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Characterizations of various curvatures on weakly symmetric Kähler manifolds

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Abstract. The objective of the present paper is to introduce a special type of semi-symmetric metric connection ∇^* on weakly symmetric Kähler manifolds to reduce the manifold, which is Einstein and Ricci flat. Also, we use a non-flat Riemannian manifold called a weakly concircular symmetric manifold and study its geometric properties on a weakly symmetric Kähler manifold endowed with a special type of semi-symmetric metric connection that satisfies several relations on 1−form, differential equations, recurrent space, and Einstein manifold. In this paper, we also study weakly symmetric Kähler manifolds with parallel Weyl conformal curvature tensor on weakly symmetric Kähler manifolds with a special type of semi-symmetric metric connection and have shown that it is an Einstein manifold and it is reduced to a recurrent space.

1. Introduction

In the discipline of mathematics, a weakly symmetric space was first proposed in the 1950s by Norwegian mathematician Atle Selberg as an extension of the idea of symmetric space. Partial differential equations are a fundamental tool in the investigation of weakly symmetric spaces. They are utilized to analyze the curvature properties, study geometric structures, and provide a deeper understanding of the intricate relationships between curvature tensors and the geometry of these spaces. Researchers often apply sophisticated techniques from differential geometry and partial differential equations theory to study and solve these equations in the context of weakly symmetric spaces. The Riemannian symmetric spaces were first developed in the 19th century by the French mathematician Cartan, who used them extensively in differential geometry. A Riemannian manifold is called locally symmetric [3] if $\nabla R = 0$, where R is the

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Riemannian curvature tensor of (M^n, g). Several authors have examined the concept of locally symmetric manifolds in various ways over the last five decades, such as Chaturvedi and Pandey, studying weakly symmetric manifolds equipped with semi-symmetric metric connections [5]. For the further studies on semi-symmetric connections, we refer to [11, 15–17]. The concept of weakly symmetric and weakly Ricci-symmetric manifolds was introduced by Tamassy and Binh in their works [2, 26]. An n-dimensional (pseudo) Riemannian manifold is characterized as weakly symmetric if its curvature tensor addresses the following condition:

$$(\nabla_{\rho_1}R)(\rho_2,\rho_3,\rho_4,\rho_5) = \nu_1(\rho_1)R(\rho_2,\rho_3,\rho_4,\rho_5) + \nu_2(\rho_2)R(\rho_1,\rho_3,\rho_4,\rho_5) + \nu_3(\rho_3)R(\rho_2,\rho_1,\rho_4,\rho_5) + \nu_4(\rho_4)R(\rho_2,\rho_3,\rho_1,\rho_5) + \nu_5(\rho_5)R(\rho_2,\rho_3,\rho_4,\rho_1), \tag{1}$$

where, $\nu_1,\nu_2,\nu_3,\nu_4,\nu_5$ are 1-forms and $\forall\ \rho_2,\rho_3,\rho_4,\rho_5 \in \chi(M)$
If the Ricci tensor S of the Riemannian manifold satisfies

$$(\nabla_{\rho_1}S)(\rho_2,\rho_3) = \nu_1(\rho_1)S(\rho_2,\rho_3) + \nu_2(\rho_2)S(\rho_1,\rho_3) + \nu_3(\rho_3)S(\rho_2,\rho_1), \tag{2}$$

then the Riemannian manifold is called a weakly Ricci-symmetric manifold; throughout the paper we denote this manifold by $(WRSM)_n$, where $\nu_1,\nu_2,\nu_3,\nu_4,\nu_5$ are simultaneously non-vanishing 1-forms and $\rho_1,\rho_2,\rho_3,\rho_4,\rho_5$ are vector fields. Studying nonlinear equations in the context of weakly Ricci symmetric manifolds involves examining differential equations that express geometric and curvature conditions specific to these spaces. A weakly Ricci symmetric manifold is a Riemannian manifold where the Ricci curvature tensor satisfies certain symmetry conditions. A common property of weakly Ricci symmetric manifolds is that the Ricci curvature tensor satisfies conditions similar to $R_{ijkl} = R_{klij}$, which is a manifestation of the weak symmetry condition. Expressing this condition in terms of the components of the Ricci tensor leads to nonlinear equations that govern the geometry of the manifold. According to Prvanovic [22], in a weakly symmetric manifold, if $\nu_2 = \nu_3 = \nu_4 = \nu_5$ then $\nu_2 = \nu_3 = \nu_4 = \nu_5 = \omega$. Then from equation (1), we have

$$(\nabla_{\rho_1}R)(\rho_2,\rho_3,\rho_4,\rho_5) = \nu_1(\rho_1)R(\rho_2,\rho_3,\rho_4,\rho_5) + \omega(\rho_2)R(\rho_1,\rho_3,\rho_4,\rho_5) + \omega(\rho_3)R(\rho_2,\rho_1,\rho_4,\rho_5) + \omega(\rho_4)R(\rho_2,\rho_3,\rho_1,\rho_5) + \omega(\rho_5)R(\rho_2,\rho_3,\rho_4,\rho_1). \tag{3}$$

Where ν_1, ω are two non zero 1-forms defined as

$$g(\rho_1,\mu) = A(\rho_1), g(\rho_1,\phi) = \omega(\rho_1). \tag{4}$$

For all vector fields ρ_1, ∇ denotes the operator of covariant differentiation with respect to the metric g.
The Weyl conformal curvature tensor holds significance in the realm of differential geometry. The Weyl conformal curvature tensor W [18], and the concircular curvature tensor C of type $(0,4)$ of the Riemannian manifold (M^n, g) are defined respectively by

$$\begin{aligned} W(\rho_2,\rho_3,\rho_4,\rho_5) = R(\rho_2,\rho_3,\rho_4,\rho_5) &- \frac{1}{n-2}[S(\rho_3,\rho_4)g(\rho_2,\rho_5) - S(\rho_2,\rho_4)g(\rho_3,\rho_5) \\ &+ g(\rho_3,\rho_4)S(\rho_2,\rho_5) - g(\rho_2,\rho_4)S(\rho_3,\rho_5)] \\ &+ \frac{r}{(n-1)(n-2)}[g(\rho_3,\rho_4)g(\rho_2,\rho_5) - g(\rho_2,\rho_4)g(\rho_3,\rho_5)], \end{aligned} \tag{5}$$

and

$$C(\rho_2,\rho_3,\rho_4,\rho_5) = R(\rho_2,\rho_3,\rho_4,\rho_5) - \frac{r}{n(n-1)}[g(\rho_3,\rho_4)g(\rho_2,\rho_5) - g(\rho_2,\rho_4)g(\rho_3,\rho_5)]. \tag{6}$$

Where r is the scalar curvature and g is the Riemannian metric on the Riemannian manifold (M^n, g).

2. Preliminaries

Various differential geometers have explored semi-symmetric metric connections. In 1970, Yano delved into Riemannian manifolds that admit a semi-symmetric metric connection with a curvature tensor that vanishes, establishing certain results. Also, De and Biswas [9] examined the semi-symmetric metric connection in a Riemannian manifold. In 2008, Chaturvedi and Pandey [4] investigated a semi-symmetric non-metric connection. Murathan and Özgür [19] explored the semi-symmetric metric connection with a unit parallel vector field ϕ and derived interesting outcomes on a Riemannian manifold. The work by Chaubey et al. [7], Choudhary et al. [8], and Chaturvedi et al. [6] also contributes to this area of study.
A Kähler manifold is an n-dimensional manifold with a complex structure F and a positive definite metric g that satisfies

$$F^2 = -I, \quad g(F\rho_1, F\rho_2) = g(\rho_1, \rho_2), \textit{ and } (\nabla_{\rho_1}F)\rho_2 = 0, \tag{7}$$

where F is a $(1, 1)$ tensor and ∇ means covariant derivative according to the Livi-Civita connection.

$$\begin{aligned} &S(\rho_1, \rho_2) = S(F\rho_1, F\rho_2),\ S(F\rho_1, \rho_2) = -S(\rho_1, F\rho_2),\\ &g(F\rho_1, \rho_2) = -g(\rho_1, F\rho_2),\ R(\rho_1, \rho_2) = R(F\rho_1, F\rho_2). \end{aligned} \tag{8}$$

Let (M^n, g) be an n-dimensional differentiable manifold with metric tensor g. A smooth linear connection ∇^* on (M^n, g) is said to be semi-symmetric if its torson tensor T of ∇^* satisfies the relation

$$T(\rho_1, \rho_2) = \omega(\rho_2)\rho_1 - \omega(\rho_1)\rho_2. \tag{9}$$

There is a 1-form associated with the torson tensor T of the connection ∇^* for any vector fields ρ_1 and ρ_2 on (M^n, g) and ω. A semi-symmetric metric connection [29], is one in which ∇^* further satisfies the requirement $\nabla^* g = 0$.
The relation between the semi-symmetric metric connection ∇^* and the Riemannian connection ∇ of (M^n, g) is given by [29].

$$\nabla^*_{\rho_1}\rho_2 = \nabla_{\rho_1}\rho_2 + \omega(\rho_2)\rho_1 - g(\rho_1, \rho_2)\phi, \tag{10}$$

where $\omega(\rho_1) = g(\rho_1, \phi)$ also, ρ_1, ρ_2 and ϕ are the vector fields on (M^n, g). Yano established a relation between R and R^*, if R and R^* are the Riemannian curvature tensors with respect to ∇ and ∇^*, respectively, then,

$$\begin{aligned} R^*(\rho_1, \rho_2, \rho_3, T) = {} & R(\rho_1, \rho_2, \rho_3, T) - \Phi(\rho_2, \rho_3)g(\rho_1, T) + \Phi(\rho_1, \rho_3)g(\rho_2, T)\\ & - g(\rho_2, \rho_3)\Phi(\rho_1, T) + g(\rho_1, \rho_3)\Phi(\rho_2, T), \end{aligned} \tag{11}$$

where

$$\Phi(\rho_1, \rho_2) = (\nabla_{\rho_1}\omega)\rho_2 - \omega(\rho_1)\omega(\rho_2) + \frac{1}{2}g(\rho_1, \rho_2). \tag{12}$$

Now if ϕ is a parallel unit vector field with respect to the connection ∇, then $\nabla\phi = 0$, which gives

$$(\nabla_{\rho_1}\omega)\rho_2 = 0. \tag{13}$$

We know that the Kulkarni-Nomizu product $\overline{\wedge}$ of the tensor of type $(0, 2)$ is defined by

$$\begin{aligned} (g \,\overline{\wedge}\, \Phi)(\rho_1, \rho_2, \rho_3, T) = {} & \Phi(\rho_2, \rho_3)g(\rho_1, T) - \Phi(\rho_1, \rho_3)g(\rho_2, T)\\ & + g(\rho_2, \rho_3)\Phi(\rho_1, T) - g(\rho_1, \rho_3)\Phi(\rho_2, T). \end{aligned} \tag{14}$$

From equation (11) and straight forward calculation, we can easily get

$$R(\rho_1, \rho_2)\phi = 0 \text{ and } \mathrm{S}(\rho_2, \phi) = 0, \tag{15}$$

where S denotes the Ricci tensor of the connection ∇.
In 2000, weak symmetries of Kähler manifolds were studied by Tamsassy, De, and Bink [27], also in 2010, quasi-conformally flat almost pseudo-Ricci symmetric manifolds were studied by Shaikh, Hui, and Bagewadi [23] and De and Ghosh studied weakly Ricci symmetric spacetime manifolds in [10]. Almost pseudo-symmetric Kähler manifold on semi-symmetic metric connection is studied by [1, 20, 21]. In continuing the study of the above developments, we plan to study admitting a special type of semi-symmetric metric connection on weakly symmetric Kähler manifolds, weakly concircular symmetric Kähler manifolds, and weakly symmetric Kähler manifolds with a parallel concircular curvature tensor as well as a conformal curvature tensor.

3. Weakly symmetric Kähler manifolds

Assume for the purposes of this section that the weakly symmetric Kähler manifold (M^n, g) can be represented as

$$R(\rho_2, \rho_3, \rho_4, \rho_5) = R(\rho_2, \rho_3, \bar{\rho_4}, \bar{\rho_5}). \tag{16}$$

Now taking covariant derivative of equation (16) on both sides, we get

$$(\nabla_{\rho_1} R)(\rho_2, \rho_3, \rho_4, \rho_5) = (\nabla_{\rho_1} R)(\rho_2, \rho_3, \bar{\rho_4}, \bar{\rho_5}), \tag{17}$$

using equation (1) in equation (17), we have

$$\nu_4(\rho_4)R(\rho_2, \rho_3, \rho_1, \rho_5) + \nu_5(\rho_5)R(\rho_2, \rho_3, \rho_4, \rho_1) = \nu_4(\bar{\rho_4})R(\rho_2, \rho_3, \rho_1, \bar{\rho_5}) + \nu_5(\bar{\rho_5})R(\rho_2, \rho_3, \bar{\rho_4}, \rho_1). \tag{18}$$

Now contracting $\rho_3 = \rho_1 = e_i$ and simplifying equation (18), we get

$$\nu_4(\rho_4)S(\rho_2, \rho_5) - \nu_5(\rho_5)S(\rho_2, \rho_4) = \nu_4(\bar{\rho_4})S(\rho_2, \bar{\rho_5}) - \nu_5(\bar{\rho_5})S(\rho_2, \bar{\rho_4}). \tag{19}$$

Again contracting $\rho_2 = \rho_5 = e_i$, we get

$$\nu_4(\rho_4)S(\rho_2, \rho_5) - \nu_5(\rho_5)S(\rho_2, \rho_4) = \nu_4(\bar{\rho_4})S(\rho_2, \bar{\rho_5}) - \nu_5(\bar{\rho_5})S(\rho_2, \bar{\rho_4}). \tag{20}$$

$$r\nu_4(\rho_4) = 2S(\rho_4, \phi), \tag{21}$$

which can be written as

$$S(\rho_4, \phi) = \frac{r}{2} g(\rho_4, \phi), \tag{22}$$

which is an Einstein manifold. Again from equation (15), we have

$$r\omega(\rho_4) = 0. \tag{23}$$

This implies that either $r = 0$ or $\omega(\rho_4) = 0$, Now if we take $\omega(\rho_4) = 0$ then the connection ∇^* defined by (10) will be changed. Hence $\omega(\rho_4)$ can't be zero.
Thus, we conclude:

Theorem 3.1. *Let (M^n, g) be a weakly symmetric Kähler manifold endowed with a special type of semi-symmetric metric connection ∇^*, then (M^n, g) is a manifold of zero scalar with respect to the Live-Civita connection ∇.*

Theorem 3.2. *Weakly symmetric Kähler manifold admitting a special type of semi-symmetric metric connection ∇^*, is an Einstein manifold.*

Theorem 3.3. *Weakly symmetric Kähler manifold admitting a special type of semi-symmetric metric connection ∇^*, with parallel unit vector ϕ, is an Ricci flat.*

4. Weakly concircular symmetric Kähler manifolds

In this section, we introduce and define weakly concircular symmetric Kähler manifolds. The study of Riemannian symmetric manifolds has been initiated with the work of Cartan in 1926. After that, in 1950, Walker [28], introduced the notion of a locally recurrent Riemannian manifold. Weakly concircular symmetric Kähler manifold allowing a special type of semi-symmetric metric connection ∇^*, weakly concircular symmetric Kähler manifolds reduced as 1-forms, recurrent space, and Einstein manifolds with semi-symmetric metric connection.

Definition 4.1. *A Riemannian manifold is said to be a weakly concircular symmetric Kähler manifold if concircular curvature tensor C of type* $(0,4)$ *satisfies the relation*

$$\begin{aligned}(\nabla_{\rho_1}C)(\bar{\rho_2},\bar{\rho_3},\rho_4,\rho_5) &= \nu_1(\rho_1)C(\bar{\rho_2},\bar{\rho_3},\rho_4,\rho_5)+\omega(\bar{\rho_2})C(\rho_1,\bar{\rho_3},\rho_4,\rho_5)+\omega(\bar{\rho_3})C(\bar{\rho_2},\rho_1,\rho_4,\rho_5)\\ &\quad+\omega(\rho_4)C(\bar{\rho_2},\bar{\rho_3},\rho_1,\rho_5)+\omega(\rho_5)C(\bar{\rho_2},\bar{\rho_3},\rho_4,\rho_1),\end{aligned} \tag{24}$$

where ν_1,ω are two non-zero 1-forms and using equation (6) in above equation, we get

$$\begin{aligned}&(\nabla_{\rho_1}C)(\bar{\rho_2},\bar{\rho_3},\rho_4,\rho_5) = \nu_1(\rho_1)\Big\{R(\bar{\rho_2},\bar{\rho_3},\rho_4,\rho_5)-\frac{r}{n(n-1)}[g(\bar{\rho_3},\rho_4)g(\bar{\rho_2},\rho_5)\\ &-g(\bar{\rho_2},\rho_4)g(\bar{\rho_3},\rho_5)]\Big\}+\omega(\bar{\rho_2})\Big\{R(\rho_1,\bar{\rho_3},\rho_4,\rho_5)-\frac{r}{n(n-1)}[g(\bar{\rho_3},\rho_4)g(\rho_1,\rho_5)\\ &-g(\rho_1,\rho_4)g(\bar{\rho_3},\rho_5)]\Big\}+\omega(\bar{\rho_3})\Big\{R(\bar{\rho_2},\rho_1,\rho_4,\rho_5)-\frac{r}{n(n-1)}[g(\rho_1,\rho_4)g(\bar{\rho_2},\rho_5)\\ &-g(\bar{\rho_2},\rho_4)g(\rho_1,\rho_5)]\Big\}+\omega(\rho_4)\Big\{R(\bar{\rho_2},\bar{\rho_3},\rho_1,\rho_5)-\frac{r}{n(n-1)}[g(\bar{\rho_3},\rho_1)g(\bar{\rho_2},\rho_5)\\ &-g(\bar{\rho_2},\rho_1)g(\bar{\rho_3},\rho_5)]\Big\}+\omega(\rho_5)\Big\{R(\bar{\rho_2},\bar{\rho_3},\rho_4,\rho_1)-\frac{r}{n(n-1)}[g(\bar{\rho_3},\rho_4)g(\bar{\rho_2},\rho_1)\\ &-g(\bar{\rho_2},\rho_4)g(\bar{\rho_3},\rho_1)]\Big\}.\end{aligned} \tag{25}$$

Again, contracting by replacing $\rho_2=\rho_5=e_i$ in equation (24) and taking summation over $i,(i=1,2,3,...,n)$, we obtain

$$\begin{aligned}&[(\nabla_{\rho_1}S)(\rho_3,\rho_4)-\frac{\nabla_{\rho_1}rg(\rho_3,\rho_4)}{n(n-1)}]\nu_1(\rho_1)=\\ &[S(\rho_3,\rho_4)-\frac{r}{n(n-1)}g(\rho_3,\rho_4)]\nu_1(\rho_1)\\ &+g(\bar{e_i},\phi)R(\rho_1,\bar{\rho_3},\rho_4,e_i)-\frac{r}{n(n-1)}[g(\rho_1,\rho_4)g(\rho_3,\phi)\\ &-g(\bar{\rho_3},\rho_4)g(\rho_1,\bar{\phi})]+g(\bar{\rho_3},\phi)[S(\bar{\rho_1},\rho_4)-\frac{r}{n(n-1)}g(\rho_1,\bar{\rho_4})]\\ &+g(\rho_4,\phi)[S(\rho_3,\rho_1)-\frac{r}{n(n-1)}g(\rho_3,\rho_1)]\\ &+g(e_i,\phi)R(e_i,\rho_3,\rho_4,\rho_1)-\frac{r}{n(n-1)}[g(\bar{\rho_4},\phi)g(\bar{\rho_3},\rho_1)\\ &-g(\bar{\rho_3},\rho_4)g(\bar{\rho_1},\phi)].\end{aligned} \tag{26}$$

Now, again replacing $\rho_3=\rho_4=e_i$, we calculate

$$(\nabla_{\rho_1}r)\frac{(n-2)}{(n-1)}=r[\frac{(n-2)}{(n-1)}\nu_1(\rho_1)-\frac{4}{n(n-1)}g(\rho_1,\phi)], \tag{27}$$

from equation (4) and equation (27), we have

$$dr(\rho_1)=r[\nu_1(\rho_1)-\frac{4\omega(\rho_1)}{n(n-2)}]. \tag{28}$$

Thus, we conclude the following

Theorem 4.2. The scalar curvature tensor r of weakly concircular symmetric Kähler manifold endowed with a special type of semi-symmetric metric connection ∇^* satisfies the relation

$$n(n-2)dr(\rho_1) = r[n(n-2)\nu_1(\rho_1) - 4\omega(\rho_1)]. \tag{29}$$

Let us consider $dr(\rho_1) = 0$, in (28) and if $r \neq 0$ then, we have

$$\nu_1(\rho_1) = \frac{4}{n(n-2)}\omega(\rho_1). \tag{30}$$

Corollary 4.3. If a weakly concircular symmetric Kähler manifold admits a special type of semi-symmetric metric connection ∇^*, then it is reduced to the 1-form $\nu_1(\rho_1) = \frac{4}{n(n-2)}\omega(\rho_1)$.

Also, using (30) and we assumed that $\omega(\rho_1) = \nu_1(\rho_1)$, we get

$$[n(n-2) - 4]\nu_1(\rho_1) = 0, \tag{31}$$

If $[n(n-2) - 4] \neq 0$ and then $\nu_1(\rho_1) = 0$.
Therefore (24) becomes

$$\begin{aligned}(\nabla_{\rho_1}C)(\bar{\rho_2}, \bar{\rho_3}, \rho_4, \rho_5) = {} & \omega(\bar{\rho_2})C(\rho_1, \bar{\rho_3}, \rho_4, \rho_5) + \omega(\bar{\rho_3})C(\bar{\rho_2}, \rho_1, \rho_4, \rho_5) \\ & + \omega(\rho_4)C(\bar{\rho_2}, \bar{\rho_3}, \rho_1, \rho_5) + \omega(\rho_5)C(\bar{\rho_2}, \bar{\rho_3}, \rho_4, \rho_1).\end{aligned} \tag{32}$$

Theorem 4.4. If weakly concircular symmetric Kähler manifold admitting a special type of semi-symmetric metric connection ∇^* then it is reduced to special type of concircular symmetric Kähler manifold with 1-forms ω.

Also, using (30) and we assumed that $\nu_1(\rho_1) = \omega(\rho_1)$, we get

$$[n(n-2) - 4]\omega(\rho_1) = 0, \tag{33}$$

If $[n(n-2) - 4] \neq 0$ and then $\omega(\rho_1) = 0$.
In weakly symmetric manifold if $\nu_2 = \nu_3 = \nu_4 = \nu_5 = \omega = 0$, then (3) becomes recurrent space i.e.,

$$(\nabla_{\rho_1}R)(\rho_2, \rho_3, \rho_4, \rho_5) = \nu_1(\rho_1)R(\rho_2, \rho_3, \rho_4, \rho_5). \tag{34}$$

Theorem 4.5. A weakly concircular symmetric Kähler manifold admitting a special type of semi-symmetric metric connection ∇^* is a recurrent space.

Also, assume that the concircular curvature of weakly symmetric Kähler manifold is parallel i.e., $\nabla C = 0$. Using the properties of Kähler manifolds and then the mentioned equation (6), can be expressed as

$$C(\rho_2, \rho_3, \bar{\rho_4}, \bar{\rho_5}) = R(\rho_2, \rho_3, \bar{\rho_4}, \bar{\rho_5}) - \frac{r}{n(n-1)}[g(\rho_3, \bar{\rho_4})g(\rho_2, \bar{\rho_5}) - g(\rho_2, \bar{\rho_4})g(\rho_3, \bar{\rho_5})]. \tag{35}$$

After contracting (35) by taking $\rho_2 = \rho_4 = e_i$, we get

$$g(C(e_i, \rho_3)\bar{e_i}, \bar{\rho_5}) = S(\rho_3, \rho_5) - \frac{r}{n(n-1)}[-g(\rho_3, \rho_5) + ng(\rho_3, \rho_5)], \tag{36}$$

now taking covarient derivative of equation (36), we get

$$(\nabla_{\rho_1}S)(\rho_3, \rho_5) = \frac{1}{n}\nabla_{\rho_1}[rg(\rho_3, \rho_5)], \tag{37}$$

using (37) and also we know that the relation $(\nabla_{\rho_1}g)(\rho_3, \rho_5) = \nabla g = 0$, in semi-symmetric metric connection then, we get differential equations

$$(\nabla_{\rho_1}S)(\rho_3, \rho_5) = \frac{1}{n}dr(\rho_1)g(\rho_3, \rho_5), \tag{38}$$

from equation (2) and (38), we get

$$\nu_1(\rho_1)S(\rho_3,\rho_5)+\nu_2(\rho_3)S(\rho_1,\rho_5)+\nu_3(\rho_5)S(\rho_3,\rho_1)=\frac{dr(\rho_1)}{n}g(\rho_3,\rho_5). \tag{39}$$

Now, replacing $\rho_1=\phi$ and using equation (15) and (39), we have differential equations

$$S(\rho_3,\rho_5)=\frac{dr(\rho_1)}{n}g(\rho_3,\rho_5). \tag{40}$$

Which is an Einstein equation.
Thus we conclude:

Theorem 4.6. The Weakly symmetric Kähler manifold with parallel concircular curvature tensor endowed with a weakly Ricci semi-symmetric manifold with special type of semi-symmetric metric connection ∇^* is an Einstein equation.

Now, putting $\rho_5=\phi$ in equation (40) and using equation (15), we get

$$\frac{dr(\rho_1)}{n}\omega(\rho_3)=0. \tag{41}$$

In weakly symmetric manifold if $\nu_2=\nu_3=\nu_4=\nu_5=\omega=0$, then by using (3), we get

$$(\nabla_{\rho_1}R)(\rho_2,\rho_3,\rho_4,\rho_5)=\nu_1(\rho_1)R(\rho_2,\rho_3,\rho_4,\rho_5), \tag{42}$$

which becomes recurrent space .
Thus, we conclude:

Theorem 4.7. The Weakly symmetric Kähler manifold with parallel concircular curvature tensor endowed with a $(WRSM)_n$ with special type of semi-symmetric metric connection ∇^* is a recurrent space.

5. Weakly symmetric Kähler manifold with parallel Weyl conformal curvature tensor

Assume that the Weyl conformal curvature of weakly symmetric Kähler manifold is parallel i.e., $\nabla W=0$. Using the properties of Kähler manifolds and using equation (5), then the mentioned equation can be expressed as

$$\begin{aligned}W(\rho_2,\rho_3,\bar{\rho_4},\bar{\rho_5}) = {} & R(\rho_2,\rho_3,\bar{\rho_4},\bar{\rho_5})-\frac{1}{n-2}[S(\rho_3,\bar{\rho_4})g(\rho_2,\bar{\rho_5})-S(\rho_2,\bar{\rho_4})g(\rho_3,\bar{\rho_5})\\ &+g(\rho_3,\bar{\rho_4})S(\rho_2,\bar{\rho_5})-g(\rho_2,\bar{\rho_4})S(\rho_3,\bar{\rho_5})]\\ &+\frac{r}{(n-1)(n-2)}[g(\rho_3,\bar{\rho_4})g(\rho_2,\bar{\rho_5})-g(\rho_2,\bar{\rho_4})g(\rho_3,\bar{\rho_5})].\end{aligned} \tag{43}$$

Now, contracting by $\rho_2=\rho_4=e_i$, we get

$$\begin{aligned}g(W(e_i,\rho_3)\bar{e_i},\bar{\rho_5}) = {} & S(\rho_3,\rho_5)+\frac{2}{n-2}S(\rho_3,\rho_5)\\ &+\frac{r}{(n-1)(n-2)}g(\rho_3,e_i)g(\bar{e_i},\bar{\rho_5}).\end{aligned} \tag{44}$$

Now, taking covarient derivative of equation (44) we get

$$(\nabla_{\rho_1}S)(\rho_3,\rho_5)=\frac{1}{n(n-1)}\nabla_{\rho_1}[rg(\rho_3,\rho_5)], \tag{45}$$

using (45) and also we know that the relation $(\nabla_{\rho_1} g)(\rho_3, \rho_2) = \nabla g = 0$, in semi-symmetric metric connection then we get

$$(\nabla_{\rho_1} S)(\rho_3, \rho_5) = \frac{1}{n(n-1)} dr(\rho_1) g(\rho_3, \rho_5), \tag{46}$$

from equation (2) and (46), we get

$$\nu_1(\rho_1) S(\rho_3, \rho_5) + \nu_2(\rho_3) S(\rho_1, \rho_5) + \nu_3(\rho_5) S(\rho_3, \rho_1) = \frac{dr(\rho_1)}{n(n-1)} g(\rho_3, \rho_5). \tag{47}$$

Now, replacing $\rho_1 = \phi$ and using equation (15) and (47) we get

$$S(\rho_3, \rho_5) = \frac{dr(\rho_1)}{n(n-1)} g(\rho_3, \rho_5). \tag{48}$$

Which is an Einstein equation.
Thus we conclude:

Theorem 5.1. The Weakly symmetric Kähler manifold with parallel Weyl conformal curvature tensor endowed with a $(WRSM)_n$ with special type of semi-symmetric metric connection ∇^* is an Einstein equation.

Now, replacing $\rho_5 = \phi$ in equation (48) also using (15), we get

$$\frac{dr(\rho_1)}{n(n-1)} \omega(\rho_3) = 0, \tag{49}$$

In weakly symmetric manifold if $\nu_2 = \nu_3 = \nu_4 = \nu_5 = \omega = 0$, then by using (3), we get

$$(\nabla_{\rho_1} R)(\rho_2, \rho_3, \rho_4, \rho_5) = \nu_1(\rho_1) R(\rho_2, \rho_3, \rho_4, \rho_5), \tag{50}$$

which becomes recurrent space.
Thus, we conclude:

Theorem 5.2. The Weakly symmetric Kähler manifold with parallel Weyl conformal curvature tensor allowing a $(WRSM)_n$ with special type of semi-symmetric metric connection ∇^* is a recurrent space.

6. Conclusion

We have made an effort to study recognizing special types of semi-symmetric metric connections on weakly symmetric Kähler manifolds, weakly concircular symmetric Kähler manifolds, weakly symmetric Kähler manifolds with parallel concircular curvature tensor, and conformal curvature tensor based on the work presented above in this paper.

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