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Crossed products of inner quasidiagonal C^*-algebras

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Abstract. In this paper, we show that if $\mathcal{A}$ is an infinite dimensional separable unital inner quasidiagonal C^*-algebra, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property, and $\mathcal{A}$ is α-simple, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a separable inner quasidiagonal C^*-algebra again.

1. Introduction

Quasidiagonal C^*-algebras have now been studied for more than 30 years. In [26] Voiculescu gave a characterization of quasidiagonal C^*-algebras as following:

Definition 1.1. *A C^*-algebra $\mathcal{A}$ is quasidiagonal if, for every $x_1, x_2, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and a finite-rank projection $p \in B(\mathcal{H})$ such that $\|p\pi(x_i) - \pi(x_i)p\| < \varepsilon$, $\|p\pi(x_i)p\| > \|x_i\| - \varepsilon$ for $1 \le i \le n$, where $B(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}$.*

In [2] Blackadar and Kirchberg study NF algebras and strong NF algebras. To see the difference between the class of NF algebras and the class of strong NF algebras, Blackadar and Kirchberg introduce the concept of inner quasidiagonal by modifying Voiculescu's characterization of quasidiagonal C^*-algebras:

Definition 1.2 (Definition 2.2 of [3]). *A C^*-algebra $\mathcal{A}$ is inner quasidiagonal if and only if for every $x_1, x_2, \cdots, x_n \in \mathcal{A}$ and $\varepsilon > 0$, there is a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a finite-rank projection $p \in \pi(\mathcal{A})'' \subset B(\mathcal{H})$ with $\|p\pi(x_j)p\| > \|x_j\| - \varepsilon$ and $\|[p, \pi(x_j)]\| < \varepsilon$ for all j, where $B(\mathcal{H})$ is the C^*-algebra of all bounded linear operators on $\mathcal{H}$.*

It was shown that a separable C^*-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal [3]. Blackadar and Kirchberg also gave examples of separable nuclear C^*-algebras which are quasidiagonal but not inner quasidiagonal, hence NF algebras can be not strong NF.

Next we recall some permanence results of inner quasidiagonal C^*-algebras.

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Theorem 1.3 (Theorem 2.3 of [28]). *If $\mathcal{A}$ is a separable inner quasidiagonal C^*-algebra, then so is $p\mathcal{A}p$ for every non-zero projection $p \in \mathcal{A}$.*

Corollary 1.4. *If $\mathcal{A}$ is a separable inner quasidiagonal C^*-algebra and $p \in \mathcal{A}$ is a non-zero projection, then $M_n \otimes p\mathcal{A}p$ is inner quasidiagonal for every $n \in \mathbb{N}$.*

Proof. It is obvious that matrix algebra M_n is inner quasidiagonal. By Proposition 6.1 of [4] and Theorem 1.3 $M_n \otimes p\mathcal{A}p$ is inner quasidiagonal. □

The Rokhlin property in ergodic theory was adopted to the context of von Neumann algebras by Connes in [6]. The Rokhlin property (see Definition 2.3) for actions on C^*-algebras appeared in [12, 14–18]. This property is useful to understand the structure of the crossed product of C^*-algebras and properties passing from the underlying algebra to the crossed product [22]. However, actions with the Rokhlin property are rare and many C^*-algebras admit no finite group actions with the Rokhlin property. Indeed, the Rokhlin property imposes severe K-theoretical obstructions on C^*-algebras. So in [23] Phillips introduced the tracial Rokhlin property for finite group actions on simple unital C^*-algebras (see Definition 2.4). The tracial Rokhlin property is generic in many cases (see [24]), and also can be used to study properties passing from the underlying algebra to the crossed product [1, 7, 9, 23]. As pointed out in [27], although Phillips' definition of tracial Rokhlin property makes sense for non-simple C^*-algebras, it may be too strong to be distinctive from the Rokhlin property. Therefore, some weak versions of tracial Rokhlin properties are introduced for non-simple C^*-algebras (see [10, 11, 13, 27]), when C^*-algebra is simple, these versions of weak tracial Rokhlin properties all imply tracial Rokhlin properties. In this paper, we adopt the definition of the weak tracial Rokhlin property (see Definition 2.5) in [27].

In [28], the following result has been proven:

Theorem 1.5 (Theorem 4.1 of [28]). *If $\mathcal{A}$ is a separable inner quasidiagonal C^*-algebra, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of a finite group G on $\mathcal{A}$ which has the Rokhlin property, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a separable inner quasidiagonal C^*-algebra again.*

As mentioned earlier, actions with the Rokhlin property are rare, while the tracial Rokhlin property is generic in many cases. So in this paper, we study the crossed products of inner quasidiagonal C^*-algebras which has the weak tracial Rokhlin action.

The organization of the paper is as follows. In Section 2, we recall some definitions and fix some notations. We also introduce strongly tracially inner quasidiagonal C^*-algebras and show that each strongly tracially inner quasidiagonal C^*-algebra is an inner quasidiagonal C^*-algebra. In Section 3, we show that if $\mathcal{A}$ is an infinite dimensional separable unital inner quasidiagonal C^*-algebra with Property (SP), and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a simple separable unital tracially inner quasidiagonal C^*-algebra when $\mathcal{A}$ is α-simple. In the last section, we prove that if $\mathcal{A}$ is an infinite dimensional separable unital inner quasidiagonal C^*-algebra, and $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a separable inner quasidiagonal C^*-algebra when $\mathcal{A}$ is α-simple.

2. Preliminaries

We will use the following convention:

(i) Let $\mathcal{A}$ be a C^*-algebra. We denote by $\mathrm{Aut}(\mathcal{A})$ the automorphism group of $\mathcal{A}$. Denote by $\mathcal{A}_+$ the set of all positive elements in $\mathcal{A}$. Denote by $\mathrm{Her}(a)$ the hereditary C^*-subalgebra generated by $a \in \mathcal{A}_+$.

(ii) Let $x \in \mathcal{A}$, $\varepsilon > 0$ and $F \subseteq \mathcal{A}$. We write $x \in_\varepsilon F$, if $\mathrm{dist}(x, F) < \varepsilon$, or there is $y \in F$ such that $\|x - y\| < \varepsilon$.

(iii) Let $\mathcal{A}$ be a C^*-algebra and $\alpha \in \mathrm{Aut}(\mathcal{A})$. We say $\mathcal{A}$ is *α-simple* if $\mathcal{A}$ does not have any non-trivial α-invariant closed two-sided ideals.

(iv) Let $\mathcal{A}$ be a unital C^*-algebra and $a, b \in \mathcal{A}_+$. We say $a \sim b$ if there exists an element $x \in \mathcal{A}$ such that $a = x^*x$ and $\mathrm{Her}(xx^*) = \mathrm{Her}(b)$. We say $a \precsim b$ if there exists $b' \in \mathrm{Her}(b)$ such that $a \sim b'$ ([21]). Here Blackadar's comparison is a generalization of Murray-von Neumann comparison for projections. In particular, let $a \in \mathcal{A}$ be a positive element and let $p \in \mathcal{A}$ be a projection. Then $p \precsim a$ if and only if there is a projection $q \in \overline{a\mathcal{A}a}$ and a partial isometry $v \in \mathcal{A}$ such that $v^*v = p$ and $vv^* = q$, i.e., p is Murray-von Neumann equivalent to a projection in $\overline{a\mathcal{A}a}$.

Let $\mathscr{C}$ be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebras in $\mathscr{C}$, denoted by TA$\mathscr{C}$, is defined as follows:

Definition 2.1 (Definition 2.2 of [8]). *A unital C^*-algebra $\mathcal{A}$ is said to belong to the class TA$\mathscr{C}$ if for any $\varepsilon > 0$, any finite set $F \subseteq \mathcal{A}$, and any non-zero $a \in \mathcal{A}_+$, there exist a non-zero projection $p \in \mathcal{A}$ and a C^*-subalgebra $C \subseteq \mathcal{A}$ such that $C \in \mathscr{C}$, $1_C = p$, and for all $x \in F$,*
(1) $\|xp - px\| < \varepsilon$,
(2) $pxp \in_\varepsilon C$,
(3) $1 - p \precsim a$.

In particular, in this paper we let $\mathscr{C}$ be the class of inner quasidiagonal C^*-algebras. We say that $\mathcal{A}$ is a *tracially inner quasidiagonal C^*-algebra* if $\mathcal{A} \in$ TA$\mathscr{C}$. We say that $\mathcal{A}$ is a *strongly tracially inner quasidiagonal C^*-algebra* if $\mathcal{A} \in$ TA$\mathscr{C}$ and it also needs to satisfy that $\|pxp\| > \|x\| - \varepsilon$ for all $x \in F$ in Definition 2.1.

Theorem 2.2. *Each strongly tracially inner quasidiagonal C^*-algebra $\mathcal{A}$ is an inner quasidiagonal C^*-algebra.*

Proof. Since $\mathcal{A}$ is a strongly tracially inner quasidiagonal C^*-algebra, then for any finite set $F = \{x_1, x_2, \cdots, x_n\} \subset \mathcal{A}$, any $\varepsilon > 0$ and any non-zero $a_0 \in \mathcal{A}_+$, there exist an inner quasidiagonal C^*-algebra $C \subseteq \mathcal{A}$ such that $1_C = p$, and for all $x \in F$,
(1) $\|xp - px\| < \frac{\varepsilon}{6}$,
(2) $pxp \in_{\frac{\varepsilon}{6}} C$, and $\|pxp\| > \|x\| - \frac{\varepsilon}{6}$,
(3) $1 - p \precsim a_0$.
Now we can find $c_1, c_2, \cdots, c_n \in C$ such that $\|px_ip - c_i\| < \frac{\varepsilon}{6}$ for $i = 1, 2, \ldots, n$. Since C is an inner quasidiagonal C^*-algebra, there exist a representation π of C on a Hilbert space $\mathcal{H}$, and a projection $q \in \pi(C)'' \subseteq B(\mathcal{H})$ such that

$$\|q\pi(c_i) - \pi(c_i)q\| < \frac{\varepsilon}{6}, \quad \|q\pi(c_i)q\| > \|c_i\| - \frac{\varepsilon}{6}.$$

Without loss of generality, we can assume that π is nondegenerate. Define $\widetilde{\pi} = \pi \oplus 0 : C + (1-p)\mathcal{A}(1-p) \to B(\mathcal{H})$ by $\widetilde{\pi}(a + (1-p)b(1-p)) = \pi(a)$ for all $a \in C$ and $b \in \mathcal{A}$. Then $\widetilde{\pi}$ is also nondegenerate and $q \in \widetilde{\pi}(C)'' \subseteq B(\mathcal{H})$. By Theorem 5.5.1 of [20] there exist a Hilbert space $\mathcal{H}_1$, a subspace $\mathcal{H}_1'$ of $\mathcal{H}_1$, a nondegenerate representation ρ of $\mathcal{A}$ and a unitary operator $U : \mathcal{H} \to \mathcal{H}_1'$ such that $\rho(a) = U\widetilde{\pi}(a)U^*$ for all $a \in C + (1-p)\mathcal{A}(1-p)$. Let $\widetilde{q} = UqU^*$. Then $\widetilde{q}$ is a projection acting on $\mathcal{H}_1'$ and it can be extended naturally to a projection acting on $\mathcal{H}_1$. To simplify the notation, we still denote the extended projection by $\widetilde{q}$.

Note that $\|x_i - (px_ip + (1-p)x_i(1-p))\| \leq \|px_i(1-p) + (1-p)x_ip\| < \frac{\varepsilon}{3}$. Then

$$\begin{aligned}
\|\widetilde{q}\rho(x_i) - \rho(x_i)\widetilde{q}\| &\leq \|\widetilde{q}\rho(x_i) - \widetilde{q}\rho(c_i)\| + \|\widetilde{q}\rho(c_i) - \rho(c_i)\widetilde{q}\| + \|\rho(c_i)\widetilde{q} - \rho(x_i)\widetilde{q}\| \\
&\leq \|x_i - c_i\| + \|\widetilde{q}\rho(c_i) - \rho(c_i)\widetilde{q}\| + \|c_i - x_i\| \\
&= 2\|x_i - c_i\| + \|UqU^*U\widetilde{\pi}(c_i)U^* - U\widetilde{\pi}(c_i)U^*UqU^*\| \\
&= 2\|x_i - c_i\| + \|Uq\pi(c_i)U^* - U\pi(c_i)qU^*\| \\
&< \frac{\varepsilon}{2} < \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\|\widetilde{q}\rho(x_i)\widetilde{q}\| &= \|\widetilde{q}\rho(c_i)\widetilde{q}+\widetilde{q}\rho(x_i)\widetilde{q}-\widetilde{q}\rho(c_i)\widetilde{q}\| \\
&\geq \|\widetilde{q}\rho(c_i)\widetilde{q}\| - \|\widetilde{q}\rho(x_i)\widetilde{q}-\widetilde{q}\rho(c_i)\widetilde{q}\| \\
&= \|\widetilde{q}\rho(c_i)\widetilde{q}\| - \|\widetilde{q}\rho(px_ip+px_i(1-p)+(1-p)x_ip+(1-p)x_i(1-p))\widetilde{q}-\widetilde{q}\rho(c_i)\widetilde{q}\| \\
&= \|\widetilde{q}\rho(c_i)\widetilde{q}\| - \|\widetilde{q}\rho(px_ip-c_i)\widetilde{q}+\widetilde{q}\rho(px_i(1-p)+(1-p)x_ip)\widetilde{q}+\widetilde{q}\rho((1-p)x_i(1-p))\widetilde{q}\| \\
&\geq \|\widetilde{q}\rho(c_i)\widetilde{q}\| - (\frac{\varepsilon}{6}+2\cdot\frac{\varepsilon}{6}+0) \\
&\geq \|Uq\widetilde{\pi}(c_i)qU^*\| - \frac{\varepsilon}{2} \\
&= \|q\pi(c_i)q\| - \frac{\varepsilon}{2} \\
&> \|c_i\| - \frac{2\varepsilon}{3} \\
&= \|px_ip+c_i-px_ip\| - \frac{2\varepsilon}{3} \\
&\geq \|px_ip\| - \|c_i-px_ip\| - \frac{2\varepsilon}{3} \\
&> \|px_ip\| - \frac{5\varepsilon}{6} \\
&> \|x_i\| - \varepsilon.
\end{aligned}$$

It is a straightforward calculation to show that $\widetilde{q} \in \rho(\mathcal{A})''$. Therefore, $\mathcal{A}$ is inner quasidiagonal. □

Definition 2.3 (Definition 1.1 of [23] or [15]). *Let $\mathcal{A}$ be a separable unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. We say that α has the Rokhlin property if for every finite set $F \subset \mathcal{A}$, and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in \mathcal{A}$ for $g \in G$ such that:*
(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,
(3) $\sum_{g\in G} e_g = 1$.

Definition 2.4 (Definition 1.2 of [23]). *Let $\mathcal{A}$ be an infinite dimensional simple separable unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. We say that α has the tracial Rokhlin property if for every finite set $F \subset \mathcal{A}$, every $\varepsilon > 0$, and every positive element $x \in \mathcal{A}$ with $\|x\| = 1$, there are mutually orthogonal projection $e_g \in \mathcal{A}$ for $g \in G$ such that:*
(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,
(3) With $e = \sum_{g\in G} e_g$, $1 - e \precsim x$,
(4) With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

It is obvious that α has the Rokhlin property implies that α has the tracial Rokhlin property.

An element a in a C^*-algebra $\mathcal{A}$ is said to be *full* if the closed two-sided ideal generated by a is the whole C^*-algebra $\mathcal{A}$. Next we give the definition of the weak tracial Rokhlin property.

Definition 2.5 (Definition 4.2 of [27]). *Let $\mathcal{A}$ be an infinite dimensional separable unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. We say that α has the weak tracial Rokhlin property if for every finite set $F \subset \mathcal{A}$, every $\varepsilon > 0$, every positive element $b \in \mathcal{A}$ with $\|b\| = 1$ and every full positive element $x \in \mathcal{A}$, there are mutually orthogonal projections $e_g \in \mathcal{A}$ for $g \in G$ such that:*
(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,
(3) With $e = \sum_{g\in G} e_g$, $1 - e \precsim x$,
(4) With e as in (3), we have $\|ebe\| > 1 - \varepsilon$.

By Proposition 4.4 of [27] the weak tracial Rokhlin property coincides with the original tracial Rokhlin property in the simple C^*-algebra case.

Lemma 2.6 (Lemma 4.3 of [27]). *Let $\mathcal{A}$ be an infinite dimensional separable unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$. If α has the weak tracial Rokhlin property, then for every finite set $F \subset \mathcal{A}$, every $\varepsilon > 0$, every positive element $b \in \mathcal{A}$ with $\|b\| = 1$ and every full positive element $x \in \mathcal{A}$, there are mutually orthogonal projections $e_g \in \mathcal{A}$ for $g \in G$ such that:*
(1) $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$,
(2) $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$,
(3) With $e = \sum_{g\in G} e_g$, e is α-invariant, $1 - e \precsim x$,
(4) With e as in (3), we have $\|ebe\| > 1 - \varepsilon$.

3. Tracially inner quasidiagonality of crossed product C^*-algebras

Let's first recall some of the results that will be used below.

Lemma 3.1 (Lemma 4.11 of [27]). *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a finite group action which has the weak tracial Rokhlin property. Then $\mathcal{A}$ is α-simple if and only if the crossed product $C^*(G, \mathcal{A}, \alpha)$ is simple.*

We say that a C^*-algebra $\mathcal{A}$ has the *Property (SP)*, if every non-zero hereditary C^*-subalgebra of $\mathcal{A}$ contains a non-zero projection.

Lemma 3.2 (Lemma 4.12 of [27]). *Let $\mathcal{A}$ be an unital C^*-algebra, and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be a finite group action which has the weak tracial Rokhlin property. If $\mathcal{A}$ is α-simple, then either $\mathcal{A}$ has Property (SP) or α has the Rokhlin property.*

Lemma 3.3 (Lemma 4.16 of [27]). *Let $\mathcal{A}$ be a C^*-algebra with Property (SP), and let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action which has the weak tracial Rokhlin property. Then the crossed product $C^*(\mathcal{A}, G, \alpha)$ also has Property (SP). Moreover, every non-zero hereditary C^*-subalgebra of $C^*(\mathcal{A}, G, \alpha)$ has a non-zero projection which is equivalent to some projection in $\mathcal{A}$ in the sense of Murray-von Neumann.*

Lemma 3.4 (Lemma 3.5.7 of [19]). *Let $\mathcal{A}$ be a non-elementary simple C^*-algebra with Property (SP). Then, for any non-zero projection p in $\mathcal{A}$ and any integer $n \geq 1$, there are n mutually orthogonal sub-projections $p_1, p_2, \cdots, p_n$ of p which are mutually Murray-von Neumann equivalent.*

Lemma 3.5 (Lemma 4.9 of [8]). *Let $\mathcal{A}$ be a simple C^*-algebra with the Property (SP). Then, for any finite set of projections $\{p_1, \ldots, p_n\}$, there is a non-zero sub-projection e of p_1 such that e is Murray-von Neumann equivalent to a sub-projection of p_i for all $1 \leq i \leq n$.*

Lemma 3.6 (Lemma 2.1 of [23]). *Let $n \in \mathbb{N}$. For every $\varepsilon > 0$, there is a $\delta > 0$ such that, whenever $(e_{j,k})_{1\leq j,k\leq n}$ is a system of matrix units for M_n, whenever $\mathcal{B}$ is a unital C^*-algebra, and whenever $w_{j,k}$, for $1 \leq j,k \leq n$, are elements of $\mathcal{B}$ such that $\|w^*_{j,k} - w_{k,j}\| < \delta$ for $1 \leq j,k \leq n$, such that $\|w_{j_1,k_1} w_{j_2,k_2} - \delta_{j_2,k_1} w_{j_1,k_2}\| < \delta$ for $1 \leq j_1, j_2, k_1, k_2 \leq n$, and such that the $w_{j,j}$ are orthogonal projections with $\sum_{j=1}^n w_{j,j} = 1$, then there exists a unital homomorphism $\varphi : M_n \to \mathcal{B}$ such that $\varphi(e_{j,j}) = w_{j,j}$ for $1 \leq j \leq n$ and $\|\varphi(e_{j,k}) - w_{j,k}\| < \varepsilon$ for $1 \leq j,k \leq n$.*

Theorem 3.7. *Let $\mathcal{A}$ be an infinite dimensional separable unital inner quasidiagonal C^*-algebra with Property (SP). Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property. If $\mathcal{A}$ is α-simple, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a simple separable unital tracially inner quasidiagonal C^*-algebra.*

Proof. Let $\mathcal{B} = C^*(G, \mathcal{A}, \alpha)$. By Lemma 3.1 $\mathcal{B}$ is simple. It suffices to consider a finite set of the form $S = F \cup \{u_g : g \in G\}$, where F is a finite set of the unit ball of $\mathcal{A}$ and $u_g \in \mathcal{B}$ is the canonical unitary implementing the automorphism α_g. So let $F \subset \mathcal{A}$ be a finite set with $\|a\| \leq 1$ for all $a \in F$, let $\varepsilon > 0$, and let $x \in \mathcal{B}$ be a non-zero positive element. Set $n = \mathrm{card}(G)$. By Lemma 3.3 $\mathcal{B}$ has Property (SP). Since $\mathcal{B}$ is infinite dimensional unital simple, it is non-elementary. By Lemma 3.4 we can find n non-zero mutually orthogonal and equivalent projections $p_1, p_2, \cdots, p_n$ in $\overline{x\mathcal{B}x}$. Then by Lemma 3.3 again, we can find a projection $q \in \mathcal{A}$ such that $q \precsim p_1$. For finite set of projections $\{\alpha_{g^{-1}}(q) \,|\, g \in G\}$, by Lemma 3.5 there is a non-zero sub-projection p of q such that p is Murray-von Neumann equivalent to a sub-projection of $\alpha_{g^{-1}}(q)$ for all $g \in G$. So $\alpha_g(p) \precsim p_1$ for all $g \in G$.

Since $\mathcal{B}$ is simple, p is full in $\mathcal{B}$, by Lemma 3.3 of [25] there are $s_1, s_2, \cdots, s_m$ in $\mathcal{B}$ such that $\sum_{j=1}^m s_j p s_j^* \geq 1_{\mathcal{B}}$. We write $s_j = \sum_{g\in G} t_{j,g} u_g$, where $t_{j,g} \in \mathcal{A}$ for all $j \in \{1, 2, \ldots, m\}$, $g \in G$. Then

$$\sum_{j=1}^m s_j p s_j^* = \sum_{j=1}^m (\sum_{g\in G} t_{j,g} u_g) p (\sum_{g'\in G} t_{j,g'} u_{g'})^* = \sum_{j=1}^m \sum_{g\in G} \sum_{g'\in G} t_{j,g} u_g p (u_{g'})^* t_{j,g'}^*.$$

Since $s^*ps + t^*pt - (s^*pt + t^*ps) = (s^* - t^*)p(s - t) \geq 0$, we have that $s^*pt + t^*ps \leq s^*ps + t^*pt$ for all $s, t \in \mathcal{B}$. Thus

$$\sum_{j=1}^m \sum_{g\in G} \sum_{g'\in G} t_{j,g} u_g p (u_{g'})^* t_{j,g'}^* \leq n \sum_{j=1}^m \sum_{g\in G} t_{j,g} u_g p (u_g)^* t_{j,g}^* = n \sum_{j=1}^m \sum_{g\in G} t_{j,g} \alpha_g(p) t_{j,g}^*.$$

Let $P = \sum_{g\in G} \alpha_g(p)$. Then

$$n \sum_{j=1}^m \sum_{g\in G} t_{j,g} P t_{j,g}^* \geq n \sum_{j=1}^m \sum_{g\in G} t_{j,g} \alpha_g(p) t_{j,g}^* \geq 1.$$

By Lemma 3.3 of [25] P is full in $\mathcal{A}$.

Set $\varepsilon_0 = \varepsilon/n$. Choose $\delta > 0$ according to Lemma 3.6 for n as given and for ε_0 in place of ε. Apply Lemma 2.6 to α, with F as given, with δ in place of ε, and with P in place of x, obtaining projections $e_g \in \mathcal{A}$ for $g \in G$. Set $e = \sum_{g\in G} e_g$. By construction, $u_g e u_g^* = \alpha_g(e) = e$ for every $g \in G$. Also, for $a \in F$ we have $\|ea - ae\| \leq \sum_{g\in G} \|e_g a - a e_g\| < n\varepsilon_0$.

Define $w_{g,h} = u_{gh^{-1}} e_h$ for $g, h \in G$. We claim that the $w_{g,h}$ form a δ-approximate system of $n \times n$ matrix units in $e\mathcal{B}e$. We estimate

$$\|w_{g,h}^* - w_{h,g}\| = \|e_h u_{gh^{-1}}^* - u_{hg^{-1}} e_g\| = \|u_{gh^{-1}} e_h u_{gh^{-1}}^* - e_g\| = \|\alpha_{gh^{-1}}(e_h) - e_g\| < \delta.$$

Also, using $e_g e_h = \delta_{g,h} e_h$ at the second step,

$$\begin{aligned} \|w_{g_1,h_1} w_{g_2,h_2} - \delta_{g_2,h_1} w_{g_1,h_2}\| &= \|u_{g_1h_1^{-1}} e_{h_1} u_{g_2h_2^{-1}} e_{h_2} - \delta_{g_2,h_1} u_{g_1h_2^{-1}} e_{h_2}\| \\ &= \|u_{g_1h_1^{-1}} e_{h_1} u_{g_2h_2^{-1}} e_{h_2} - u_{g_1h_1^{-1}g_2h_2^{-1}} e_{h_2g_2^{-1}h_1} e_{h_2}\| \\ &= \|u_{g_1h_1^{-1}g_2h_2^{-1}} (u_{g_2h_2^{-1}}^* e_{h_1} u_{g_2h_2^{-1}} - e_{h_2g_2^{-1}h_1}) e_{h_2}\| < \delta. \end{aligned}$$

Finally, $\sum_{g\in G} w_{g,g} = \sum_{g\in G} e_g = e$.

Let $(v_{g,h})_{g,h\in G}$ be a system of matrix units for M_n. By the choice of δ, there exists a unital homomorphism $\varphi_0 : M_n \to e\mathcal{B}e$ such that $\|\varphi_0(v_{g,h}) - w_{g,h}\| < \varepsilon_0$ for all $g, h \in G$, and $\varphi_0(v_{g,g}) = e_g$ for all $g \in G$. Now define a unital homomorphism $\varphi : M_n \otimes e_1\mathcal{A}e_1 \to e\mathcal{B}e$ by $\varphi(v_{g,h} \otimes a) = \varphi_0(v_{g,1}) a \varphi_0(v_{1,h})$ for $g, h \in G$ and $a \in e_1\mathcal{A}e_1$. Since $M_n \otimes e_1\mathcal{A}e_1$ is an inner quasidiagonal C^*-algebra by Corollary 1.4, and φ is injective, so $D := \varphi(M_n \otimes e_1\mathcal{A}e_1)$ is also an inner quasidiagonal C^*-algebra.

For any $a \in S$, choosing $c \in D$ such that $\|eae - c\| < \varepsilon$ and $\|ea - ae\| < n\varepsilon_0 \leq \varepsilon$. Finally, note that $1 - e \precsim P$ in $\mathcal{A}$, and $P \precsim p_1 + p_2 + \cdots + p_n$ in $\mathcal{B}$, by Lemma 2.11 and Proposition 2.13 of [27] we have $1 - e \precsim p_1 + p_2 + \cdots + p_n \in \overline{x\mathcal{B}x}$. Thus $\mathcal{B}$ is a tracially inner quasidiagonal C^*-algebra. □

4. Inner quasidiagonality of crossed product C^*-algebras

We recall that a unital C^*-algebra $\mathcal{A}$ is said to be *finite*, if $x^*x = 1$ implies that $xx^* = 1$. $\mathcal{A}$ is said to be *stably finite*, if $M_n(\mathcal{A})$ is finite for all $n \in \mathbb{N}$.

Theorem 4.1 (Theorem 4.1 of [8]). *Let $\mathscr{C}$ be a class of finite unital C^*-algebras. Then any simple C^*-algebra in the class $TA\mathscr{C}$ is finite. Moreover, if C^*-algebras in $\mathscr{C}$ are stably finite, then simple C^*-algebras in the class $TA\mathscr{C}$ are also stably finite.*

It is known by definition that inner quasidiagonal C^*-algebras are quasidiagonal C^*-algebras. Note that quasidiagonal C^*-algebras are stably finite by Proposition 3.19 of [5]. Thus we have that inner quasidiagonal C^*-algebras are stably finite.

By combining Lemma 3.3, Theorem 3.7 and Theorem 4.1, we get the following:

Theorem 4.2. *Let $\mathcal{A}$ be an infinite dimensional separable unital inner quasidiagonal C^*-algebra with Property (SP). Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property. If $\mathcal{A}$ is α-simple, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a simple separable unital finite tracially inner quasidiagonal C^*-algebra with the Property (SP).*

Lemma 4.3 (Lemma 1.15 of [23]). *Let $\mathcal{A}$ be an infinite dimensional finite unital C^*-algebra with Property (SP), let $x \in \mathcal{A}$ be a positive element with $\|x\| = 1$, and let $\varepsilon > 0$. Then there exists a non-zero projection $q \in \overline{x\mathcal{A}x}$ such that, whenever $e \in \mathcal{A}$ is a projection such that $1 - e \precsim q$, then $\|exe\| > 1 - \varepsilon$.*

Lemma 4.4. *Let $\mathcal{A}$ be an infinite dimensional separable unital inner quasidiagonal C^*-algebra with Property (SP). Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property. If $\mathcal{A}$ is α-simple, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a strongly tracially inner quasidiagonal C^*-algebra.*

Proof. Let $\mathcal{B} = C^*(G, \mathcal{A}, \alpha)$. For any given $\varepsilon > 0$ and any finite set $F = \{x_1, x_2, \ldots, x_n\} \subseteq \mathcal{B}$ with $\|x_i\| = 1, i = 1, 2, \ldots, n$, and any non-zero $e \in \mathcal{B}_+$. Let $H = \{x_1, x_2, \ldots, x_n, x_1^*x_1, x_2^*x_2, \ldots, x_n^*x_n\}$ and $H_1 = \{x_1^*x_1, x_2^*x_2, \ldots, x_n^*x_n\}$. Let ε_0 satisfy $1 + \varepsilon_0 - (1 - \varepsilon_0)^{\frac{1}{2}} = \varepsilon$ and $\delta = \min\{\varepsilon_0, \varepsilon\}$. By Theorem 4.2 $\mathcal{B}$ is a simple separable unital finite tracially inner quasidiagonal C^*-algebra with the Property (SP). So we can find non-zero projections $p_i \in \overline{(x_i^*x_i)\mathcal{B}(x_i^*x_i)}$ for $i = 1, 2, \ldots, n$. By Lemma 3.5 there is a non-zero sub-projection e of p_1 such that e is Murray-von Neumann equivalent to a sub-projection of p_i for all $1 \le i \le n$.

Note that $\mathcal{B}$ is a tracially inner quasidiagonal C^*-algebra, there exist a non-zero projection $p \in \mathcal{B}$ and an inner quasidiagonal C^*-subalgebra $C \subset \mathcal{B}$ with $1_C = p$ such that:
(1) $\|xp - px\| < \delta$ for all $x \in H$,
(2) $pxp \in_\delta C$ for all $x \in H$, and
(3) $1 - p \precsim e$.
Since $\mathcal{B}$ is finite and has Property (SP), by Lemma 4.3 we have $\|pbp\| > 1 - \delta$ for all $b \in H_1$, i.e. $\|px_i^*x_ip\| > 1 - \delta$ for $i = 1, 2, \ldots, n$. Now we compute that

$$\begin{aligned}
\|px_ip\| &\ge \|x_ip\| - \|px_ip - x_ip\| \\
&> \|x_ip\| - \delta = \|px_i^*x_ip\|^{\frac{1}{2}} - \delta \\
&\ge (1 - \delta)^{\frac{1}{2}} - \delta \\
&\ge (1 - \varepsilon_0)^{\frac{1}{2}} - \varepsilon_0 \\
&= 1 - (1 + \varepsilon_0 - (1 - \varepsilon_0)^{\frac{1}{2}}) \\
&= 1 - \varepsilon.
\end{aligned}$$

Thus we have
(1) $\|xp - px\| < \varepsilon$ for all $x \in F$,
(2) $pxp \in_\varepsilon C$ and $\|pxp\| > 1 - \varepsilon$ for all $x \in F$,
(3) $1 - p \precsim e$.
Finally, for any $F = \{x_1, x_2, \ldots, x_n\}$, let $M = \max\{\|x_1\|, \|x_2\|, \ldots, \|x_n\|\}$. In the above proof, we can obtain a general conclusion by replacing ε with $\frac{\varepsilon}{M}$.
□

Next we will prove our main theorem.

Theorem 4.5. *Let $\mathcal{A}$ be an infinite dimensional separable unital inner quasidiagonal C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$ which has the weak tracial Rokhlin property. If $\mathcal{A}$ is α-simple, then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a separable inner quasidiagonal C^*-algebra.*

Proof. Since α has the weak tracial Rokhlin property, by Lemma 3.2 $\mathcal{A}$ has Property (SP) or α has the Rokhlin property. If α has the Rokhlin property, by Theorem 1.5 $C^*(G, \mathcal{A}, \alpha)$ is a separable inner quasidiagonal C^*-algebra.

Next we assume that $\mathcal{A}$ has Property (SP). By Lemma 4.4 $C^*(G, \mathcal{A}, \alpha)$ is a strongly tracially inner quasidiagonal C^*-algebra, then we have that $C^*(G, \mathcal{A}, \alpha)$ is an inner quasidiagonal C^*-algebra by Theorem 2.2. □

Corollary 4.6. *Let $\mathcal{A}$ be an infinite dimensional simple separable unital inner quasidiagonal C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ be an action of a finite group G on $\mathcal{A}$ which has the tracial Rokhlin property. Then the crossed product $C^*(G, \mathcal{A}, \alpha)$ is a separable inner quasidiagonal C^*-algebra.*

Proof. As pointed out in [27] that the weak tracial Rokhlin property and the tracial Rokhlin property are the same when $\mathcal{A}$ is simple. $\mathcal{A}$ is automatically α-simple when $\mathcal{A}$ is simple. Thus we get the conclusion by Theorem 4.5. □

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