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Null Legendre curves in Lorentzian hypersurfaces of 5-dimensional cosymplectic B-metric manifolds

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To Professor Svetislav Minčić, on his 95th birthday

Abstract. The object of study in the present paper is a class of null curves in Lorentzian hypersurfaces M of a 5-dimensional cosymplectic B-metric manifold $\overline{M}$, which are Legendre curves in the ambient manifold. We construct a basis along the examined curves through the almost contact B-metric structure of $\overline{M}$ and the induced objects in M. By using this basis, we prove that there exists a unique Cartan frame for the curves belonging to the investigated class. We show that if the Lorentzian hypersurface M is totally geodesic (resp. totally umbilical), then the curve is geodesic (resp. non-geodesic). Special attention is paid to the case when M is totally umbilical. We obtain that if M is an extrinsic sphere, then the studied curves are helices. We construct an example of a helix belonging to the considered class of null curves in a 4-dimensional anti-de Sitter space H_1^4, which is a Lorentzian hypersurface of $\mathbb{R}_2^5$, endowed with a cosymplectic B-metric structure.

1. Introduction

The study of null curves is of special interest from the point of view of both mathematical physics and differential geometry. This study is different from that of space-like and time-like curves. A distinguishing feature of null curves is that the length of any arc vanishes. For this reason, a new parameter (called the pseudo-arc), which normalizes the derivative of the tangent vector, is introduced. Also, contrary to the case of non-null curves, the normal bundle $TC^{\perp}$ of a null curve C in a proper semi-Riemannian manifold M contains the tangent bundle TC while $TC^{\perp}$ is also a null subbundle of TM. Thus, the sum of TC and $TC^{\perp}$ is not the whole of TM along a null curve C. In [9] Bejancu and Duggal developed the general theory of null curves considering TM as a sum of three non-intersecting complementary (but non-orthogonal) vector bundles - TC, the screen vector bundle $S(TC^{\perp})$, which is non-degenerate and finally the unique null vector bundle ntr(C) for a given $S(TC^{\perp})$.

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The general Frenet frame and its general Frenet equations of a null curve C in a Lorentzian manifold are given in [9, 10]. Since they depend on the parameter on C and the screen vector bundle, there exist different Frenet frames and equations of C. Bonnor dealt with this non-uniqueness problem by introducing a unique Frenet frame (called the Cartan frame) along a null curve in $\mathbb{R}^4_1$, parameterized by a pseudo-arc parameter. The Cartan frame consists of the minimum number of curvature functions, called the Cartan curvatures. The results of Bonnor were generalized by Ferrández-Giménez-Lucas in [2], where the authors examined null curves $C(t)$ in a Lorentzian manifold (M^m_1, g) for which t is a pseudo-arc parameter and $\{\dot{C}(t), \ddot{C}(t), \dots, C^{(m)}(t)\}$ is a basis of $T_{C(t)}M^m_1$ for all t. They proved that for a null curve belonging to this class there exists a unique Cartan frame expressed in terms of the considered basis. We deal here with 4-dimensional Lorentz manifolds, but 3-dimensional Lorentz manifolds are also of interest for investigation [13].

After the work [2], the study of null curves in 4-dimensional Lorentzian manifolds is focused on null curves in 4-dimensional Minkowski spaces (see [14], [1], [3], and the references therein). This fact motivate us to investigate null curves in 4-dimensional Lorentzian manifolds. On the other hand, Legendre curves in contact manifolds are important because a diffeomorphism of a contact manifold is a contact transformation if and only if it maps a Legendre curve to a Legendre curve. In [12], Belkhelfa et al. have examined Legendre curves in Riemannian and Lorentzian manifolds.

The main goal of the present paper is to study null curves in Lorentzian hypersurfaces of a 5-dimensional cosymplectic B-metric manifold, which are Legendre curves in the ambient manifold.

The paper is organized as follows. Section 2 contains some preliminaries about almost contact B-metric manifolds and geometry of null curves in 4-dimensional Lorentzian manifolds. In Section 3 we consider a Lorentzian hypersurface (M, g) of a 5-dimensional almost contact B-metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ with a unit time-like normal vector field $\overline{N}$ such that $\overline{\xi}$, $\overline{\varphi}\overline{N}$ and $\overline{\varphi}X$ $(X \in TM)$ are in a general position with respect to M, i.e. they have a tangent and a normal part. We prove (Proposition 3.3) that for a null curve $C(t)$ in M, which is a Legendre curve in $\overline{M}$, the vector fields $\{\dot{C}, \varphi\dot{C}, \xi_0, \xi_1\}$ form a basis of $T_{C(t)}M$ for all t, where $\varphi\dot{C}$, ξ_0 and ξ_1 are the tangent parts of $\overline{\varphi}\dot{C}$, $\overline{\xi}$ and $\overline{\varphi}\overline{N}$, respectively. Also, we show that three classes of the considered curves are interesting to be investigated, with respect to the functions $n = \overline{g}(\dot{C}, \overline{\varphi}\dot{C})$ and $\alpha(\dot{C}) = -g(\dot{C}, \xi_1)$. One of these classes, in case M is a Lorentzian hypersurface of a 5-dimensional cosymplectic B-metric manifold $\overline{M}$, we study in Section 4. This class consists of null curves C in M, which are Legendre curves in $\overline{M}$, such that any integral curve $\overline{C}$ of $\overline{\varphi}\dot{C}$ is also curve in M. Thus, $\overline{C}$ belongs to the same class as C, i.e. $\overline{C}$ is a null curve in M, which is a Legendre curve in $\overline{M}$. We give necessary and sufficient conditions for the examined curves to be geodesic (Theorem 4.4, Proposition 4.6). We establish that if M is totally geodesic, then C is geodesic. The main result in this section is Theorem 4.7, where we prove that for C, parameterized by the pseudo-arc, there exists a unique Cartan frame up to an orientation, which is expressed by the basis $\{\dot{C}, \varphi\dot{C}, \xi_0, \xi_1\}$ along C. We note that in this theorem we do not suggest the derivative vectors of C form a basis as in Theorem 3.1 [2, p. 5]. The last Section 5 is devoted to the study of the curves C from Section 4 when the Lorentzian hypersurface M of the cosymplectic B-metric manifold $\overline{M}$ is totally umbilical. We show (Corollary 5.2) that if M is totally umbilical, then C is non-geodesic. We find the unique Cartan frame and the Cartan curvatures $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$. In Proposition 5.4 we give equivalent conditions to the condition $\widetilde{\sigma}_2$ is a constant. We prove (Theorem 5.8) that if the function n is constant or M is an extrinsic sphere, then C is a helix (i.e. C has constant Cartan curvatures). Moreover, if M is an extrinsic sphere and $\overline{\xi}$ is tangent to M, then both $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ vanish (Corollary 5.9). At the end of Section 5, we construct a family of the studied curves in a 4-dimensional anti-de Sitter space H^4_1, which is a Lorentzian hypersurface of $\mathbb{R}^5_2$, endowed with a cosymplectic B-metric structure. We find a Cartan curve $\hat{C}$ belonging to this family and its Cartan frame. The obtained curve $\hat{C}$ is a helix.

2. Preliminaries

Let $\overline{M}$ be a $(2n + 1)$-dimensional smooth manifold, which is endowed with an almost contact structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta})$. Here $\overline{\varphi}$ is an endomorphism of the tangent bundle $T\overline{M}$, $\overline{\xi}$ is a Reeb vector field whoose dual 1-form is $\overline{\eta}$ and finally $\overline{\varphi}$, $\overline{\xi}$, $\overline{\eta}$ satisfy the following relations:

$$\overline{\varphi}^2\overline{X} = -\overline{X} + \overline{\eta}(\overline{X})\overline{\xi}, \qquad \overline{\eta}(\overline{\xi}) = 1. \tag{1}$$

If $\overline{M}$ is equipped with a pseudo-Riemannian metric $\overline{g}$ (known as *a B-metric*), satisfying

$$\overline{g}(\overline{\varphi}\overline{X}, \overline{\varphi}\overline{Y}) = -\overline{g}(\overline{X}, \overline{Y}) + \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y}), \tag{2}$$

then $\overline{M}$ is called *an almost contact B-metric manifold* [7] and it is denoted by $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. The term of B-metric used here is known in literature also as Norden metric. Here and further $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are tangent vector fields on $\overline{M}$. Immediate consequences of (1) and (2) are:

$$\begin{aligned} &\overline{\eta}\circ\overline{\varphi} = 0, \qquad \overline{\varphi}\overline{\xi} = 0, \qquad \overline{g}(\overline{\varphi}\overline{X}, Y) = \overline{g}(\overline{X}, \overline{\varphi}\overline{Y}), \\ &\overline{\eta}(\overline{X}) = \overline{g}(\overline{X}, \overline{\xi}), \qquad \overline{g}(\overline{\xi}, \overline{\xi}) = 1. \end{aligned} \tag{3}$$

The distribution $\mathbb{D} : x \in \overline{M} \longrightarrow \mathbb{D}_x \subset T_x\overline{M}$, where

$$\mathbb{D}_x = \operatorname{Ker}\overline{\eta} = \{\overline{X}_x \in T_x\overline{M} : \overline{\eta}(\overline{X}_x) = 0\}$$

is called *a contact distribution* generated by $\overline{\eta}$. Then the tangent space $T_x\overline{M}$ at each $x \in \overline{M}$ splits into the following orthogonal direct sum

$$T_x\overline{M} = \mathbb{D}_x \oplus span_{\mathbb{R}}\{\overline{\xi}_x\}.$$

The tensor field $\overline{\overline{g}}$ of type $(0,2)$ given by $\overline{\overline{g}}(\overline{X}, \overline{Y}) = \overline{g}(\overline{X}, \overline{\varphi}\overline{Y}) + \overline{\eta}(\overline{X})\overline{\eta}(\overline{Y})$ is also a B-metric, called *an associated metric* to $\overline{g}$. Both metrics $\overline{g}$ and $\overline{\overline{g}}$ are necessarily of signature $(n+1, n)$ $(+\ldots+-\ldots-)$.

The tensor field $\overline{F}$ of type $(0,3)$ on $\overline{M}$ is defined by

$$\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = \overline{g}((\overline{\nabla}_{\overline{X}}\overline{\varphi})\overline{Y}, \overline{Z}), \tag{4}$$

where $\overline{\nabla}$ is the Levi-Civita connection of the metric $\overline{g}$. It has the following properties:

$$\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = \overline{F}(\overline{X}, \overline{Z}, \overline{Y}) = \overline{F}(\overline{X}, \overline{\varphi}\overline{Y}, \overline{\varphi}\overline{Z}) + \overline{\eta}(\overline{Y})\overline{F}(\overline{X}, \overline{\xi}, \overline{Z}) + \overline{\eta}(\overline{Z})\overline{F}(\overline{X}, \overline{Y}, \overline{\xi}).$$

Moreover, we have

$$\overline{F}(\overline{X}, \overline{\varphi}\overline{Y}, \overline{\xi}) = (\overline{\nabla}_{\overline{X}}\overline{\eta})\overline{Y} = \overline{g}(\overline{\nabla}_{\overline{X}}\overline{\xi}, \overline{Y}). \tag{5}$$

In [7] a classification of the almost contact B-metric manifolds with respect to the tensor $\overline{F}$ is made and eleven basic classes $\mathcal{F}_i (i = 1, 2, \ldots, 11)$ are obtained. The special class $\mathcal{F}_0$ is the intersection of all basic classes and it is determined by the condition $\overline{F}(\overline{X}, \overline{Y}, \overline{Z}) = 0$. The class $\mathcal{F}_0$ is known as the class of the *cosymplectic B-metric manifolds.* By using (4) and (5), for a cosymplectic B-metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ we get

$$\overline{\nabla}\overline{\varphi} = 0, \qquad \overline{\nabla}\,\overline{\xi} = 0, \qquad \overline{\nabla}\overline{\eta} = 0. \tag{6}$$

In the remaining part of this section we provide basic concepts about null curves in a 4-dimensional Lorentzian manifold M_1^4 that we need in the following sections.

A *Lorentzian scalar product* on an n-dimensional real vector space V is a nondegenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ of index 1. This means one can find a basis $\{e_1, e_2, \ldots, e_n\}$ of V such that

$$\langle e_1, e_1\rangle = -1; \quad \langle e_i, e_i\rangle = 1,\ i \in \{2, \ldots, n\}; \quad \left\langle e_i, e_j\right\rangle = 0,\ i \neq j.$$

A *Lorentzian manifold* is a pair (M_1^n, g), where M is an n-dimensional smooth manifold and g is a Lorentzian metric, i.e. g_x is a Lorentzian scalar product on the tangent space T_xM at each point $x \in M$.

Let (M_1^4, g) be a 4-dimensional Lorentzian manifold and $C : I \longrightarrow M_1^4$ be a smooth curve in M_1^4 given locally by

$$x_i = x_i(t), \quad t \in I \subseteq \mathbb{R}, \quad i \in \{1, 2, 3, 4\}$$

for a coordinate neighborhood U of C. The tangent vector field is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4) = \dot{C},$$

where we denote $\frac{\mathrm{d}x_i}{\mathrm{d}t}$ by $\dot{x}_i$ for $i \in \{1, 2, 3, 4\}$.

The smooth curve C is said to be *a null (lightlike) curve* in (M_1^4, g), if at each point x of C we have

$$g(\dot{C}, \dot{C}) = 0, \qquad \dot{C} \neq 0 \quad \text{for} \quad \forall t \in I. \tag{7}$$

Consider a smooth null curve C immersed in an $(m + 2)$-dimensional proper semi-Riemannian manifold (M_q^{m+2}, g) of a constant index $q \geq 1$. It is known [9, 10] that the tangent bundle TM along C splits into a sum of the following three non-intersecting complementary (but non-orthogonal) vector bundles:

$$TM_{|C} = \{TC \oplus \mathrm{ntr}(C)\} \oplus_{orth} S(TC^{\perp}). \tag{8}$$

The m-dimensional vector bundle $S(TC^{\perp})$ is called *a screen vector bundle* of C in M. It is semi-Riemannian of index $(q - 1)$ and a complementary vector bundle to TC in the normal bundle $TC^{\perp}$ of C, i.e. $TC^{\perp} = TC \oplus_{orth} S(TC^{\perp})$. Moreover, given a $S(TC^{\perp})$ for a null curve C, there exists a unique null vector bundle $\mathrm{ntr}(C)$ of rank 1 which is called *a null transversal bundle*.

Based on the decomposition (8), there exists a quasi-orthonormal basis $\mathbf{F} = \{\dot{C}, N, W_1, W_2\}$ along a null curve C on a 4-dim Lorentzian manifold M_1^4, which means that the vector fields in $\mathbf{F}$ satisfy the equalities:

$$\begin{aligned} &g(N, N) = g(N, W_1) = g(N, W_2) = g(\dot{C}, W_1) = g(\dot{C}, W_2) = 0, \\ &g(\dot{C}, N) = g(W_1, W_1) = g(W_2, W_2) = 1. \end{aligned} \tag{9}$$

Also, from the decomposition (8), we have

$$TC = \mathrm{span}\{\dot{C}\}, \quad \mathrm{ntr}(C) = \mathrm{span}\{N\}, \quad S(TC^{\perp}) = \mathrm{span}\{W_1, W_2\}.$$

In [9, 10] the following *general Frenet equations* of a null curve C in M_1^4 with respect to $\mathbf{F}$ and the Levi-Civita connection ∇ on M_1^4 were obtained:

$$\begin{aligned} &\nabla_{\dot{C}}\dot{C} = h\dot{C} + k_1 W_1, \\ &\nabla_{\dot{C}}N = -hN + k_2 W_1 + k_3 W_2, \\ &\nabla_{\dot{C}}W_1 = -k_2\dot{C} - k_1 N + k_4 W_2, \\ &\nabla_{\dot{C}}W_2 = -k_3\dot{C} - k_4 W_1, \end{aligned} \tag{10}$$

where h and $\{k_1, k_2, k_3, k_4\}$ are smooth functions on a coordinate neighborhood U of C. The frame $\mathbf{F} = \{\dot{C}, N, W_1, W_2\}$ is called *a general Frenet frame* on M_1^4 along C with respect to the screen vector bundle $S(TC^{\perp}) = \mathrm{span}\{W_1, W_2\}$. The functions $\{k_1, k_2, k_3, k_4\}$ are *the curvature functions* of C with respect to $\mathbf{F}$.

The general Frenet frame $\mathbf{F}$ and its general Frenet equations (10) depend on the parameter and the choice of the screen vector bundle $S(TC^{\perp})$ of C and therefore they are not unique (see [9, 10]). In [15] Bonnor introduced a unique Frenet frame along a null curve in $\mathbb{R}_1^4$. This Frenet frame consists of the minimum number of curvature functions. It is called the Cartan frame and the null curve - a Cartan curve. Ferrández-Giménez-Lucas [2] studied null Cartan curves C in a Lorentzian manifold (M_1^{m+2}, g) which are parameterized by a *pseudo-arc parameter*, that is, $g(\nabla_{\dot{C}}\dot{C}, \nabla_{\dot{C}}\dot{C}) = 1$. From Theorem 3.1 [2, p. 5] it is known that for a null curve $C(t)$ in M_1^4 parameterized by the pseudo-arc such that $\{\dot{C}, \ddot{C}, C^{(3)}, C^{(4)}\}$ is a basis of $T_{C(t)}M_1^4$ for all t, there exists a unique Cartan frame satisfying the following Cartan equations:

$$\begin{aligned}
&\nabla_{\dot{C}}\dot{C} = W_1,\\
&\nabla_{\dot{C}}N = \sigma_1 W_1 + \sigma_2 W_2,\\
&\nabla_{\dot{C}}W_1 = -\sigma_1\dot{C} - N,\\
&\nabla_{\dot{C}}W_2 = -\sigma_2\dot{C},
\end{aligned} \tag{11}$$

where σ_1 and σ_2 are called the Cartan curvatures.

3. Null curves in Lorentzian hypersurfaces of a 5-dimensional almost contact B-metric manifold, which are Legendre curves in the ambient manifold

Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be a 5-dimensional almost contact B-metric manifold and M be a hypersurface of $\overline{M}$. We assume that there exists a unit time-like vector field $\overline{N}$, defined globally over M, i.e.

$$\overline{g}(\overline{N}, \overline{N}) = -1. \tag{12}$$

We denote by g the restriction of $\overline{g}$ on M. Then (M, g) is a 4-dimensional Lorentzian manifold. In what follows, we use the notations $\mathcal{F}(M)$ and $\chi(M)$ for the set of all smooth real functions and vector fields on M, respectively. Also, X, Y, Z, W stand for vector fields belonging to $\chi(M)$.

Let us consider the following decomposition for $\overline{\xi}$, $\overline{\varphi}X$, $\overline{\varphi}N$ with respect to TM and $\overline{N}$:

$$\overline{\xi} = \xi_0 + a\overline{N}, \tag{13}$$

$$\overline{\varphi}X = \varphi X + \alpha(X)\overline{N}, \quad X \in \chi(M) \tag{14}$$

$$\overline{\varphi}\overline{N} = \xi_1 + b\overline{N}, \tag{15}$$

where: ξ_0, $\xi_1 \in \chi(M)$; $a, b \in \mathcal{F}(M)$; φ is a tensor field of type $(1,1)$ on M and α is a 1-form on M. By using the latter three equalities and (2), (3), (12), we obtain

$$a = -\overline{\eta}(\overline{N}), \qquad b = -\overline{g}(\overline{N}, \overline{\varphi}\overline{N}), \tag{16}$$

$$\alpha(X) = -\overline{g}(X, \overline{\varphi}\overline{N}) = -g(X, \xi_1), \tag{17}$$

$$g(\xi_0, \xi_0) = 1 + a^2, \qquad g(\xi_0, \xi_1) = ab, \qquad g(\xi_1, \xi_1) = 1 + a^2 + b^2. \tag{18}$$

We note that contrary to the case of almost contact metric and almost paracontact metric manifolds, the function b is not zero in general. The equalities (1), (13), (14) and (15) imply

$$\begin{aligned}
&\varphi^2 X = -X - \alpha(X)\xi_1 + \overline{\eta}(X)\xi_0,\\
&\alpha(\varphi X) + b\alpha(X) = a\overline{\eta}(X).
\end{aligned} \tag{19}$$

By using (2) and (14) we get

$$\begin{aligned}
&g(X, \varphi Y) = g(\varphi X, Y),\\
&g(\varphi X, \varphi Y) = -g(X, Y) + \alpha(X)\alpha(Y) + \overline{\eta}(X)\overline{\eta}(Y).
\end{aligned} \tag{20}$$

From $\overline{\varphi}\overline{\xi} = 0$ and $\overline{\varphi}^2\overline{N} = -\overline{N} + \overline{\eta}(\overline{N})\overline{\xi}$, taking into account (13)÷(16), we derive

$$\varphi\xi_0 = -a\xi_1 \qquad \text{and} \qquad \varphi\xi_1 = -a\xi_0 - b\xi_1, \tag{21}$$

respectively.
Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections of the metrics $\overline{g}$ and g on $\overline{M}$ and M, respectively. Then the Gauss-Weingarten formulas are:

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y),$$
$$\overline{\nabla}_X \overline{N} = -A_{\overline{N}}X + \nabla^{\perp}_X \overline{N}.$$

Here, B is the second fundamental form, $A_{\overline{N}}$ is the shape operator with respect to $\overline{N}$ and $\nabla^{\perp}$ is the normal connection on the normal bundle $TM^{\perp}$. For B, $A_{\overline{N}}$ and $\nabla^{\perp}$ we obtain

$$B(X,Y) = -g(A_{\overline{N}}X,Y)\overline{N} = -g(X,A_{\overline{N}}Y)\overline{N}, \quad \nabla^{\perp}_X \overline{N} = 0. \tag{22}$$

Hence, the Gauss-Weingarten formulas become:

$$\overline{\nabla}_X Y = \nabla_X Y - g(A_{\overline{N}}X,Y)\overline{N},$$
$$\overline{\nabla}_X \overline{N} = -A_{\overline{N}}X. \tag{23}$$

D. E. Blair introduced in [6] a Legendre curve γ as an integral curve in the contact distribution $D = \operatorname{Ker}\eta$ of a contact manifold (M', φ, ξ, η). Having in mind that a Frenet curve γ in M' is a Legendre curve if and only if $\eta(\dot{\gamma}) = 0$ (see [6]), a Legendre curve in an almost contact B-metric manifold is defined as follows:

Definition 3.1. *A smooth curve C in an almost contact B-metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is said to be a Legendre curve if $\overline{\eta}(\dot{C}) = 0$ at each point of C.*

Let C be a null curve in (M, g), which is a Legendre curve in $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. We put

$$\overline{g}(\dot{C}, \overline{\varphi}\dot{C}) = n, \tag{24}$$

where n is a smooth function on a curve C in M. By using (13), (14), (17), (20), (21) and (24) we find

$$\overline{\eta}(\dot{C}) = g(\dot{C}, \xi_0) = 0, \tag{25}$$

$$g(\varphi\dot{C}, \varphi\dot{C}) = \alpha((\dot{C}))^2, \quad g(\varphi\dot{C}, \xi_0) = a\alpha(\dot{C}), \quad g(\varphi\dot{C}, \xi_1) = b\alpha(\dot{C}). \tag{26}$$

Proposition 3.2. *Let $C : I \longrightarrow M$ be a null curve in M, which is a Legendre curve in $\overline{M}$. For every $t \in I$ at least one of $n(t)$ and $\alpha(\dot{C}(t))$ is not zero.*

Proof. Let us assume that there exists $t_1 \in I$ such that $n(t_1) = 0$ and $\alpha(\dot{C}(t_1)) = 0$. The condition $\alpha(\dot{C}(t_1)) = 0$ and (14) imply $\overline{\varphi}\dot{C}(t_1)$ is a tangent vector field to M. Moreover, taking into account (2), we have $\overline{g}(\overline{\varphi}\dot{C}, \overline{\varphi}\dot{C}) = 0$ along C. Hence, $\dot{C}(t_1)$ and $\overline{\varphi}\dot{C}(t_1)$ are orthogonal lightlike tangent vector fields to M. From a well known fact in the Lorentzian geometry it follows that $\dot{C}(t_1)$ and $\overline{\varphi}\dot{C}(t_1)$ are linearly dependent. Then $\overline{\varphi}\dot{C}(t_1) = u\dot{C}(t_1)$, $u \in \mathbb{R}$. Acting with $\overline{\varphi}$ to the both sides of this equality, we get $-\dot{C}(t_1) = u^2\dot{C}(t_1)$. The latter leads to a contradiction. □

Proposition 3.3. *Let $C : I \longrightarrow M$ be a null curve in M, which is a Legendre curve in $\overline{M}$. Then the vector fields $\{\dot{C}, \varphi\dot{C}, \xi_0, \xi_1\}$ form a basis of $T_{C(t)}M$ for all $t \in I$.*

Proof. In a standard way, using (18), (25) and (26), we obtain that the vector fields $\{\dot{C}, \varphi\dot{C}, \xi_0, \xi_1\}$ are linearly independent along C if and only if $\Delta \neq 0$, where

$$\Delta = -n^2\left(1 + a^2\right)^2 - \left(nb + (\alpha(\dot{C}))^2\right)^2.$$

Now, applying Proposition 3.2, we complete the proof. □

4. Null curves in Lorentzian hypersurfaces of a 5-dimensional cosymplectic B-metric manifold, which are Legendre curves in the ambient manifold

From now on, $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a 5-dimensional cosymplectic B-metric manifold and (M, g) is the Lorentzian hypersurface of $\overline{M}$, defined by (12). Further, by straightforward computations, we establish that the induced objects φ, α, ξ_0, ξ_1 on M satisfy the following conditions:

- by using $(\overline{\nabla}_X\overline{\varphi})Y = 0$, (14), (15) and (23), we obtain

$$(\nabla_X\varphi)Y = -g(A_{\overline{N}}X, Y)\xi_1 + \alpha(Y)A_{\overline{N}}X, \qquad (\nabla_X\alpha)Y = g(A_{\overline{N}}X, \varphi Y) - bg(A_{\overline{N}}X, Y); \tag{27}$$

- the equalities $(\overline{\nabla}_X\overline{\varphi})\overline{N} = 0$, (14), (15), (17) and (23) imply

$$\nabla_X\xi_1 = bA_{\overline{N}}X - \varphi(A_{\overline{N}}X), \tag{28}$$

$$X(b) = 2g(A_{\overline{N}}X, \xi_1); \tag{29}$$

- from $\overline{\nabla}_X\overline{\xi} = 0$, (13) and (23) we get

$$\nabla_X\xi_0 = aA_{\overline{N}}X, \tag{30}$$

$$X(a) = g(A_{\overline{N}}X, \xi_0). \tag{31}$$

Next, we study the considered null curves in Section 3 using the basis constructed in Proposition 3.3. According to Proposition 3.2, we can investigate the following three classes of such curves: both $\alpha(\dot{C(t)})$ and $n(t)$ are not zero for all t; $\alpha(\dot{C(t)}) \neq 0$ and $n(t) = 0$ for all t; $\alpha(\dot{C(t)}) = 0$ and $n(t) \neq 0$ for all t.

The condition $\alpha(\dot{C(t)}) = 0$ along C has a clear geometric meaning, namely $\overline{\varphi}\dot{C(t)} \in \chi(M)$. Moreover, if $\overline{C}$ is an integral curve of $\overline{\varphi}\dot{C}$ in M, then $\overline{C}$ is also a null curve in M, which is a Legendre curve in $\overline{M}$. Motivated by this fact, in the present paper we begin with the examination of curves of the third class.

In the remaining part of the paper, C will stand for a null curve in M, which is a Legendre curve in $\overline{M}$, such that $\alpha(\dot{C}) = 0$ and $n \neq 0$ along C. Then, taking into account that $\overline{\varphi}\dot{C} = \varphi\dot{C}$, (26) becomes:

$$g(\overline{\varphi}\dot{C}, \overline{\varphi}\dot{C}) = g(\overline{\varphi}\dot{C}, \xi_0) = g(\overline{\varphi}\dot{C}, \xi_1) = 0. \tag{32}$$

Further, for the sake of brevity, we use the notations $Q = g(\nabla_{\dot{C}}\dot{C}, \xi_0)$, $P = g(\nabla_{\dot{C}}\dot{C}, \xi_1)$ and $\dot{C}(f) = \dot{f}$ for any $f \in \mathcal{F}(M)$. We give the following

Lemma 4.1. *For C the following equalities are fulfilled:*

$$g(\nabla_{\dot{C}}\dot{C}, \dot{C}) = 0, \tag{33}$$

$$g(\nabla_{\dot{C}}\dot{C}, \overline{\varphi}\dot{C}) = g(\nabla_{\dot{C}}\overline{\varphi}\dot{C}, \dot{C}) = \frac{\dot{n}}{2}, \tag{34}$$

$$Q = -ag(A_{\overline{N}}\dot{C}, \dot{C}), \tag{35}$$

$$P = \left(\nabla_{\dot{C}}\alpha\right)\dot{C} = g(A_{\overline{N}}\dot{C}, \overline{\varphi}\dot{C}) - bg(A_{\overline{N}}\dot{C}, \dot{C}). \tag{36}$$

Proof. As an immediate consequence from (7) we obtain (33). The first equality in (27), $\alpha(\dot{C}) = 0$ and (24) imply (34). By virtue of (30) and (25) we get (35). By using (17) and the second equality in (27) we receive (36). □

Proposition 4.2. *With respect to the basis* $\{\dot{C}, \overline{\varphi}\dot{C}, \xi_0, \xi_1\}$ *of* $T_{C(t)}M$ *the vector field* $\nabla_{\dot{C}}\dot{C}$ *is given by*

$$\nabla_{\dot{C}}\dot{C} = \lambda_1\dot{C} + \mu_1\overline{\varphi}\dot{C} + \nu_1\xi_0 + \delta_1\xi_1, \tag{37}$$

where λ_1, μ_1, ν_1, δ_1 *are the following functions along* C

$$\lambda_1 = \frac{\dot{n}}{2n}, \qquad \mu_1 = 0, \tag{38}$$

$$\nu_1 = \frac{(1+a^2+b^2)Q - abP}{(1+a^2)^2 + b^2}, \tag{39}$$

$$\delta_1 = \frac{-abQ + (1+a^2)P}{(1+a^2)^2 + b^2}. \tag{40}$$

For the curvature k_1 *we have*

$$k_1^2 = \frac{(1+a^2+b^2)Q^2 - 2abQP + (1+a^2)P^2}{(1+a^2)^2 + b^2}. \tag{41}$$

Proof. With respect to the basis $\{\dot{C}, \overline{\varphi}\dot{C}, \xi_0, \xi_1\}$ of $T_{C(t)}M$, $\nabla_{\dot{C}}\dot{C}$ has the decomposition (37). Using (33), (34) we obtain the following linear system of equations for λ_1, μ_1, ν_1, δ_1

$$\left| \begin{array}{l} g(\nabla_{\dot{C}}\dot{C}, \dot{C}) = 0 = n\mu_1 \\ g(\nabla_{\dot{C}}\dot{C}, \overline{\varphi}\dot{C}) = \dfrac{\dot{n}}{2} = n\lambda_1 \\ Q = (1+a^2)\nu_1 + ab\delta_1 \\ P = ab\nu_1 + (1+a^2+b^2)\delta_1. \end{array} \right.$$

The determinant of the above system is $\Delta_1 = -n^2\left((1+a^2)^2 + b^2\right)$. Since $n \neq 0$, the system has a unique solution given by (38), (39) and (40). From the first equality in (10) it follows that

$$k_1^2 = g(\nabla_{\dot{C}}\dot{C}, \nabla_{\dot{C}}\dot{C}) = (1+a^2)\nu_1^2 + 2ab\nu_1\delta_1 + (1+a^2+b^2)\delta_1^2.$$

Substituting (39) and (40) in the latter equality we obtain (41). □

As an immediate consequence of Proposition 4.2, we state

Corollary 4.3. *The original parameter* t *is a pseudo-arc parameter of* $C(t)$ *if and only if*

$$\frac{(1+a^2+b^2)Q^2 - 2abQP + (1+a^2)P^2}{(1+a^2)^2 + b^2} = 1. \tag{42}$$

It is known [9] that a null curve is geodesic if and only if the curvature k_1 vanishes. Using (41) we obtain that $C(t)$ is geodesic if and only if $Q = P = 0$ for all t. Now, taking into account (13), (35) and (36), we state

Theorem 4.4. *Let* $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *be a 5-dimensional cosymplectic B-metric manifold and* M *be the Lorentzian hypersurface of* $\overline{M}$*, defined by* (12). *Then for* $C(t)$ *in* M *the following assertions are equivalent:*
(i) $C(t)$ *is geodesic in* M;
(ii) $\overline{\xi}$ *is tangent to* M *along* C *or* $g(A_{\overline{N}}\dot{C}, \dot{C}) = 0$ *for all* t *and* α *is parallel along* C.

Definition 4.5. *[11] A submanifold S in a (semi-) Riemannian manifold $(\overline{S}, h)$ is said to be:*

(i) *totally geodesic if its shape operator A vanishes identically, that is, $A = 0$ or equivalently the second fundamental form B vanishes identically;*

(ii) *umbilical with respect to the normal vector field V to S if $A_V = fI$ (I is the identity transformation) for some function f;*

(iii) *totally umbilical if S is umbilical with respect to every normal vector field to S.*

Proposition 4.6. *Each of the following statements*

(i) M is totally geodesic;

(ii) ξ_0 is parallel along C and $\overline{\xi}$ is not tangent to M along C;

(iii) $\overline{\xi}$ is tangent to M and α is parallel along C.

(iv) ξ_1 is parallel along C;

is sufficient to guarantee that C is geodesic.

Proof. With the help of (13), (30), (35), (36) and (41), we easy check that any of the statements (i), (ii) and (iii) is sufficient to guarantee that C is geodesic.

(iv) Since $\nabla_{\dot{C}}\xi_1 = 0$, from (28) we have $bA_{\overline{N}}\dot{C} = \varphi(A_{\overline{N}}\dot{C})$. Acting with φ to the both sides of this equality and using the first equality in (19), we obtain $(b^2 + 1)(A_{\overline{N}}\dot{C}) = -\alpha(A_{\overline{N}}\dot{C})\xi_1 + \overline{\eta}(A_{\overline{N}}\dot{C})\xi_0$. Now, taking into account (32), (35), (36) and (41), we complete the proof. □

Theorem 4.7. *Let the original parameter t of $C(t)$ be a pseudo-arc parameter. Then there exists a unique Cartan frame $\{\dot{C}, \mathcal{N}, \mathcal{W}_1, \mathcal{W}_2\}$ up to an orientation, which with respect to the basis $\{\dot{C}, \overline{\varphi}\dot{C}, \xi_0, \xi_1\}$ of $T_{C(t)}M$ is given by*

$$\mathcal{W}_1 = \nabla_{\dot{C}}\dot{C} = \lambda_1\dot{C} + \nu_1\xi_0 + \delta_1\xi_1, \tag{43}$$

$$\mathcal{W}_2 = \lambda_2\dot{C} + \mu_2\overline{\varphi}\dot{C} + \nu_2\xi_0 + \delta_2\xi_1, \tag{44}$$

$$\mathcal{N} = \alpha\dot{C} + \beta\overline{\varphi}\dot{C} + \gamma\xi_0 + \delta\xi_1, \tag{45}$$

where λ_1, ν_1, δ_1 are given by (38), (39), (40) *and λ_2, μ_2, ν_2, δ_2, α, β, γ, δ are the following functions along C*

$$\begin{aligned}\lambda_2 = {} & \frac{\epsilon}{[(1+a^2)^2+b^2]^{\frac{3}{2}}}\Big\{(P\dot{Q} - Q\dot{P})\left[(1+a^2)^2+b^2\right] \\ & +\dot{a}b\left[(1+b^2)Q^2 - 2abQP + a^2P^2\right] \\ & +\frac{\dot{b}}{2}\left[a(1+a^2-b^2)Q^2 + 2(1+a^2)bQP - a(1+a^2)P^2\right]\Big\},\end{aligned} \tag{46}$$

$$\mu_2 = 0, \quad \nu_2 = \frac{\epsilon P}{\sqrt{(1+a^2)^2+b^2}}, \quad \delta_2 = \frac{\epsilon Q}{\sqrt{(1+a^2)^2+b^2}}, \tag{47}$$

$$\alpha = -\frac{1}{2}\left(\lambda_2^2 + \frac{\dot{n}^2}{4n^2}\right), \qquad \beta = \frac{1}{n}, \tag{48}$$

$$\gamma = -\frac{\epsilon\lambda_2 P}{\sqrt{(1+a^2)^2+b^2}} + \frac{\dot{n}[abP - (1+a^2+b^2)Q]}{2n\,[(1+a^2)^2+b^2]}, \tag{49}$$

$$\delta = \frac{\epsilon\lambda_2 Q}{\sqrt{(1+a^2)^2+b^2}} - \frac{\dot{n}[(1+a^2)P - abQ]}{2n\,[(1+a^2)^2+b^2]}, \qquad \epsilon = \pm 1. \tag{50}$$

Proof. A frame $\{\dot{C}, N, W_1, W_2\}$ along a null curve C, parameterized by the pseudo-arc parameter, is a Cartan frame in the sense of Theorem 3.1 and Definition 3.2 [2, p. 5] if it satisfies (9) and (11). Note that (11) are obtained from (9) by $h = 0$, $k_1 = 1$, $k_4 = -g(\nabla_{\dot{C}} W_2, W_1) = 0$ and labeling $k_2 = \sigma_1$, $k_3 = \sigma_2$.

The vector field $\mathcal{W}_1$, defined by (43), satisfies the first equation in (11). Since t is a pseudo-arc parameter of $C(t)$, we have $g(\mathcal{W}_1, \mathcal{W}_1) = 1$. From Corollary 4.3 it follows that for the functions a, b, Q and P the equality (42) holds. Next, we look for a vector field $\mathcal{W}_2$ such that $g(\mathcal{W}_2, \dot{C}) = g(\mathcal{W}_2, \mathcal{W}_1) = 0$, $g(\mathcal{W}_2, \mathcal{W}_2) = 1$ and $k_4 = -g(\nabla_{\dot{C}} \mathcal{W}_2, \mathcal{W}_1) = 0$. By using (18), (24), (25), (32), we obtain the following system for the functions λ_2, μ_2, ν_2, δ_2 in (44):

$$\left| \begin{array}{l} n\mu_2 = 0 \\ Q\nu_2 + P\delta_2 = 0 \\ (1 + a^2)\nu_2^2 + 2ab\nu_2\delta_2 + (1 + a^2 + b^2)\delta_2^2 = 1 \\ \lambda_2 + \dot{a}(a\nu_1\nu_2 + b\nu_1\delta_2 + a\delta_1\delta_2) \\ +\dfrac{\dot{b}}{2}(a\delta_1\nu_2 + a\nu_1\delta_2 + 2b\delta_1\delta_2) - \dot{Q}\nu_2 - \dot{P}\delta_2 = 0. \end{array} \right. \tag{51}$$

Taking into account that $n \neq 0$, from the first equation in (51) we get $\mu_2 = 0$. Since $C(t)$ is non-geodesic, we have $(Q, P) \neq (0, 0)$ along $C(t)$. Let us assume that $P \neq 0$. Then $\delta_2 = -\dfrac{Q\nu_2}{P}$, which we substitute in the third equation in (51) and obtain the expressions for ν_2 and δ_2 in (47). The function λ_2 we find from the last equation in (51), using (39), (40) and (47). The unique null transversal bundle ntr(C) of C with respect to the screen vector bundle $S(TC^{\perp}) = \text{span}\{\mathcal{W}_1, \mathcal{W}_2\}$ is spanned by the vector field $\mathcal{N}$, which satisfies the conditions: $g(\mathcal{N}, \dot{C}) = 1$, $g(\mathcal{N}, \mathcal{W}_1) = g(\mathcal{N}, \mathcal{W}_2) = g(\mathcal{N}, \mathcal{N}) = 0$. The functions α, β, γ, δ in (45) we determine from the system

$$\left| \begin{array}{l} n\beta = 1 \\ Q\gamma + P\delta + \dfrac{\dot{n}}{2n} = 0 \\ \lambda_2 + [(1 + a^2)\nu_2 + ab\delta_2]\gamma + [ab\nu_2 + (1 + a^2 + b^2)\delta_2]\delta = 0 \\ (1 + a^2)\gamma^2 + 2ab\gamma\delta + (1 + a^2 + b^2)\delta^2 + 2\alpha = 0. \end{array} \right. \tag{52}$$

From the first and the second equation in (52) we obtain $\beta = \dfrac{1}{n}$ and $\delta = -\dfrac{Q}{P}\gamma - \dfrac{\dot{n}}{2nP}$, respectively. By using the latter equality and (47), from the third equation in (52) we get (49), (50). Substituting (49) and (50) in the last equation in (52), we find the function α, which is given in (48).

After substituting the obtained expression for λ_2 in (48), (49) and (50), we see that the functions α, γ and δ do not depend on ϵ, which means that $\mathcal{N}$ is unique. If we replace ϵ with 1 (respectively -1) in (46) and (47), we obtain two opposite vector fields, which we denote by $\mathcal{W}_2^+$ (respectively $\mathcal{W}_2^-$). Hence, both frames $\{\dot{C}, \mathcal{N}, \mathcal{W}_1, \mathcal{W}_2^+\}$ and $\{\dot{C}, \mathcal{N}, \mathcal{W}_1, \mathcal{W}_2^-\}$ are Cartan frames of C that differ only in having opposite orientations. □

Remark 4.8. *In Theorem 4.4, we proved that there exists a unique Cartan frame for C without the condition $\{\dot{C}, \ddot{C}, C^{(3)}, C^{(4)}\}$ to be linearly independent. Let us note that in Theorem 3.1 [2, p. 5], Ferrández et al. obtained a unique Cartan frame by the assumption that the derivative vectors of the curve form a basis.*

5. Null curves in totally umbilical Lorentzian hypersurfaces of a 5-dimensional cosymplectic B-metric manifold, which are Legendre curves in the ambient manifold

In Proposition 4.6 we established that if M is totally geodesic, then C is geodesic. In this section we consider the case when M is totally umbilical. Then $A_{\overline{N}} = fI$, where $f = \dfrac{\mathrm{tr}A_{\overline{N}}}{4}$.

Lemma 5.1. *If M is totally umbilical, we have:*
(i) the functions a and b are constant along C;
(ii) $\nabla_{\dot{C}}\xi_0 = af\dot{C}$, $\nabla_{\dot{C}}\xi_1 = bf\dot{C} - f\overline{\varphi}\dot{C}$;
(iii) $Q = 0$, $P = fn$;
(iv) the original parameter t of $C(t)$ is a pseudo-arc parameter if and only if

$$f = \epsilon_1 \frac{1}{n}\sqrt{\frac{(1+a^2)^2+b^2}{1+a^2}}, \quad \epsilon_1 = \pm 1. \tag{53}$$

Proof. (*i*) We substitute X with $\dot{C}$ in (29) and (31). Taking into account that $A_{\overline{N}}\dot{C} = f\dot{C}$, $\alpha(\dot{C}) = 0$, $g(\dot{C}, \xi_0) = 0$, we get $\dot{b} = 0$ and $\dot{a} = 0$, respectively. Hence, the functions a and b are constant along C.

In case M is totally umbilical, (30), (28) and (35), (36) become the equalities in (*ii*) and (*iii*), respectively.

(*iv*) According to Corollary 4.3, the original parameter t of $C(t)$ is a pseudo-arc parameter if and only if (42) holds. Substituting $Q = 0$ and $P = fn$ in (42) we obtain (53). □

The assertion (*iii*) in Lemma 5.1 and (41) imply

Corollary 5.2. *If M is totally umbilical, then C is non-geodesic.*

By using Theorem 4.7 and Lemma 5.1, we state the following theorem:

Theorem 5.3. *Let the original parameter t of $C(t)$ be a pseudo-arc parameter, M totally umbilical and $fn > 0$ (respectively $fn < 0$) for all t. Then there exists a unique Cartan frame $\{\dot{C}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{W}}_1, \widetilde{\mathcal{W}}_2\}$ up to an orientation, which with respect to the basis $\{\dot{C}, \overline{\varphi}\dot{C}, \xi_0, \xi_1\}$ of $T_{C(t)}M$ is given by*

$$\begin{aligned}
\widetilde{\mathcal{W}}_1 &= \frac{\dot{n}}{2n}\dot{C} - \frac{\epsilon_1 ab}{\sqrt{(1+a^2)\left((1+a^2)^2+b^2\right)}}\xi_0 + \frac{\epsilon_1\sqrt{1+a^2}}{\sqrt{(1+a^2)^2+b^2}}\xi_1, \\
\widetilde{\mathcal{W}}_2 &= \frac{\epsilon\epsilon_1}{\sqrt{1+a^2}}\xi_0, \\
\widetilde{\mathcal{N}} &= -\frac{\dot{n}^2}{8n^2}\dot{C} + \frac{1}{n}\overline{\varphi}\dot{C} + \frac{\epsilon_1 ab\dot{n}}{2n\sqrt{(1+a^2)\left((1+a^2)^2+b^2\right)}}\xi_0 \\
&\quad - \frac{\epsilon_1\sqrt{1+a^2}\,\dot{n}}{2n\sqrt{(1+a^2)^2+b^2}}\xi_1,
\end{aligned} \tag{54}$$

where $\epsilon = \pm 1$ and $\epsilon_1 = 1$ (respectively $\epsilon_1 = -1$) if $fn > 0$ (respectively $fn < 0$) for all t.

Further, using (11), for the Cartan curvatures $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ of C in a totally umbilical Lorentzian hypersurface M we have

$$\widetilde{\sigma}_1 = -g(\nabla_{\dot{C}}\widetilde{\mathcal{W}}_1, \widetilde{\mathcal{N}}), \qquad \widetilde{\sigma}_2 = -g(\nabla_{\dot{C}}\widetilde{\mathcal{W}}_2, \widetilde{\mathcal{N}}).$$

By standard calculations, taking into account Lemma 5.1 and (54), we obtain

$$\widetilde{\sigma}_1 = \frac{1}{n}\left(-\frac{\ddot{n}}{2} + \frac{3\dot{n}^2}{8n} - \frac{b}{1+a^2}\right), \tag{55}$$

$$\widetilde{\sigma}_2 = -\frac{\epsilon a\sqrt{(1+a^2)^2+b^2}}{n(1+a^2)}, \quad \epsilon = \pm 1. \tag{56}$$

As an immediate consequence of (56), Lemma 5.1 and (13), we establish

Proposition 5.4. *Let M be totally umbilical and let $\widetilde{\sigma}_2$ be the Cartan curvature of C. Then the following assertions are equivalent:*

(i) $\widetilde{\sigma}_2$ is a constant function;

(ii) n is a constant function;

(iii) f is a constant function along C.

Moreover, $\widetilde{\sigma}_2 = 0$ if and only if $\overline{\xi}$ is tangent to M along C.

Corollary 5.5. *If M is totally umbilical and $\overline{\xi}$ is tangent to M, then $\widetilde{\sigma}_2 = 0$.*

Let us recall that the mean curvature vector H of a submanifold S in $\overline{S}$ is defined by $H = \frac{\mathrm{tr}B}{\mathrm{dim}S}$, where B stands for the second fundamental form. If $\nabla^\perp$ is the normal connection on $TS^\perp$, then the mean curvature vector H is called parallel if $\nabla^\perp H = 0$ identically.

Definition 5.6. *[5] A totally umbilical submanifold with a non-zero parallel mean curvature vector is said to be an extrinsic sphere.*

Definition 5.7. *[2] A null curve is said to be a helix if it has constant Cartan curvatures.*

Theorem 5.8. *Each of the following statements*

(i) M is totally umbilical and n is a constant function;

(ii) M is an extrinsic sphere;

is sufficient to guarantee that C is a helix, whose Cartan curvatures $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ satisfy

$$\widetilde{\sigma}_1 = \frac{\epsilon b\widetilde{\sigma}_2}{a\sqrt{(1+a^2)^2+b^2}}, \quad \epsilon = \pm 1. \tag{57}$$

Proof. (i) If n is a constant function, then from (55), (56) and a, b are constant functions along C it follows that the Cartan curvatures $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are also constant, for which (57) holds.

(*ii*) The mean curvature vector H of M is given by $H = -\frac{(\mathrm{tr}A_{\overline{N}})}{4}\overline{N}$. Since $\nabla^\perp_X\overline{N} = 0$, from $\nabla^\perp_X H = 0$ we obtain that $X(\mathrm{tr}A_{\overline{N}}) = 0$. The latter implies $\mathrm{tr}A_{\overline{N}}$ is a constant on M and hence f is also a constant on M. From (53) we deduce that n is a constant function, completing the proof. □

Corollary 5.9. *Let M be an extrinsic sphere.*

(i) If $\overline{\varphi}\overline{N}$ is tangent to M, then $\widetilde{\sigma}_1 = 0$;

(ii) If $\overline{\xi}$ is tangent to M, then $\widetilde{\sigma}_1 = \widetilde{\sigma}_2 = 0$.

Example 5.10. *Let $\overline{M} = \mathbb{R}^5_2 = \{u = (z^1, z^2, z^3, z^4, z^5) \mid z^i \in \mathbb{R}\}$. We define an almost contact B-metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on $\mathbb{R}^5_2$ in the following way:*

$$\begin{gathered}\overline{\varphi}\left(\frac{\partial}{\partial z^i}\right) = \frac{\partial}{\partial z^{i+2}}, \qquad \overline{\varphi}\left(\frac{\partial}{\partial z^{i+2}}\right) = -\frac{\partial}{\partial z^i}, \quad i = 1, 2; \\ \overline{\varphi}\left(\frac{\partial}{\partial z^5}\right) = 0, \qquad \overline{\xi} = \frac{\partial}{\partial z^5}; \qquad \overline{\eta} = \mathrm{d}z^5\end{gathered} \tag{58}$$

and $\overline{g}$ is a pseudo-Euclidean scalar product, determined by the equality

$$\langle u, u\rangle = \sum_{i=1}^{2}\left\{-(z^i)^2 + (z^{i+2})^2\right\} + (z^5)^2. \tag{59}$$

It is easy to see that $\overline{\nabla}\,\overline{\varphi} = 0$. *Hence,* $\overline{M} = (\mathbb{R}^5_2, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *is a cosymplectic B-metric manifold.*

Identifying the point $(z^1, z^2, z^3, z^4, z^5) \in (\mathbb{R}^5_2, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *with its position vector* Z, *we define a hypersurface* $M = H^4_1$ *by*

$$\overline{g}(Z, Z) = -1.$$

We consider the following parametric equations of H^4_1:

$$H^4_1 : \begin{cases} z^1 = \cos u \cosh v \\ z^2 = \sin u \cosh v \\ z^3 = \sinh v \sin w \sin q \\ z^4 = \sinh v \sin w \cos q \\ z^5 = \sinh v \cos w \end{cases}, \quad v \neq 0, \quad w \neq 0. \tag{60}$$

The position vector Z *is normal to* $TH^4_1 = \operatorname{span}\left\{\frac{\partial Z}{\partial u}, \frac{\partial Z}{\partial v}, \frac{\partial Z}{\partial w}, \frac{\partial Z}{\partial q}\right\}$, *where*

$$\begin{cases} \frac{\partial Z}{\partial u} = (-\cosh v \sin u, \cosh v \cos u, 0, 0, 0)\,, \\ \frac{\partial Z}{\partial v} = (\cos u \sinh v, \sin u \sinh v, \cosh v \sin w \sin q, \\ \qquad \cosh v \sin w \cos q, \cosh v \cos w)\,, \\ \frac{\partial Z}{\partial w} = (0, 0, \sinh v \cos w \sin q, \sinh v \cos w \cos q, -\sinh v \sin w)\,, \\ \frac{\partial Z}{\partial q} = (0, 0, \sinh v \sin w \cos q, -\sinh v \sin w \sin q, 0) \end{cases}.$$

The induced metric g *of* $\overline{g}$ *on* H^4_1 *is given by*

$$g\left(\frac{\partial Z}{\partial u}, \frac{\partial Z}{\partial u}\right) = -\cosh^2 v, \quad g\left(\frac{\partial Z}{\partial v}, \frac{\partial Z}{\partial v}\right) = 1,$$

$$g\left(\frac{\partial Z}{\partial w}, \frac{\partial Z}{\partial w}\right) = \sinh^2 v, \quad g\left(\frac{\partial Z}{\partial q}, \frac{\partial Z}{\partial q}\right) = \sinh^2 v \sin^2 w,$$

$$g\left(\frac{\partial Z}{\partial u}, \frac{\partial Z}{\partial u}\right) = g\left(\frac{\partial Z}{\partial v}, \frac{\partial Z}{\partial v}\right) = g\left(\frac{\partial Z}{\partial w}, \frac{\partial Z}{\partial w}\right) = g\left(\frac{\partial Z}{\partial q}, \frac{\partial Z}{\partial q}\right) = 0,$$

which means that g *is a Lorentzian metric on* H^4_1. *Hence,* (H^4_1, g) *is a Lorentzian hypersurface of* $(\mathbb{R}^5_2, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. *Note that* (H^4_1, g) *is a 4-dimensional anti-de Sitter space [8].*

By using (16), (58), (59) *and* (60), *we have*

$$a = -\sinh v \cos w, \quad b = -\sinh 2v \sin w \sin(u + q). \tag{61}$$

Since the Levi-Civita connection $\overline{\nabla}$ *of* $\overline{g}$ *is flat, we have* $\overline{\nabla}_X Z = X$ *for any* $X \in \chi(H^4_1)$. *Then the Gauss-Weingarten formulas are*

$$\overline{\nabla}_X Y = \nabla_X Y + g(X, Y)\overline{Z},$$
$$\overline{\nabla}_X Z = X, \quad X, Y \in \chi(H^4_1).$$

Thus, $A_Z = -I$ *and* $f = -1$*. Hence,* H_1^4 *is totally umbilical. Moreover,* $\mathrm{tr}A_Z = -2$ *implies that* H_1^4 *is an extrinsic sphere.*

Let $w(t) : I \subset \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$ *be a non-constant function. Consider the family of curves* $C_w : I \longrightarrow H_1^4$ *given locally by*
$C_w(t) = (z^1(t), z^2(t), z^3(t), z^4(t), z^5(t))$*, where*

$$\begin{cases} u = \ln\left(\dfrac{\sin w(t) + \sqrt{1+\cos^2 w(t)}}{\cos w(t)}\right) \\ v = \ln\left(\dfrac{1 + \sqrt{1+\cos^2 w(t)}}{\cos w(t)}\right) \\ w = w(t) \\ q = -u \end{cases} . \tag{62}$$

By straightforward computations, using (58), (59), (60), (62) *and the components of the metric* g*, we see that* C_w *satisfy the conditions:*

$$g(\dot{C}_w, \dot{C}_w) = -\cosh^2 u(\dot{u})^2 + (\dot{v})^2 + \sinh^2 v(\dot{w})^2 + \sinh^2 v \sin^2 w(\dot{q})^2 = 0,$$

$$\overline{\eta}(\dot{C}_w) = \cosh v \cos w \dot{v} - \sinh v \sin w \dot{w} = 0,$$

$$\alpha(\dot{C}_w) = -(z^1\dot{z}^3 + z^2\dot{z}^4 + z^3\dot{z}^1 + z^4\dot{z}^2) = (z^1z^3 + z^2z^4)^{\cdot}$$

$$= (\cosh v \sinh v \sin w \sin(u+q))^{\cdot} = 0,$$

$$n_w = \overline{g}(\dot{C}_w, \overline{\varphi}C_w) = \frac{4(\dot{w})^2}{\cos^2 w(1+\cos^2 w)} \neq 0, \quad t \in I. \tag{63}$$

Hence, the curves C_w *belong to the studied type of curves in this paper.*

Since H_1^4 *is totally umbilical, from Lemma 5.1 it follows that the original parameter* t *of* $C_w(t)$ *is a pseudo-arc parameter if and only if* (53) *holds. By using* (61) *and* (62) *we get* $a = -1$ *and* $b = 0$ *along* C_w*. Now, taking into account that* $f = -1$*, from* (53) *we obtain* $n_w = -\epsilon_1\sqrt{2}$*,* $\epsilon_1 \pm 1$*. The equality* (63) *implies* $\epsilon_1 = -1$*. Thus, the original parameter* t *of* $C_w(t)$ *is a pseudo-arc parameter if and only if the following condition is fulfilled*

$$\sqrt{2} = \frac{4(\dot{w})^2}{\cos^2 w(1+\cos^2 w)}. \tag{64}$$

The function

$$\hat{w}(t) = \arctan\left(\frac{e^{\sqrt[4]{2}t} - 2}{2e^{\frac{t}{\sqrt[4]{8}}}}\right)$$

is a solution of the ordinary differential equation (64)*. Substituting* $w(t)$ *with* $\hat{w}(t)$ *in* (62)*, we obtain*

$$\begin{cases} \hat{u} = \dfrac{t}{\sqrt[4]{8}} \\ \hat{v} = \ln\left(\sqrt{e^{2\sqrt[4]{2}t} + 4} + e^{\sqrt[4]{2}t} + 2\right) - \dfrac{t}{\sqrt[4]{8}} - \ln 2 \\ \hat{w} = \arctan\left(\dfrac{e^{\sqrt[4]{2}t} - 2}{2e^{\frac{t}{\sqrt[4]{8}}}}\right) \\ \hat{q} = -\hat{u} \end{cases} . \tag{65}$$

According to Theorem 5.3, for the curve $\hat{C}(t) = (z^1(t), z^2(t), z^3(t), z^4(t), z^5(t))$ *given by* (65)*, there exists only one Cartan frame* $\{\dot{\hat{C}}, \hat{\mathcal{N}}, \hat{\mathcal{W}}_1, \hat{\mathcal{W}}_2\}$ *up to an orientation. In* (54) *we replace* ϵ, n, a, b *with* -1, $\sqrt{2}$, -1, 0, *respectively, and using* (60), (65) *we get*

$$\dot{\hat{C}} = \frac{1}{2\sqrt[4]{8}e^{\frac{t}{\sqrt[4]{8}}}}\left\{\left(e^{\sqrt[4]{2}t} - 2\right)\cos\frac{t}{\sqrt[4]{8}} - \left(e^{\sqrt[4]{2}t} + 2\right)\sin\frac{t}{\sqrt[4]{8}},\right.$$

$$\left(e^{\sqrt[4]{2}t} + 2\right)\cos\tfrac{t}{\sqrt[4]{8}} + \left(e^{\sqrt[4]{2}t} - 2\right)\sin\tfrac{t}{\sqrt[4]{8}},$$

$$-\left(e^{\sqrt[4]{2}t} - 2\right)\cos\tfrac{t}{\sqrt[4]{8}} - \left(e^{\sqrt[4]{2}t} + 2\right)\sin\tfrac{t}{\sqrt[4]{8}},$$

$$\left.\left(e^{\sqrt[4]{2}t} + 2\right)\cos\tfrac{t}{\sqrt[4]{8}} - \left(e^{\sqrt[4]{2}t} - 2\right)\sin\tfrac{t}{\sqrt[4]{8}}, 0\right\}$$

$$\hat{\mathcal{W}}_1 = -\frac{\sqrt{2}}{2}\xi_1|_{\hat{C}} = -\frac{\sqrt{2}}{4e^{\frac{t}{\sqrt[4]{8}}}}\left\{\left(e^{\sqrt[4]{2}t} - 2\right)\sin\frac{t}{\sqrt[4]{8}}, -\left(e^{\sqrt[4]{2}t} - 2\right)\cos\frac{t}{\sqrt[4]{8}},\right.$$

$$\left.\left(e^{\sqrt[4]{2}t} + 2\right)\cos\tfrac{t}{\sqrt[4]{8}}, \left(e^{\sqrt[4]{2}t} + 2\right)\sin\tfrac{t}{\sqrt[4]{8}}, 0\right\}$$

$$\hat{\mathcal{W}}_2 = -\frac{\epsilon\sqrt{2}}{2}\xi_0|_{\hat{C}} = -\frac{\epsilon\sqrt{2}}{4e^{\frac{t}{\sqrt[4]{8}}}}\left\{\left(e^{\sqrt[4]{2}t} + 2\right)\cos\frac{t}{\sqrt[4]{8}}, \left(e^{\sqrt[4]{2}t} + 2\right)\sin\frac{t}{\sqrt[4]{8}},\right.$$

$$\left.-\left(e^{\sqrt[4]{2}t} - 2\right)\sin\tfrac{t}{\sqrt[4]{8}}, \left(e^{\sqrt[4]{2}t} - 2\right)\cos\tfrac{t}{\sqrt[4]{8}}, 2\right\},$$

$$\hat{\mathcal{N}} = \frac{\sqrt{2}}{2}\overline{\varphi}\dot{\hat{C}}|_{\hat{C}} = \frac{1}{4\sqrt[4]{2}e^{\frac{t}{\sqrt[4]{8}}}}\left\{\left(e^{\sqrt[4]{2}t} - 2\right)\cos\frac{t}{\sqrt[4]{8}} + \left(e^{\sqrt[4]{2}t} + 2\right)\sin\frac{t}{\sqrt[4]{8}},\right.$$

$$-\left(e^{\sqrt[4]{2}t} + 2\right)\cos\tfrac{t}{\sqrt[4]{8}} + \left(e^{\sqrt[4]{2}t} - 2\right)\sin\tfrac{t}{\sqrt[4]{8}},$$

$$\left(e^{\sqrt[4]{2}t} - 2\right)\cos\tfrac{t}{\sqrt[4]{8}} - \left(e^{\sqrt[4]{2}t} + 2\right)\sin\tfrac{t}{\sqrt[4]{8}},$$

$$\left.\left(e^{\sqrt[4]{2}t} + 2\right)\cos\tfrac{t}{\sqrt[4]{8}} + \left(e^{\sqrt[4]{2}t} - 2\right)\sin\tfrac{t}{\sqrt[4]{8}}, 0\right\}.$$

For the Cartan curvatures $\hat{\sigma}_1$ *and* $\hat{\sigma}_2$ *of* $\hat{C}$ *we find*

$$\hat{\sigma}_1 = 0, \qquad \hat{\sigma}_2 = \frac{\epsilon\sqrt{2}}{2},$$

i.e. $\hat{C}$ *is a helix in the extrinsic sphere* H_1^4.

This example is relevant to Proposition 5.4 and Theorem 5.8.

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