\}$	$l_n = l_{n-2} + l_{n-3}$
Pell-Padovan	$\{m_n\} = \{T_n(1,1,1;0,2,1)\}$	$m_n = 2m_{n-2} + m_{n-3}$
Pell-Perrin	$\{o_n\} = \{T_n(3,0,2;0,2,1)\}$	$o_n = 2o_{n-2} + o_{n-3}$
T. Fibonacci-Pell	$\{r_n\} = \{T_n(1,0,2;0,2,1)\}$	$r_n = 2r_{n-2} + r_{n-3}$
T. Lucas-Pell	$\{s_n\} = \{T_n(3,0,4;0,2,1)\}$	$s_n = 2s_{n-2} + s_{n-3}$
A. Pell-Padovan	$\{t_n\} = \{T_n(0,1,0;0,2,1)\}$	$t_n = 2t_{n-2} + t_{n-3}$
T. Pell	$\{u_n\} = \{T_n(0,1,2;2,1,1)\}$	$u_n = 2u_{n-1} + u_{n-2} + u_{n-3}$
T. Pell-Lucas	$\{v_n\} = \{T_n(3,2,6;2,1,1)\}$	$v_n = 2v_{n-1} + v_{n-2} + v_{n-3}$
T. modified Pell	$\{T_n\} = \{T_n(0,1,1;2,1,1)\}$	$w_n = 2w_{n-1} + w_{n-2} + w_{n-3}$
T. Pell-Perrin	$\{z_n\} = \{T_n(3,0,2;2,1,1)\}$	$z_n = 2z_{n-1} + z_{n-2} + z_{n-3}$
T. Jacobsthal	$\{\tau_n\} = \{T_n(0,1,1;1,1,2)\}$	$\tau_n = \tau_{n-1} + \tau_{n-2} + 2\tau_{n-3}$
T. Jacobsthal-Lucas	$\{\gamma_n\} = \{T_n(2,1,5;1,1,2)\}$	$\gamma_n = \gamma_{n-1} + \gamma_{n-2} + 2\gamma_{n-3}$
M. T. Jacobsthal	$\{\delta_n\} = \{T_n(3,1,3;1,1,2)\}$	$\delta_n = \delta_{n-1} + \delta_{n-2} + 2\delta_{n-3}$
T. Jacobsthal-Perrin	$\{\epsilon_n\} = \{T_n(3,0,2;1,1,2)\}$	$\epsilon_n = \epsilon_{n-1} + \epsilon_{n-2} + 2\epsilon_{n-3}$
Jacobsthal-Padovan	$\{\varepsilon_n\} = \{T_n(1,1,1;0,1,2)\}$	$\varepsilon_n = \varepsilon_{n-2} + 2\varepsilon_{n-3}$
Jacobsthal-Perrin	$\{\eta_n\} = \{T_n(3,0,2;0,1,2)\}$	$\eta_n = \eta_{n-2} + 2\eta_{n-3}$
A. Jacobsthal-Padovan	$\{\Gamma_n\} = \{T_n(0,1,0;0,1,2)\}$	$\Gamma_n = \Gamma_{n-2} + 2\Gamma_{n-3}$
M. Jacobsthal-Padovan	$\{\Omega_n\} = \{T_n(3,1,3;0,1,2)\}$	$\Omega_n = \Omega_{n-2} + 2\Omega_{n-3}$
Narayana	$\{\Delta_n\} = \{T_n(0,1,1;1,0,1)\}$	$\Delta_n = \Delta_{n-1} + \Delta_{n-3}$
Narayana-Lucas	$\{\Theta_n\} = \{T_n(3,1,1;1,0,1)\}$	$\Theta_n = \Theta_{n-1} + \Theta_{n-3}$
Narayana-Perrin	$\{\theta_n\} = \{T_n(3,0,2;1,0,1)\}$	$\theta_n = \theta_{n-1} + \theta_{n-3}$
3-primes	$\{\vartheta_n\} = \{T_n(0,1,2;2,3,5)\}$	$\vartheta_n = 2\vartheta_{n-1} + 3\vartheta_{n-2} + 5\vartheta_{n-3}$
Lucas 3-primes	$\{\nu_n\} = \{T_n(3,2,10;2,3,5)\}$	$\nu_n = 2\nu_{n-1} + 3\nu_{n-2} + 5\nu_{n-3}$
M. 3-primes	$\{\rho_n\} = \{T_n(0,1,1;2,3,5)\}$	$\rho_n = 2\rho_{n-1} + 3\rho_{n-2} + 5\rho_{n-3}$
Reverse 3-primes	$\{\Phi_n\} = \{T_n(0,1,5;5,3,2)\}$	$\Phi_n = 5\Phi_{n-1} + 3\Phi_{n-2} + 2\Phi_{n-3}$
Reverse Lucas 3-primes	$\{\Upsilon_n\} = \{T_n(3,5,31;5,3,2)\}$	$\Upsilon_n = 5\Upsilon_{n-1} + 3\Upsilon_{n-2} + 2\Upsilon_{n-3}$
Reverse M. 3-primes	$\{\varrho_n\} = \{T_n(0,1,4;5,3,2)\}$	$\varrho_n = 5\varrho_{n-1} + 3\varrho_{n-2} + 2\varrho_{n-3}$

*A.: Adjusted, M.: Modified, T.: Third order

Table 4: A Brief Classification for Generalized Tribonacci Numbers

Name	$\{T_n\} = \{T_n(T_0, T_1, T_2; r, s, t)\}$	**Recurrence Relation**
G. Tribonacci (usual)	$\{A_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 1)\}$	$A_n = A_{n-1} + A_{n-2} + A_{n-3}$
G. Padovan	$\{H_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 1)\}$	$H_n = H_{n-2} + H_{n-3}$
G. Pell-Padovan	$\{M_n\} = \{T_n(T_0, T_1, T_2; 0, 2, 1)\}$	$M_n = 2M_{n-2} + M_{n-3}$
G. T. Pell	$\{U_n\} = \{T_n(T_0, T_1, T_2; 2, 1, 1)\}$	$U_n = 2U_{n-1} + U_{n-2} + U_{n-3}$
G. T. Jacobsthal	$\{\tau_n\} = \{T_n(T_0, T_1, T_2; 1, 1, 2)\}$	$\tau_n = \tau_{n-1} + \tau_{n-2} + 2\tau_{n-3}$
G. Jacobsthal-Padovan	$\{\varepsilon_n\} = \{T_n(T_0, T_1, T_2; 0, 1, 2)\}$	$\varepsilon_n = \varepsilon_{n-2} + 2\varepsilon_{n-3}$
G. Narayana	$\{\Delta_n\} = \{T_n(T_0, T_1, T_2; 1, 0, 1)\}$	$\Delta_n = \Delta_{n-1} + \Delta_{n-3}$
G. 3-primes	$\{\vartheta_n\} = \{T_n(T_0, T_1, T_2; 2, 3, 5)\}$	$\vartheta_n = 2\vartheta_{n-1} + 3\vartheta_{n-1} + 5\vartheta_{n-3}$
G. Reverse 3-primes	$\{\Phi_n\} = \{T_n(T_0, T_1, T_2; 5, 3, 2)\}$	$\Phi_n = 5\Phi_{n-1} + 3\Phi_{n-1} + 2\Phi_{n-3}$

*G.: Generalized, T.: Third Order

3. 3-Parameter Generalized Quaternions with Generalized Tribonacci Numbers Components

In this section, we determine the 3-parameter generalized quaternions with generalized Tribonacci numbers components (for short; 3PGQs with GTN components). Then, we also examine some special cases of this new type special recurrence sequence. We construct a Maple code to find elements of this special number family. Moreover, we obtain some new equations and classical well-known equations such as; Binet formulas, generating function, exponential generating function, Poisson generating function, summation formulas, polar representation, and matrix equation. Additionally, we construct determinant, characteristic polynomials, characteristic equations, eigenvalues, and eigenvectors concerning the matrix representation of 3PGQs with GTN components.

Definition 3.1. *Let $\mathcal{T}_n$ be the nth 3PGQ with GTN components. Then, it is determined as follows:*

$$\mathcal{T}_n = T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3 \quad \text{for all} \quad n \in \mathbb{N}, \tag{7}$$

where T_n is the nth nonnegative subscripted GTN. Let $\mathcal{T}_{-n}$ be the $-n$th negative subscripted 3PGQ with GTN components. Then, it is defined as follows:

$$\mathcal{T}_{-n} = T_{-n} + T_{-n+1}e_1 + T_{-n+2}e_2 + T_{-n+3}e_3 \quad \text{for all} \quad n \in \mathbb{Z}^+, \tag{8}$$

where T_{-n} is the $-n$th negative subscripted GTN. Also, e_1, e_2, e_3 hold rules in Table 1. For nonnegative and negative subscripted 3PGQ with GTN components, the following initial values are respectively given:

$$\begin{cases} \mathcal{T}_0 = a + be_1 + ce_2 + (rc + sb + ta)\, e_3, \\ \mathcal{T}_1 = b + ce_1 + (rc + sb + ta)e_2 + \left[\left(r^2 + s\right)c + (rs + t)\, b + rta\right] e_3, \\ \mathcal{T}_2 = c + (rc + sb + ta)\, e_1 + \left[\left(r^2 + s\right)c + (rs + t)\, b + rta\right] e_2 + \left[\left(r^3 + 2rs + t\right)c + \left(r^2 s + s^2 + rt\right)b + \left(r^2 t + st\right)a\right] e_3, \end{cases}$$

and

$$\begin{cases} \mathcal{T}_{-1} = \frac{c}{t} - \frac{r}{t}b - \frac{s}{t}a + ae_1 + be_2 + ce_3, \\ \mathcal{T}_{-2} = \frac{b}{t} - \frac{r}{t}a - \frac{s}{t}\left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right) + \left(\frac{c}{t} - \frac{r}{t}b - \frac{s}{t}a\right)e_1 + ae_2 + be_3, \\ \mathcal{T}_{-3} = \frac{a}{t} - \frac{r}{t}\left(\frac{c}{t} - \frac{r}{t}b - \frac{s}{t}a\right) - \frac{s}{t}\left[\frac{b}{t} - \frac{r}{t}a - \frac{s}{t}\left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right)\right] + \left[\frac{b}{t} - \frac{r}{t}a - \frac{s}{t}\left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right)\right]e_1 \\ \qquad + \left(\frac{c}{t} - \frac{s}{t}a - \frac{r}{t}b\right)e_2 + ae_3. \end{cases}$$

As a generalization, it ought to be written that the definition in Equation (7) holds for all $n \in \mathbb{Z}$. Throughout this article, while some equations and properties are written separately for nonnegative subscripted and negative subscripted 3PGQ with GTN components, some of them are considered for all $n \in \mathbb{Z}$ for the sake of brevity. The classification of the 3PGQs with GTN components can be seen in Table 5.

Table 5: Classification of 3PGQs with GTN Components

For	*Type of 3PGQs with GTN Components*
$\lambda_1 = 1, \quad \lambda_2, \lambda_3 \in \mathbb{R}$	2PGQs with GTN components [6, 18, 19, 26, 27, 32–34, 78]
$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = -1$	Split quaternions with GTN components [19]
$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 1$	Hamilton quaternions with GTN components [6, 18, 26, 78]
$\lambda_1 = 1, \quad \lambda_2 = 1, \quad \lambda_3 = 0$	Semi-quaternions with GTN components
$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 0$	Split semi-quaternions with GTN components
$\lambda_1 = 1, \quad \lambda_2 = 0, \quad \lambda_3 = 0$	1/4-quaternions with GTN components

Let us examine operations over 3PGQs with GTN components. For all $n, m \in \mathbb{Z}$, let $\mathcal{T}_n$ and $\mathcal{T}_m$ be the nth and mth 3PGQ with GTN components, respectively. Equality, addition/subtraction, and multiplication by scalar are performed in a familiar way. Additionally, scalar and vector parts, multiplication of any elements of this family, conjugation, inner product, norm and inverse are determined as follows:

- ✻ *Scalar and vector parts:* The scalar part of $\mathcal{T}_n$ is represented as $S_{\mathcal{T}_n}$ and $S_{\mathcal{T}_n} = T_n$. The vector part of $\mathcal{T}_n$ is denoted by $V_{\mathcal{T}_n}$ and $V_{\mathcal{T}_n} = T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3$. Therefore, $S_{\mathcal{T}_n \pm \mathcal{T}_m} = T_n \pm T_m = S_{\mathcal{T}_n} \pm S_{\mathcal{T}_m}$ and $V_{\mathcal{T}_n \pm \mathcal{T}_m} = V_{\mathcal{T}_n} \pm V_{\mathcal{T}_m}$.
- ✻ *Multiplication:* $\mathcal{T}_n \mathcal{T}_m = S_{\mathcal{T}_n} S_{\mathcal{T}_m} - f\left(V_{\mathcal{T}_n}, V_{\mathcal{T}_m}\right) + S_{\mathcal{T}_n} V_{\mathcal{T}_n} + S_{\mathcal{T}_m} V_{\mathcal{T}_m} + V_{\mathcal{T}_n} \wedge V_{\mathcal{T}_m}$,
  where
  $$f(V_{\mathcal{T}_n}, V_{\mathcal{T}_m}) = \lambda_1\lambda_2 T_{n+1}T_{m+1} + \lambda_1\lambda_3 T_{n+2}T_{n+2} + \lambda_2\lambda_3 T_{n+3}T_{n+3}$$
  and
  $$V_{\mathcal{T}_n} \wedge V_{\mathcal{T}_m} = \begin{vmatrix} \lambda_3 e_1 & \lambda_2 e_2 & \lambda_1 e_3 \\ T_{n+1} & T_{n+2} & T_{n+3} \\ T_{m+1} & T_{m+2} & T_{m+3} \end{vmatrix} = \begin{array}{l} \lambda_3 \left(T_{n+2}T_{m+3} - T_{n+3}T_{m+2}\right) e_1 \\ +\lambda_2 \left(T_{n+3}T_{m+1} - T_{n+1}T_{m+3}\right) e_2 \\ +\lambda_1 \left(T_{n+1}T_{m+2} - T_{n+2}T_{m+1}\right) e_3. \end{array}$$
  By taking into account Table 1, we can write the following form of multiplication as:
  $$\begin{aligned} \mathcal{T}_n \mathcal{T}_m = & T_n T_m - \lambda_1\lambda_2 T_{n+1}T_{m+1} - \lambda_1\lambda_3 T_{n+2}T_{m+2} - \lambda_2\lambda_3 T_{n+3}T_{m+3} \\ & + \left(T_n T_{m+1} + T_m T_{n+1} + \lambda_3 \left(T_{n+2}T_{m+3} - T_{n+3}T_{m+2}\right)\right) e_1 \\ & + \left(T_n T_{m+2} + T_m T_{n+2} + \lambda_2 \left(T_{n+3}T_{m+1} - T_{n+1}T_{m+3}\right)\right) e_2 \\ & + \left(T_n T_{m+3} + T_m T_{n+3} + \lambda_1 \left(T_{n+1}T_{m+2} - T_{n+2}T_{m+1}\right)\right) e_3. \end{aligned}$$
- ✻ *Conjugation:* $\overline{\mathcal{T}}_n = T_n - T_{n+1}e_1 - T_{n+2}e_2 - T_{n+3}e_3$ in which $\overline{\mathcal{T}}_n$ represents the conjugation of $\mathcal{T}_n$.
- ✻ *Inner product:* $\langle \mathcal{T}_n, \mathcal{T}_m \rangle = T_n T_m + \lambda_1\lambda_2 T_{n+1}T_{m+1} + \lambda_1\lambda_3 T_{n+2}T_{m+2} + \lambda_2\lambda_3 T_{n+3}T_{m+3}$.
- ✻ *Norm:* $N_{\mathcal{T}_n} = \mathcal{T}_n \overline{\mathcal{T}}_n = \overline{\mathcal{T}}_n \mathcal{T}_n = T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2$.
- ✻ *Inverse:* $\mathcal{T}_n^{-1} = \dfrac{\overline{\mathcal{T}}_n}{N_{\mathcal{T}_n}} = \dfrac{T_n - T_{n+1}e_1 - T_{n+2}e_2 - T_{n+3}e_3}{T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2}, \quad N_{\mathcal{T}_n} \neq 0.$

Theorem 3.2 (Recurrence Relation). *Let $\mathcal{T}_n$ be the nth 3PGQ with GTN components. The following recurrence relation is satisfied*

$$\mathcal{T}_n = r\mathcal{T}_{n-1} + s\mathcal{T}_{n-2} + t\mathcal{T}_{n-3} \quad \text{for all} \quad n \geq 3. \tag{9}$$

Also, the following recurrence relation of negative subscripted 3PGQs with GTN components is satisfied:

$$\mathcal{T}_{-n} = \frac{1}{t}\mathcal{T}_{-(n-3)} - \frac{s}{t}\mathcal{T}_{-(n-1)} - \frac{r}{t}\mathcal{T}_{-(n-2)} \quad \text{for all} \quad n \in \mathbb{Z}^+. \tag{10}$$

Proof. With the help of Equations (2) and (8), we get:

$$\begin{aligned}\frac{1}{t}\mathcal{T}_{-(n-3)}-\frac{s}{t}\mathcal{T}_{-(n-1)}-\frac{r}{t}\mathcal{T}_{-(n-2)}=&\frac{1}{t}\Big(T_{-(n-3)}+T_{-(n-3)+1}e_1+T_{-(n-3)+2}e_2+T_{-(n-3)+3}e_3\Big)\\&-\frac{s}{t}\Big(T_{-(n-1)}+T_{-(n-1)+1}e_1+T_{-(n-1)+2}e_2+T_{-(n-1)+3}e_3\Big)\\&-\frac{r}{t}\Big(T_{-(n-2)}+T_{-(n-2)+1}e_1+T_{-(n-2)+2}e_2+T_{-(n-2)+3}e_3\Big)\\=&\left(\frac{1}{t}T_{-(n-3)}-\frac{s}{t}T_{-(n-1)}-\frac{r}{t}T_{-(n-2)}\right)+\left(\frac{1}{t}T_{-(n-4)}-\frac{s}{t}T_{-(n-2)}-\frac{r}{t}T_{-(n-3)}\right)e_1\\&+\left(\frac{1}{t}T_{-(n-5)}-\frac{s}{t}T_{-(n-3)}-\frac{r}{t}T_{-(n-4)}\right)e_2+\left(\frac{1}{t}T_{-(n-6)}-\frac{s}{t}T_{-(n-4)}-\frac{r}{t}T_{-(n-5)}\right)e_3\\=&T_{-n}+T_{-(n-1)}e_1+T_{-(n-2)}e_2+T_{-(n-3)}e_3\\=&\mathcal{T}_{-n}.\end{aligned}$$

The recurrence relation in Equation (9) can be proved similarly. □

Also, it should be noted that the recurrence relation in Equation (9) holds for all $n \in \mathbb{Z}$.

We shall give some special subfamilies of nonnegative and negative subscripted 3PGQs with GTN components in Table 6. With the help of the r, s, t values which are given in Table 4 and Equations (7), (8), (9), we obtain the following classification in Table 6 for 3PGQs with GTN components. It is not necessary to rewrite Table 6 for negative subscripted 3PGQs with GTN components for the sake of brevity, for why by putting $n \to -n$ into these equalities, clarity appears.

Table 6: Several Classification for Special Cases of 3PGQs with GTN Components

3PGQ with "Number Family"	*Definition*	*Recurrence Relation*
G. Tribonacci (usual) C.	$\breve{A}_n = A_n + A_{n+1}e_1 + A_{n+2}e_2 + A_{n+3}e_3$	$\breve{A}_n = \breve{A}_{n-1} + \breve{A}_{n-2} + \breve{A}_{n-3}$
G. Padovan C.	$\breve{H}_n = H_n + H_{n+1}e_1 + H_{n+2}e_2 + H_{n+3}e_3$	$\breve{H}_n = \breve{H}_{n-2} + \breve{H}_{n-3}$
G. Pell-Padovan C.	$\breve{M}_n = M_n + M_{n+1}e_1 + M_{n+2}e_2 + M_{n+3}e_3$	$\breve{M}_n = 2\breve{M}_{n-2} + \breve{M}_{n-3}$
G. T. Pell C.	$\breve{U}_n = U_n + U_{n+1}e_1 + U_{n+2}e_2 + U_{n+3}e_3$	$\breve{U}_n = 2\breve{U}_{n-1} + \breve{U}_{n-2} + \breve{U}_{n-3}$
G. T. Jacobsthal C.	$\breve{\boldsymbol{\tau}}_n = \boldsymbol{\tau}_n + \boldsymbol{\tau}_{n+1}e_1 + \boldsymbol{\tau}_{n+2}e_2 + \boldsymbol{\tau}_{n+3}e_3$	$\breve{\boldsymbol{\tau}}_n = \breve{\boldsymbol{\tau}}_{n-1} + \breve{\boldsymbol{\tau}}_{n-2} + 2\breve{\boldsymbol{\tau}}_{n-3}$
G. Jacobsthal-Padovan C.	$\breve{\boldsymbol{\varepsilon}}_n = \boldsymbol{\varepsilon}_n + \boldsymbol{\varepsilon}_{n+1}e_1 + \boldsymbol{\varepsilon}_{n+2}e_2 + \boldsymbol{\varepsilon}_{n+3}e_3$	$\breve{\boldsymbol{\varepsilon}}_n = \breve{\boldsymbol{\varepsilon}}_{n-2} + 2\breve{\boldsymbol{\varepsilon}}_{n-3}$
G. Narayana C.	$\breve{\boldsymbol{\Delta}}_n = \boldsymbol{\Delta}_n + \boldsymbol{\Delta}_{n+1}e_1 + \boldsymbol{\Delta}_{n+2}e_2 + \boldsymbol{\Delta}_{n+3}e_3$	$\breve{\boldsymbol{\Delta}}_n = \breve{\boldsymbol{\Delta}}_{n-1} + \breve{\boldsymbol{\Delta}}_{n-3}$
G. 3-primes C.	$\breve{\boldsymbol{\vartheta}}_n = \boldsymbol{\vartheta}_n + \boldsymbol{\vartheta}_{n+1}e_1 + \boldsymbol{\vartheta}_{n+2}e_2 + \boldsymbol{\vartheta}_{n+3}e_3$	$\breve{\boldsymbol{\vartheta}}_n = 2\breve{\boldsymbol{\vartheta}}_{n-1} + 3\breve{\boldsymbol{\vartheta}}_{n-2} + 5\breve{\boldsymbol{\vartheta}}_{n-3}$
G. Reverse 3-primes C.	$\breve{\boldsymbol{\Phi}}_n = \boldsymbol{\Phi}_n + \boldsymbol{\Phi}_{n+1}e_1 + \boldsymbol{\Phi}_{n+2}e_2 + \boldsymbol{\Phi}_{n+3}e_3$	$\breve{\boldsymbol{\Phi}}_n = 5\breve{\boldsymbol{\Phi}}_{n-1} + 3\breve{\boldsymbol{\Phi}}_{n-2} + 2\breve{\boldsymbol{\Phi}}_{n-3}$

*C: Component, G.: Generalized, T.: Third Order

The other table which includes the special cases concerning both the r, s, t and initial values can be constructed quickly via Table 3. We omit them for the sake of brevity. As an example, one can see some properties for *3PGQs with third-order Pell numbers components* in Corollary 3.17. Namely, recurrence relations and definitions for special cases with respect to both r, s, t, and initial values can be written similarly as in Table 6.

Now, let us construct the following Maple 12 code in order to calculate nonnegative and negative subscripted 3PGQs with GTN components:

```
> restart: with(LinearAlgebra):with(linalg):
> T(n):
> T:=proc(n)
> if n=0 then return a:
> elif n=1 then return b:
> elif n=2 then return c:
> elif n=3 then return r*c+s*b+t*a:
> else return r*T(n-1)+s*T(n-2)+t*T(n-3)
> end if;
> end proc;
> T_star(n):
> T_star:=proc(n)
> if n=0 then return T(0)+T(1)*e[1]+T(2)*e[2]+T(3)*e[3]:
> elif n=1 then return T(1)+T(2)*e[1]+T(3)*e[2]+T(4)*e[3]:
> elif n=2 then return T(2)+T(3)*e[1]+T(4)*e[2]+T(5)*e[3]:
> elif n=3 then return r*(T(2)+T(3)*e[1]+T(4)*e[2]+T(5)*e[3])+ s*(T(1)+T(2)*e[1]+T(3)*e
    ↪ [2]+T(4)*e[3])+ t*(T(0)+T(1)*e[1]+T(2)*e[2]+T(3)*e[3]):
> else return r*T_star(n-1)+s*T_star(n-2)+t*T_star(n-3)
> end if;
> end proc;
> T_negative(n):
> T_negative:=proc(n)
> if n=-1 then return (1/t)*c+(-r/t)*b+(-s/t)*a:
> elif n=-2 then return (1/t)*T(4)+(-r/t)*T(3)+(-s/t)*T(2):
> elif n=-3 then return (1/t)*T(5)+(-r/t)*T(4)+(-s/t)*T(3):
> else return (1/t)*T(-n+3)+(-r/t)*T(-n+2)+(-s/t)*T(-n+1)
> end if;
> end proc;
> T_starnegative(n):
> T_starnegative:=proc(n)
> if n=-1 then return T_negative(-1)+T(0)*e[1]+T(1)*e[2]+T(2)*e[3]:
> elif n=-2 then return T_negative(-2)+ T_negative(-1)*e[1] +T(0)*e[2]+T(1)*e[3]:
> elif n=-3 then return (1/t)*(T(0)+T(1)*e[1]+ T(2)*e[2]+T(3)*e[3])+(-r/t)*(T_negative
    ↪ (-1)+ T(0)*e[1]+T(1)*e[2]+ T(2)*e[3])+(-s/t)*(T_negative(-2)+ T_negative(-1)*e[1]+
    ↪  T(0)*e[2]+T(1)*e[3]):
> else return (1/t)*T_starnegative(-n+3)+(-r/t)*T_starnegative(-n+2)+(-s/t)*
    ↪ T_starnegative(-n+1)
> end if;
> end proc;
```

*The notations T, T_star, T_negative and T_starnegative represent, nonnegative subscripted GTN, nonnegative subscripted 3PGQs with GTN components, negative subscripted GTN and negative subscripted 3PGQs with GTN components, respectively.

From here on, $\mathcal{T}_n$ and $\mathcal{T}_{-n}$ represent the nth and $-n$th nonnegative and negative subscripted 3PGQs with GTN components, respectively.

Theorem 3.3. *For all $m, n \in \mathbb{Z}$, the following properties hold for 3PGQs with GTN components:*

(a) $\mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3 = 2\mathcal{T}_n - (T_n + \lambda_1\lambda_2 T_{n+2} + \lambda_1\lambda_3 T_{n+4} + \lambda_2\lambda_3 T_{n+6})$,

(b) $\overline{\mathcal{T}}_n = 2T_n - \mathcal{T}_n$,

(c) $\mathcal{T}_n^2 + \overline{\mathcal{T}}_n^2 = 2\left(T_n^2 - \lambda_1\lambda_2 T_{n+1}^2 - \lambda_1\lambda_3 T_{n+2}^2 - \lambda_2\lambda_3 T_{n+3}^2\right)$ *and* $\mathcal{T}_n^2 - \overline{\mathcal{T}}_n^2 = 4T_n\mathcal{T}_n - 4T_n^2$,

(d) $\overline{\mathcal{T}}_n\overline{\mathcal{T}}_m - \mathcal{T}_n\mathcal{T}_m = -2\left[(T_n\mathcal{T}_{m+1} + T_{n+1}\mathcal{T}_m)e_1 + (T_n\mathcal{T}_{m+2} + T_{n+2}\mathcal{T}_m)e_2 + (T_n\mathcal{T}_{m+3} + T_{n+3}\mathcal{T}_m)e_3\right]$,

(e) $$\begin{aligned}\overline{\mathcal{T}}_n\overline{\mathcal{T}}_m + \mathcal{T}_n\mathcal{T}_m = 2\,[&T_nT_m - \lambda_1\lambda_2 T_{n+1}T_{m+1} - \lambda_1\lambda_3 T_{n+2}T_{m+2} - \lambda_2\lambda_3 T_{n+3}T_{m+3} + \lambda_3(T_{n+2}T_{m+3} - T_{n+3}T_{m+2})e_1\\ &+ \lambda_2(T_{n+3}T_{m+1} - T_{n+1}T_{m+3})e_2 + \lambda_1(T_{n+1}T_{m+2} - T_{n+2}T_{m+1})e_3],\end{aligned}$$

(f) $\overline{\mathcal{T}}_n\mathcal{T}_m - \mathcal{T}_n\overline{\mathcal{T}}_m = -2\left[(T_{n+1}T_m - T_nT_{m+1})e_1 + (T_{n+2}T_m - T_nT_{m+2})e_2 + (T_{n+3}T_m - T_nT_{m+3})e_3\right]$,

(g) $$\begin{aligned}\overline{\mathcal{T}}_n\mathcal{T}_m + \mathcal{T}_n\overline{\mathcal{T}}_m = +2[&T_nT_m + \lambda_1\lambda_2 T_{n+1}T_{m+1} + \lambda_1\lambda_3 T_{n+2}T_{m+2} + \lambda_2\lambda_3 T_{n+3}T_{m+3}\\ &+ \lambda_3(T_{n+3}T_{m+2} - T_{n+2}T_{m+3})e_1 + \lambda_2(T_{n+1}T_{m+3} - T_{n+3}T_{m+1})e_2\\ &+ \lambda_1(T_{n+2}T_{m+1} - T_{n+1}T_{m+2})e_3].\end{aligned}$$

Proof. (a) By using Table 1 and Equation (7), we have:

$$\begin{aligned}\mathcal{T}_n + \mathcal{T}_{n+1}e_1 + \mathcal{T}_{n+2}e_2 + \mathcal{T}_{n+3}e_3 =&T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3\\ &+ (T_{n+1} + T_{n+2}e_1 + T_{n+3}e_2 + T_{n+4}e_3)e_1\\ &+ (T_{n+2} + T_{n+3}e_1 + T_{n+4}e_2 + T_{n+5}e_3)e_2\\ &+ (T_{n+3} + T_{n+4}e_1 + T_{n+5}e_2 + T_{n+6}e_3)e_3\\ =&2\mathcal{T}_n - (T_n + \lambda_1\lambda_2 T_{n+2} + \lambda_1\lambda_3 T_{n+4} + \lambda_2\lambda_3 T_{n+6}).\end{aligned}$$

Besides for the last term, we can write

$$T_n + \lambda_1\lambda_2 T_{n+2} + \lambda_1\lambda_3 T_{n+4} + \lambda_2\lambda_3 T_{n+6} = \mathcal{T}_n - \mathcal{T}_{n+1}e_1 - \mathcal{T}_{n+2}e_2 - \mathcal{T}_{n+3}e_3.$$

(b) By means of Equation (1), Equation (7), and conjugation of $\mathcal{T}_n$, we obtain:

$$2T_n - \mathcal{T}_n = 2T_n - (T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3) = T_n - T_{n+1}e_1 - T_{n+2}e_2 - T_{n+3}e_3 = \overline{\mathcal{T}}_n.$$

It is also clear by using $\mathcal{T}_n + \overline{\mathcal{T}}_n = 2T_n$ and $\mathcal{T}_n - \overline{\mathcal{T}}_n = 2\mathcal{T}_n - 2T_n$.

(c) By using Table 1 and Equation (7) and conjugate of $\mathcal{T}_n$, we achieve:

$$\begin{aligned}\mathcal{T}_n^2 =&(T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3)(T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3)\\ =&2T_n\mathcal{T}_n - \left(T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2\right).\end{aligned}$$

and

$$\begin{aligned}\overline{\mathcal{T}}_n^2 =&(T_n - T_{n+1}e_1 - T_{n+2}e_2 - T_{n+3}e_3)(T_n - T_{n+1}e_1 - T_{n+2}e_2 - T_{n+3}e_3)\\ =&-2T_n\mathcal{T}_n - \left(-3T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2\right).\end{aligned}$$

Then, we get the followings:

$$\begin{aligned}\mathcal{T}_n^2 + \overline{\mathcal{T}}_n^2 =&2T_n\mathcal{T}_n - \left(T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2\right)\\ &- 2T_n\mathcal{T}_n - \left(-3T_n^2 - \lambda_1\lambda_2 T_{n+1}^2 - \lambda_1\lambda_3 T_{n+2}^2 - \lambda_2\lambda_3 T_{n+3}^2\right)\\ =&2\left(T_n^2 - \lambda_1\lambda_2 T_{n+1}^2 - \lambda_1\lambda_3 T_{n+2}^2 - \lambda_2\lambda_3 T_{n+3}^2\right)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}_n^2-\overline{\mathcal{T}}_n^2 =&2T_n\mathcal{T}_n-\left(T_n^2+\lambda_1\lambda_2T_{n+1}^2+\lambda_1\lambda_3T_{n+2}^2+\lambda_2\lambda_3T_{n+3}^2\right)\\&+2T_n\mathcal{T}_n+\left(-3T_n^2+\lambda_1\lambda_2T_{n+1}^2+\lambda_1\lambda_3T_{n+2}^2+\lambda_2\lambda_3T_{n+3}^2\right)\\=&4T_n\mathcal{T}_n-4T_n^2.\end{aligned}$$

(d) By means of Table 1 and Equation (7), conjugation and multiplication properties, we get:

$$\begin{aligned}\overline{\mathcal{T}}_n\overline{\mathcal{T}}_m-\mathcal{T}_n\mathcal{T}_m =&(T_n-T_{n+1}e_1-T_{n+2}e_2-T_{n+3}e_3)(T_m-T_{m+1}e_1-T_{m+2}e_2-T_{m+3}e_3)\\&-(T_n+T_{n+1}e_1+T_{n+2}e_2+T_{n+3}e_3)(T_m+T_{m+1}e_1+T_{m+2}e_2+T_{m+3}e_3)\\=&-2[(T_n\mathcal{T}_{m+1}+T_{n+1}\mathcal{T}_m)e_1+(T_n\mathcal{T}_{m+2}+T_{n+2}\mathcal{T}_m)e_2+(T_n\mathcal{T}_{m+3}+T_{n+3}\mathcal{T}_m)e_3].\end{aligned}$$

(e) By using Table 1, Equation (7), conjugation and multiplication properties, we have:

$$\begin{aligned}\overline{\mathcal{T}}_n\overline{\mathcal{T}}_m+\mathcal{T}_n\mathcal{T}_m =&(T_n-T_{n+1}e_1-T_{n+2}e_2-T_{n+3}e_3)(T_m-T_{m+1}e_1-T_{m+2}e_2-T_{m+3}e_3)\\&+(T_n+T_{n+1}e_1+T_{n+2}e_2+T_{n+3}e_3)(T_m+T_{m+1}e_1+T_{m+2}e_2+T_{m+3}e_3)\\=&2[T_nT_m-\lambda_1\lambda_2T_{n+1}T_{m+1}-\lambda_1\lambda_3T_{n+2}T_{m+2}-\lambda_2\lambda_3T_{n+3}T_{m+3}+\lambda_3(T_{n+2}T_{m+3}-T_{n+3}T_{m+2})e_1\\&+\lambda_2(T_{n+3}T_{m+1}-T_{n+1}T_{m+3})e_2+\lambda_1(T_{n+1}T_{m+2}-T_{n+2}T_{m+1})e_3.\end{aligned}$$

(f) By utilizing Table 1, Equation (7), conjugation and multiplication properties, we obtain:

$$\begin{aligned}\overline{\mathcal{T}}_n\mathcal{T}_m-\mathcal{T}_n\overline{\mathcal{T}}_m =&(T_n-T_{n+1}e_1-T_{n+2}e_2-T_{n+3}e_3)(T_m+T_{m+1}e_1+T_{m+2}e_2+T_{m+3}e_3)\\&-(T_n+T_{n+1}e_1+T_{n+2}e_2+T_{n+3}e_3)(T_m-T_{m+1}e_1-T_{m+2}e_2-T_{m+3}e_3)\\=&-2[(T_{n+1}T_m-T_nT_{m+1})e_1+(T_{n+2}T_m-T_nT_{m+2})e_2+(T_{n+3}T_m-T_nT_{m+3})e_3].\end{aligned}$$

(g) With the help of Table 1, Equation (7), conjugation and multiplication properties, we achieve:

$$\begin{aligned}\overline{\mathcal{T}}_n\mathcal{T}_m+\mathcal{T}_n\overline{\mathcal{T}}_m =&(T_n-T_{n+1}e_1-T_{n+2}e_2-T_{n+3}e_3)(T_m+T_{m+1}e_1+T_{m+2}e_2+T_{m+3}e_3)\\&+(T_n+T_{n+1}e_1+T_{n+2}e_2+T_{n+3}e_3)(T_m-T_{m+1}e_1-T_{m+2}e_2-T_{m+3}e_3)\\=&+2[T_nT_m+\lambda_1\lambda_2T_{n+1}T_{m+1}+\lambda_1\lambda_3T_{n+2}T_{m+2}+\lambda_2\lambda_3T_{n+3}T_{m+3}\\&+\lambda_3(T_{n+3}T_{m+2}-T_{n+2}T_{m+3})e_1+\lambda_2(T_{n+1}T_{m+3}-T_{n+3}T_{m+1})e_2\\&+\lambda_1(T_{n+2}T_{m+1}-T_{n+1}T_{m+2})e_3].\end{aligned}$$

Hence, we completed the proof. □

Theorem 3.4. *For all $n \in \mathbb{Z}$, Binet formula for 3PGQs with GTN components is satisfied as follows:*

$$\mathcal{T}_n=\frac{\widehat{P}x_1^n\widehat{x_1}}{(x_1-x_2)(x_1-x_3)}+\frac{\widehat{R}x_2^n\widehat{x_2}}{(x_2-x_1)(x_2-x_3)}+\frac{\widehat{S}x_3^n\widehat{x_3}}{(x_3-x_1)(x_3-x_2)}, \tag{11}$$

where x_1, x_2, x_3 and $\widehat{P}, \widehat{R}, \widehat{S}$ are written in Equations (4) *and* (6), *respectively and*

$$\widehat{x_1}=1+x_1e_1+x_1^2e_2+x_1^3e_3,\quad \widehat{x_2}=1+x_2e_1+x_2^2e_2+x_2^3e_3,\quad \widehat{x_3}=1+x_3e_1+x_3^2e_2+x_3^3e_3.$$

Proof. By using Equations (5) and (7), the proof is done:

$$
\begin{aligned}
\mathcal{T}_n =& T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3 \\
=& \frac{\widehat{P}x_1^n}{(x_1-x_2)(x_1-x_3)} + \frac{\widehat{R}x_2^n}{(x_2-x_1)(x_2-x_3)} + \frac{\widehat{S}x_3^n}{(x_3-x_1)(x_3-x_2)} \\
&+ \left(\frac{\widehat{P}x_1^{n+1}}{(x_1-x_2)(x_1-x_3)} + \frac{\widehat{R}x_2^{n+1}}{(x_2-x_1)(x_2-x_3)} + \frac{\widehat{S}x_3^{n+1}}{(x_3-x_1)(x_3-x_2)}\right)e_1 \\
&+ \left(\frac{\widehat{P}x_1^{n+2}}{(x_1-x_2)(x_1-x_3)} + \frac{\widehat{R}x_2^{n+2}}{(x_2-x_1)(x_2-x_3)} + \frac{\widehat{S}x_3^{n+2}}{(x_3-x_1)(x_3-x_2)}\right)e_2 \\
&+ \left(\frac{\widehat{P}x_1^{n+3}}{(x_1-x_2)(x_1-x_3)} + \frac{\widehat{R}x_2^{n+3}}{(x_2-x_1)(x_2-x_3)} + \frac{\widehat{S}x_3^{n+3}}{(x_3-x_1)(x_3-x_2)}\right)e_3 \\
=& \frac{\widehat{P}x_1^n\widehat{x_1}}{(x_1-x_2)(x_1-x_3)} + \frac{\widehat{R}x_2^n\widehat{x_2}}{(x_2-x_1)(x_2-x_3)} + \frac{\widehat{S}x_3^n\widehat{x_3}}{(x_3-x_1)(x_3-x_2)}.
\end{aligned}
$$

□

Theorem 3.5. *For all $n \in \mathbb{N}$, the generating functions for nonnegative and negative subscripted 3PGQs with GTN components are satisfied, respectively:*

$$\sum_{n=0}^{\infty} \mathcal{T}_n x^n = \frac{\mathcal{T}_0 + (\mathcal{T}_1 - r\mathcal{T}_0)\,x + (\mathcal{T}_2 - r\mathcal{T}_1 - s\mathcal{T}_0)\,x^2}{1 - rx - sx^2 - tx^3}, \tag{12}$$

$$\sum_{n=0}^{\infty} \mathcal{T}_{-n} x^n = \frac{\mathcal{T}_0 + \left(\mathcal{T}_{-1} + \frac{s}{t}\mathcal{T}_0\right)x + \left(\mathcal{T}_{-2} + \frac{s}{t}\mathcal{T}_{-1} + \frac{r}{t}\mathcal{T}_0\right)x^2}{1 - \frac{1}{t}x^3 + \frac{s}{t}x + \frac{r}{t}x^2}. \tag{13}$$

Proof. Let $\sum\limits_{n=0}^{\infty} \mathcal{T}_n x^n = \mathcal{T}_0 + \mathcal{T}_1 x + \mathcal{T}_2 x^2 + \ldots + \mathcal{T}_n x^n + \ldots$ be generating function of 3PGQs with GTN components. Then, let us multiply both sides of this equality by rx, sx^2, tx^3:

$$
\begin{aligned}
rx\sum_{n=0}^{\infty} \mathcal{T}_n x^n &= r\mathcal{T}_0 x + r\mathcal{T}_1 x^2 + r\mathcal{T}_2 x^3 + \ldots + r\mathcal{T}_n x^{n+1} + \ldots \\
sx^2\sum_{n=0}^{\infty} \mathcal{T}_n x^n &= s\mathcal{T}_0 x^2 + s\mathcal{T}_1 x^3 + s\mathcal{T}_2 x^4 + \ldots + s\mathcal{T}_n x^{n+2} + \ldots \\
tx^3\sum_{n=0}^{\infty} \mathcal{T}_n x^n &= t\mathcal{T}_0 x^3 + t\mathcal{T}_1 x^4 + t\mathcal{T}_2 x^5 + \ldots + t\mathcal{T}_n x^{n+3} + \ldots
\end{aligned}
$$

After these, by using Equation (9), we have:

$$(1 - rx - sx^2 - tx^3)\sum_{n=0}^{\infty} \mathcal{T}_n x^n = \mathcal{T}_0 + (\mathcal{T}_1 - r\mathcal{T}_0)x + (\mathcal{T}_2 - r\mathcal{T}_1 - s\mathcal{T}_0)x^2.$$

Finally, we attain Equation (12). By using the same manner, the proof of Equation (13) can be completed. □

Theorem 3.6. *The exponential generating functions of nonnegative and negative subscripted 3PGQs with GTN components are written as follows, respectively:*

$$\sum_{n=0}^{\infty} \mathcal{T}_n \frac{y^n}{n!} = \frac{\widehat{P}\widehat{x_1}e^{x_1 y}}{(x_1-x_2)(x_1-x_3)} + \frac{\widehat{R}\widehat{x_2}e^{x_2 y}}{(x_2-x_1)(x_2-x_3)} + \frac{\widehat{S}\widehat{x_3}e^{x_3 y}}{(x_3-x_1)(x_3-x_2)}, \tag{14}$$

$$\sum_{n=0}^{\infty}\mathcal{T}_{-n}\frac{y^n}{n!}=\frac{\widehat{P}\widehat{x_1}e^{\frac{y}{x_1}}}{(x_1-x_2)(x_1-x_3)}+\frac{\widehat{R}\widehat{x_2}e^{\frac{y}{x_2}}}{(x_2-x_1)(x_2-x_3)}+\frac{\widehat{S}\widehat{x_3}e^{\frac{y}{x_3}}}{(x_3-x_1)(x_3-x_2)}. \tag{15}$$

Proof. With the help of Equation (11), we get:

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_{n}\frac{y^n}{n!}&=\sum_{n=0}^{\infty}\left(\frac{\widehat{P}x_1^n\widehat{x_1}}{(x_1-x_2)(x_1-x_3)}+\frac{\widehat{R}x_2^n\widehat{x_2}}{(x_2-x_1)(x_2-x_3)}+\frac{\widehat{S}x_3^n\widehat{x_3}}{(x_3-x_1)(x_3-x_2)}\right)\frac{y^n}{n!}\\&=\sum_{n=0}^{\infty}\frac{\widehat{P}x_1^n\widehat{x_1}}{(x_1-x_2)(x_1-x_3)}\frac{y^n}{n!}+\sum_{n=0}^{\infty}\frac{\widehat{R}x_2^n\widehat{x_2}}{(x_2-x_1)(x_2-x_3)}\frac{y^n}{n!}+\sum_{n=0}^{\infty}\frac{\widehat{S}x_3^n\widehat{x_3}}{(x_3-x_1)(x_3-x_2)}\frac{y^n}{n!}\\&=\frac{\widehat{P}\widehat{x_1}}{(x_1-x_2)(x_1-x_3)}\sum_{n=0}^{\infty}\frac{(x_1y)^n}{n!}+\frac{\widehat{R}\widehat{x_2}}{(x_2-x_1)(x_2-x_3)}\sum_{n=0}^{\infty}\frac{(x_2y)^n}{n!}+\frac{\widehat{S}\widehat{x_3}}{(x_3-x_1)(x_3-x_2)}\sum_{n=0}^{\infty}\frac{(x_3y)^n}{n!}\\&=\frac{\widehat{P}\widehat{x_1}e^{x_1y}}{(x_1-x_2)(x_1-x_3)}+\frac{\widehat{R}\widehat{x_2}e^{x_2y}}{(x_2-x_1)(x_2-x_3)}+\frac{\widehat{S}\widehat{x_3}e^{x_3y}}{(x_3-x_1)(x_3-x_2)}.\end{aligned}$$

We completed the proof of Equation (14). Since Equation (11) is valid for all integers, by substituting $n\to -n$ we can show Equation (15) similarly. □

Theorem 3.7. *The Poisson generating functions of 3PGQs with GTN components are written as:*

$$e^{-y}\sum_{n=0}^{\infty}\mathcal{T}_{n}\frac{y^n}{n!}=\frac{\widehat{P}\widehat{x_1}e^{x_1y}}{e^y(x_1-x_2)(x_1-x_3)}+\frac{\widehat{R}\widehat{x_2}e^{x_2y}}{e^y(x_2-x_1)(x_2-x_3)}+\frac{\widehat{S}\widehat{x_3}e^{x_3y}}{e^y(x_3-x_1)(x_3-x_2)},$$

$$e^{-y}\sum_{n=0}^{\infty}\mathcal{T}_{-n}\frac{y^n}{n!}=\frac{\widehat{P}\widehat{x_1}e^{\frac{y}{x_1}}}{e^y(x_1-x_2)(x_1-x_3)}+\frac{\widehat{R}\widehat{x_2}e^{\frac{y}{x_2}}}{e^y(x_2-x_1)(x_2-x_3)}+\frac{\widehat{S}\widehat{x_3}e^{\frac{y}{x_3}}}{e^y(x_3-x_1)(x_3-x_2)}.$$

Proof. By using Equation (14) and (15), we get the desired results, since Poisson generating function is written as multiplying the exponential generating function by e^{-y} (cf. also [77]). □

By using the study [72], we construct the following sum formulas for 3PGQs with GTN components in Theorem 3.8 and Theorem 3.10.

Theorem 3.8. *For every $m,n\in\mathbb{N}$, the following summation formulas for 3PGQs with GTN components satisfied:*

(a) $$\sum_{n=0}^{m}\mathcal{T}_n=\frac{\mathcal{T}_{m+3}+(1-r)\mathcal{T}_{m+2}+(1-r-s)\mathcal{T}_{m+1}-\mathcal{T}_2+(r-1)\mathcal{T}_1+(r+s-1)\mathcal{T}_0}{r+s+t-1},$$

(b) $$\sum_{n=0}^{m}\mathcal{T}_{2n}=\frac{(1-s)\mathcal{T}_{2m+2}+(t+rs)\mathcal{T}_{2m+1}+(t^2+rt)\mathcal{T}_{2m}+(s-1)\mathcal{T}_2+(-t-rs)\mathcal{T}_1+(r^2-s^2+rt+2s-1)\mathcal{T}_0}{(r+s+t-1)(r-s+t+1)},$$

(c) $$\sum_{n=0}^{m}\mathcal{T}_{2n+1}=\frac{\begin{array}{l}(r+t)\mathcal{T}_{2m+2}+(s-s^2+t^2+rt)\mathcal{T}_{2m+1}+(t-st)\mathcal{T}_{2m}+(-r-t)\mathcal{T}_2\\+(-1+s+r^2+rt)\mathcal{T}_1+(-t+st)\mathcal{T}_0\end{array}}{(r-s+t+1)(r+s+t-1)},$$

where $r+s+t-1\neq 0$ and $(r-s+t+1)(r+s+t-1)\neq 0$.

Particular Cases: If $s=1$ and $r+t\neq 0$, we get the following summation formulas for special cases of parts (b) and (c) of Theorem 3.8:

(a) $\sum\limits_{n=0}^{m} \mathcal{T}_{2n} = \dfrac{\mathcal{T}_{2m+1} + t\mathcal{T}_{2m} - \mathcal{T}_1 + r\mathcal{T}_0}{r+t}$, (b) $\sum\limits_{n=0}^{m} \mathcal{T}_{2n+1} = \dfrac{\mathcal{T}_{2m+2} + t\mathcal{T}_{2m+1} - \mathcal{T}_2 + r\mathcal{T}_1}{r+t}$.

Theorem 3.9. *For every $m, n \in \mathbb{N}$, the summing formula for 3PGQs with GTN components can be given:*

$$\sum_{n=0}^{m} \mathcal{T}_n = \frac{\mathcal{T}_{m+2} + (1-r)\mathcal{T}_{m+1} + t\mathcal{T}_m + \xi}{\varphi},$$

where

$$\begin{cases} \varphi = r+s+t-1, \\ \Psi = (r+s-1)\,a + (r-1)\,b - c, \\ \xi = \Psi + (\Psi - \varphi a)\,e_1 + (\Psi - \varphi\,(a+b))\,e_2 + (\Psi - \varphi\,(a+b+c))\,e_3. \end{cases}$$

Proof. By using Equation (7) and the Lemma 2.3 on page 6 in the study [6], then the followings are constructed:

$$\begin{aligned} \sum_{n=0}^{m} \mathcal{T}_n &= \sum_{n=0}^{m} T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3 \\ &= \sum_{n=0}^{m} T_n + \sum_{n=0}^{m} T_{n+1}e_1 + \sum_{n=0}^{m} T_{n+2}e_2 + \sum_{n=0}^{m} T_{n+3}e_3 \\ &= \frac{1}{\varphi}\begin{bmatrix} T_{m+2} + (1-r)T_{m+1} + tT_m + \Psi + (T_{m+3} + (1-r)T_{m+2} + tT_{m+1} + \Psi - \varphi a)\,e_1 \\ + (T_{m+4} + (1-r)T_{m+3} + tT_{m+2} + \Psi - \varphi\,(a+b))\,e_2 + (T_{m+5} + (1-r)T_{m+4} + tT_{m+3} + \Psi - \varphi\,(a+b+c))\,e_3 \end{bmatrix} \\ &= \frac{\mathcal{T}_{m+2} + (1-r)\mathcal{T}_{m+1} + t\mathcal{T}_m + \xi}{\varphi}. \end{aligned}$$

□

Theorem 3.10. *For all $m, n \in \mathbb{N}$, the summation formulas for 3PGQs with GTN components are satisfied:*

(a) $\sum\limits_{n=1}^{m} \mathcal{T}_{-n} = \dfrac{(s+t-r)\mathcal{T}_{-m-1} - (s+t)\mathcal{T}_{-m-2} - t\mathcal{T}_{-m-3} + \mathcal{T}_2 + (1-r)\mathcal{T}_1 + (1-r-s)\mathcal{T}_0}{r+s+t-1}$,

(b) $\sum\limits_{n=1}^{m} \mathcal{T}_{-2n} = \dfrac{\begin{array}{l} -(r+t)\mathcal{T}_{-2m+1} + (r^2+rt+s-1)\mathcal{T}_{-2m} + (st-t)\mathcal{T}_{-2m-1} + (1-s)\mathcal{T}_2 + (t+rs)\mathcal{T}_1 \\ +(1-rt-2s-r^2+s^2)\mathcal{T}_0 \end{array}}{(r+s+t-1)(r-s+t+1)}$,

(c) $\sum\limits_{n=1}^{m} \mathcal{T}_{-2n+1} = \dfrac{(s-1)\mathcal{T}_{-2m+1} - (t+rs)\mathcal{T}_{-2m} - (t^2+rt)\mathcal{T}_{-2m-1} + (r+t)\mathcal{T}_2 + (1-r^2-rt-s)\mathcal{T}_1 + (t-st)\mathcal{T}_0}{(r-s+t+1)(r+s+t-1)}$,

where $r+s+t-1 \neq 0$ and $(r-s+t+1)(r+s+t-1) \neq 0$.

Particular Cases: If $s \neq 1$ and $r+t=0$, the following summation formulas for special cases of properties (b) and (c) of Theorem 3.10 satisfied.

(a) $\sum\limits_{n=0}^{m} \mathcal{T}_{-2n} = \dfrac{-\mathcal{T}_{-2m} - t\mathcal{T}_{-2m-1} + \mathcal{T}_2 + t\mathcal{T}_1 + (1-s)\mathcal{T}_0}{s-1}$,

(b) $\sum\limits_{n=0}^{m} \mathcal{T}_{-2n+1} = \dfrac{-\mathcal{T}_{-2m+1} - t\mathcal{T}_{-2m} + \mathcal{T}_1 + t\mathcal{T}_0}{s-1}$.

The following theorem constructs some matrix formulas for 3PGQs with GTN components.

Theorem 3.11. *For every $n \in \mathbb{N}$, the following matrix properties hold for 3PGQs with GTN components:*

(a) $\begin{pmatrix} \mathcal{T}_{n+2} \\ \mathcal{T}_{n+1} \\ \mathcal{T}_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{T}_2 \\ \mathcal{T}_1 \\ \mathcal{T}_0 \end{pmatrix}$, (b) $\begin{pmatrix} \mathcal{T}_{-n} \\ \mathcal{T}_{-n-1} \\ \mathcal{T}_{-n-2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{t} & -\frac{r}{t} & -\frac{s}{t} \end{pmatrix}^n \begin{pmatrix} \mathcal{T}_0 \\ \mathcal{T}_{-1} \\ \mathcal{T}_{-2} \end{pmatrix}$.

Proof. Via the mathematical induction method, we show the proofs for both parts (a) and (b). □

Now, by taking advantage of Theorem 5 on page 5 in study [41], we achieve the following Theorem 3.12 that allows to find the nth and $-(n+1)$th terms of the 3PGQ with GTN components. We skip the proof for the sake of the brevity since it can be done easily by the mathematical induction method via Equations (9) and (10), respectively.

Theorem 3.12. *For every $n \geq 0$, the following $(n+1)\times(n+1)$ determinant equalities*[2)] *satisfied:*

$$\text{(a)}\ \mathcal{T}_n = \begin{vmatrix} \mathcal{T}_0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{T}_1 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{T}_2 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & t & s & r & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & s & r \end{vmatrix}, \qquad \text{(b)}\ \mathcal{T}_{-(n+1)} = \begin{vmatrix} \mathcal{T}_{-1} & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{T}_{-2} & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ \mathcal{T}_{-3} & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{t} & -\frac{r}{t} & -\frac{s}{t} & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{s}{t} & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{r}{t} & -\frac{s}{t} \end{vmatrix}.$$

Definition 3.13. *For $N_{\mathcal{T}_n} > 0$ and $\lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2 \neq 0$, polar representation of 3PGQ with GTN components is written as follows:*

$$\mathcal{T}_n = \sqrt{N_{\mathcal{T}_n}}\left(\cos\theta + \hat{\mathcal{T}}_n \sin\theta\right),$$

where

$$\hat{\mathcal{T}}_n = \frac{1}{\sqrt{\lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2}}(T_{n+1}, T_{n+2}, T_{n+3})$$

and

$$\cos\theta = \frac{T_n}{\sqrt{N_{\mathcal{T}_n}}}, \ \sin\theta = \sqrt{\frac{\lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2}{N_{\mathcal{T}_n}}},$$

where $\hat{\mathcal{T}}_n$ is called generalized unit vector of 3PGQ with GTN components.

Theorem 3.14. *The matrix representation of $\mathcal{T}_n$ can be written as follows:*

$$\mathcal{M}_{\mathcal{T}_n} = \begin{pmatrix} T_n & -\lambda_1\lambda_2 T_{n+1} & -\lambda_1\lambda_3 T_{n+2} & -\lambda_2\lambda_3 T_{n+3} \\ T_{n+1} & T_n & -\lambda_3 T_{n+3} & \lambda_3 T_{n+2} \\ T_{n+2} & \lambda_2 T_{n+3} & T_n & -\lambda_2 T_{n+1} \\ T_{n+3} & -\lambda_1 T_{n+2} & \lambda_1 T_{n+1} & T_n \end{pmatrix}.$$

Here the matrix $\mathcal{M}_{\mathcal{T}_n}$ is called fundamental matrix for 3PGQ with GTN components.

[2)] The Laplace expansion along the last column is used to calculate the determinant, bearing in mind the noncommutativity of 3PGQs, i.e., for any $\left[a_{ij}\right]_{n\times n}$, $\det\left(\left[a_{ij}\right]_{n\times n}\right) = \sum_{i=1}^{n} b_{in}a_{in}$, with $b_{in} = (-1)^{i+n}\det C_{in}$ where $\det C_{in}$ is the i,n minor of $\left[a_{ij}\right]_{n\times n}$.

Proof. Multiplying $\mathcal{T}_n = T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3$ with $1, e_1, e_2, e_3$ from the left side and using Table 1, we get:

$$\begin{aligned}
\mathcal{T}_n 1 &= T_n + T_{n+1}e_1 + T_{n+2}e_2 + T_{n+3}e_3,\\
\mathcal{T}_n e_1 &= -\lambda_1\lambda_2 T_{n+1} + T_n e_1 + \lambda_2 T_{n+3}e_2 - \lambda_1 T_{n+2}e_3,\\
\mathcal{T}_n e_2 &= -\lambda_1\lambda_3 T_{n+2} - \lambda_3 T_{n+3}e_1 + T_n e_2 + \lambda_1 T_{n+1}e_3,\\
\mathcal{T}_n e_3 &= -\lambda_2\lambda_3 T_{n+3} + \lambda_3 T_{n+2}e_1 - \lambda_2 T_{n+1}e_2 + T_n e_3.
\end{aligned}$$

Then, writing the coefficients of $\{1, e_1, e_2, e_3\}$ of the above equations as columns constructs the matrix $\mathcal{M}_{\mathcal{T}_n}$. □

According to the values of $\lambda_{i\in\{1,2,3\}}$, $\mathcal{M}_{\mathcal{T}_n}$ can be classified. For $\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$, the fundamental matrix for 2PGQ with GTN components is given. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$, then the fundamental matrix for split quaternions with GTN components is given. Also, for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then the fundamental matrix for Hamilton quaternions with GTN components is given.

Remark 3.15. *Let $\mathcal{T}_n$ and $\mathcal{T}_m$ be nth and mth 3PGQ with GTN components for al $m, n \in \mathbb{Z}$. Then, the following matrix computation can be given:*

$$\mathcal{M}_{\mathcal{T}_n}(\mathcal{T}_m^*)^{t} = \mathcal{M}_{\mathcal{T}_m}(\mathcal{T}_n^*)^{t} = (\mathcal{T}_n\mathcal{T}_m)^*,$$

where the superscript * represents column matrix forms. Therefore, here $\mathcal{T}_n^* = \begin{pmatrix} T_n & T_{n+1} & T_{n+2} & T_{n+3}\end{pmatrix}^{t}$, $\mathcal{T}_m^* = \begin{pmatrix} T_m & T_{m+1} & T_{m+2} & T_{m+3}\end{pmatrix}^{t}$ *and*

$$(\mathcal{T}_n\mathcal{T}_m)^* = \begin{pmatrix} T_nT_m - \lambda_1\lambda_2 T_{n+1}T_{m+1} - \lambda_1\lambda_3 T_{n+2}T_{m+2} - \lambda_2\lambda_3 T_{n+3}T_{m+3}\\ T_nT_{m+1} + T_{n+1}T_m + \lambda_3 T_{n+2}T_{m+3} - \lambda_3 T_{n+3}T_{m+2}\\ T_nT_{m+2} + T_{n+2}T_m - \lambda_2 T_{n+1}T_{m+3} + \lambda_2 T_{n+3}T_{m+1}\\ T_nT_{m+3} + T_nT_{m+3} + \lambda_1 T_{n+1}T_{n+2} - \lambda_1 T_{n+2}T_{m+1}\end{pmatrix}.$$

Thanks to the Şentürk and Ünal [76], we can give the following Definition 3.16:

Definition 3.16. *Let $\mathcal{T}_n$ be the nth 3PGQ with GTN components for all $n \in \mathbb{Z}$. Then, the following mathematical equations are satisfied:*

- ✻ *The determinant of $\mathcal{M}_{\mathcal{T}_n}$:* $\det(\mathcal{M}_{\mathcal{T}_n}) = N^2_{\mathcal{T}_n}$.
- ✻ *The characteristic polynomial of $\mathcal{M}_{\mathcal{T}_n}$:*

$$P_{\mathcal{M}_{\mathcal{T}_n}}(\mathrm{u}) = \left(\mathrm{u}^2 - 2\mathrm{u}T_n + T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2\right)^2.$$

- ✻ *The characteristic equation of $\mathcal{M}_{\mathcal{T}_n}$:*

$$\det(\mathcal{M}_{\mathcal{T}_n} - \mathrm{u}I_4) = 0 \Leftrightarrow P_{\mathcal{M}_{\mathcal{T}_n}}(\mathrm{u}) = \left(\mathrm{u}^2 - 2\mathrm{u}T_n + T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2\right)^2 = 0.$$

- ✻ *The eigenvalues of $\mathcal{M}_{\mathcal{T}_n}$:*

$$\mathfrak{T}_{1,2} = T_n + \sqrt{-\lambda_1\lambda_2 T_{n+1}^2 - \lambda_1\lambda_3 T_{n+2}^2 - \lambda_2\lambda_3 T_{n+3}^2},\ \mathfrak{T}_{3,4} = T_n - \sqrt{-\lambda_1\lambda_2 T_{n+1}^2 - \lambda_1\lambda_3 T_{n+2}^2 - \lambda_2\lambda_3 T_{n+3}^2}.$$

- ✻ *Multiplication of the eigenvalues of $\mathcal{M}_{\mathcal{T}_n}$:*

$$\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = T_n^2 + \lambda_1\lambda_2 T_{n+1}^2 + \lambda_1\lambda_3 T_{n+2}^2 + \lambda_2\lambda_3 T_{n+3}^2 = N_{\mathcal{T}_n}.$$

✻ *The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ of $\mathcal{M}_{\mathcal{T}_n}$:*

$$\begin{pmatrix} \frac{\lambda_1 T_{n+2}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}-\lambda_1\lambda_2 T_{n+1}T_{n+3}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ \frac{T_{n+3}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}+\lambda_1 T_{n+1}T_{n+2}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\lambda_2 T_{n+3}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}+\lambda_1\lambda_2 T_{n+1}T_{n+2}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ -\frac{T_{n+2}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}-\lambda_2 T_{n+1}T_{n+3}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

✻ *The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ of $\mathcal{M}_{\mathcal{T}_n}$:*

$$\begin{pmatrix} \frac{\lambda_1 T_{n+2}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}+\lambda_1\lambda_2 T_{n+1}T_{n+3}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ -\frac{T_{n+3}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}-\lambda_1 T_{n+1}T_{n+2}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{\lambda_2 T_{n+3}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}-\lambda_1\lambda_2 T_{n+1}T_{n+2}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ \frac{T_{n+2}\sqrt{-\lambda_1\lambda_2 T_{n+1}^2-\lambda_1\lambda_3 T_{n+2}^2-\lambda_2\lambda_3 T_{n+3}^2}+\lambda_2 T_{n+1}T_{n+3}}{\lambda_1 T_{n+2}^2+\lambda_2 T_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

According to the previous expressions, we take over the 3PGQs with third-order Pell numbers components in the following Corollary 3.17. The other special cases will be omitted for brevity since they can be constructed in detail like this corollary.

Corollary 3.17. *Let $\widetilde{u}_n$ and $\widetilde{u}_{-n}$ be the nth and −nth nonnegative and negative subscripted 3PGQ with third-order Pell numbers components such that*

$$\widetilde{u}_n = u_n + u_{n+1}e_1 + u_{n+2}e_2 + u_{n+3}e_3, \quad \widetilde{u}_{-n} = u_{-n} + u_{-n+1}e_1 + u_{-n+2}e_2 + u_{-n+3}e_3, \quad \text{for all} \quad n \in \mathbb{N},$$

with initial values:

$$\begin{cases} \widetilde{u}_0 = e_1 + 2e_2 + 5e_3, \\ \widetilde{u}_1 = 1 + 2e_1 + 5e_2 + 13e_3, \\ \widetilde{u}_2 = 2 + 5e_1 + 13e_2 + 33e_3, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{u}_{-1} = e_2 + 2e_3, \\ \widetilde{u}_{-2} = 1 + 2e_3, \\ \widetilde{u}_{-3} = -1 + e_1. \end{cases}$$

Then the followings can be given:

(a) *The recurrence relations:*

$$\widetilde{u}_n = 2\widetilde{u}_{n-1} + \widetilde{u}_{n-2} + \widetilde{u}_{n-3}, \quad \widetilde{u}_{-n} = -\widetilde{u}_{-(n-1)} - 2\widetilde{u}_{-(n-2)} + \widetilde{u}_{-(n-3)}.$$

(b) *The Binet formula:* $\widetilde{u}_n = \dfrac{x_1^{n+1}\widehat{x_1}}{(x_1-x_2)(x_1-x_3)} + \dfrac{x_2^{n+1}\widehat{x_2}}{(x_2-x_1)(x_2-x_3)} + \dfrac{x_3^{n+1}\widehat{x_3}}{(x_3-x_1)(x_3-x_2)}.$

(c) *The generating functions:*

$$\sum_{n=0}^{\infty} \widetilde{u}_n x^n = \frac{e_1 + 2e_2 + 5e_3 + (1 + e_2 + 3e_3)x + (e_2 + 2e_3)x^2}{1 - 2x - x^2 - x^3},$$

$$\sum_{n=0}^{\infty} \widetilde{u}_{-n} x^n = \frac{e_1 + 2e_2 + 5e_3 + (e_1 + 3e_2 + 7e_3)x + (1 + 2e_1 + 5e_2 + 14e_3)x^2}{1 + x + 2x^2 - x^3}.$$

(d) *The exponential generating functions:*

$$\sum_{n=0}^{\infty}\widetilde{u}_n\frac{y^n}{n!}=\frac{x_1\widehat{x_1}e^{x_1y}}{(x_1-x_2)(x_1-x_3)}+\frac{x_2\widehat{x_2}e^{x_2y}}{(x_2-x_1)(x_2-x_3)}+\frac{x_3\widehat{x_3}e^{x_3y}}{(x_3-x_1)(x_3-x_2)},$$

$$\sum_{n=0}^{\infty}\widetilde{u}_{-n}\frac{y^n}{n!}=\frac{x_1\widehat{x_1}e^{\frac{y}{x_1}}}{(x_1-x_2)(x_1-x_3)}+\frac{x_2\widehat{x_2}e^{\frac{y}{x_2}}}{(x_2-x_1)(x_2-x_3)}+\frac{x_3\widehat{x_3}e^{\frac{y}{x_3}}}{(x_3-x_1)(x_3-x_2)}.$$

(e) *The Poisson generating functions:*

$$e^{-y}\sum_{n=0}^{\infty}\widetilde{u}_n\frac{y^n}{n!}=\frac{x_1\widehat{x_1}e^{x_1y}}{e^y(x_1-x_2)(x_1-x_3)}+\frac{x_2\widehat{x_2}e^{x_2y}}{e^y(x_2-x_1)(x_2-x_3)}+\frac{x_3\widehat{x_3}e^{x_3y}}{e^y(x_3-x_1)(x_3-x_2)},$$

$$e^{-y}\sum_{n=0}^{\infty}\widetilde{u}_{-n}\frac{y^n}{n!}=\frac{x_1\widehat{x_1}e^{\frac{y}{x_1}}}{e^y(x_1-x_2)(x_1-x_3)}+\frac{x_2\widehat{x_2}e^{\frac{y}{x_2}}}{e^y(x_2-x_1)(x_2-x_3)}+\frac{x_3\widehat{x_3}e^{\frac{y}{x_3}}}{e^y(x_3-x_1)(x_3-x_2)}.$$

(f) *The following summation formulas hold:*

- ✻ $\sum_{n=0}^{m}\widetilde{u}_n=\frac{1}{3}(\widetilde{u}_{m+3}-\widetilde{u}_{m+2}-2\widetilde{u}_{m+1}-1-e_2-4e_2-10e_3)$,
- ✻ $\sum_{n=0}^{m}\widetilde{u}_{2n}=\frac{1}{3}(\widetilde{u}_{2m+1}+\widetilde{u}_{2m}-1-e_2-3e_3)$,
- ✻ $\sum_{n=0}^{m}\widetilde{u}_{2n+1}=\frac{1}{3}(\widetilde{u}_{2m+2}+\widetilde{u}_{2m+1}-e_1-3e_2-7e_3)$,
- ✻ $\sum_{n=1}^{m}\widetilde{u}_{-n}=\frac{1}{3}(-4\widetilde{u}_{-m-1}-2\widetilde{u}_{-m-2}-\widetilde{u}_{-m-3}+1+e_1+4e_2+10e_3)$,
- ✻ $\sum_{n=1}^{m}\widetilde{u}_{-2n}=\frac{1}{9}(-\widetilde{u}_{-2m+1}+2\widetilde{u}_{-2m}+1+e_2+3e_3)$,
- ✻ $\sum_{n=1}^{m}\widetilde{u}_{-2n+1}=\frac{1}{9}(-\widetilde{u}_{-2m}-\widetilde{u}_{-2m-1}+e_1+3e_2+7e_3)$.

(g) *For all $m,n\in\mathbb{N}$, the following summation property hold:* $\sum_{n=0}^{m}\widetilde{u}_n=\dfrac{\widetilde{u}_{m+2}-\widetilde{u}_{m+1}+\widetilde{u}_m+(-1-e_1-4e_2-10e_3)}{3}$.

(h) *For all $n\in\mathbb{N}$, the following matrix formulas are satisfied:*

- ✻ $\begin{pmatrix}\widetilde{u}_{n+2}\\ \widetilde{u}_{n+1}\\ \widetilde{u}_n\end{pmatrix}=\begin{pmatrix}2&1&1\\1&0&0\\0&1&0\end{pmatrix}^n\begin{pmatrix}2+5e_1+13e_2+33e_3\\1+2e_1+5e_2+13e_3\\e_1+2e_2+5e_3\end{pmatrix}$,
- ✻ $\begin{pmatrix}\widetilde{u}_{-n}\\ \widetilde{u}_{-n-1}\\ \widetilde{u}_{-n-2}\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&-2&-1\end{pmatrix}^n\begin{pmatrix}e_1+2e_2+5e_3\\e_2+2e_3\\1+2e_3\end{pmatrix}$.

(i) *For all $n\in\mathbb{N}$, the following $(n+1)\times(n+1)$ determinant equalities are satisfied:*

$$\circledast\ \widetilde{u}_n = \begin{vmatrix} e_1+2e_2+5e_3 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1+2e_1+5e_2+13e_3 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 2+5e_1+13e_2+33e_3 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 2 \end{vmatrix},$$

$$\circledast\ \widetilde{u}_{-(n+1)} = \begin{vmatrix} e_2+2e_3 & -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1+2e_3 & 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1+e_1 & 0 & 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ddots & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \ddots & -2 & -1 \end{vmatrix}.$$

(j) *Under the conditions* $N_{\widetilde{u}_n} > 0$ *and* $\lambda_1\lambda_2 u_{n+1}^2 + \lambda_1\lambda_3 u_{n+2}^2 + \lambda_2\lambda_3 u_{n+3}^2 \neq 0$, *polar representation of* $\widetilde{u}_n$ *is written as follows:*

$$\widetilde{u}_n = \sqrt{N_{\widetilde{u}_n}}\left(\cos\theta + \hat{\widetilde{u}}_n \sin\theta\right), \tag{16}$$

where $\hat{\widetilde{u}}_n$ *is called generalized unit vector of 3PGQ with third-order Pell numbers components and defined as*

$$\hat{\widetilde{u}}_n = \frac{1}{\sqrt{\lambda_1\lambda_2 u_{n+1}^2 + \lambda_1\lambda_3 u_{n+2}^2 + \lambda_2\lambda_3 u_{n+3}^2}}(u_{n+1}, u_{n+2}, u_{n+3})$$

and also $\cos\theta = \dfrac{u_n}{\sqrt{N_{\widetilde{u}_n}}}$, $\sin\theta = \sqrt{\dfrac{\lambda_1\lambda_2 u_{n+1}^2 + \lambda_1\lambda_3 u_{n+2}^2 + \lambda_2\lambda_3 u_{n+3}^2}{N_{\widetilde{u}_n}}}$.

(k) *The matrix representation of* $\widetilde{u}_n$ *can be written as follows:*

$$\mathcal{M}_{\widetilde{u}_n} = \begin{pmatrix} u_n & -\lambda_1\lambda_2 u_{n+1} & -\lambda_1\lambda_3 u_{n+2} & -\lambda_2\lambda_3 u_{n+3} \\ u_{n+1} & u_n & -\lambda_3 u_{n+3} & \lambda_3 u_{n+2} \\ u_{n+2} & \lambda_2 u_{n+3} & u_n & -\lambda_2 u_{n+1} \\ u_{n+3} & -\lambda_1 u_{n+2} & \lambda_1 u_{n+1} & u_n \end{pmatrix}.$$

Here the matrix $\mathcal{M}_{\widetilde{u}_n}$ *is called the fundamental matrix for 3PGQ with third-order Pell numbers components.*

(l) *For all* $n \in \mathbb{Z}$, *the following mathematical equations are satisfied:*

- ✻ *The determinant of* $\mathcal{M}_{\widetilde{u}_n}$: $\det\left(\mathcal{M}_{\widetilde{u}_n}\right) = N_{\widetilde{u}_n}^2$.
- ✻ *The characteristic polynomial of* $\mathcal{M}_{\widetilde{u}_n}$:

$$P_{\mathcal{M}_{\widetilde{u}_n}}(\mathfrak{u}) = \left(\mathfrak{u}^2 - 2\mathfrak{u}u_n + u_n^2 + \lambda_1\lambda_2 u_{n+1}^2 + \lambda_1\lambda_3 u_{n+2}^2 + \lambda_2\lambda_3 u_{n+3}^2\right)^2.$$

- ✻ *The characteristic equation of* $\mathcal{M}_{\widetilde{u}_n}$:

$$\det\left(\mathcal{M}_{\widetilde{u}_n} - \mathfrak{u}I_4\right) = 0 \Leftrightarrow P_{\mathcal{M}_{\widetilde{u}_n}}(\mathfrak{u}) = \left(\mathfrak{u}^2 - 2\mathfrak{u}u_n + u_n^2 + \lambda_1\lambda_2 u_{n+1}^2 + \lambda_1\lambda_3 u_{n+2}^2 + \lambda_2\lambda_3 u_{n+3}^2\right)^2 = 0.$$

- ✻ *The eigenvalues of* $\mathcal{M}_{\widetilde{u}_n}$:

$$\mathfrak{T}_{1,2} = u_n + \sqrt{-\lambda_1\lambda_2 u_{n+1}^2 - \lambda_1\lambda_3 u_{n+2}^2 - \lambda_2\lambda_3 u_{n+3}^2},\ \mathfrak{T}_{3,4} = u_n - \sqrt{-\lambda_1\lambda_2 u_{n+1}^2 - \lambda_1\lambda_3 u_{n+2}^2 - \lambda_2\lambda_3 u_{n+3}^2}.$$

✻ *Multiplication of the eigenvalues of $\mathcal{M}_{\widetilde{u}_n}$:*

$$\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = u_n^2 + \lambda_1\lambda_2 u_{n+1}^2 + \lambda_1\lambda_3 u_{n+2}^2 + \lambda_2\lambda_3 u_{n+3}^2 = N_{\widetilde{u}_n}.$$

✻ *The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ of $\mathcal{M}_{\widetilde{u}_n}$:*

$$\begin{pmatrix} \frac{\lambda_1 u_{n+2}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}-\lambda_1\lambda_2 u_{n+1}u_{n+3}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ \frac{u_{n+3}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}+\lambda_1 u_{n+1}u_{n+2}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\lambda_2 u_{n+3}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}+\lambda_1\lambda_2 u_{n+1}u_{n+2}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ -\frac{u_{n+2}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}-\lambda_2 u_{n+1}u_{n+3}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

✻ *The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ of $\mathcal{M}_{\widetilde{u}_n}$:*

$$\begin{pmatrix} \frac{\lambda_1 u_{n+2}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}+\lambda_1\lambda_2 u_{n+1}u_{n+3}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ -\frac{u_{n+3}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}-\lambda_1 u_{n+1}u_{n+2}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\frac{\lambda_2 u_{n+3}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}-\lambda_1\lambda_2 u_{n+1}u_{n+2}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ \frac{u_{n+2}\sqrt{-\lambda_1\lambda_2 u_{n+1}^2-\lambda_1\lambda_3 u_{n+2}^2-\lambda_2\lambda_3 u_{n+3}^2}+\lambda_2 u_{n+1}u_{n+3}}{\lambda_1 u_{n+2}^2+\lambda_2 u_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

Now, let us give a numerical example as follows:

Example 3.18. *Let $\widetilde{u}_6$ be the 6th 3PGQ with third-order Pell numbers components.*

✻ *According to Equation* (16)*, polar representation of $\widetilde{u}_6$ is as follows:*

$$\widetilde{u}_6 = \sqrt{7056 + 45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3}\left(\cos\theta + \hat{\widetilde{u}}_6 \sin\theta\right),$$

where

$$\hat{\widetilde{u}}_6 = \frac{(214, 545, 1388)}{\sqrt{45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3}}$$

is 3*-parameter generalized unit vector of $\widetilde{u}_6$ and*

$$\begin{cases} \cos\theta = \dfrac{84}{\sqrt{7056 + 45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3}}, \\ \sin\theta = \sqrt{\dfrac{45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3}{7056 + 45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3}}. \end{cases}$$

✻ *The following matrix can be constructed as follows:*

$$\mathcal{M}_{\widetilde{u}_6} = \begin{pmatrix} 84 & -214\lambda_1\lambda_2 & -545\lambda_1\lambda_3 & -1388\lambda_2\lambda_3 \\ 214 & 84 & -1388\lambda_3 & 545\lambda_3 \\ 545 & 1388\lambda_2 & 84 & -214\lambda_2 \\ 1388 & -545\lambda_1 & 214\lambda_1 & 84 \end{pmatrix}.$$

✻ *The determinant of $\mathcal{M}_{\widetilde{u}_6}$ is written as:*

$$\det\left(\mathcal{M}_{\widetilde{u}_6}\right) = (7056 + 45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3)^2 = \left(N_{\widetilde{u}_6}\right)^2.$$

✻ *The characteristic polynomial of $\mathcal{M}_{\widetilde{u}_6}$ is given:*

$$P_{\widetilde{u}_6}(\mathfrak{u}) = \left(\mathfrak{u}^2 - 168\mathfrak{u} + 7056 + 45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3\right)^2.$$

✻ *The eigenvalues of $\mathcal{M}_{\widetilde{u}_6}$ are determined:*

$$\mathfrak{T}_{1,2} = 84 + \sqrt{-45796\lambda_1\lambda_2 - 297025\lambda_1\lambda_3 - 1926544\lambda_2\lambda_3}$$

and

$$\mathfrak{T}_{3,4} = 84 - \sqrt{-45796\lambda_1\lambda_2 - 297025\lambda_1\lambda_3 - 1926544\lambda_2\lambda_3}.$$

✻ *The multiplication of the eigenvalues of $\mathcal{M}_{\widetilde{u}_6}$ is:*

$\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = 7056 + 45796\lambda_1\lambda_2 + 297025\lambda_1\lambda_3 + 1926544\lambda_2\lambda_3 = N_{\widetilde{u}_6}$.

✻ *The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ of $\mathcal{M}_{\widetilde{u}_6}$ are expressed:*

$$\begin{pmatrix} \frac{\lambda_1\left(-297032\lambda_2+545\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}\right)}{297025\lambda_1+1926544\lambda_2} & \frac{116630\lambda_1+1388\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}}{297025\lambda_1+1926544\lambda_2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{\lambda_2\left(116630\lambda_1+1388\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}\right)}{297025\lambda_1+1926544\lambda_2} & -\frac{297032\lambda_2+545\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}}{297025\lambda_1+1926544\lambda_2} & 0 & 1 \end{pmatrix}^t.$$

✻ *The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ of $\mathcal{M}_{\widetilde{u}_6}$ are expressed:*

$$\begin{pmatrix} \frac{\lambda_1\left(116630\lambda_2+545\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}\right)}{297025\lambda_1+1926544\lambda_2} & -\frac{-297032\lambda_1+1388\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}}{297025\lambda_1+1926544\lambda_2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} -\frac{\lambda_2\left(-116630\lambda_1+1388\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_1\lambda_2}\right)}{297025\lambda_1+1926544\lambda_2} & \frac{297032\lambda_2+545\sqrt{-45796\lambda_1\lambda_2-297025\lambda_1\lambda_3-1926544\lambda_2\lambda_3}}{297025\lambda_1+1926544\lambda_2} & 0 & 1 \end{pmatrix}^t.$$

Example 3.19. *Let $\widetilde{u}_{-9}$ be the -9th 3PGQ with third-order Pell numbers components.*

✻ *According to Equation* (16)*, polar representation of $\widetilde{u}_{-9}$ is as follows:*

$$\widetilde{u}_{-9} = \sqrt{49 + 256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3}\left(\cos\theta + \hat{\widetilde{u}}_{-9}\sin\theta\right),$$

where

$$\hat{\widetilde{u}}_{-9} = \frac{(16, -6, -3)}{\sqrt{256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3}},$$

is 3*-parameter generalized unit vector of $\widetilde{u}_{-9}$ and*

$$\cos\theta = \frac{-7}{\sqrt{49 + 256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3}}, \quad \sin\theta = \sqrt{\frac{256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3}{49 + 256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3}}.$$

✻ *The following matrix can be obtained as follows:*

$$\mathcal{M}_{\widetilde{u}_{-9}} = \begin{pmatrix} -7 & -16\lambda_1\lambda_2 & 6\lambda_1\lambda_3 & 3\lambda_2\lambda_3 \\ 16 & -7 & 3\lambda_3 & -6\lambda_3 \\ -6 & -3\lambda_2 & -7 & -16\lambda_2 \\ -3 & 6\lambda_1 & 16\lambda_1 & -7 \end{pmatrix}.$$

✻ *The determinant of* $\mathcal{M}_{\widetilde{u}_{-9}}$ *is:* $\det\left(\mathcal{M}_{\widetilde{u}_{-9}}\right) = (49 + 256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3)^2 = \left(N_{\widetilde{u}_{-3}}\right)^2$.

✻ *The characteristic polynomial of* $\mathcal{M}_{\widetilde{u}_{-9}}$ *is written:*

$$P_{\widetilde{u}_{-9}}(\mathfrak{u}) = \left(\mathfrak{u}^2 + 14\mathfrak{u} + 49 + 256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3\right)^2.$$

✻ *The eigenvalues of* $\mathcal{M}_{\widetilde{u}_{-9}}$ *are written:*

$$\mathfrak{T}_{1,2} = -7 + \sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}, \quad \mathfrak{T}_{3,4} = -7 - \sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3},$$

✻ *The multiplication of the eigenvalues of* $\mathcal{M}_{\widetilde{u}_{-9}}$ *is:* $\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = 49 + 256\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 9\lambda_2\lambda_3 = N_{\widetilde{u}_{-9}}$.

✻ *The eigenvectors corresponding to the eigenvalue* $\mathfrak{T}_{1,2}$ *of* $\mathcal{M}_{\widetilde{u}_{-9}}$ *are calculated:*

$$\begin{pmatrix} \frac{6\lambda_1\left(8\lambda_2 - \sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}\right)}{36\lambda_1 + 9\lambda_2} & \frac{96\lambda_1 - 3\sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}}{36\lambda_1 + 9\lambda_2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{3\lambda_2\left(-32\lambda_1 - \sqrt{-25\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}\right)}{36\lambda_1 + 9\lambda_2} & \frac{-48\lambda_2 + 6\sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}}{36\lambda_1 + 9\lambda_2} & 0 & 1 \end{pmatrix}^t.$$

✻ *The eigenvectors corresponding to the eigenvalue* $\mathfrak{T}_{3,4}$ *of* $\mathcal{M}_{\widetilde{u}_{-9}}$ *are expressed:*

$$\begin{pmatrix} \frac{-6\lambda_1\left(8\lambda_2 + \sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}\right)}{36\lambda_1 + 9\lambda_2} & -\frac{-96\lambda_1 + 3\sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}}{36\lambda_1 + 9\lambda_2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{3\lambda_2\left(32\lambda_1 + \sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}\right)}{36\lambda_1 + 9\lambda_2} & \frac{48\lambda_2 - 6\sqrt{-256\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 9\lambda_2\lambda_3}}{36\lambda_1 + 9\lambda_2} & 0 & 1 \end{pmatrix}^t.$$

4. Conclusions

In this study, we investigated the nonnegative and negative subscripted 3PGQs with GTN components and also scrutinized some special cases of them. Additionally, we gave the Maple code of this special number family, determined both some new and classical well-known equations such as; Binet formulas, generating function, exponential generating function, Poisson generating function, summation formulas, polar representation, and matrix equation. Then, we obtained determinant, characteristic polynomial, characteristic equation, eigenvalues, and eigenvectors concerning the matrix representation of 3PGQs with GTN components.

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