Filomat 40:1 (2026), 133–142
https://doi.org/10.2298/FIL2601133J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On construction of various range equalities for mixed operations of operators and their generalized inverses

Bo Jiang[a]

[a]*College of Mathematics and Information Science, Shandong Technology and Business University, Yantai, Shandong, China*

Abstract. This article is concerned with a classic problem on the constructions and classifications of operator range equalities for algebraic operations of operators and their generalized inverses on a Hilbert space. We shall establish miscellaneous novel formulas and facts associated with ranges of mixed products of two or three operators and their Moore–Penrose inverses, and present some new formulas and facts about the well-known the reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ for the Moore–Penrose inverse of two operator product and its variations by using the operator range methodology.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be two infinite dimensional complex Hilbert spaces and let $\mathcal{B}(\mathcal{K}, \mathcal{H})$ be the set of all bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$ and abbreviate $\mathcal{B}(\mathcal{K}, \mathcal{H})$ to $\mathcal{B}(\mathcal{H})$ to denote the C^*-algebra in $\mathcal{H}$ if $\mathcal{K} = \mathcal{H}$. It is well known that C^*-algebras are the study of operators acting on a Hilbert space with algebraic methods. In this note, we use $\mathscr{R}(T)$ to denote the range of $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. The adjoint of $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is denoted by $T^* \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The four Penrose equations associated T are defined to be

$$(1)\ TXT = T, \quad (2)\ XTX = X, \quad (3)\ (TX)^* = TX, \quad (4)\ (XT)^* = XT. \tag{1.1}$$

If the four equations have a common solution, then the solution is unique, denoted by $X = T^{\dagger}$, and is called the Moore–Penrose generalized inverse of T. It is now a well-recognized fact in operator theory that the Moore–Penrose inverse of an operator T exists if and only if the range of T is closed. Also as we know in the current operator theory, generalized inverses have been taken as one of the fundamental concepts with essential applications in dealing with many theoretical problems. The most significant fact is that we can use generalized inverses of matrices and operators in the case when ordinary inverses do not exist in order to solve some matrix and operator equations. Hence, generalized inverses now play important roles in the theoretical and numerical analysis of many matrix and operator problems. As demonstrated above, generalized inverses are defined to be common solutions of one or more algebraic equations as certain extensions of the ordinary inverses of invertible operators. In comparison, the Moore–Penrose inverses and the group inverses of operators are two highly-recognized generalized inverses, which are known to have many remarkable algebraic and computational properties, and thus have been extensively studied in

2020 *Mathematics Subject Classification*. Primary 47A05; Secondary 47B15, 47B20.
Keywords. Hilbert space, Moore–Penrose inverse, operator, range, reverse order law.
Received: 17 June 2025; Accepted: 01 October 2025
Communicated by Dragana Cvetković-Ilić
Email address: jiangboliumengyu@163.com (Bo Jiang)
ORCID iD: https://orcid.org/0009-0007-8350-0009 (Bo Jiang)

mathematics and applications. In particular, the Moore–Penrose inverse of an operator was defined and approached in operator algebra and applications in 1920s (cf. [1, 21]), and it was reputed as a core and influential part in the discipline of generalized inverses. Now the two kinds of generalized inverses are regarded as the main bodies that are built around theory of generalized inverses of operators. Interested readers may wish to consult [2–4, 6, 8, 12, 14–16, 19, 20] for more expositions regarding generalized inverses of operators.

Recall the range of an operator is one of the fundamental characteristics associated with the operator, which can be used to describe a variety of algebraic properties and performances of operators and their operations. In view of this fact, algebraists are interested in establishing various equalities for the range of an operators and using them in to deal with various operator problems. As one of such work, we show how construct and characterize different kinds of range equalities for expressions that involve operator and their generalized inverses.

To illustrate a motivation of this study, let us recall a fundamental range equality in operator theory: a $T \in \mathcal{B}(\mathcal{H})$ is said to be EP (range-Hermitian) if and only if

$$\mathscr{R}(T) = \mathscr{R}(T^*) \tag{1.2}$$

holds. The concept of EP matrix was first introduced by Schwerdtfeger in [25], and there has been a long-lasting interest in the exploration of the characterizations and performances of EP objects from theoretical and applied points of view. A broad range of operators are known to be in the class of EP objects. For example, if A is invertible, then it is trivially an EP operator; if A is self-adjoint/skew-self-adjoint ($A = \pm A^*$), then it is EP; if A is normal ($AA^* = A^*A$), then it is EP; if A is bi-normal ($(AA^*)(A^*A) = (A^*A)(AA^*)$ and $\mathscr{R}(A^2) = \mathscr{R}(A)$), then it is EP. One of the most important applications of the EP property of operator is to characterize various reverse order laws for the Moore–Penrose inverses of operator products. Here, we mention the following three well-known mutual implication facts:

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A) \ \text{(cf. [2, p. 161])}, \tag{1.3}$$

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathscr{R}(A^\dagger ABB^*) = \mathscr{R}(BB^*A^\dagger A) \ \text{and} \ \mathscr{R}(A^*ABB^\dagger) = \mathscr{R}(BB^\dagger A^*A) \ \text{(cf. [17])}, \tag{1.4}$$

$$(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger \Leftrightarrow \mathscr{R}(AB) = \mathscr{R}(AA^*AB) \ \text{and} \ \mathscr{R}((AB)^*) = \mathscr{R}(B^*B(AB)^*) \ \text{(cf. [5])} \tag{1.5}$$

associated with operator range equalities, as well as the following equivalent reverse order laws

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow (AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger \ \text{and} \ (A^\dagger ABB^\dagger)^\dagger = BB^\dagger A^\dagger A \ \text{(cf. [12])}, \tag{1.6}$$

where $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and A, B, AB, and $A^\dagger ABB^\dagger$ are assumed to have closed ranges. As basic operation rules, the above reverse order laws and their variations can be utilized to construct and simplify various concrete expressions and equalities that are composed of generalized inverses of products of operators, as well as to approach properties and performances of generalized inverse operations under various assumptions. In fact, it is generally recognized that the above reverse order law problems have been attractive issues in the theory of generalized inverses, while the corresponding investigations and contributions have given rise to greater progresses and cross-fertilizations of ideas in the theory of generalized inverses of matrices and operators in the past several decades, see, e.g., [2, 5, 7, 9–13, 17, 18, 22–24, 27–35] for some earlier and recent references related. In a recent paper [32], it was shown that many kinds of range equalities can reasonably be established for algebraic expressions that involve matrices and their generalized inverses. Motivated by the study in [32], we shall present an extended approach to the constructions and classifications of nontrivial and meaningful operator range equalities for algebraic expressions that involve products of operators and their generalized inverses.

The rest of this article is organized as follows. In Section 2, we introduce some preliminary facts and results concerning ranges and generalized inverses of matrices. In Section 3, we propose and study miscellaneous operator range equalities associated with mixed products of two and three operators and their Moore–Penrose generalized inverses. Section 4 presents some new facts related to the reverse order law in (1.3) and (1.4) and its variations. Concluding remarks are given in Section 5.

2. Preliminaries

In this section, we first present some existing simple general formulas and facts addressing basic operations of operators, as well ranges of operators on Hilbert space (cf. [2, 6]), which we shall exploit in the derivations of the main results in the article.

Lemma 2.1. *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ with closed range. Then, the following operator equalities hold:*

$$(T^{\dagger})^{*} = (T^{*})^{\dagger}, \quad (T^{\dagger})^{\dagger} = T, \tag{2.1}$$

$$T^{\dagger} = T^{*}(TT^{*})^{\dagger} = (T^{*}T)^{\dagger}T^{*} = T^{*}(T^{*}TT^{*})^{\dagger}T^{*}, \tag{2.2}$$

$$(T^{*})^{\dagger}T^{*} = (TT^{\dagger})^{*} = TT^{\dagger}, \;\; T^{*}(T^{*})^{\dagger} = (T^{\dagger}T)^{*} = T^{\dagger}T, \tag{2.3}$$

$$(TT^{*})^{\dagger} = (T^{\dagger})^{*}T^{\dagger}, \;\; (T^{*}T)^{\dagger} = T^{\dagger}(T^{\dagger})^{*}, \;\; (TT^{*}T)^{\dagger} = T^{\dagger}(T^{\dagger})^{*}T^{\dagger}, \tag{2.4}$$

and the following operator range equalities hold:

$$\mathscr{R}(TT^{*}T) = \mathscr{R}(TT^{*}) = \mathscr{R}(TT^{\dagger}) = \mathscr{R}((T^{\dagger})^{*}) = \mathscr{R}(T), \tag{2.5}$$

$$\mathscr{R}(T^{*}TT^{*}) = \mathscr{R}(T^{*}T) = \mathscr{R}(T^{\dagger}T) = \mathscr{R}(T^{\dagger}) = \mathscr{R}(T^{*}). \tag{2.6}$$

Lemma 2.2. *Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $Q \in \mathcal{B}(\mathcal{K}, \mathcal{L})$ with closed range. Then, the following operator range equalities hold:*

$$\mathscr{R}(TQ^{\dagger}Q) = \mathscr{R}(TQ^{\dagger}) = \mathscr{R}(TQ^{*}Q) = \mathscr{R}(TQ^{*}), \tag{2.7}$$

and the following implicating facts

$$\mathscr{R}(T) \subseteq \mathscr{R}(S) \Rightarrow \mathscr{R}(PT) \subseteq \mathscr{R}(PS), \tag{2.8}$$

$$\mathscr{R}(T) = \mathscr{R}(S) \Rightarrow \mathscr{R}(PT) = \mathscr{R}(PS) \tag{2.9}$$

hold for general operators $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $S \in \mathcal{B}(\mathcal{L}, \mathcal{H})$, and $P \in \mathcal{B}(\mathcal{H}, \mathcal{G})$.

3. Miscellaneous operator range equalities associated with mixed products of two operators and their Moore–Penrose inverses

As usual in the investigation of certain specified algebraic objects, there is an enough interest to create various required products of operators and to approach their properties from theoretical and applied points of view. Correspondingly, we are able to create many types of range equalities for the products of operators and their generalized inverses and to approach their algebraic properties and performances. In this section, our main attention is focussed on the constructions and classifications of various operator range equalities that are composed mixed operations of Moore–Penrose inverse of operators. Here are the main results that we shall prove on a wide selection of specified operator range equalities associated with some regular mixed products of two operators and their Moore–Penrose inverses.

Theorem 3.1. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$. Then, the following range equalities hold:*

$$\mathscr{R}(ABB^{*}A^{*}AB) = \mathscr{R}(ABB^{*}A^{*}) = \mathscr{R}(ABB^{*}) = \mathscr{R}(AB), \tag{3.1}$$

$$\mathscr{R}(B^{*}A^{*}ABB^{*}A^{*}) = \mathscr{R}(B^{*}A^{*}AB) = \mathscr{R}(B^{*}A^{*}A) = \mathscr{R}(B^{*}A^{*}). \tag{3.2}$$

Proof. By (2.5) and (2.6), the following range inclusions

$$\mathscr{R}(AB) = \mathscr{R}(ABB^{*}A^{*}AB) \subseteq \mathscr{R}(ABB^{*}A^{*}) \subseteq \mathscr{R}(ABB^{*}) \subseteq \mathscr{R}(AB),$$
$$\mathscr{R}(B^{*}A^{*}) = \mathscr{R}(B^{*}A^{*}ABB^{*}A^{*}) \subseteq \mathscr{R}(B^{*}A^{*}AB) \subseteq \mathscr{R}(B^{*}A^{*}A) \subseteq \mathscr{R}(B^{*}A^{*})$$

hold, which naturally imply the range equalities in (3.1) and (3.2). □

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and assume that A, B and AB have closed ranges. Then, the following operator range equalities hold*:

$$\mathscr{R}(ABB^{\dagger}A^{\dagger}AB) = \mathscr{R}(ABB^{\dagger}A^{\dagger}) = \mathscr{R}(AB), \tag{3.3}$$

$$\mathscr{R}(B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}A^{\dagger}AB) = \mathscr{R}(B^{\dagger}A^{\dagger}), \tag{3.4}$$

$$\mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}) = \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}) = \mathscr{R}((B^{\dagger}A^{\dagger})^{*}), \tag{3.5}$$

$$\mathscr{R}((AB)^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}) = \mathscr{R}((AB)^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}) = \mathscr{R}((AB)^{*}). \tag{3.6}$$

Proof. The following fact

$$\mathscr{R}(ABB^{\dagger}A^{\dagger}AB) \subseteq \mathscr{R}(ABB^{\dagger}A^{\dagger}) \subseteq \mathscr{R}(AB) \tag{3.7}$$

is obvious according to the orders of the products of the operators. On the other hand,

$$\begin{aligned}\mathscr{R}(ABB^{\dagger}A^{\dagger}AB) &= \mathscr{R}(ABB^{\dagger}A^{\dagger}ABB^{\dagger})\\ &= \mathscr{R}(A(BB^{\dagger}A^{\dagger}A)(BB^{\dagger}A^{\dagger}A)^{*})\\ &= \mathscr{R}(ABB^{\dagger}A^{\dagger}A)\\ &= \mathscr{R}(A(A^{\dagger}ABB^{\dagger})(A^{\dagger}ABB^{\dagger})^{*})\\ &= \mathscr{R}(AA^{\dagger}ABB^{\dagger}) = \mathscr{R}(AB).\end{aligned} \tag{3.8}$$

In this case, combining (3.7) and (3.8) leads to (3.3). Replacing A and B in (3.3) with $B^{\dagger}$ and $A^{\dagger}$, respectively, yields (3.4). Also by (2.7),

$$\begin{aligned}\mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}) &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}(B^{\dagger}A^{\dagger})^{*})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}((AB)^{\dagger})^{*}(AB)^{*}(A^{\dagger})^{*}(B^{\dagger})^{*})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}((AB)^{\dagger})^{*}B^{*}A^{\dagger}AB)\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger}((AB)^{\dagger})^{*}B^{*}A^{*})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}(AB)^{\dagger})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}((B^{\dagger}A^{\dagger})^{\dagger})^{*}(B^{\dagger}A^{\dagger})^{*}B^{*}A^{*})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}((B^{\dagger}A^{\dagger})^{\dagger})^{*}(A^{\dagger})^{*}A^{\dagger}ABB^{\dagger}A^{*})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}((B^{\dagger}A^{\dagger})^{\dagger})^{*}(A^{\dagger})^{*}A^{\dagger}AB)\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}((B^{\dagger}A^{\dagger})^{\dagger})^{*}(B^{\dagger}A^{\dagger})^{*})\\ &= \mathscr{R}((B^{\dagger}A^{\dagger})^{\dagger}),\end{aligned}$$

as required for (3.5). Replacing A and B in (3.5) with $B^{\dagger}$ and $A^{\dagger}$, respectively, and simplifying yields (3.6). □

Theorem 3.3. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and assume that A, B and AB have closed ranges. Then, the following operator range equalities hold*:

$$\mathscr{R}(B(AB)^{\dagger}A) = \mathscr{R}(B(AB)^{*}A), \tag{3.9}$$

$$\mathscr{R}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger}) = \mathscr{R}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger}) = \mathscr{R}((A^{*}A)^{\dagger}(BB^{*})^{\dagger}), \tag{3.10}$$

$$\mathscr{R}((B(AB)^{\dagger}A)^{\dagger}) = \mathscr{R}((B(AB)^{\dagger}A)^{*}) = \mathscr{R}((B(AB)^{*}A)^{\dagger}) = \mathscr{R}((B(AB)^{*}A)^{*}) = \mathscr{R}(A^{*}ABB^{*}), \tag{3.11}$$

$$\mathscr{R}((A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}) = \mathscr{R}((A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}) = \mathscr{R}((A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}) = \mathscr{R}((A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}) = \mathscr{R}((BB^{*})^{\dagger}(A^{*}A)^{\dagger}). \tag{3.12}$$

In particular, if $A, B \in \mathcal{B}(\mathcal{H})$ are two projections, then the following operator range equalities hold:

$$\mathscr{R}(B(AB)^{\dagger}A) = \mathscr{R}(BA), \quad \mathscr{R}((B(AB)^{\dagger}A)^{\dagger}) = \mathscr{R}(AB). \tag{3.13}$$

Proof. By the definition of the range and (2.2), the following facts

$$\mathscr{R}(B(AB)^{\dagger}A) \supseteq \mathscr{R}(B(AB)^{\dagger}AB(AB)^{*}) = \mathscr{R}(B(AB)^{*}) \supseteq \mathscr{R}(B(AB)^{*}A), \tag{3.14}$$

and

$$\mathscr{R}(B(AB)^{*}A) \supseteq \mathscr{R}(B(AB)^{*}((AB)^{\dagger})^{*}(AB)^{\dagger}) = \mathscr{R}(B(AB)^{\dagger}) \supseteq \mathscr{R}(B(AB)^{\dagger}A) \tag{3.15}$$

hold. Combining (3.14) and (3.15) leads to the range equality in (3.9). Replacing A and B in (3.9) with $B^{\dagger}$ and $A^{\dagger}$, respectively, and performing simplification operations yields (3.10). Applying (2.5), (2.6), (3.9), and (3.10) to $(B(AB)^{\dagger}A)^{\dagger}$ and $(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}$ leads to the range equalities in (3.11) and (3.12), respectively. Eq. (3.13) follows immediately from (3.9) and (3.11). □

Obviously, the right-hand sides of (3.9)–(3.12) illustrate some attractive extrusion properties in contrast with the left-hand sides. Hence, they can be named as operator range extrusion equalities. Since the appearance and derivation of (3.9)–(3.12) are all with the ordinary algebraic operations of operators and their generalized inverses, we believe intuitively that there are many possible variations and extensions in response to range extrusion equalities associated with various mixed operator products. In what follows, we present several groups of explicit range extrusion equalities as such concrete examples without proofs.

Theorem 3.4. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and assume that A, B and AB have closed ranges. Then, the following operator range equalities hold:*

$$\begin{aligned}\mathscr{R}(A(B(AB)^{\dagger}A)^{\dagger}B) &= \mathscr{R}(A(B(AB)^{*}A)^{\dagger}B) = \mathscr{R}(A(B(AB)^{\dagger}A)^{*}B)\\ &= \mathscr{R}(A(B(AB)^{*}A)^{*}B) = \mathscr{R}(AA^{*}ABB^{*}B),\end{aligned}$$

$$\begin{aligned}\mathscr{R}((A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}) &= \mathscr{R}((A(B(AB)^{*}A)^{\dagger}B)^{\dagger}) = \mathscr{R}((A(B(AB)^{\dagger}A)^{*}B)^{\dagger})\\ &= \mathscr{R}((A(B(AB)^{\dagger}A)^{\dagger}B)^{*}) = \mathscr{R}((A(B(AB)^{*}A)^{*}B)^{\dagger})\\ &= \mathscr{R}((A(B(AB)^{*}A)^{\dagger}B)^{*}) = \mathscr{R}((A(B(AB)^{\dagger}A)^{*}B)^{*})\\ &= \mathscr{R}((A(B(AB)^{*}A)^{*}B)^{*}) = \mathscr{R}(B^{*}BB^{*}A^{*}AA^{*}),\end{aligned}$$

$$\begin{aligned}\mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger}) &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger}) = \mathscr{R}((BB^{*}B)^{\dagger}(AA^{*}A)^{\dagger}),\end{aligned}$$

$$\begin{aligned}\mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}) &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}A^{\dagger})^{\dagger})\\ &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{*}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{\dagger})\\ &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{*}) = \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{*}A^{\dagger})^{*})\\ &= \mathscr{R}((B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}) = \mathscr{R}((A^{*}AA^{*})^{\dagger}(B^{*}BB^{*})^{\dagger}).\end{aligned}$$

In particular, if $A, B \in \mathcal{B}(\mathcal{H})$ are two projections, then the following operator range equalities hold:

$$\begin{aligned}&\mathscr{R}(A(B(AB)^{\dagger}A)^{\dagger}B) = \mathscr{R}(A(B(AB)^{*}A)^{*}B) = \mathscr{R}(AB),\\ &\mathscr{R}((A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}) = \mathscr{R}((A(B(AB)^{*}A)^{*}B)^{*}) = \mathscr{R}(BA).\end{aligned}$$

Theorem 3.5. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and assume that A, B and AB have closed ranges. Then, the following operator range equalities hold:*

$$\begin{aligned}\mathscr{R}((B(A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}A)^{\dagger}) &= \mathscr{R}((B(A(B(AB)^{*}A)^{\dagger}B)^{\dagger}A)^{\dagger})\\ &= \mathscr{R}((B(A(B(AB)^{*}A)^{*}B)^{\dagger}A)^{\dagger})\\ &= \mathscr{R}((B(A(B(AB)^{*}A)^{*}B)^{*}A)^{\dagger})\\ &= \mathscr{R}((B(A(B(AB)^{*}A)^{*}B)^{*}A)^{*})\\ &= \mathscr{R}((A^{*}A)^{2}(BB^{*})^{2}),\end{aligned}$$

$$\begin{aligned}\mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}) &= \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger})\\ &= \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger})\\ &= \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{\dagger})\\ &= \mathscr{R}((A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{*})\\ &= \mathscr{R}(((BB^{*})^{2})^{\dagger}((A^{*}A)^{2})^{\dagger}).\end{aligned}$$

Theorem 3.6. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and assume that A, B and AB have closed ranges. Also define $\widehat{A} = AA^{*}AA^{*}A$ and $\widehat{B} = BB^{*}BB^{*}B$. Then, the following two groups of operator range equalities hold:*

$$\begin{aligned}\mathscr{R}(A(B(A(B(AB)^{\dagger}A)^{\dagger}B)^{\dagger}A)^{\dagger}B) &= \mathscr{R}(A(B(A(B(AB)^{*}A)^{\dagger}B)^{\dagger}A)^{\dagger}B)\\ &= \mathscr{R}(A(B(A(B(AB)^{*}A)^{*}B)^{\dagger}A)^{\dagger}B)\\ &= \mathscr{R}(A(B(A(B(AB)^{*}A)^{*}B)^{*}A)^{\dagger}B)\\ &= \mathscr{R}(A(B(A(B(AB)^{*}A)^{*}B)^{*}A)^{*}B)\\ &= \mathscr{R}(\widehat{\widehat{A}\widehat{B}}),\end{aligned}$$

$$\begin{aligned}\mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger}) &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{\dagger}B^{\dagger})^{\dagger}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{\dagger}A^{\dagger})\\ &= \mathscr{R}(B^{\dagger}(A^{\dagger}(B^{\dagger}(A^{\dagger}(B^{\dagger}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})^{*}B^{\dagger})^{*}A^{\dagger})\\ &= \mathscr{R}(\widehat{\widehat{B^{\dagger}}\widehat{A^{\dagger}}}).\end{aligned}$$

The remarkable facts in the above theorems suggest us how to construct different kinds of range equalities that involve mixed products of operators s and their generalized inverses. Continuingly, considering the perspective of establishing range equalities for multiple operator products, it is possible to extend the previous results to mixed product by a similar approach, Below, we give, as natural extensions of the above results and facts, two groups of examples for the purpose of showing the establishments of the range equalities for triple operator products without proofs.

Theorem 3.7. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and $C \in \mathcal{B}(\mathcal{J}, \mathcal{L})$, and assume that A, B, C, AB, BC and ABC have closed ranges. Then, the following operator range equalities hold:*

$$\begin{aligned}\mathscr{R}(C(ABC)^{\dagger}A) &= \mathscr{R}(C(ABC)^{*}A),\\ \mathscr{R}((C(ABC)^{\dagger}A)^{\dagger}) &= \mathscr{R}((C(ABC)^{*}A)^{*}) = \mathscr{R}(A^{*}ABCC^{*}),\\ \mathscr{R}(A(C(ABC)^{\dagger}A)^{\dagger}C) &= \mathscr{R}(A(C(ABC)^{*}A)^{*}C) = \mathscr{R}(AA^{*}ABCC^{*}C),\end{aligned}$$

the following operator range equalities hold:

$$\begin{aligned}\mathscr{R}((A(C(ABC)^{\dagger}A)^{\dagger}C)^{\dagger}) &= \mathscr{R}((A(C(ABC)^{*}A)^{*}C)^{*}) = \mathscr{R}(C^{*}C(ABC)^{*}AA^{*}),\\ \mathscr{R}(C(A(C(ABC)^{\dagger}A)^{\dagger}C)^{\dagger}A) &= \mathscr{R}(C(A(C(ABC)^{*}A)^{*}C)^{*}A) = \mathscr{R}(CC^{*}C(ABC)^{*}AA^{*}A),\\ \mathscr{R}((C(A(C(ABC)^{\dagger}A)^{\dagger}C)^{\dagger}A)^{\dagger}) &= \mathscr{R}((C(A(C(ABC)^{*}A)^{*}C)^{*}A)^{*}) = \mathscr{R}((A^{*}A)^{2}B(CC^{*})^{2}),\\ \mathscr{R}(A(C(A(C(ABC)^{\dagger}A)^{\dagger}C)^{\dagger}A)^{\dagger}C) &= \mathscr{R}(A(C(A(C(ABC)^{*}A)^{*}C)^{*}A)^{*}C) = \mathscr{R}((AA^{*})^{2}ABC(C^{*}C)^{2}),\\ \mathscr{R}((A(C(A(C(ABC)^{\dagger}A)^{\dagger}C)^{\dagger}A)^{\dagger}C)^{\dagger}) &= \mathscr{R}((A(C(A(C(ABC)^{*}A)^{*}C)^{*}A)^{*}C)^{*}) = \mathscr{R}((C^{*}C)^{2}(ABC)^{*}(AA^{*})^{2}),\\ \mathscr{R}(BC(ABC)^{\dagger}AB) &= \mathscr{R}(BC(ABC)^{*}AB),\\ \mathscr{R}((BC(ABC)^{\dagger}AB)^{\dagger}) &= \mathscr{R}((BC(ABC)^{*}AB)^{*}) = \mathscr{R}((AB)^{*}ABC(BC)^{*}),\\ \mathscr{R}(B(BC(ABC)^{\dagger}AB)^{\dagger}B) &= \mathscr{R}(B(BC(ABC)^{*}AB)^{*}B) = \mathscr{R}(B(AB)^{*}ABC(BC)^{*}B),\\ \mathscr{R}(AB(BC(ABC)^{\dagger}AB)^{\dagger}BC) &= \mathscr{R}(AB(BC(ABC)^{*}AB)^{*}BC) = \mathscr{R}((AB)(AB)^{*}ABC(BC)^{*}BC),\end{aligned}$$

and the following operator range equalities hold:

$$\begin{aligned}\mathscr{R}(B(BC(AB(BC(ABC)^{\dagger}AB)^{\dagger}BC)^{\dagger}AB)^{\dagger}B) &= \mathscr{R}(B(BC(AB(BC(ABC)^{*}AB)^{*}BC)^{*}AB)^{*}B)\\ &= \mathscr{R}(B(AB)^{*}AB(AB)^{*}ABC(BC)^{*}BC(BC)^{*}B),\end{aligned}$$

$$\begin{aligned}\mathscr{R}(AB(BC(AB(BC(ABC)^{\dagger}AB)^{\dagger}BC)^{\dagger}AB)^{\dagger}BC) &= \mathscr{R}(AB(BC(AB(BC(ABC)^{*}AB)^{*}BC)^{*}AB)^{*}BC)\\ &= \mathscr{R}((AB(AB)^{*})^{2}ABC((BC)^{*}BC)^{2}).\end{aligned}$$

Moreover, it is expected that the operator range expansion equalities for the following mixed operations

$$C(BC(BCD(ABCD)^{\dagger}ABC)^{\dagger}BC)^{\dagger}B, \quad C(CD(BCD(BCDE(ABCDE)^{\dagger}ABCD)^{\dagger}BCD)^{\dagger}BC)^{\dagger}C,$$

etc. can also be established routinely for the given operators A, B, C, D, and E with closed ranges such that the products $ABCD$ and $ABCDE$ can be defined with closed ranges as well.

4. The reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ and its equivalent forms

As indicated in Section 1, a main subject in operator theory that algebraists are interested in is the constructions and classifications of various operator equalities, as well as the characterizations of their properties and performances. In fact, algebraic operator equalities have been a classical and attractive research topic for a very longtime and have widespread applications in the discipline of operator theory. Recall that reverse order laws, as theoretical basis of operations of generalized inverses, have been traditional but attractive research issues with fruitful outcomes during the development course of current operator theory. In view of this fact, there seems to be at present and future a continuing interest in the research of reverse order laws and related issues. Notice that the range equalities in (1.3) and (1.5) involve no operations of the Moore–Penrose generalized inverses of the given operators. It is thereby convenient to gain the inherent properties of the reverse order law and to use this range equality in the characterization of the reverse order law and its variation forms under various situations. As intriguing applications of the equalities and facts in the preceding section, we derive in this section some new equivalent facts in relation to the reverse order law in (1.3) and (1.4) using the ideas and techniques associated with operator range equalities.

Theorem 4.1. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$ and assume that A, B and AB have closed ranges. Then, the following five statements are equivalent*:

(i) $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.
(ii) $(ABB^{*}A^{*}AB)^{\dagger} = (A^{*}AB)^{\dagger}(ABB^{*})^{\dagger}$.
(iii) $((A^{*}ABB^{*})^{2})^{\dagger} = ((A^{*}ABB^{*})^{\dagger})^{2}$.
(iv) $((A^{*}A)^{1/2}(BB^{*})^{1/2})^{\dagger} = ((BB^{*})^{1/2})^{\dagger}((A^{*}A)^{1/2})^{\dagger}$, *where $(A^{*}A)^{1/2}$ and $(BB^{*})^{1/2}$ are the square roots of the positive operators $A^{*}A$ and BB^{*}, respectively*.
(v) $\mathscr{R}(B(AB)^{\dagger}A) = \mathscr{R}(A^{*}(B^{*}A^{*})^{\dagger}B^{*})$, *i.e., $B(AB)^{\dagger}A$ is an EP operator*.

In particular, let $A \in \mathcal{B}(\mathcal{H})$ with closed range. Then, the following five statements are equivalent:

(i′) $(A^{2})^{\dagger} = (A^{\dagger})^{2}$, *namely, A is bi-dagger*.
(ii′) $(A^{2}(A^{*})^{2}A^{2})^{\dagger} = (A^{*}A^{2})^{\dagger}(A^{2}A^{*})^{\dagger}$.
(iii′) $((A^{*}A^{2}A^{*})^{2})^{\dagger} = ((A^{*}A^{2}A^{*})^{\dagger})^{2}$.
(iv′) $((A^{*}A)^{1/2}(AA^{*})^{1/2})^{\dagger} = ((AA^{*})^{1/2})^{\dagger}((A^{*}A)^{1/2})^{\dagger}$.
(v′) $\mathscr{R}(A(A^{2})^{\dagger}A) = \mathscr{R}(A^{*}((A^{*})^{2})^{\dagger}A^{*})$, *i.e., $A(A^{2})^{\dagger}A$ is an EP operator*.

Proof. It is easy to obtain by (2.5) and (2.6) that

$$\mathscr{R}((A^{*}ABB^{*})^{2}) \subseteq \mathscr{R}(A^{*}ABB^{*}), \tag{4.1}$$

$$\mathscr{R}((A^{*}ABB^{*})^{2}) \supseteq \mathscr{R}((A^{*}ABB^{*})^{2}A^{*}A) = \mathscr{R}(A^{*}ABB^{*}A^{*}(A^{*}ABB^{*}A^{*})^{*}) = \mathscr{R}(A^{*}ABB^{*}A^{*}) = \mathscr{R}(A^{*}ABB^{*}) \tag{4.2}$$

hold. The two facts in (4.1) and (4.2) imply that the following range equality

$$\mathscr{R}((A^*ABB^*)^2) = \mathscr{R}(A^*ABB^*) \tag{4.3}$$

holds. Similarly, we are able to obtain the following range equality

$$\mathscr{R}((BB^*A^*A)^2) = \mathscr{R}(BB^*A^*A). \tag{4.4}$$

Now let $P = ABB^*$ and $Q = A^*AB$. Then, $P^*PQQ^* = (BB^*A^*A)^3$ and $QQ^*P^*P = (A^*ABB^*)^3$ hold. In this case, we obtain from (4.3) and (4.4) the following range equalities

$$\mathscr{R}(P^*PQQ^*) = \mathscr{R}((BB^*A^*A)^3) = \mathscr{R}((BB^*A^*A)^2) = \mathscr{R}(BB^*A^*A), \tag{4.5}$$

$$\mathscr{R}(QQ^*P^*P) = \mathscr{R}((A^*ABB^*)^3) = \mathscr{R}((A^*ABB^*)^2) = \mathscr{R}(A^*ABB^*). \tag{4.6}$$

By (1.3), the equality $(PQ)^\dagger = Q^\dagger P^\dagger$ holds if and only if $\mathscr{R}(P^*PQQ^*) = \mathscr{R}(QQ^*P^*P)$, which is equivalent to $\mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A)$ by (4.5) and (4.6), thus establishing the equivalence of (i) and (ii).

Letting $M = A^*ABB^*$ and applying (1.3) to M^2, we see that the bi-dagger operator equality $(M^2)^\dagger = (M^\dagger)^2$ holds if and only if

$$\mathscr{R}(M^*M^2M^*) = \mathscr{R}(M(M^*)^2M), \tag{4.7}$$

where by (2.5)–(2.7), (4.3), and (4.4), the following range equalities

$$\mathscr{R}(M^*M^2M^*) = \mathscr{R}(M^*M^2) = \mathscr{R}(M^*M) = \mathscr{R}(M^*),$$
$$\mathscr{R}(M(M^*)^2M) = \mathscr{R}(M(M^*)^2) = \mathscr{R}(MM^*) = \mathscr{R}(M)$$

hold. These two groups of range equalities imply that (4.7) is equivalent to $\mathscr{R}(M) = \mathscr{R}(M^*)$, the range equality in (1.3), thus establishing the equivalence of (i) and (iii).

The square root of a positive self-adjoint operator was defined in [26]. Applying (1.3) to the product $(A^*A)^{1/2}(BB^*)^{1/2}$, and noting that

$$((A^*A)^{1/2})^*(A^*A)^{1/2} = A^*A \ \text{ and } \ (BB^*)^{1/2}((BB^*)^{1/2})^* = BB^*$$

leads to the equivalence of the two operator equalities in (i) and (iv) via the range equality in (1.3).

The equivalence of (i) and (v) follows from (1.3), (3.9) and (3.11). The equivalences of (i')–(v') follow immediately from (i)–(v) by letting $A = B$. □

Concluding this section, we give our second group of results associated with the first reverse order law in (1.3) and (1.4).

Theorem 4.2. *Let $A \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{K})$, and assume that A, B, and AB have closed ranges. Then, the following four statements are equivalent*:

(i) $(AB)^\dagger = B^\dagger A^\dagger$.

(ii) $(A^*ABB^*)^\dagger = (BB^*)^\dagger(A^*A)^\dagger$ *and* $(AA^*ABB^*B)^\dagger = (BB^*B)^\dagger(AA^*A)^\dagger$.

(iii) $\mathscr{R}((A^*A)^2(BB^*)^2) = \mathscr{R}((BB^*)^2(A^*A)^2)$ *and* $\mathscr{R}((A^*A)^3(BB^*)^3) = \mathscr{R}((BB^*)^3(A^*A)^3)$, *i.e.*, $(A^*A)^2(BB^*)^2$ *and* $(A^*A)^3(BB^*)^3$ *are EP operators*.

(iv) $\mathscr{R}(A^*A(BB^*A^*A)^\dagger BB^*) = \mathscr{R}(BB^*(A^*ABB^*)^\dagger A^*A)$ *and* $\mathscr{R}(A^*AA^*(B^*BB^*A^*AA^*)^\dagger B^*BB^*)$ $= \mathscr{R}(BB^*B(AA^*ABB^*B)^\dagger AA^*A)$, *i.e.*, $A^*A(BB^*A^*A)^\dagger BB^*$ *and* $A^*AA^*(B^*BB^*A^*AA^*)^\dagger B^*BB^*$ *are EP operators*.

Proof. Under (1.3), we are able to obtain the following range equalities

$$\mathscr{R}((A^*A)^2(BB^*)^2) = \mathscr{R}((A^*A)^2BB^*) = \mathscr{R}(A^*ABB^*A^*A) = \mathscr{R}(A^*ABB^*),$$
$$\mathscr{R}((A^*A)^3(BB^*)^3) = \mathscr{R}((A^*A)^3BB^*) = \mathscr{R}((A^*A)^2BB^*) = \mathscr{R}(A^*ABB^*).$$

Similarly, we are able to establish the following range equalities

$$\mathscr{R}((BB^*)^3(A^*A)^3) = \mathscr{R}((BB^*)^2(A^*A)^2) = \mathscr{R}(BB^*A^*A).$$

The combination of these facts with (1.3) leads to

$$\mathscr{R}((A^*A)^2(BB^*)^2) = \mathscr{R}((BB^*)^2(A^*A)^2), \ \ \mathscr{R}((A^*A)^3(BB^*)^3) = \mathscr{R}((BB^*)^3(A^*A)^3).$$

Thus, (i) implies (iii).

The equivalence of (ii) and (iii) follows immediately from (1.3).

If (iii) holds, then it is easy to verify that

$$\mathscr{R}((BB^*)^3(A^*A)^3) = \mathscr{R}(BB^*(BB^*)^2(A^*A)^2) = \mathscr{R}(BB^*(A^*A)^2(BB^*)^2) = \mathscr{R}(BB^*A^*A),$$
$$\mathscr{R}((A^*A)^3(BB^*)^3) = \mathscr{R}(A^*A(A^*A)^2(BB^*)^2) = \mathscr{R}(A^*A(BB^*)^2(A^*A)^2) = \mathscr{R}(A^*ABB^*)$$

hold. Combining these facts leads to the range equality in (1.3). Thus, (iii) implies (i) by (1.3).

The equivalence of (ii) and (iv) follows immediately from Theorem 4.1(i) and (v). □

Apparently, it is easy to construct a great variety of algebraic equalities involving products of operators and their generalized inverses according to various conventional algebraic operation rules of operators. In addition to the basic reverse order laws considered in this section, more complicated reverse order laws for the Moore–Penrose inverses of multiple operator products can be formulated, and therefore we are able to make further profound study on these operator equality problems in the theory of generalized inverses of operators by using the classic but powerful operator range methodology.

5. Conclusion

We constructed, as an extended study of range equalities for matrix expressions that involve matrices and their generalized inverses, a wide variety of operator range equalities that are composed of mixed products of operators and their generalized inverses using various ordinary algebraic operations of operators and their generalized inverses. Most of these research findings are given in simple general forms and have not been described in details in the existing literature on operator operations. Since the main results and their derivations are obtained within the core domain of the Hilbert space operator theory, we hope that this new and insightful study can actively encourage and cultivate deep-doing and fruitful approaches and advances in the constructions and classifications of various essential and useful operator range equalities so as to explore new horizons in this area of research.

Finally, we claim that the rigorous and resultful analysis in the preceding sections establishes a variety of essential connections among different operator expressions and operator equalities that involve algebraic operations of operators and their generalized inverses, and thus they can help readers know what is new and exciting on research fronts related to the well-known classic reverse order law problems. As is well known, one of the main concerns in operator theory is to construct and classify various algebraic equalities that are composed of operators and their generalized inverses from theoretical and applied point of view. Generally speaking, to describe the range properties of these operator equalities is an important motivational attempt, but this kind of research is challenging in most cases, and more technical and helpful preparations are required for substantially dealing with the corresponding classification and simplification problems on algebraic operations of operators and their different types of generalized inverses.

Declarations

Ethical Approval: Not applicable.

Data availability: There is no data used in the article.

Funding Declaration: No Funding.

Competing interests: The author declares that there are no competing interests.

References

[1] A. Ben-Israel, *The Moore of the Moore–Penrose inverse*, Electron. J. Linear Algebra **9**(2002), 150–157.
[2] A. Ben–Israel, T.N.E. Greville, *Generalized Inverses: Theory and Applications*, 2nd ed., Springer, New York, 2003.
[3] E. Boasso, *On the Moore–Penrose inverse in C^*-algebras*, Extra. Math. **21**(2006), 93–106.
[4] E. Boasso, *On the Moore–Penrose inverse, EP Banach space operators, and EP Banach algebra elements*, J. Math. Anal. Appl. **339**(2008) 1003–1014.
[5] R. Bouldin, *The pseudo-inverse of a product*, SIAM J. Appl. Math. **24**(1973), 489–495.
[6] S.L. Campbell, C.D. Meyer, *Generalized Inverses of Linear Transformations*, Pitman Publishing Limited, 1979; Dover Publications, 1991; SIAM, Philadelphia, 2009.
[7] E. Cline, *Note on the generalized inverse of the product of matrices*, SIAM Rev. **6**(1964), 57–58.
[8] N. Č. Dinčić, D.S. Djordjević, *Identities concerning the reverse order law for the Moore–Penrose inverse*, Appl. Math. Comput. **220**(2013), 439–445.
[9] D.S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix. Anal. Appl. **29**(2007), 1242–1246.
[10] D.S. Djordjević, N. Č. Dinčić, *Reverse order law for the Moore–Penrose inverse*, J. Math. Anal. Appl. **361**(2010), 252–261.
[11] I. Erdelyi, *On the "reverse order law" related to the generalized inverse of matrix products*, J. Assoc. Comp. Mach. **13**(1966), 439–443.
[12] A.M. Galperin, Z. Waksman, *On pseudoinverse of operator products*, Linear Algebra Appl. **33**(1980), 123–131.
[13] T.N.E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. **8**(1966), 518–521.
[14] C.W. Groetsch, *Inclusions and identities for the Moore–Penrose inverse of a closed linear operator*, Math. Nachr. **171**(1995), 157–164.
[15] R.E. Harte, M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. **103**(1992), 71–77.
[16] R.E. Harte, M. Mbekhta, *On generalized inverses in C^*-algebras*, II. Studia Math. **106**(1993), 129–138.
[17] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tôhoku Math. J. **34**(1982), 43–52.
[18] M.M. Karizaki, J. Farokhi–Ostad, *New identities for Moore–Penrose inverses of some operator products and their reverse-order laws*, Acta Math. Sin. Engl. Ser. **39**(2023), 2414–2426.
[19] J.J. Koliha, *The Drazin and Moore–Penrose inverse in C^*-algebras*, Math. Proc. R. Ir. Acad. A **99**(1999), 17–27.
[20] X. Liu, S. Huang, D.S. Cvetković-Ilić, *Mixed-type reverse-order laws for {1, 3, 4}-generalized inverses over Hilbert spaces*, Appl. Math. Comput. **218**(2012), 8570–8577.
[21] E.H. Moore, *On the reciprocal of the general algebraic matrix*, Bull. Amer. Math. Soc. **26**(1920), 394–395.
[22] D. Mosić, D.S. Djordjević, *Further results on the reverse order law for the Moore–Penrose inverse in rings with involution*, Appl. Math. Comput. **218**(2011), 1478–1483.
[23] D. Mosić, D.S. Djordjević, *Reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution*, Filomat **28**(2014), 1791–1815.
[24] D. Mosić, D.S. Djordjević, *The reverse order law $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution*, RACSAM **109**(2015), 257–265.
[25] H. Schwerdtfeger, *Introduction to Linear Algebra and the Theory of Matrices*, Noordhoff, Groningen, 1950.
[26] Z. Sebestyén, Z. Tarcsay, *On the square root of a positive selfadjoint operator*, Period. Math. Hung. **75**(2017), 268–272.
[27] N. Shinozaki, M. Sibuya, *Further results on the reverse order law*, Linear Algebra Appl. **27**(1979), 9–16.
[28] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products*, Linear Algebra Appl. **211**(1994), 85–100.
[29] Y. Tian, *The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalent equalities*, J. Math. Kyoto Univ. **45**(2005), 841–850.
[30] Y. Tian, *The equivalence between $(AB)^\dagger = B^\dagger A^\dagger$ and other mixed-type reverse-order laws*, Internat. J. Math. Edu. Sci. Tech. **37**(2007), 331–339.
[31] Y. Tian, *A family of 512 reverse order laws for generalized inverses of a matrix product: a review*, Heliyon **6**(2020), e04924.
[32] Y. Tian, *A study of range equalities for matrix expressions that involve matrices and their generalized inverses*, Comput. Appl. Math. **41**(2022), 384.
[33] Y. Tian, *Constructions and characterizations of mixed reverse-order laws for products of Moore–Penrose inverses and group inverses*, Georgian Math. J. **31**(2024), 1049–1062.
[34] Y. Tian, *Towards finding equalities involving mixed products of the Moore–Penrose and group inverses by matrix rank methodology*, Demonstratio Math. **58**(2025), 20240089.
[35] Z. Xiong, Y. Qin, *Mixed-type reverse-order laws for the generalized inverses of an operator product*, Arab. J. Sci. Eng. **36**(2011), 475–486.