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Moore-Penrose m-weak group inverses in rings with involution

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Abstract. In 2024, Mosić et al. defined the Moore-Penrose m-weak group inverse (MP-m-WGI) of a complex matrix by combining the Moore-Penrose inverse with m-weak group inverse in an appropriate way. In this paper, we generalize it to rings with involution and define the MP-m-WGI of an element in rings with involution. Some expressions and characterizations for this generalized inverse are presented. Then, we establish the relationship between the MP-m-WGI and (b, c)-inverse. Finally, we give some equivalent characterizations when the MP-m-WGI coincides with other generalized inverses, such as the Drazin inverse and the pseudo core inverse.

1. Introduction

As a classical generalized inverse, the Moore-Penrose inverse (MP inverse) was introduced by Moore [15] and latter rediscovered independently by Bjerhammar [2] and Penrose [22]. The m-weak group inverse (m-WGI) introduced in [30] is a new type of generalized inverses. The m-WGI covers the core-EP inverse [13], the weak group inverse [25] and the generalized group inverse (or GGI) [6]. For more results of the MP inverse and the m-WGI, readers can see [9, 16–18, 22, 23].

Using the MP inverse and the m-WGI, Mosić et al.[19] defined the Moore-Penrose m-weak group inverse (MP-m-WGI) of a complex matrix, which is very significant as a generalization for the MP weak group inverse [24], the MPD inverse [12, 19] and the dual core inverse [1]. For a complex matrix A and $m \in \mathbb{N}$, the symbols $A^\dagger$, $A^{\textcircled{w}_m}$ and $A^{\textcircled{\dagger}}$ stand for the MP inverse, the m-WGI and the core-EP inverse [13] of A, respectively. The MP-m-WGI of A is defined as

$$A^{\dagger,\textcircled{w}_m} = A^\dagger A^{\textcircled{w}_m} A$$

and presents uniquely determined solution to matrix equations

$$XAX = X, \quad AX = (A^{\textcircled{\dagger}})^{m+1} A^{m+1}, \quad XA = A^\dagger (A^{\textcircled{\dagger}})^{m+1} A^{m+2}.$$

A number of expressions and characterizations of the MP-m-WGI were given.

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Motivated by the work of Mosić above, we put forward the notion of MP-m-WGI in rings with involution as a generalization for both m-WGI in rings and MP-m-WGI for complex matrices.

This paper is organized as follows. In Section 2, we present some necessary definitions and auxiliary lemmas. In Section 3, we define the MP-m-WGI in rings with involution and give some expressions for MP-m-WGI. In Section 4, we investigate the relationship between the MP-m-WGI and other generalized inverses in rings, such as the (b, c)-inverse, the inverse along an element, the Drazin inverse and the pseudo core inverse.

2. Preliminaries

Let R be a ring with involution. An involution $*$ in R is an anti-isomorphism of degree 2, i.e. for any $r, s \in R$,

$$(r^*)^* = r, \quad (rs)^* = s^*r^*, \quad (r+s)^* = r^* + s^*.$$

Definition 2.1. [22] *An element $a \in R$ is said to be Moore-Penrose invertible if there exists $x \in R$ satisfying the following equations*

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa.$$

Such an x is unique when it exists, and is called the Moore-Penrose inverse (MP inverse) of a and denoted by $a^\dagger$.

Moreover, x is called a $\{1\}$-inverse of a (or a is regular) if the equation (1) holds. If x satisfies equations (1) and (3), then x is called a $\{1,3\}$-inverse of a and denoted by $a^{(1,3)}$. If x satisfies equations (1) and (4), then x is called a $\{1,4\}$-inverse of a and denoted by $a^{(1,4)}$.

Definition 2.2. [3] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k\ , \ \ ax^2 = x, \ \ xa = ax,$$

then a is called Drazin invertible. Such an x is unique and denoted by a^D when it exists.

The smallest positive integer k satisfying above equations is called the Drazin index of a, denoted by $\mathrm{i}(a)$. In particular, if $\mathrm{i}(a) = 1$, x is called the group inverse of a and denoted by $a^\#$.

Definition 2.3. [7] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (ax)^* = ax,$$

then x is called the pseudo core inverse of a. It is unique and denoted by $a^{\circledD}$ when the pseudo core inverse exists.

The smallest positive integer k satisfying above equations is called the pseudo core index of a. If a is pseudo core invertible, then it must be Drazin invertible, and the pseudo core index coincides with the Drazin index [7]. In particular, x is called the core inverse of a and denoted by $a^{\circledast}$ when $k = 1$ [1, 23].

The dual pseudo core inverse [7] was defined similarly.

Definition 2.4. [30] *Let $a \in R$ and $m \in \mathbb{N}$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (a^k)^*a^{m+1}x = (a^k)^*a^m,$$

then x is called the m-weak group inverse (m-WGI) of a. When the m-WGI of a exists and is unique, it is denoted by $a^{\circledW_m}$.

The smallest positive integer k satisfying above equations is called the m-weak group index of a. If a is m-weak group invertible, then a is Drazin invertible and the m-weak group index is equal to the Drazin index.

The symbols $R^{\{1\}}$, $R^{\{1,3\}}$, $R^{\{1,4\}}$, $R^{\dagger}$, R^{D}, $R^{ⓦ_m}$, $R^{ⓓ}$, $R_{ⓓ}$ denote sets of all regular, $\{1,3\}$-invertible, $\{1,4\}$-invertible, Moore-Penrose invertible, Drazin invertible, m-weak group invertible, pseudo core invertible and dual pseudo core invertible elements in R, respectively.

Recall that $x \in R$ is a minimal weak Drazin inverse [27] of $a \in R$ if $xa^{k+1} = a^k$ for some $k \in \mathbb{N}$ and $ax^2 = x$. Many generalized inverses such as Drazin inverse, pseudo core inverse, m-WGI and DMP inverse [12] are special cases of minimal weak Drazin inverses. So the following Lemmas 2.5 and 2.6 can efficiently simplify some proofs.

Lemma 2.5. [7] *Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}$ such that*

$$xa^{k+1} = a^k, \quad ax^2 = x,$$

then we have

(1) *$ax = a^m x^m$ for arbitrary positive integer m;*

(2) *$xax = x$;*

(3) *a is Drazin invertible, $a^D = x^{k+1}a^k$ and $\mathrm{i}(a) \leq k$.*

Lemma 2.6. [29] *Let $a \in R^D$ and $k_1, \ldots, k_n, s_1, \ldots, s_n \in \mathbb{N}$. If $x_1, \ldots, x_n$ are minimal weak Drazin inverses of a and $s_n \neq 0$, then*

$$\prod_{i=1}^{n} a^{k_i} x_i^{s_i} = a^k x_n^s, \tag{1}$$

where $k = \sum_{i=1}^{n} k_i$ and $s = \sum_{i=1}^{n} s_i$.

Lemma 2.7. [7] *Let $a \in R$ and $l, k \in \mathbb{N}^+$ with $l \geqslant k$. Then $a \in R^{ⓓ}$ with $\mathrm{i}(a) = k$ if and only if $a \in R^D$ with $\mathrm{i}(a) = k$ and $a^l \in R^{\{1,3\}}$. In this case, $a^{ⓓ} = a^D a^l (a^l)^{(1,3)}$.*

Applying Lemmas 2.6 and 2.7, we get the following corollary immediately.

Corollary 2.8. [20] *Let $a \in R^{ⓓ}$ with $\mathrm{i}(a) = k$ and $l \in \mathbb{N}^+$ with $l \geqslant k$. Then*

$$(a^{ⓓ})^m = (a^D)^m a^l (a^l)^{(1,3)} \text{ for } m \in \mathbb{N}^+.$$

Lemma 2.9. [30] *Let $a \in R$ and $m \in \mathbb{N}$. If $a \in R^{ⓓ}$, then*

$$a^{ⓦ_m} = (a^{ⓓ})^{m+1} a^m. \tag{2}$$

Proof. It follows by [30, Corollaries 4.3, 4.9 and 4.11]. □

Lemma 2.10. [9] *Let $a \in R$. Then*

(1) *$Ra = Ra^*a$ if and only if $a \in R^{\{1,3\}}$;*

(2) *$aR = aa^*R$ if and only if $a \in R^{\{1,4\}}$.*

3. MP-*m*-WGI in rings with involution

In this section, we introduce the MP-m-WGI in R using the MP inverse and the m-WGI, which generalize the MP-m-WGI of a complex matrix.

Theorem 3.1. *Let $a \in R^{\dagger} \cap R^{\textcircled{D}}$ and $m \in \mathbb{N}$. The system of equations*

$$xax = x, \quad ax = (a^{\textcircled{D}})^{m+1}a^{m+1}, \quad xa = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2} \tag{3}$$

has a unique solution: $x = a^{\dagger}a^{\textcircled{W}_m}a = a^{\dagger}aa^{\textcircled{W}_{m+1}} = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}$.

Proof. First, by [30, Proposition 4.8], $(a^{\textcircled{W}_m})^2a = a^{\textcircled{W}_{m+1}}$, then we have

$$a^{\dagger}a^{\textcircled{W}_m}a = a^{\dagger}a(a^{\textcircled{W}_m})^2a = a^{\dagger}aa^{\textcircled{W}_{m+1}}.$$

In addition, it follows from Lemma 2.9 that

$$a^{\dagger}a^{\textcircled{W}_m}a \overset{(2)}{=} a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}.$$

Take $x = a^{\dagger}a^{\textcircled{W}_m}a$. Then by Lemmas 2.5 and 2.6,

$$ax = aa^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1} \overset{(1)}{=} aa^{\dagger}aa^{D}(a^{\textcircled{D}})^{m+1}a^{m+1} = aa^{D}(a^{\textcircled{D}})^{m+1}a^{m+1} \overset{(1)}{=} (a^{\textcircled{D}})^{m+1}a^{m+1},$$

$$xax = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}(a^{\textcircled{D}})^{m+1}a^{m+1} = a^{\dagger}(a^{\textcircled{D}})^{m+1}aa^{\textcircled{D}}a^{m+1} \overset{(1)}{=} a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}$$

and

$$xa = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2}.$$

Therefore, $x = a^{\dagger}a^{\textcircled{W}_m}a = a^{\dagger}aa^{\textcircled{W}_{m+1}} = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}$ is a solution to the system (3).

Next, we prove the uniqueness of the solution. Suppose that x is a solution to the system (3). Then by Lemmas 2.5 and 2.9, we have

$$\begin{aligned} x &= xax = (xa)x = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2}x = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}(ax) \\ &= a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}(a^{\textcircled{D}})^{m+1}a^{m+1} = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1} \overset{(2)}{=} a^{\dagger}a^{\textcircled{W}_m}a. \end{aligned}$$

□

Definition 3.2. *Let $a \in R^{\dagger} \cap R^{\textcircled{D}}$ and $m \in \mathbb{N}$. The* Moore-Penrose m-weak group inverse *(*MP-m-WGI *for short) of a is defined as*

$$a^{\dagger,\textcircled{W}_m} = a^{\dagger}a^{\textcircled{W}_m}a.$$

Similar to the cases of complex matrices in [19], many generalized inverses are special cases of MP-m-WGI in R:

- For $m = 1$, $a^{\dagger,\textcircled{W}_1} = a^{\dagger}a^{\textcircled{W}}a$ is the MPWGI [24];
- For $m = 2$, $a^{\dagger,\textcircled{W}_2} = a^{\dagger}a^{\textcircled{W}_2}a$ is the MP-2-WGI (MPGGI) ;
- For $m \geqslant \mathrm{i}(a)$, $a^{\textcircled{W}_m} = a^{D}$ by [30], $a^{\dagger,\textcircled{W}_m} = a^{\dagger}aa^{D} = a^{\dagger,D}$ is the MPD inverse;
- For $m \geqslant 1 = \mathrm{i}(a)$, $a^{\textcircled{W}_m} = a^{\#}$ and $a^{\dagger,\textcircled{W}_m} = a^{\dagger}aa^{\#}$ is the dual core inverse [28];

The following proposition gives a expression for the MP-$(m+1)$-WGI using the MP-m-WGI and the MPD inverse in R.

Proposition 3.3. *Let $a \in R^{\dagger} \cap R^{\textcircled{D}}$ and $m \in \mathbb{N}$. Then*

$$a^{\dagger,\textcircled{W}_{m+1}} = a^{\dagger,D}a^{\dagger,\textcircled{W}_m}a.$$

Proof. Since $a^D a a^{\textcircled{W}_{m+1}} \overset{(1)}{=} a(a^{\textcircled{W}_{m+1}})^2 = a^{\textcircled{W}_{m+1}}$, it follows that

$$\begin{aligned} a^{\dagger,\textcircled{W}_{m+1}} &= a^\dagger a^{\textcircled{W}_{m+1}} a = a^\dagger a^D a a^{\textcircled{W}_{m+1}} a \\ &= a^\dagger a^D a a^\dagger a a^{\textcircled{W}_{m+1}} a = (a^\dagger a^D a)(a^\dagger a a^{\textcircled{W}_{m+1}})a \\ &= a^{\dagger,D} a^{\dagger,\textcircled{W}_m} a. \end{aligned}$$

□

The following result gives a expression of the MP-m-WGI in R in terms of {1}-inverse.

Proposition 3.4. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then $(a^k)^* a^{k+m+1} \in R^{\{1\}}$ and*

$$a^{\dagger,\textcircled{W}_m} = a^\dagger a^k ((a^k)^* a^{k+m+1})^- (a^k)^* a^{m+1}.$$

Proof. First, since

$$\begin{aligned} &(a^k)^* a^{k+m+1} (a^{k+m+1})^{(1,3)} ((a^k)^{(1,3)})^* (a^k)^* a^{k+m+1} \\ &= (a^k)^* (a^{k+m+1} (a^{k+m+1})^{(1,3)})^* (a^k (a^k)^{(1,3)})^* a^{k+m+1} \\ &= (a^{k+m+1} (a^{k+m+1})^{(1,3)} a^k)^* a^k (a^k)^{(1,3)} a^{k+m+1} = (a^k)^* a^{k+m+1}, \end{aligned}$$

it follows that $(a^k)^* a^{k+m+1} \in R^{\{1\}}$.

Next, taking $p = ((a^k)^* a^{k+m+1})^- (a^k)^* a^{k+m+1}$, we have $p^2 = p$ and

$$\begin{aligned} Rp &= R((a^k)^* a^{k+m+1})^- (a^k)^* a^{k+m+1} = R(a^k)^* a^{k+m+1} \\ &= R(a^k)^* a^k a^{m+1} = R a^k a^{m+1} = R a^k, \end{aligned}$$

where $R(a^k)^* a^k = R a^k$ is obtained from $a^k \in R^{\{1,3\}}$ by Lemmas 2.7 and 2.10. So, $a^k = a^k p$.

Therefore, by Lemma 2.5, we have

$$\begin{aligned} a^{\dagger,\textcircled{W}_m} &= a^\dagger a a^{\textcircled{W}_{m+1}} = a^\dagger a^k (a^{\textcircled{W}_{m+1}})^k = a^\dagger a^k p (a^{\textcircled{W}_{m+1}})^k \\ &= a^\dagger a^k ((a^k)^* a^{k+m+1})^- (a^k)^* a^{k+m+1} (a^{\textcircled{W}_{m+1}})^k \\ &= a^\dagger a^k ((a^k)^* a^{k+m+1})^- (a^k)^* a^{m+2} a^{\textcircled{W}_{m+1}} \\ &= a^\dagger a^k ((a^k)^* a^{k+m+1})^- (a^k)^* a^{m+1}. \end{aligned}$$

□

The following result gives a expression of the MP-m-WGI in R in terms of Drazin inverse and {1, 3}-inverse.

Proposition 3.5. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. If $l \in \mathbb{N}^+$ with $l \geqslant k$, then*

$$a^{\dagger,\textcircled{W}_m} = a^\dagger (a^D)^{m+1} a^l (a^l)^{(1,3)} a^{m+1} = a^\dagger a^l (a^{l+m+1})^{(1,3)} a^{m+1}.$$

Proof. Since $a \in R^{\textcircled{D}}$ and $l \geqslant k$, it follows from Lemma 2.7 that $a^l, a^{l+m+1} \in R^{\{1,3\}}$. Then by Corollary 2.8, we have

$$a^{\textcircled{W}_m} \overset{(2)}{=} (a^{\textcircled{D}})^{m+1} a^m = (a^D)^{m+1} a^l (a^l)^{(1,3)} a^m.$$

In addition, since

$$a^l (a^l)^{(1,3)} R = a^{l+m+1} (a^{l+m+1})^{(1,3)} R,$$

it follows that

$$a^{\textcircled{W}_m} = (a^D)^{m+1}a^l(a^l)^{(1,3)}a^m = (a^D)^{m+1}a^{l+m+1}(a^{l+m+1})^{(1,3)}a^m = a^l(a^{l+m+1})^{(1,3)}a^m.$$

Therefore,

$$a^{\dagger,\textcircled{W}_m} = a^\dagger a^{\textcircled{W}_m} a = a^\dagger(a^D)^{m+1}a^l(a^l)^{(1,3)}a^{m+1} = a^\dagger a^l(a^{l+m+1})^{(1,3)}a^{m+1}.$$

□

A new expression for $a^{\dagger,\textcircled{W}_m}$ can be given in terms of idempotents $e = 1 - aa^{\dagger,\textcircled{W}_m}$ and $f = 1 - a^{\dagger,\textcircled{W}_m}a$.

Theorem 3.6. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ and $m \in \mathbb{N}$. For elements $e = 1 - aa^{\dagger,\textcircled{W}_m} = 1 - aa^{\textcircled{W}_{m+1}}$ and $f = 1 - a^{\dagger,\textcircled{W}_m}a$, the following statements hold:*

(1) $a \pm e \in R^{-1}$ *and* $a \pm f \in R^{-1}$;

(2) $a^{\dagger,\textcircled{W}_m} = (1 - f)(a \pm e)^{-1}(1 - e)$.

Proof. (1) Let $i(a) = k$. First, we have

$$e = 1 - aa^{\dagger,\textcircled{W}_m} = 1 - aa^\dagger aa^{\textcircled{W}_{m+1}} = 1 - aa^{\textcircled{W}_{m+1}}.$$

Notice that $a^{\textcircled{W}_{m+1}}$ is a minimal weak Drazin inverse of a. Then by [27, Theorem 3.10], we have $a \pm e \in R^{-1}$ with

$$(a + e)^{-1} = (a + (1 - aa^{\textcircled{W}_{m+1}}))^{-1} = a^{\textcircled{W}_{m+1}} + (1 - a^{\textcircled{W}_{m+1}}a)\sum_{i=0}^{k-1}(-a)^i,$$

$$(a - e)^{-1} = (a - (1 - aa^{\textcircled{W}_{m+1}}))^{-1} = a^{\textcircled{W}_{m+1}} - (1 - a^{\textcircled{W}_{m+1}}a)\sum_{i=0}^{k-1}a^i.$$

Now, recall the Jacobson's lemma [10]: Let $a, b \in R$. If $1 - ab$ is invertible, then so is $1 - ba$ and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$. Thus, by Jacobson's lemma, $a \pm f \in R^{-1}$ with

$$\begin{aligned}
(a + f)^{-1} &= (a + (1 - a^{\dagger,\textcircled{W}_m}a))^{-1} \\
&= 1 + (a^{\dagger,\textcircled{W}_m} - 1)(a + (1 - aa^{\dagger,\textcircled{W}_m}))^{-1}a \\
&= 1 + (a^{\dagger,\textcircled{W}_m} - 1)(a + e)^{-1}a, \\
(a - f)^{-1} &= (a - (1 - a^{\dagger,\textcircled{W}_m}a))^{-1} \\
&= -1 + (a^{\dagger,\textcircled{W}_m} + 1)(a - (1 - aa^{\dagger,\textcircled{W}_m}))^{-1}a \\
&= -1 + (a^{\dagger,\textcircled{W}_m} + 1)(a - e)^{-1}a.
\end{aligned}$$

(2) It is direct to verify that

$$\begin{aligned}
&(1 - f)(a + e)^{-1}(1 - e) \\
&= a^{\dagger,\textcircled{W}_m}a\Big(a^{\textcircled{W}_{m+1}} + (1 - a^{\textcircled{W}_{m+1}}a)\sum_{i=0}^{k-1}(-a)^i\Big)aa^{\dagger,\textcircled{W}_m} \\
&= a^\dagger aa^{\textcircled{W}_{m+1}}a\Big(a^{\textcircled{W}_{m+1}} + (1 - a^{\textcircled{W}_{m+1}}a)\sum_{i=0}^{k-1}(-a)^i\Big)aa^{\textcircled{W}_{m+1}} \\
&= a^\dagger aa^{\textcircled{W}_{m+1}}aa^{\textcircled{W}_{m+1}}aa^{\textcircled{W}_{m+1}} \\
&= a^\dagger aa^{\textcircled{W}_{m+1}} = a^{\dagger,\textcircled{W}_m},
\end{aligned}$$

where $a^{\textcircled{W}_{m+1}}aa^{\textcircled{W}_{m+1}} = a^{\textcircled{W}_{m+1}}$ is obtained from Lemma 2.5.

Similarly, it can be verified that $(1 - f)(a - e)^{-1}(1 - e) = a^{\dagger,\textcircled{W}_m}$. □

Theorem 3.1 indicates that $a^{\dagger,\textcircled{w}_m}$ is a solution to the system (3). Motivated by [19, Corollary 2.2, Theorem 2.2], the following theorem shows that $a^{\dagger,\textcircled{w}_m}$ is also a solution to the following systems of equations.

Theorem 3.7. *Let $a \in R^{\dagger} \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $x = a^{\dagger,\textcircled{w}_m}$;

(2) $xax = x$, $xa = a^{\dagger}(a^D)^{m+1}a^l(a^l)^{(1,3)}a^{m+2}$ and $ax = (a^D)^{m+1}a^l(a^l)^{(1,3)}a^{m+1}$ for $l \in \mathbb{N}^+$ with $l \geqslant k$;

(3) $xax = x$, $xa = a^{\dagger}a^l(a^{l+m+1})^{(1,3)}a^{m+2}$ and $ax = a^l(a^{l+m+1})^{(1,3)}a^{m+1}$ for $l \in \mathbb{N}^+$ with $l \geqslant k$;

(4) $xax = x$, $axa = (a^{\textcircled{D}})^{m+1}a^{m+2}$, $ax = (a^{\textcircled{D}})^{m+1}a^{m+1}$, $xa = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2}$;

(5) $a^{\dagger}ax = x$, $ax = (a^{\textcircled{D}})^{m+1}a^{m+1}$;

(6) $a^{\dagger}ax = x$, $a^{\dagger}ax = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}$;

(7) $xa^{\dagger}a = x$, $xa^{\dagger} = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}a^{\dagger}$;

(8) $x(a^{\textcircled{D}})^{m+1}a^{m+1} = x$, $xa = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2}$;

(9) $a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2}x = x$, $ax = (a^{\textcircled{D}})^{m+1}a^{m+1}$.

Proof. (1) ⇔ (2) ⇔ (3) follows by Theorem 3.1 and Proposition 3.5.

(1) ⇒ (4) : Suppose $x = a^{\dagger,\textcircled{w}_m}$. Then by Theorem 3.1, x satisfies $xax = x$, $ax = (a^{\textcircled{D}})^{m+1}a^{m+1}$, $xa = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+2}$, and thus $axa = (a^{\textcircled{D}})^{m+1}a^{m+2}$.

(4) ⇒ (1) : It is obvious by Theorem 3.1.

(4) ⇒ (5) : Since $ax = (a^{\textcircled{D}})^{m+1}a^{m+1}$, it follows that $a^{\dagger}ax = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1} = a^{\dagger,\textcircled{w}_m} = x$.

(5) ⇒ (6) : Obviously.

(6) ⇒ (1) : Suppose $a^{\dagger}ax = x$ and $a^{\dagger}ax = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}$. Then

$$x = a^{\dagger}ax = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1} = a^{\dagger,\textcircled{w}_m}.$$

The rest part can be proved similarly. □

Remark 3.8. *Recall from [23, Theorem 2.8] that $Ra^{\dagger} = Ra^*$ and $a^{\dagger}R = a^*R$. So, we obtain more equivalent characterizations for $x = a^{\dagger,\textcircled{w}_m}$ in Theorem 3.7 immediately. For example:*

(6′) $a^{\dagger}ax = x$, $a^*ax = a^*(a^{\textcircled{D}})^{m+1}a^{m+1}$;

(7′) $xa^{\dagger}a = x$, $xa^* = a^{\dagger}(a^{\textcircled{D}})^{m+1}a^{m+1}a^*$.

4. Relationships with other generalized inverses

In this section, we wish to investigate the relationships between the MP-m-WGI and other generalized inverses in R. Before that, recall the following two known definitions.

Definition 4.1. [14] *Let $a, d, x \in R$. Then x is the inverse of a along d if*

$$xad = d = dax \quad \text{and} \quad Rx \subseteq Rd,\ xR \subseteq dR.$$

Definition 4.2. [4] *Let $a, b, c, x \in R$. Then x is called a (b, c)-inverse of a if*

$$x \in bRx \cap xRc \quad \text{and} \quad xab = b,\ \ cax = x.$$

Actually, [4, Proposition 6.1] provided the following equivalent characterization for (b, c)-inverse.

Lemma 4.3. [4] *Let $a, b, c, x \in R$. Then x is a (b, c)-inverse of a if and only if*

$$xax = x,\ \ xR = bR,\ \ Rx = Rc.$$

As proved in [4], the inverse along an element is a particular case of (b,c)-inverse when $b = c$. So according to Lemma 4.3, we obtain the following immediately.

Lemma 4.4. *Let $a, d, x \in R$. Then x is the inverse of a along d if and only if*

$$xax = x, \quad xR = dR, \quad Rx = Rd.$$

The right annihilator of a is denoted by a° and is defined by $a^\circ = \{x \in R : ax = 0\}$. Similarly, the left annihilator of a is the set ${}^\circ a = \{x \in R : xa = 0\}$. The following theorem reveals the relationship between the MP-m-WGI and the (b,c)-inverse in R.

Theorem 4.5. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $x = a^{\dagger,\textcircled{W}_m}$;

(2) *x is the $(a^\dagger a^k, (a^k)^* a^{m+1})$-inverse of a;*

(3) $xax = x, \quad xR = a^\dagger a^k R, \quad Rx = R(a^k)^* a^{m+1}$;

(4) $xax = x, \quad {}^\circ x = {}^\circ(a^\dagger a^k), \quad x^\circ = ((a^k)^* a^{m+1})^\circ$.

Proof. (2) ⇔ (3) follows by Lemma 4.3.

(1) ⇒ (3) : Suppose $x = a^{\dagger,\textcircled{W}_m}$. Then by Theorem 3.1, $xax = x$.

Recall that if $y \in R$ is a minimal weak Drazin inverse of a, then $yR = a^k R$ and $Ry^* = R(a^k)^*$ by [27]. Since $a^{\textcircled{W}_{m+1}}$ and $a^{\textcircled{D}}$ are both minimal weak Drazin inverses of a, it follows that $a^{\textcircled{W}_{m+1}} R = a^k R$ and $R(a^{\textcircled{D}})^* = R(a^k)^*$. Then we have

$$xR = a^{\dagger,\textcircled{W}_m} R = a^\dagger a a^{\textcircled{W}_{m+1}} R = a^\dagger a a^k R = a^\dagger a^k R$$

and

$$\begin{aligned} Rx &= Ra^{\dagger,\textcircled{W}_m} = Ra^\dagger (a^{\textcircled{D}})^{m+1} a^{m+1} = R(a^{\textcircled{D}})^{m+1} a^{m+1} \\ &= Raa^{\textcircled{D}} a^{m+1} = R(a^{\textcircled{D}})^* a^* a^{m+1} = R(a^k)^* a^* a^{m+1} = R(a^k)^* a^{m+1}. \end{aligned}$$

(3) ⇒ (1) : Suppose $xax = x$, $xR = a^\dagger a^k R$ and $Rx = R(a^k)^* a^{m+1}$. From the above proof, we have $a^{\dagger,\textcircled{W}_m}$ satisfies these three equations. Then by the uniqueness of (b,c)-inverse [4], $x = a^{\dagger,\textcircled{W}_m}$.

(3) ⇔ (4) : First, we get that x is regular by $xax = x$. In addition,

$$\begin{aligned} &(a^k)^* a^{m+1} (a^D)^{m+1} ((a^k)^{(1,3)})^* (a^k)^* a^{m+1} \\ &= (a^k)^* a a^D (a^k (a^k)^{(1,3)})^* a^{m+1} \\ &= (a^k)^* a (a^D a^k (a^k)^{(1,3)}) a^{m+1} \\ &= (a^k)^* a a^{\textcircled{D}} a^{m+1} = (a^k)^* a^{m+1}, \end{aligned}$$

which implies that $(a^k)^* a^{m+1}$ is regular. Also, since $a^\dagger a^k (a^D)^k a a^\dagger a^k = a^\dagger a^k (a^D)^k a^k = a^\dagger a^k$, it follows that $a^\dagger a^k$ is regular. Thus, by [23, Lemmas 2.5 and 2.6], the proof is completed. □

Inspired by Theorem 4.5, the following results provide the relationship between the idempotent $aa^{\dagger,\textcircled{W}_m}$ and the (b,c)-inverse, as well as the idempotent $a^{\dagger,\textcircled{W}_m} a$ and the (b,c)-inverse.

Proposition 4.6. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $x = aa^{\dagger,\textcircled{W}_m}$;

(2) *x is the $(a^k, (a^k)^* a^{m+1})$-inverse of 1;*

(3) $x^2 = x, \quad xR = a^k R, \quad Rx = R(a^k)^* a^{m+1}$;

(4) $x^2 = x$, $^{\circ}x = {}^{\circ}(a^k)$, $x^{\circ} = ((a^k)^* a^{m+1})^{\circ}$.

Proof. The proof is similar to Theorem 4.5. □

Proposition 4.7. *Let $a \in R^{\dagger} \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $x = a^{\dagger,\textcircled{W}_m} a$;

(2) *x is the $(a^{\dagger}a^k, (a^k)^* a^{m+2})$-inverse of* 1;

(3) $x^2 = x$, $xR = a^{\dagger}a^k R$, $Rx = R(a^k)^* a^{m+2}$;

(4) $x^2 = x$, $^{\circ}x = {}^{\circ}(a^{\dagger}a^k)$, $x^{\circ} = ((a^k)^* a^{m+2})^{\circ}$.

Proof. The proof is similar to Theorem 4.5. □

Notice that $a \in R^{\dagger} \cap R^{\textcircled{D}}$ in Theorem 4.5, Furthermore, if $a \in R^{\dagger} \cap R^{\textcircled{D}} \cap R_{\textcircled{D}}$, we obtain the following relationship between the MP-m-WGI and the inverse along an element in R.

Theorem 4.8. *Let $a \in R^{\dagger} \cap R^{\textcircled{D}} \cap R_{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $x = a^{\dagger,\textcircled{W}_m}$;

(2) *x is the inverse of a along $a^{\dagger}a^k(a^k)^* a^{m+1}$;*

(3) $xax = x$, $xR = a^{\dagger}a^k(a^k)^* a^{m+1}R$, $Rx = Ra^{\dagger}a^k(a^k)^* a^{m+1}$;

(4) $xax = x$, $^{\circ}x = {}^{\circ}(a^{\dagger}a^k(a^k)^* a^{m+1})$, $x^{\circ} = (a^{\dagger}a^k(a^k)^* a^{m+1})^{\circ}$.

Proof. (2) ⇔ (3) follows by Lemma 4.4.

(1) ⇒ (3) : Suppose $x = a^{\dagger,\textcircled{W}_m}$. Then by Theorem 3.1, $xax = x$.

Since $a \in R^{\textcircled{D}} \cap R_{\textcircled{D}}$, it follows that $a^k \in R^{\{1,3\}} \cap R^{\{1,4\}}$ by Lemma 2.7. Moreover, by Lemma 2.10, we have $Ra^k = R(a^k)^* a^k$ and $Ra^k(a^k)^* = R(a^k)^*$. Thus,

$$a^{\dagger}a^k(a^k)^* a^{m+1}R = a^{\dagger}a^k(a^k)^* R = a^{\dagger}a^k R$$

and

$$Ra^{\dagger}a^k(a^k)^* a^{m+1} = Ra^k(a^k)^* a^{m+1} = R(a^k)^* a^{m+1}.$$

Thus, by Theorem 4.5, $xR = a^{\dagger}a^k(a^k)^* a^{m+1}R$ and $Rx = Ra^{\dagger}a^k(a^k)^* a^{m+1}$.

(3) ⇒ (1) : Suppose $xax = x$, $xR = a^{\dagger}a^k(a^k)^* a^{m+1}R$, $Rx = Ra^{\dagger}a^k(a^k)^* a^{m+1}$. From the above proof, we have $a^{\dagger,\textcircled{W}_m}$ satisfies these three equations. Thus, $x = a^{\dagger,\textcircled{W}_m}$ follows by the uniqueness of the inverse along an element [14] .

(3) ⇔ (4) : First, x is regular by $xax = x$. Moreover, it is direct to verify that

$$a^{\dagger}a^k(a^k)^* a^{m+1}((a^D)^{m+1}((a^k)^{(1,3)})^*(a^k)^{(1,4)}a)a^{\dagger}a^k(a^k)^* a^{m+1} = a^{\dagger}a^k(a^k)^* a^{m+1},$$

which implies that $a^{\dagger}a^k(a^k)^* a^{m+1}$ is regular. Thus, by [23, Lemmas 2.5 and 2.6], the proof is completed. □

Let A be a complex matrix with index k. Recall that A is called k-EP [11] if it satisfies $A^{\dagger}A^k = A^k A^{\dagger}$. Some equivalent characterizations of k-EP matrices are presented in [5]. In addition, Zou et al.[31] proved that A is k-EP if and only if $A^{\dagger}A^{k+1} = A^k = A^{k+1}A^{\dagger}$. As one side case of k-EP matrix, it was proved in [26] that A is left k-EP (or left power-EP) if and only if $A^{\dagger}A^{k+1} = A^k$. Now we have the following results in the ring context.

Lemma 4.9. *Let $a \in R^{\dagger} \cap R^D$ with $\mathrm{i}(a) = k$. If $x, y \in R$ are minimal weak Drazin inverses of a, then $a^{\dagger}ax = y$ if and only if $a^{\dagger}a^{k+1} = a^k$ and $x = y$.*

Proof. Suppose that $a^\dagger ax = y$. Then $a^\dagger a^{k+1} = a^\dagger aa^k = a^\dagger axa^{k+1} = ya^{k+1} = a^k$. In addition, since $ax = aa^\dagger ax = ay$, it follows that $x \overset{(1)}{=} a^D ax = a^D ay = y$.

Conversely, suppose that $a^\dagger a^{k+1} = a^k$ and $x = y$. Then by Lemma 2.5, we have $a^\dagger ax = a^\dagger a^{k+1}x^{k+1} = a^k x^{k+1} = x = y$. □

Applying Lemma 4.9, some equivalent characterizations are given in the following proposition when the MP-m-WGI coincides with the Drazin inverse.

Proposition 4.10. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m, n \in \mathbb{N}$ with $m + 1 < n$. Then the following statements are equivalent:*

(1) $a^{\dagger,\textcircled{w}_m} = a^D$;

(2) $a^\dagger a^{k+1} = a^k$ *and* $a^{\textcircled{w}_{m+1}} = a^D$;

(3) $a^{\dagger,D} = a^D$ *and* $a^{\textcircled{w}_{m+1}}a = aa^{\textcircled{w}_{m+1}}$;

(4) $a^{\dagger,D} = a^{\textcircled{w}_{m+1}}$;

(5) $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{w}_n}$;

(6) $a^{\dagger,\textcircled{w}_{n-1}} = a^{\textcircled{w}_{m+1}}$.

In this case, $a^{\dagger,\textcircled{w}_l} = a^D$ for $l \in \mathbb{N}$ with $l \geqslant m$.

Proof. (1) ⇔ (2) ⇔ (4) follows by Lemma 4.9.

(2) ⇔ (3) : By Lemma 4.9, we get that $a^{\dagger,D} = a^\dagger aa^D = a^D$ is equivalent to $a^\dagger a^{k+1} = a^k$. Then by [30, Theorem 4.13], $a^{\textcircled{w}_{m+1}} = a^D$ is equivalent to $a^{\textcircled{w}_{m+1}}a = aa^{\textcircled{w}_{m+1}}$.

(1) ⇔ (5) : By [30, Theorem 4.13], $a^{\textcircled{w}_{m+1}} = a^D$ is equivalent to $a^{\textcircled{w}_{m+1}} = a^{\textcircled{w}_n}$, which implies that $a^{\dagger,\textcircled{w}_m} = a^D$ is equivalent to $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{w}_n}$ by Lemma 4.9.

(5) ⇔ (6) : It follows from Lemma 4.9 that $a^{\dagger,\textcircled{w}_m} = a^\dagger aa^{\textcircled{w}_{m+1}} = a^{\textcircled{w}_n}$ is equivalent to $a^{\textcircled{w}_{m+1}} = a^{\textcircled{w}_n}$ and $a^\dagger a^{k+1} = a^k$. Similarly, $a^{\textcircled{w}_{m+1}} = a^{\textcircled{w}_n}$ and $a^\dagger a^{k+1} = a^k$ is also equivalent to $a^{\dagger,\textcircled{w}_{n-1}} = a^\dagger aa^{\textcircled{w}_n} = a^{\textcircled{w}_{m+1}}$. Thus, the proof is completed.

In this case, since $a^{\textcircled{w}_{m+1}} = a^D$, it follows from [30, Proposition 4.8] that $a^{\textcircled{w}_{l+1}} = a^D$ for $l \in \mathbb{N}$ with $l \geqslant m$. Thus, $a^{\dagger,\textcircled{w}_l} = a^D$ for $l \in \mathbb{N}$ with $l \geqslant m$. □

Remark 4.11. *For a complex matrix A with index k, it follows from Proposition 4.10 that $A^{\dagger,\textcircled{w}_m} = A^D$ if and only if A is left k-EP (or left power-EP) and $A^{\textcircled{w}_{m+1}} = A^D$. More equivalent conditions are omitted in the complex matrix context.*

Recall that an element $a \in R$ is called $*$-DMP [21] with index k if k is the smallest positive integer such that $(a^k)^\#$ and $(a^k)^\dagger$ exist with $(a^k)^\# = (a^k)^\dagger$. The following proposition presents conditions under which the MP-m-WGI coincides with the pseudo core inverse.

Proposition 4.12. *Let $a \in R^\dagger \cap R^{\textcircled{D}}$ with $\mathrm{i}(a) = k$ and $m \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{D}}$;

(2) *a is $*$-DMP;*

(3) $a^\dagger aa^{\textcircled{D}} = a^{\textcircled{w}_n}$ *for some positive integer n.*

Proof. (1) ⇒ (2) : Suppose $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{D}}$. First, by Lemma 4.9, $a^{\dagger,\textcircled{w}_m} = a^\dagger aa^{\textcircled{w}_{m+1}} = a^{\textcircled{D}}$ if and only if $a^\dagger a^{k+1} = a^k$ and $a^{\textcircled{w}_{m+1}} = a^{\textcircled{D}}$. Then by [30, Corollary 4.14], $a^{\textcircled{w}_{m+1}} = a^{\textcircled{D}}$ if and only if $a^{\textcircled{D}} = a^D$. Thus, a is $*$-DMP by [8, Lemma 2.3].

(2) ⇒ (1) : Suppose a is $*$-DMP. Then by [8, Lemma 2.3], $a^{\textcircled{D}} = a^D$, which is equivalent to $a^{\textcircled{D}} = a^{\textcircled{w}_{m+1}}$ by [30, Corollary 4.14]. Moreover, since $a \in R^\dagger$ and a is $*$-DMP, it follows that a is k-EP by [31, Theorem 3.19], which implies that $a^\dagger a^{k+1} = a^k$. Thus, $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{D}}$.

(2) ⇔ (3) is similar to (1) ⇔ (2).

The proof is completed. □

From the proof of Proposition 4.12, we know that $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{D}}$(or a is $*$-DMP) can imply $a^{\dagger,\textcircled{w}_m} = a^D$. So it is natural to consider whether they are equivalent. However, the following example shows that $a^{\dagger,\textcircled{w}_m} = a^D$ may not imply $a^{\dagger,\textcircled{w}_m} = a^{\textcircled{D}}$(or a is $*$-DMP).

Example 4.13. *Let $R = M_4(\mathbb{Z})$ and take the involution as the transpose, where $\mathbb{Z}$ stands for the set of all integers. Set* $a = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$ *with* $\mathrm{i}(a) = 2$. *By computation, we have*

$$a^{\dagger} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad a^{D} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a^{\textcircled{D}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a^{\textcircled{W}} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $a^{\dagger}a^3 = a^2$ and $a^{\textcircled{W}} = a^D$, which implies that $a^{\dagger,\textcircled{w}_m} = a^D$ for $m \in \mathbb{N}$ by Proposition 4.10. However, since $a^{\textcircled{D}} \neq a^D$, it follows that a is not $*$-DMP by [8, Lemma 2.3].

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