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Pointwise semi-slant semi-Riemannian maps

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Abstract. We use the first variational formula on the fibers to deduce the necessary and sufficient conditions for the harmonicity of pointwise semi-slant semi Riemannian maps, which are defined on Lorentzian para-Sasakian manifolds. We define the sets of Legendre, Hamiltonian and Harmonic variations for any fibre of the map. Moreover, we address the characterization theorem for pointwise semi-slant semi Riemannian maps from Lorentzian para Saskian manifold to a semi-Riemannian metric manifold by considering the vertical Reeb vector field and investigate the properties of totally umbilical fibers. Beside from the peculiarities of pointwise semi-slant semi Riemannian maps, geometry of the distributions associated with the map such as integrability and totally geodesicness are also studied. In the end, we discuss a number of examples illustrating the existence of such maps.

1. Introduction

Watson introduced the almost Hermitian submersion concept in 1958, which revealed that in most cases, the base manifold and each fiber have the same structure as the total space [41]. The theory of semi-Riemannian metric manifolds was subsequently developed in 1966 by O'Neill [26, 27] and Gray [13]. In [14], Garcia and Kupeli studied the more geometric properties related to semi-Riemannian maps and their applications, which has since become a thriving research field in mathematics, mathematical physics and physics in space science with applications in Kaluza-Klein theory [17, 19], Yang-Mills theory [3], supergravity [24], and superstring theories [18]. Sahin later introduced several other submersions, such as anti-invariant, semi-invariant, and slant Riemannian maps using the base manifold as Hermitian manifolds, in [30, 35–37]. Further research on semi-Riemannian submersions has been conducted, including those with para-contact para-complex manifolds [10–12, 26], real and complex pseudo-hyperbolic spaces [4], almost para-cosymplectic manifolds [32], anti-invariant [7, 40], semi-invariant [1] semi-Riemannian submersions studied, The integrability of distributions and geodesic properties on slant submersions in paracontact geometry studied in [15]. The pointwise slant lightlike submersions investigated in [20]. Fischer developed

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the theory of Riemannian maps in [27], which extended the concepts of isometric immersions and course of Riemannian submersions.

In the sequence of study on para geometry para-Kaehler manifolds were defined by Libermann [23] in 1952, and the properties of para-Kaehler manifolds were introduced by Rashevskij in [34]. There has been a great deal of research in recent years into the geometry of Riemannian maps between various structures, including slant [29], hemi-slant [31], semi-slant [22, 30] Riemannian maps. Some important properties on harmonic maps between semi-Riemannian spheres have been obtained in [25], and with respect to these studies nowadays authors extended the above study on point-wise concepts such as point-wise slant Riemannian maps [2, 16]. For a deeper understanding and exploration of the concepts discussed, we highly recommend to go through these articles [5–8, 10, 11, 20, 29, 37, 38], which offer valuable insights and contribute significantly to the literature in this field. Our study in this paper is based on the notion of pointwise semi-slant semi-Riemannian maps from Loretz para-Sasakian manifolds to semi-Riemannian manifold which will extend the theory of point-wise slant submersions and pointwise slant Riemannian maps.

The study uses the following abbreviations in its subsequent sections: *PSSSRM* for pointwise semi-slant semi-Riemannian map and *LPS* manifold for Lorentzian para-Sasakian manifold.

2. Preliminaries

LPS manifold is a differentiable manifold of dimension $(2n + 1)$ defined based on a set of geometric structures. These structures include Ω a $(1, 1)$-tensor field , ξ a Reeb vector field, ω a 1-form, and a Lorentzian metric g_m. The conditions that classify a manifold as an *LPS* manifold involve specific relationships among these structures.

$$\wedge(\xi) = -1, \tag{1}$$

$$\Omega^2 = I + \wedge \otimes \xi, \tag{2}$$

$$g_m(\Omega X_1, \Omega X_2) = g_m(X_1, X_2) + \wedge(X_1) \wedge (X_2), \tag{3}$$

$$g_m(X_1, \xi) = \wedge(X_1), \ \nabla_{X_1}\xi = \Omega X_1, \tag{4}$$

$$\Omega(X_1, X_2) = g_m(X_1, \Omega X_2) = g_m(\Omega X_1, X_2) = \Omega(X_2, X_1), \tag{5}$$

$$(\nabla_{X_1}\Omega)(X_2, X_3) = g_m(X_2, (\nabla_{X_1}\Omega)X_3) = (\nabla_{X_1}\Omega)(X_3, X_2), \tag{6}$$

The covariant differentiation with respect to g_m is denoted by ∇. The Reeb vector field ξ is timelike, that is, $g_m(\xi, \xi) = -1$, and the triple $(\Omega, \xi, \wedge)$ defines an almost para contact structure on Σ_m according to Sato's definition. The Lorentzian metric defined in Equation (4) is analogous to the almost para contact semi-Riemannian metric. The structure $(\Omega, \xi, \wedge, g_m)$ is a Lorentzian para contact manifold if

$$\Omega(X_1, X_2) = \frac{1}{2}((\nabla_{X_1}\wedge)X_2 + (\nabla_{X_2}\wedge)X_1).$$

The structure $(\Omega, \xi, \wedge, g_m)$ is called *LPS* manifold if

$$(\nabla_{X_1}\Omega)X_2 = g_m(\Omega X_1, \Omega X_2)\xi + \wedge(X_2)\Omega^2 X_1.$$
$$= g_m(X_1, X_2)\xi + \wedge(X_2)X_1 + 2\wedge(X_1)\wedge(X_2)\xi \tag{7}$$

In a Lorentzian para-Sasakian (*LPS*) manifold, the 1-form $\wedge$ is closed. It has been demonstrated that if there exists a timelike unit vector field ξ in an odd-dimensional Lorentzian manifold (Σ_m, g_m) such that the associated 1-form $\wedge$ is closed and fulfills the condition:

$(\nabla_{X_1}\nabla_{X_2}\wedge)X_3 = g_m(X_1, X_2)\wedge(X_3) + g_m(X_1, X_3)\wedge(X_2) + 2\wedge(X_1)\wedge(X_2)\wedge(X_3)$.

Then Σ_m admits an *LPS* structure.

Consider two semi-Riemannian metric manifolds (Σ_m, g_m) and (Σ_n, g_n) . Let $\Psi : (\Sigma_m, g_m) \to (\Sigma_n, g_n)$ be a C^∞-map. The map Ψ is called a semi-Riemannian submersion if it is surjective and $(\Psi_*)p$ has maximal rank having length preserving properties for any $p \in \Sigma m$ means, $(\Psi_*)p : ((ker(\Psi*)p)^\perp, (g_m)p) \to (T_{\Psi(p)}\Sigma_n, (g_n)_{\Psi(p)})$ is a linear isometry for each $p \in \Sigma m$. Here, $(ker(\Psi_*)p)^\perp$ is the orthogonal to $ker(\Psi*)p$ in the tangent space $T_p\Sigma_m$ of Σ_m at p. Finally, Ψ is a semi-Riemannian map if $(\Psi_*)p : ((ker(\Psi*)p)^\perp, (g_m)p) \to ((range\Psi*)\Psi(p), (g_n)\Psi(p))$ is a linear isometry for each $p \in \Sigma m$, where $(range\Psi_*)\Psi(p) := (\Psi*)p((ker(\Psi*)p)^\perp)$ for $p \in \Sigma_m$.

Let (Σ_m, Ω, g_m) be an almost para Hermitian manifold, and (Σ_n, g_n) be a semi-Riemannian metric manifold, where Ω is an almost para complex structure on Σ_m. Consider Ψ is semi-Riemannian map between two manifolds (Σ_m, Ω, g_m) and (Σ_n, g_n) such that $\Psi : (\Sigma_m, \Omega, g_m) \to (\Sigma_n, g_n)$. The map Ψ is called a slant semi-Riemannian map if the angle $\theta = \theta(X_1)$ between ΩX_1 and the space $ker(\Psi_*)p$ is invariant for nonzero $X_1 \in ker(\Psi_*)p$ and $p \in \Sigma_m$. The angle θ is referred to as the slant angle of slant semi-Riemannian map.

The second fundamental form of Ψ can be expressed as:

$$(\nabla\Psi_*)(X_1, X_2) := \nabla^{\Psi}_{X_1}\Psi_* X_2 - \Psi_*(\nabla_{X_1}X_2) \;\; for \; X_1, X_2 \in \amalg(T\Sigma_m). \tag{8}$$

Where $X, Y \in \mathfrak{X}(\Sigma_m)$, ∇ is the Levi-Civita connection, and $\top$ denotes projection onto the horizontal space.

Remind that Ψ is said to be harmonic if we have the tension field $\tau(\Psi) := trace(\nabla\Psi_*) = 0$ and we call the map Ψ a totally geodesic map if $(\nabla\Psi_*)(X_1, X_2) = 0$ for $X_1, X_2 \in \amalg(T\Sigma_m)$. Denote the range of Ψ_* by $range\Psi_*$ as a subset of the pullback bundle $F^{-1}T\Sigma_n$. With its orthogonal complement $(range\Psi_*)^\perp$ we have the following decomposition $F^{-1}T\Sigma_n = range\Psi_* \oplus (range\Psi_*)^\perp$. Moreover, we get $T\Sigma_m = ker\Psi_* \oplus (ker\Psi_*)^\perp$. Then we easily have

Lemma 2.1. *Let $\Psi : (\Sigma_m, g_m) \longrightarrow (\Sigma_n, g_n)$ be a semi-Riemannian map from a semi-Riemannian metric manifold (Σ_m, g_m), where metric is g_m into a semi-Riemannian metric manifold (Σ_n, g_n), where metric is g_n . Then $(\nabla\Psi_*)(Y_1, Y_2) \in \amalg((range\Psi_*)^\perp)$ for $Y_1, Y_2 \in \amalg((ker\Psi_*)^\perp)$.*

Lemma 2.2. *Let $\Psi : (\Sigma_m, g_m) \longrightarrow (\Sigma_n, g_n)$ be a semi-Riemannian map from semi-Riemannian manifold (Σ_m, g_m), where metric is g_m into a semi-Riemannian metric manifold (Σ_n, g_n), where metric is g_n. Then, the tension field τ of Ψ is*

$$\tau = -m_1\Psi_*(H) + m_2H_2, \tag{9}$$

where $m_1 = dim(ker\Psi_*)$, $m_2 = rank\Psi$, H and H_2 are the mean curvature vector fields of the distributions $ker\Psi_*$ and $range\Psi_*$, respectively.

Let $\Psi : (\Sigma_m, g_m) \longrightarrow (\Sigma_n, g_n)$ be a semi-Riemannian map from a semi-Riemannian metric manifold (Σ_m, g_m), where metric is g_m into a semi-Riemannian metric manifold (Σ_n, g_n), where metric is g_n . Then we define $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_P Q = \mathcal{H}\nabla_{\mathcal{H}P}\mathcal{V}Q + \mathcal{V}\nabla_{\mathcal{H}P}\mathcal{H}Q, \tag{10}$$

$$\mathcal{T}_P Q = \mathcal{H}\nabla_{\mathcal{V}P}\mathcal{V}Q + \mathcal{V}\nabla_{\mathcal{V}P}\mathcal{H}Q, \tag{11}$$

for vector fields P, Q on Σ_m, where ∇ is the Levi-Civita connection of semi-Riemannian metric g_m. In fact one can see that these tensor fields are O'Neill's tensor fields which are defined for semi-Riemannian submersions. For any $P \in \amalg(T\Sigma_m)$, $\mathcal{A}$ is anti-symmetric on horizontal distribution and $\mathcal{T}$ is symmetric on vertical distribution $(\amalg(T\Sigma_m), g_m)$. It is also easy to see that, $\mathcal{T}_P = \mathcal{T}_{\mathcal{V}P}$ and $\mathcal{A}\,\mathcal{A} = \mathcal{A}_{\mathcal{H}P}$. We note that the tensor field $\mathcal{T}$ satisfies

$$\mathcal{T}_{X_1}X_2 = \mathcal{T}_{X_2}X_1, \tag{12}$$

$$\mathcal{A}_{X_1}Y_2 = -\mathcal{A}_{X_1}Y_2 = \frac{1}{2}\mathcal{V}[X_1, Y_2],$$

for $X_1, X_2 \in \amalg(ker\Psi_*)$ and $Y_1, Y_2 \in \amalg(ker\Psi_*)^\perp$.
On the other hand, from (11) and (12), we obtain

$$\nabla_{X_1}X_2 = \mathcal{T}_{X_1}X_2 + \hat{\nabla}_{X_1}X_2, \tag{13}$$

$$\nabla_{Y_2}X_1 = \mathcal{H}\nabla_{Y_2}X_1 + \mathcal{T}_{Y_2}X_1, \tag{14}$$

$$\nabla_{X_1}Y_2 = \mathcal{A}_{X_1}Y_2 + \mathcal{V}_2\nabla_{X_1}Y_2, \tag{15}$$

$$\nabla_{X_1}Y_2 = \mathcal{H}\nabla_{X_1}Y_2 + \mathcal{A}_{X_1}Y_2, \tag{16}$$

for $Y_1, Y_2 \in \amalg((ker\Psi_*)^\perp)$ and $X_1, X_2 \in \amalg(ker\Psi_*)$, where $\hat{\nabla}_{X_1}X_2 = \mathcal{V}\nabla_{X_1}X_2$.

3. Geometry of foliations of *PSSSRM*

This section introduces the *PSSSRM* and investigate the conditions under which *PSSSRM* is totally geodesic map. A characterization theorem for *PSSSRM* with totally umbilical fibres is also given.

Definition 3.1. *Let Ψ be a semi-Riemannian map from an almost para-contact manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ with semi-Riemannian metric g_m into a semi-Riemannian metric manifold (Σ_n, g_n) with semi-Riemannian metric g_n. Ψ is a pointwise slant map defined as, for a given point $x \in \Sigma_m$, the angle $\theta(X_1)$ between ΩX_1 is independent of the choice for $X_1 \neq 0 \in (ker\Psi_*)_x - \{\xi\})$. In this case, the angle θ is treated as function then it is called the slant function of the pointwise slant map Ψ.*

Definition 3.2. *Let $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ be an almost para contact manifold and (Σ_n, g_n) a semi-Riemannian metric manifold. A semi-Riemannian map $\Psi : (\Sigma, \Omega, \xi, \wedge, g_m) \to (\Sigma_n, g_n)$ is called a semi-slant semi-Riemannian map if there is a distribution $\mathfrak{D} \subset ker\Psi_*$ such that $ker\Psi_* = \mathfrak{D} \oplus \mathfrak{D}_\theta \oplus <\xi>$, $\Omega(\mathfrak{D}) = \mathfrak{D}$, and the angle $\theta = \theta(X_1)$ between ΩX_1 and the space $(\mathfrak{D}_\theta)_p$ is independent of the choice for $X_1 \neq 0 \in (\mathfrak{D}_\theta)_p$ and $p \in \Sigma_m$, where $\mathfrak{D}$ and $\mathfrak{D}_\theta$ are orthogonal in $ker\Psi_*$.*

The term "semi-slant angle" refers to the angle θ. A point p in a *PSSSRM* is regarded as totally real if its semi-slant is $\theta = \frac{\pi}{2}$ at p. Conversely, if a point p in a *PSSSRM* has a semi-slant function of $\theta = 0$, it is classified as a complex point. A *PSSSRM* is said to be proper if $\theta = 0, \frac{\pi}{2}$.

If P-SS semi-Riemannian map is classified as semi-slant then function θ is overall constant, indicating that it is independent of the point on Σ_m. The constant θ in this case is named as the semi-slant angle of the semi-slant Riemannian map.

As a result, we define a new type of semi-Riemannian map as follows:

Definition 3.3. *Consider $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ be LPS manifold and (Σ_n, g_n)be a semi-Riemannian metric manifold. A semi-Riemannian map $\Psi : (\Sigma, \Omega, \xi, \wedge, g_m) \to (\Sigma_n, g_n)$ is called a PSSSRM, if distribution $\mathfrak{D} \subset ker\pi_*$ such that*

$$ker\Psi_* = \mathfrak{D} \oplus \mathfrak{D}_\theta \oplus <\xi>, \;\; \Omega\mathfrak{D} = \mathfrak{D}, \tag{17}$$

where $\mathfrak{D}_\theta$ and $\mathfrak{D}$ are orthogonal in $ker\Psi_*$ and the angle $\theta = \theta(X_1)$ between ΩX_1 and the space $(\mathfrak{D}_\theta)_p$ is free of the choice of $X_1 \neq 0 \in \amalg(\mathfrak{D}_\theta)_p$ for $p \in M$ i.e. θ is a function on Σ_m, which is named as slant function of the*PSSSRM*. We call Ψ is proper if the slant function is $\theta \neq 0, \frac{\pi}{2}$.

Let $\Psi : (\Sigma, \Omega, \xi, \wedge, g_m) \to (\Sigma_n, g_n)$ be a *PSSSRM* from *LPS* manifold into a semi-Riemannian metric manifold.

The angle θ is a P-wise SS angle of semi-Riemannian map. Then distribution $\mathfrak{D} \subset ker\Psi_*$ such that $ker\Psi_* = \mathfrak{D} \oplus \mathfrak{D}_\theta \oplus < \xi >$, $\Omega(\mathfrak{D}) = \mathfrak{D}$, and the angle $\theta = \theta(X_1)$ between ΩX_1 and the space $(\mathfrak{D}_\theta)_x$ is constant for nonzero $X_1 \in (\mathfrak{D}_\theta)_x$ and $x \in \Sigma_m$, where $\mathfrak{D}_\theta$ and $\mathfrak{D}$ are orthogonal in $ker\Psi_*$.

Then for $X_1 \in \amalg(ker\Psi_*)$, we infer

$$X_1 = PX_1 + QX_1 - \wedge(X_1)\xi, \tag{18}$$

where $PX_1 \in \amalg(\mathfrak{D})$ and $QX_1 \in \amalg(\mathfrak{D}_\theta)$,

for $X_1 \in \amalg(ker\Psi_*)$. We put

$$\Omega X_1 = \omega_1 X_1 + \omega_2 X_1, \tag{19}$$

where $\omega_1 X_1 \in \amalg(ker\Psi_*)$ and $\omega_2 X_1 \in \amalg(ker\Psi_*)^\perp$,

for $Y_1 \in \amalg(ker\Psi_*)^\perp$, we infer

$$\Omega Y_1 = BY_1 + CY_1, \tag{20}$$

where $BY_1 \in \amalg(ker\Psi_*)$ and $CY_1 \in \amalg(ker\Psi_*)^\perp$,

for $X \in \amalg(T\Sigma_m)$, we get

$$X = VX + HX, \tag{21}$$

where $VX \in \amalg(ker\Psi_*)$ and $HX \in \amalg(ker\Psi_*)^\perp$,

for $Y \in \amalg(\Psi^{-1}T\Sigma_n)$, we write

$$Y = \bar{P}Y + \bar{Q}Y, \tag{22}$$

where $\bar{P}Y \in \amalg(range\Psi_*)$ and $\bar{Q}Y \in \amalg((range\Psi_*)^\perp)$.

Then

$$(ker\Psi_*)^\perp = \omega_2\mathfrak{D}_\theta \oplus \mu, \tag{23}$$

where μ is the orthogonal complement of $\omega_2\mathfrak{D}_\theta$ in $(ker\Psi_*)^\perp$ and is invariant under Ω.

Lemma 3.4. *Let Ψ be a PSSSRM from LPS $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian manifold (Σ_n, g_n). Then, we obtain*

(1) $\omega_1^2 + B\omega_2 = I + \wedge \otimes \xi$,
(2) $\omega_2\omega_1 + C\omega_2 = 0$,
(3) $\omega_1 B + BC = 0$,
(4) $\omega_2 B + C^2 = I$,

Lemma 3.5. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian manifold (Σ_n, g_n). Then, we get*

(1) $\omega_1 \mathfrak{D} = \mathfrak{D}$,
(2) $\omega_1 \mathfrak{D}_\theta \subset \mathfrak{D}_\theta$,
(3) $\omega_2 \mathfrak{D} = \{0\}$.

Lemma 3.6. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, we have*

(a) $B(\Omega \mathfrak{D}_\theta) = \mathfrak{D}_\theta$,
(b) $B\mu = \{0\}$,
(c) $C(\Omega \mathfrak{D}_\theta) = \omega_2 \mathfrak{D}_\theta$,
(d) $C\mu = \mu$.

Now, we obtain the effect of Ω on the tensors $\mathcal{T}$ and $\mathcal{A}$ of a *PSSSRM* $\Psi : (\Sigma, \Omega, \xi, \wedge, g_m) \to (\Sigma_n, g_n)$.

Lemma 3.7. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \wedge, \xi, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, we have*

$$\hat{\nabla}_{X_1} \omega_1 X_2 + \mathcal{T}_{X_1} \omega_2 X_2 = \omega_1 \hat{\nabla}_{X_1} X_2 + B\mathcal{T}_{X_1} X_2, \tag{24}$$

$$\mathcal{T}_{X_1} \omega_1 X_2 + \mathcal{H}\nabla_{X_1} \omega_2 X_2 = \omega_2 \hat{\nabla}_{X_1} X_2 + CT_{X_1} X_2, \tag{25}$$

$$\mathcal{V}\nabla_{Y_1} BY_2 + \mathcal{A}_{Y_1} CY_2 = \omega_1 \mathcal{A}_{Y_1} Y_2 + B\mathcal{H}\nabla_{Y_1} Y_2, \tag{26}$$

$$\mathcal{A}_{Y_1} BY_2 + \mathcal{H}\nabla_{Y_1} CY_2 = \omega_2 \mathcal{A}_{Y_1} Y_2 + C\mathcal{H}\nabla_{Y_1} Y_2, \tag{27}$$

$$\hat{\nabla}_{X_1} BY_1 + \mathcal{T}_{X_1} CY_1 = \omega_1 \mathcal{T}_{X_1} Y_1 + B\mathcal{H}\nabla_{X_1} Y_1, \tag{28}$$

$$\mathcal{T}_{X_1} BY_1 + \mathcal{H}\nabla_{X_1} CY_1 = \omega_2 \mathcal{T}_{X_1} Y_1 + C\mathcal{H}\nabla_{X_1} Y_1, \tag{29}$$

$$\mathcal{V}\nabla_{Y_1} \omega_1 X_1 + \mathcal{A}_{Y_1} \omega_2 X_1 = B\mathcal{A}_{Y_1} X_1 + \omega_1 X_1 \nabla_{Y_1} X_1, \tag{30}$$

$$\mathcal{A}_{Y_1} \omega_1 X_1 + \mathcal{H}\nabla_{Y_1} \omega_2 X_1 = C\mathcal{A}_{Y_1} X_1 + \omega_2 X_1 \nabla_{Y_1} X_1, \tag{31}$$

where $X_1, X_2 \in \amalg(ker\pi_*)$, and $Y_1, Y_2 \in \amalg(ker\pi_*^\perp)$.

Proposition 3.8. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, we obtain*

$$\omega_1^2 X_1 = cos^2\theta X_1, \tag{32}$$

for $X_1 \in \amalg(\mathfrak{D}_\theta)$, where θ denotes the slant function.

Proof. If $X_1 \in \amalg(\mathfrak{D}_\theta)$ is vanishing, then done. For any nonzero $X_1 \in \amalg(\mathfrak{D}_\theta)$, we obtain

$$cos\theta = \frac{g_m(\omega_1 X_1, \Omega X_1)}{\|\omega_1 X_1\|\|\Omega X_1\|} = \frac{\|\omega_1 X_1\|}{\|\Omega X_1\|}, \tag{33}$$

so that $g_m(\omega_1 X_1, \omega_1 X_1) = g_m(\omega_1^2 X_1, X_1) = cos^2\theta g_m(\Omega X_1, \Omega X_1)$. Substituting X_1 by $X_1 + X_2, X_2 \in \mathfrak{D}_\theta$, at the above equation, we induce

$$g_m((\omega_1^2 - cos^2\theta(I + \wedge \otimes \xi))(X_1), X_2) + g_m(X_1, (\omega_1^2 - cos^2\theta(I + \wedge \otimes \xi))(X_2)) = 0 \tag{34}$$

$\omega_1^2 - cos^2\theta(I + \wedge \otimes \xi)$ is symmetric so that $g_m((\omega_1^2 - cos^2\theta(I + \wedge \otimes \xi))(X_1), (X_2)) = 0$, we obtain
$\omega_1^2 X_1 = cos^2\theta X_1 \quad X_1 \in \mathfrak{D}_\theta$ □

Easily, we observe that Proposition 3.8 is also true in its converse.

Theorem 3.9. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). The invariant distribution $\mathfrak{D}$ is integrable iff*

$$\omega_1(\hat{\nabla}_{X_3} X_4 - \hat{\nabla}_{X_4} X_3) \in \mathfrak{D}, \tag{35}$$

for $X_3, X_4 \in \amalg(\mathfrak{D})$.

Proof. $X_3, X_4 \in \mathfrak{D}$ and $X_1 \in \mathfrak{D}_\theta$ we know $[X_3, X_4] \in \mathfrak{D}$ iff $\Omega[X_3, X_4] \in \mathfrak{D}$, from (19), we infer,

$$\begin{aligned} g_m(\Omega[X_3, X_4], X_1) &= g_m(\Omega(\nabla_{X_3} X_4 - \nabla_{X_4} X_3), X_1), \\ &= g_m(\Omega(T_{X_3} X_4 + \hat{\nabla}_{X_3} X_4 - T_{X_4} X_3 - \hat{\nabla}_{X_4} X_3), X_1), \\ &= g_m(\omega_1(\hat{\nabla}_{X_3} X_4 - \omega_1(\hat{\nabla}_{X_4} X_3), X_1), . \end{aligned}$$

Thus, $[X_3, X_4] \in \mathfrak{D}$ iff $\omega_1(\hat{\nabla}_{X_3} X_4 - \omega_1(\hat{\nabla}_{X_4} X_3) \in \mathfrak{D}$. □

The proof of the next theorem is similar to the above theorem.

Theorem 3.10. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, the slant distribution $\mathfrak{D}_\theta$ is integrable iff*

$$\omega_1(\hat{\nabla}_{X_1} X_2 - \hat{\nabla}_{X_2} X_1) \in \mathfrak{D}_\theta,$$

for $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$.

Lemma 3.11. *Let Ψ be a proper PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, we obtain*

$$g_m(\nabla_{X_3} X_4, X_1) = cosec^2\theta\{g_m(T_{X_3} X_4, \omega_2\omega_1 X_1) + g_m(T_{X_3}\omega_1 X_4, \omega_2 X_1)\}, \tag{36}$$

$$g_m(\nabla_{X_1} X_2, X_3) = cosec^2\theta\{g_m(T_{X_1}\omega_2\omega_1 X_2, X_3) + g_m(T_{X_1}\omega_2 X_2, \omega_1 X_3)\}, \tag{37}$$

where θ is so call slant fuction, $X_3, X_4 \in \amalg(\mathfrak{D})$ and $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$.

Proof. Let $X_3, X_4 \in \amalg(\mathfrak{D})$ and $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$. Then, from (3) and (19), we obtain

$$\begin{aligned} g_m(\nabla_{X_3} X_4, X_1) &= g_m(\Omega\nabla_{X_3} X_4, \Omega X_1) - \wedge(\nabla_{X_3} X_4) \wedge (X_1) \\ &= g_m(\nabla_{X_3}\Omega X_4 - (\nabla_{X_3}\Omega)X_4, \Omega X_1) \\ &= g_m(\nabla_{X_3}\Omega X_4, \omega_1 X_1) + g_m(\nabla_{X_3}\Omega X_4, \omega_2 X_1) - g_m((\nabla_{X_3}\Omega)X_4, \Omega X_1) \\ &= g_m(\nabla_{X_3} X_4, (\omega_1{}^2 X_1 + \omega_2\omega_1 X_1)) + g_m(\nabla_{X_3}\omega_1 X_4, \omega_2 X_1). \end{aligned}$$

From (32), (13) and (14), we infer

$$\begin{aligned} g_m(\nabla_{X_3}X_4, X_1) &= g_m(\nabla_{X_3}X_4, cos^2\theta X_1) + g_m(\nabla_{X_3}X_4, \omega_2\omega_1X_1) + g_m(\nabla_{X_3}\omega_1X_4, \omega_2X_1) \\ &= cos^2\theta g_m(\nabla_{X_3}X_4, X_1) + g_m(T_{X_3}X_4, \omega_2\omega_1X_1) + g_m(T_{X_3}\omega_1X_4, \omega_2X_1) \\ sin^2\theta g_m(\nabla_{X_3}X_4, X_1) &= g_m(T_{X_3}X_4, \omega_2\omega_1X_1) + g_m(T_{X_3}\omega_1X_4, \omega_2X_1), \end{aligned}$$

We obtain our first result.

For the another result of theorem, we follow the similar pattern as in first part.
Let $X_1, X_2 \in \amalg\mathfrak{D}_\theta$ and $X_3 \in \amalg\mathfrak{D}$. Then from (3) and (19) we get

$$\begin{aligned} g_m(\nabla_{X_1}X_2, X_3) &= g_m(\Omega\nabla_{X_1}X_2, \Omega X_3) - \wedge(\nabla_{X_1}X_2) \wedge (X_3) \\ &= g_m(\nabla_{X_1}\Omega X_2 - g_m(X_1, X_2)\xi, \Omega X_3) \\ &= g_m(\nabla_{X_1}\Omega X_2, \Omega X_3) \\ &= g_m(\nabla_{X_1}\Omega\omega_1X_2 - g_m(X_1, \omega_1X_2)\xi, X_3) + g_m(\nabla_{X_1}\omega_2X_2, \Omega X_3) \\ &= g_m(\nabla_{X_1}({\omega_1}^2X_2 + \omega_2\omega_1X_2), X_3) + g_m(\nabla_{X_1}\omega_2X_2, \Omega X_3). \end{aligned}$$

If we consider (32), (13) and (14), then we get

$$\begin{aligned} g_m(\nabla_{X_1}X_2, X_3) &= g_m(\nabla_{X_1}cos^2\theta X_2, X_3) + g_m(\nabla_{X_1}\omega_2\omega_1X_2, X_3) + g_m(\nabla_{X_1}\omega_2X_2, \Omega X_3) \\ &= g_m((-sin2\theta)(X_1\theta)X_2, X_3) + g_m(cos^2\theta\nabla_{X_1}X_2, X_3) \\ &\quad + g_m(T_{X_1}\omega_2\omega_1X_2, X_3) + g_m(T_{X_1}\omega_2X_2, \Omega X_3) \\ sin^2\theta g_m(\nabla_{X_1}X_2, X_3) &= g_m((-sin2\theta)(X_1\theta)X_2, X_3) + g_m(T_{X_1}\omega_2\omega_1X_2, X_3) + g_m(T_{X_1}\omega_2X_2, \omega_1X_3). \end{aligned}$$

Therefore, since $g_m((-sin2\theta)(X_1\theta)X_2, X_3) = 0$.
The proof of Lemma is complete. □

Theorem 3.12. *Let Ψ be a proper PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian manifold (Σ_n, g_n). Then, the invariant distribution $\mathfrak{D}$ is integrable iff*

$$g_m(T_{X_3}\omega_1X_4 - T_{X_4}\omega_1X_3, \omega_2X_1) = 0,$$

for $X_3, X_4 \in \amalg(\mathfrak{D})$ and $X_1 \in \amalg(\mathfrak{D}_\theta)$.

Proof. Let $X_3, X_4 \in \amalg(\mathfrak{D})$ and $X_1 \in \amalg(\mathfrak{D}_\theta)$. Then, from Lemma 3.11 and equation (12), we get

$$\begin{aligned} g_m([X_3, X_4], X_1) &= g_m(\nabla_{X_3}X_4 - \nabla_{X_4}X_3, X_1) \\ &= cosec^2\theta\{g_m(T_{X_3}X_4, \omega_2\omega_1X_1) + g_m(T_{X_3}\omega_1X_4, \omega_2X_1)\} \\ &\quad - cosec^2\theta\{g_m(T_{X_4}X_3, \omega_2\omega_1X_1) + g_m(T_{X_4}\omega_1X_3, \omega_2X_1)\} \\ &= cosec^2\theta g_m(T_{X_3}\omega_1X_4 - T_{X_4}\omega_1X_3, \omega_2X_1). \end{aligned}$$

Therefore, $\mathfrak{D}$ is integrable then $g(T_{X_3}\omega_1X_4 - T_{X_4}\omega_1X_3, \omega_2X_1) = 0$ and if $g(T_{X_3}\omega_1X_4 - T_{X_4}\omega_1X_3, \omega_2X_1) = 0$, then $\mathfrak{D}$ is integrable. □

In the same pattern, we investigate an integrability of slant distribution $\mathfrak{D}_\theta$.

Theorem 3.13. *Let Ψ be a proper PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$, into a semi-Riemannian manifold (Σ_n, g_n), where g_m and g_n are Lorentzian metric and semi-Riemannian metric respectively. Then, the slant distribution $\mathfrak{D}_\theta$ is integrable iff*

$$g_m(T_{X_1}\omega_2\omega_1X_2 - T_{X_2}\omega_2\omega_1X_1, X_3) = g_m(T_{X_2}\omega_2X_1 - T_{X_1}\omega_2X_2, \omega_1X_3),$$

for $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$ and $X_3 \in \amalg(\mathfrak{D})$.

Proof. Let $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$ and $X_3 \in \amalg(\mathfrak{D})$ then on similar way according to Theorem 3.12 and using Lemma 3.11, we have-

$$\begin{aligned} g_m([X_1, X_2], X_3) &= g_m(\nabla_{X_1} X_2 - \nabla_{X_2} X_1, X_3) \\ &= cosec^2\theta\{g_m(T_{X_1}\omega_2\omega_1 X_2, X_3) + g_m(T_{X_1}\omega_2 X_2, \omega_1 X_3)\} \\ &- cosec^2\theta\{g_m(T_{X_2}\omega_2\omega_1 X_1, X_3) + g_m(T_{X_2}\omega_2 X_1, \omega_1 X_3)\} \\ &= cosec^2\theta\{g_m(T_{X_1}\omega_2\omega_1 X_2, X_3) - g_m(T_{X_2}\omega_2\omega_1 X_1, X_3) \\ &+ g_m(T_{X_1}\omega_2 X_2, \omega_1 X_3) - g_m(T_{X_2}\omega_2 X_1, \omega_1 X_3)\}. \end{aligned}$$

Thus, slant distribution $\mathfrak{D}_\theta$ is integrable iff

$$g_m(T_{X_1}\omega_2\omega_1 X_2 - T_{X_2}\omega_2\omega_1 X_1, X_3) = g_m(T_{X_2}\omega_2 X_1 - T_{X_1}\omega_2 X_2, \omega_1 X_3).$$

□

Now, we study to obtain conditions for totally geodesic foliation of distributions involved in the definition of*PSSSRM*s.

Proposition 3.14. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, $ker\Psi_*$ defines a totally geodesic foliation iff*

$$C(T_{X_1}\omega_1 X_2 + \mathcal{H}\nabla_{X_1}\omega_2 X_2) + \omega_2(\hat{\nabla}_{X_1}\omega_1 X_2 + T_{X_1}\omega_2 X_2) = 0, \tag{38}$$

for, $X_1, X_2 \in \amalg(ker\pi_*)$.

Proof. For, $X_1, X_2 \in \amalg(ker\pi_*)$, from (13), (14) and (19), we get

$$\begin{aligned} \nabla_{X_1} X_2 &= \Omega^2\nabla_{X_1} X_2 - \wedge(\nabla_{X_1} X_2)\xi \\ &= \Omega^2\nabla_{X_1} X_2 \\ &= \Omega\nabla_{X_1}\Omega X_2 \\ &= \Omega T_{X_1}\omega_1 X_2 + \Omega\hat{\nabla}_{X_1}\omega_1 X_2 + \Omega(T_{X_1}\omega_2 X_2) + \Omega(\mathcal{H}\nabla_{X_1}\omega_2 X_2) \\ &= BT_{X_1}\omega_1 X_2 + CT_{X_1}\omega_1 X_2 + \omega_1\hat{\nabla}_{X_1}\omega_1 X_2 + \omega_2\hat{\nabla}_{X_1}\omega_1 X_2 + \omega_1 T_{X_1}\omega_2 X_2 \\ &+ \omega_2 T_{X_1}\omega_2 X_2 + B\mathcal{H}\nabla_{X_1}\omega_2 X_2 + C\mathcal{H}\nabla_{X_1}\omega_2 X_2. \end{aligned}$$

Therefore $ker\pi_*$ defines a totally geodesic foliation iff
$C(T_{X_1}\omega_1 X_2 + \mathcal{H}\nabla_{X_1}\omega_2 X_2) + \omega_2(\hat{\nabla}_{X_1}\omega_1 X_2 + T_{X_1}\omega_2 X_2) = 0.$ □

Proposition 3.15. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian manifold (Σ_n, g_n). Then, $ker\Psi_*^\perp$ defines a totally geodesic foliation iff*

$$B(A_{Y_1}BY_2 + \mathcal{H}\nabla_{Y_1}CY_2) + \omega_1(V\nabla_{Y_1}BY_2 + A_{Y_1}CY_2) = 0, \tag{39}$$

for, $Y_1, Y_2 \in \amalg(ker\Psi_*^\perp)$.

Proof. This proof is similar to Proposition 3.14. □

We have new following results after combining Proposition 3.14 and Proposition 3.15.

Corollary 3.16. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, Σ_m is a locally product $\Sigma_{mker\pi_*} \times \Sigma_{mker\pi_*^\perp}$ iff (38) and (39) hold, where $\Sigma_{mker\pi_*}$ and $\Sigma_{mker\pi_*^\perp}$ are defined as integral manifolds of $ker\pi_*$ and $ker\pi_*^\perp$ respectively.*

Proposition 3.17. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, the invariant distribution $\mathfrak{D}$ defines a totally geodesic foliation on $ker\pi_*$ iff for $X, X_3 \in \amalg(\mathfrak{D})$,*

$$Q(BT_X\omega_1X_3 + \omega_1\hat{\nabla}_X\omega_2X_3) = 0 \quad and \quad (CT_X\omega_1X_3 + \omega_2\hat{\nabla}_X\omega_2X_3) = 0. \tag{40}$$

Proof. For $X, X_3 \in \amalg(\mathfrak{D})$, from (13), (14), (19) and (20) we obtain

$$\begin{aligned}\nabla_X X_3 &= \Omega^2\nabla_X X_3 - \wedge(\nabla_X X_3)\xi \\ &= \Omega\nabla_X(\omega_1X_3 + \omega_2X_3) \\ &= \Omega\nabla_X\omega_1X_3 + \Omega\nabla_X\omega_2X_3 \\ &= \Omega(T_X\omega_1X_3 + \hat{\nabla}_X\omega_2X_3) \\ &= BT_X\omega_1X_3 + CT_X\omega_1X_3 + \omega_1\hat{\nabla}_X\omega_2X_3 + \omega_2\hat{\nabla}_X\omega_2X_3.\end{aligned}$$

Now, the proof is over. □

Proposition 3.18. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian manifold (Σ_n, g_n). Then, $\mathfrak{D}_\theta$ is totally geodesic foliation on $ker\pi_*$ iff for $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$,*

$$P(B(T_{X_1}\omega_1X_2 + \mathcal{H}\nabla_{X_1}\omega_2X_2) + \omega_1(\hat{\nabla}_{X_1}\omega_1X_2 + T_{X_1}\omega_2X_2)) = 0, \tag{41}$$

and

$$\omega_2(\hat{\nabla}_{X_1}\omega_1X_2 + T_{X_1}\omega_2X_2) + C(T_{X_1}\omega_1X_2 + \mathcal{H}\nabla_{X_1}\omega_2X_2) = 0. \tag{42}$$

Proof. The proof of Proposition 3.18 is the same as Proposition 3.17. □

From Proposition 3.17 and Proposition 3.18, we obtain the next result.

Corollary 3.19. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian manifold (Σ_n, g_n). Then, the vertical distribution $ker\Psi_*$ is a locally product $\Sigma_{m\mathfrak{D}} \times \Sigma_{m\mathfrak{D}_\theta}$ iff (40) and (41) true, where $\Sigma_{m\mathfrak{D}}$ and $\Sigma_{m\mathfrak{D}_\theta}$ are integral manifolds of invariant distribution $\mathfrak{D}$ and slant distribution $\mathfrak{D}_\theta$.*

Theorem 3.20. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian manifold (Σ_n, g_n). Then, Ψ is a totally geodesic map iff*

$$\omega_2(\hat{\nabla}_{X_3}\omega_1X_4 + T_{X_3}\omega_2X_4) + C(T_{X_3}\omega_1X_4 + \mathcal{H}\nabla_{X_3}\omega_2X_4) = 0 \tag{43}$$

and

$$\omega_2(\hat{\nabla}_{X_3}BY_3 + T_{X_3}CY_3) + C(T_{X_3}BY_3 + \mathcal{H}\nabla_{X_3}CY_3) = 0 \tag{44}$$

for $X_3, X_4 \in \amalg(ker\pi_*)$ and $Y_3 \in \amalg(ker\pi_*^\perp)$

Proof. Since Ψ is a semi-Riemannian map, we have

$$(\nabla\pi_*)(Y_3, Y_4) = 0, \quad for \ \ Y_3, Y_4 \in \amalg(ker\pi_*^\perp).$$

For $X_3, X_4 \in \amalg(ker\pi_*)$, we obtain

$$
\begin{aligned}
(\nabla\pi_*)(X_3, X_4) &= \nabla^{\pi}_{X_3}(\pi_* X_4) - \pi_*\nabla_{X_3}X_4 \\
&= -\pi_*\nabla_{X_3}X_4 = -\pi_*(\Omega^2\nabla_{X_3}X_4 - \wedge(\nabla_{X_3}X_4)\xi) \\
&= -\pi_*\Omega^2\nabla_{X_3}X_4 = -\pi_*(\Omega(-(\nabla_{X_3}\Omega)X_4 + \nabla_{X_3}\Omega X_4)) \\
&= -\pi_*\Omega\nabla_{X_3}\Omega X_4) = -\pi_*(\Omega(\nabla_{X_3}\omega_1 X_4 + \nabla_{X_3}\omega_2 X_4) \\
&= -\pi_*\Omega(T_{X_3}\omega_1 X_4 + \hat{\nabla}_{X_3}\omega_1 X_4 + T_{X_3}\omega_2 X_4 + \mathcal{H}\nabla_{X_3}\omega_2 X_4) \\
&= -\pi_*(BT_{X_3}\omega_1 X_4 + CT_{X_3}\omega_1 X_4 + \omega_1\hat{\nabla}_{X_3}\omega_1 X_4 + \omega_2\hat{\nabla}_{X_3}\omega_1 X_4 \\
&+ \omega_1 T_{X_3}\omega_2 X_4 + \omega_2 T_{X_3}\omega_2 X_4 + B\mathcal{H}\nabla_{X_3}\omega_2 X_4 + C\mathcal{H}\nabla_{X_3}\omega_2 X_4) \\
&= -\pi_*(CT_{X_3}\omega_1 X_4 + \omega_2\hat{\nabla}_{X_3}\omega_1 X_4 + \omega_2 T_{X_3}\omega_2 X_4 + C\mathcal{H}\nabla_{X_3}\omega_2 X_4).
\end{aligned}
$$

Thus,

$(\nabla\pi_*)(X_3, X_4) = 0 \Leftrightarrow \omega_2(\hat{\nabla}_{X_3}\omega_1 X_4 + T_{X_3}\omega_2 X_4) + C(T_{X_3}\omega_1 X_4 + \mathcal{H}\nabla_{X_3}\omega_2 X_4) = 0$. In same way as above, for $X_3 \in \amalg(ker\pi_*)$ and $Y_3 \in \amalg(ker\pi_*^{\perp})$, we infer

$$(\nabla\pi_*)(X_3, Y_3) = 0 \Leftrightarrow \omega_2(\hat{\nabla}_{X_3}BY_3 + T_{X_3}CY_3) + C(T_{X_3}BY_3 + \mathcal{H}\nabla_{X_3}CY_3) = 0.$$

□

The fibers of a semi-Riemannian map $\pi : (\Sigma_m, g_m) \to (\Sigma_n, g_n)$ is said to be a totally umbilical if

$$T_X X_3 = g_m(X, X_3)H, \tag{45}$$

for any $X, X_3 \in \amalg(ker\pi_*)$, where H denotes the mean curvature vector field of the fiber.

Let Ψ be a *PSSSRM* from *LPS* manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian manifold (Σ_n, g_n), where g_m and g_n are Lorentzian metric and semi-Riemannian metric. Now, we are able to define some useful equations needed for further study

$$(\nabla_X\omega_1)X_3 = \hat{\nabla}_X\omega_1 X_3 - \omega_1\hat{\nabla}_X X_3, \tag{46}$$

$$(\nabla_X\omega_2)X_3 = \mathcal{H}\nabla_X\omega_2 X_3 - \omega_2\hat{\nabla}_X X_3, \tag{47}$$

$$(\nabla_X B)Y_3 = \hat{\nabla}_X BY_3 - B\mathcal{H}\nabla_X Y_3, \tag{48}$$

$$(\nabla_X C)Y_3 = \mathcal{H}\nabla_X CY_3 - C\mathcal{H}\nabla_X Y_3, \tag{49}$$

where $X, X_3 \in \amalg(ker\pi_*)$ and $Y_3 \in \amalg(ker\pi_*^{\perp})$.

We say that ω_1 (resp. ω_2, B or C) is parallel if $\nabla\omega_1 = 0$ (resp. $\nabla\omega_2 = 0$, $\nabla B = 0$ or $\nabla C = 0$).

Lemma 3.21. *Let Ψ be a PSSSRM with canonical parallel structures from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian metric manifold (Σ_n, g_n), where g_m and g_n are Lorentzian metric and semi-Riemannian metric respectively. Then for any $X, X_3 \in \amalg(ker\pi_*)$ and $Y_3 \in \amalg(ker\pi_*^{\perp})$, we infer*

$$(\nabla_X \omega_1)X_3 = BT_X X_3 - T_X \omega_2 X_3, \tag{50}$$

$$(\nabla_X \omega_2)X_3 = CT_X X_3 - T_X \omega_1 X_3, \tag{51}$$

$$(\nabla_X B)Y_3 = \omega_1 T_X Y_3 - T_X C Y_3, \tag{52}$$

$$(\nabla_X C)Y_3 = \omega_2 T_X Y_3 - T_X B Y_3, \tag{53}$$

Proof. All the equations follow from Lemma 3.11 and (46) to (49) . □

Theorem 3.22. *Suppose Ψ be a PSSSRM with totally umbilical fibers from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). If dimension of slant distribution $\mathfrak{D}_\theta$ is greater than 2 and ω_1 holds same direction condition, then we obtain fibers of Ψ are totally geodesic or the mean curvature H is a member of μ.*

Proof. If the fibers of Ψ are totally geodesic then it is stateforward. Now, we consider the second state. Since $\dim(\mathfrak{D}_\theta) \geq 2$, then we can suppose $X_1, X_2 \in \amalg(\mathfrak{D}_\theta)$ such that the set $\{X_1, X_2\}$ is orthonormal. From (3), (7), (19), (20), (13) and (14), we observe

$$\begin{aligned}
\nabla_{X_1}\Omega X_2 &= (\nabla_{X_1}\Omega)X_2 + \Omega\nabla_{X_1}X_2 \\
&= g_m(X_1, X_2)\xi + \Omega\nabla_{X_1}X_2 \\
\nabla_{X_1}(\omega_1 X_2 + \omega_2 X_2) &= \Omega\nabla_{X_1}X_2 + g_m(X_1, X_2)\xi \\
&= \Omega(T_{X_1}X_2 + \hat{\nabla}_{X_1}X_2) + g_m(X_1, X_2)\xi \\
T_{X_1}\omega_1 X_2 + \hat{\nabla}_{X_1}\omega_1 X_2 + T_{X_1}\omega_2 X_2 + \mathcal{H}\nabla_{X_1}\omega_2 X_2 &= BT_{X_1}X_2 + CT_{X_1}X_2 + \omega_1\hat{\nabla}_{X_1}X_2 \\
&\quad + \omega_2\hat{\nabla}_{X_1}X_2 + g_m(X_1, X_2)\xi.
\end{aligned}$$

Taking inner product with X_1

$$\begin{aligned}
g_m(\hat{\nabla}_{X_1}\omega_1 X_2, X_1) + g_m(T_{X_1}\omega_2 X_2, X_1) &= g_m(BT_{X_1}X_2, X_1) + g_m(\omega_1\hat{\nabla}_{X_1}X_2, X_1) \\
g_m(\hat{\nabla}_{X_1}\omega_1 X_2 - \omega_1\hat{\nabla}_{X_1}X_2, X_1) &= g_m(T_{X_1}\omega_2 X_2 - BT_{X_1}X_2, X_1) \\
g_m((\hat{\nabla}_{X_1}\omega_1)X_2, X_1) &= g_m(T_{X_1}\omega_2 X_2 - BT_{X_1}X_2, X_1).
\end{aligned}$$

Since, $\nabla_{X_1}\omega_1 = 0$, we get

$$\begin{aligned}
g_m(T_{X_1}\omega_2 X_2 - BT_{X_1}X_2, X_1) &= 0, \\
g_m(\Omega T_{X_1}X_2 - T_{X_1}\Omega X_2, X_1) &= 0,
\end{aligned}$$

$$g_m(\Omega T_{X_1}X_2, X_1) = g_m(T_{X_1}\Omega X_2, X_1). \tag{54}$$

Thus using (45) and (54), we have

$$\begin{aligned}
g_m(H, \Omega X_2) = g_m(T_{X_1}X_1, \Omega X_2) = g_m(T_{X_1}\Omega X_2, X) &= g_m(\Omega T_{X_1}X_2, X_1) = g_m(T_{X_1}X_2, \Omega X_1) \\
&= g_m(T_{X_1}X_1, \Omega X_1) = g_m(X_1, X_1)g_m(H, \Omega X_1) = 0.
\end{aligned}$$

Since, $g(X_1, X_1) \neq 0$ so $g_m(H, \Omega X_1) = 0$. So, we see $H \perp \omega_2\mathfrak{D}_\theta$. Therefore, it follows H is a member of μ from (23). □

Corollary 3.23. *Suppose Ψ be a PSSSRM with totally umbilical fibers from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). If $(ker\Psi_*)^{\perp} = \omega_2\mathfrak{D}_\theta$, i.e, $\mu = \{0\}$ and ω_1 is in same direction, then it is sure that fibers of Ψ are totally geodesic.*

Assume Ψ be a *PSSSRM* from *LPS* manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). Then, the fibers of Ψ are said to be mixed geodesic, if $T_X X_4 = 0$, for all $X \in \amalg(\mathfrak{D}_\theta)$, $X_4 \in \amalg(\mathfrak{D})$.

Theorem 3.24. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). If ω_2 is in same direction , i.e, $\nabla\omega_2 = 0$, then the fibers of Ψ must be mixed geodesic.*

Proof. Assume ω_2 is in same direction, then for any $X, X_3 \in \amalg(ker\Psi_*)$ from (51), we observe

$$CT_X X_3 = T_X\omega_1 X_3. \tag{55}$$

Using (55), we obtain

$$C^2T_X X_3 = T_X\omega_1^2 X_3. \tag{56}$$

If we put $X = X_4 \in \amalg(\mathfrak{D})$ and $X_3 = X \in \amalg(\mathfrak{D}_\theta)$ in (56) and using (32), we get

$$C^2T_{X_4}X = cos^2\theta T_{X_4}X. \tag{57}$$

Since T is symmetric on $\amalg(ker\Psi_*)$ and from (55), we infer

$$C^2T_{X_4}X = C^2T_X X_4 = T_X\omega_1^2 X_4 = T_X X_4, \tag{58}$$

$$C^2T_{X_4}X = T_X X_4. \tag{59}$$

From (57) and (59), we obtain

$$T_X X_4 = 0. \tag{60}$$

□

4. The first variational form of *PSSSRM*

The purpose of this section is to present an alternative method for verifying the harmonicity of a map, and to define the first variational form of a *PSSSRM* from a Lorentzian para-Sasakian manifold into a semi-Riemannian metric manifold.

Let Ψ be a *PSSSRM* from *LPS* manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ into a semi-Riemannian metric manifold (Σ_n, g_n). We define the 1-form dual to the vector field FY_3, for $Y_3 \in \amalg(ker\Psi_*^{\perp})$, such that $\sigma Y_3 : \amalg(ker\Psi*) \to F(\Psi q^{-1})$, where $q \in N$ and $X_3 \to \sigma Y_3(X_3) = g_m(\Omega Y_3, X_3)$, for all $X_3 \in \amalg(ker\Psi^{\perp}*)$. We define the sets of Legendre, Hamiltonian, and harmonic variations of any fiber of Ψ as $\Gamma_3 = Y_3 \in \amalg(ker\Psi^{\perp}*) : d\sigma Y_3 = 0$, $\Gamma_1 = Y_3 \in \amalg(ker\Psi^{\perp}*) : \exists f \in \Omega(\Psi^{-1}q) \Rightarrow \sigma Y_3 = df$, and $\Gamma_2 = Y_3 \in \amalg(ker\Psi_*^{\perp}) : \Delta\sigma Y_3 = 0$, respectively. It should be noted that $\Gamma_1 \subset \Gamma_3$, $\Gamma_2 \subset \Gamma_3$, and $\Gamma_1 \cap \Gamma_2 = 0$ by the definitions of differential and co-differential operators.

The study is focused on identifying the conditions under which the 1-form σ_{Y_3} defined in the previous content can be considered as a Legendre variation.

Lemma 4.1. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian metric manifold (Σ_n, g_n). The 1-form $\sigma Y3$ is a Legendre variation iff*

$$g_m(T_X Y_2, \omega_1 X_2) - g_m(T_{X_2} Y_2, \omega_1 X) = g_m(A_{Y_2} X_2, \omega_2 X) - g_m(A_{Y_2} X, \omega_2 X_2), \tag{61}$$

for all $X, X_2 \in \amalg(ker\Psi_*)$.

Proof. Let $X, X_2 \in \amalg(ker\Psi_*)$. Then, by the definition of differential, (14) and (3), we obtain

$$\begin{aligned}(d\sigma_{Y_2})(X, X_2) &= Xg_m(\Omega Y_2, X_2) - X_2 g_m(\Omega Y_2, X) - g_m(\Omega Y_2, [X, X_2]) \\ &= \nabla_X g_m(Y_2, \Omega X_2) + g_m(\nabla_X Y_2, \Omega X_2) + g_m(Y_2, \nabla_X \Omega X_2) - \nabla_{X_2} g_m(Y_2, \Omega X) \\ &- g_m(\nabla_{X_2} Y_2, \Omega X) - g_m(Y_2, \nabla_{X_2} \Omega X) - g_m(Y_2, \Omega \nabla_X X_2 - \Omega \nabla_{X_2} X) \\ &= g_m(\nabla_X Y_2, \Omega X_2) - g_m(\nabla_{X_2} Y_2, \Omega X) \\ &= g_m(T_X Y_2 + \mathcal{H}\nabla_X Y_2, \omega_1 X_2) + g_m(T_X Y_2 + \mathcal{H}\nabla_X Y_2, \omega_2 X_2) \\ &- g_m(T_{X_2} Y_2 + \mathcal{H}\nabla_{X_2} Y_2, \omega_1 X) \\ &- g_m(T_{X_2} Y_2 + \mathcal{H}\nabla_{X_2} Y_2, \omega_2 X) \\ &= g_m(T_X Y_2, \omega_1 X_2) + g_m(\mathcal{H}\nabla_X Y_2, \omega_2 X_2) - g_m(T_{X_2} Y_2, \omega_1 X) - g_m(\mathcal{H}\nabla_{X_2} Y_2, \omega_2 X).\end{aligned}$$

Since, we suppose Y_2 is basic, we get

$$(d\sigma_{Y_2})(X, X_2) = g_m(T_X Y_2, \omega_1 X_2) + g_m(A_{Y_2} X, \omega_2 X_2) - g_m(T_{X_2} Y_2, \omega_1 X) - g_m(AY_2 X_2, \omega_2 X).$$

Since $(d\sigma_{Y_2})(X, X_2) = 0$. Therefore,

$$g_m(T_X Y_2, \omega_1 X_2) - g_m(T_{X_2} Y_2, \omega_1 X) = g_m(A_{Y_2} X_2, \omega_2 X) - g_m(A_{Y_2} X, \omega_2 X_2).$$

□

Lemma 4.2. *For $Y_2 \in \amalg(\mu)$, $\sigma_{Y_2} \equiv 0$.*

Proof. Let $Y_3 \in \amalg(\mu)$ then $\Omega Y_3 \in \amalg(\mu)$ for any $X_2 \in \amalg(ker\Psi_*)$, we get

$$\sigma_{Y_3}(X_2) = g(\Omega Y_3, X_2) = 0,$$

so, $\sigma_{Y_3} \equiv 0$ for all $X_2 \in \amalg(ker\Psi_*)$. □

Remark 4.3. *By virtue of Lemma (4.2), the assumption that $\mathcal{H}$ belongs to $\amalg(\omega_2\mathfrak{D}\theta)$ is made throughout this paper.*

Proposition 4.4. *Let Ψ be a PSSSRM that maps LPS manifold $(\Sigma m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian metric manifold (Σ_n, g_n), and let f be a smooth function on a fiber. We then assert that $\Omega(grad(f)|\omega_2\mathfrak{D}_\theta) \in \Gamma_1$.*

Proof. On fibers suppose f is a smooth function, then for $Y_3 = \Omega(grad(f)|\omega_2\mathfrak{D}_\theta)$ and any $X_2 \in \amalg(ker\Psi_*)$. We obtain $\sigma_{Y_3}(X_2) = g_m(\Omega Y_3, X_2) = g_m(grad(f), X_2) = X_2[f] = df(X_2)$. Hence, $\sigma_{Y_3} = df$, implying $Y_3 \in \Gamma_1$. □

Moreover, for $Y_3 \in \amalg(\ker\Psi_*^\perp)$, the first variation of the volume of a fiber Ψ_q^{-1}, for $q \in \Sigma_n$ is defined from [22].

$$\varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(Y_3, \Gamma_2) * 1. \tag{62}$$

In this equation $k = dim(\Psi_q^{-1})$. In this context, we define the fibers as follows:

- If $\varphi'(Y_3) = 0$ for all $Y_3 \in \Gamma_3$, then Ψ_q^{-1} is Γ_3-minimal.

- If $\varphi'(Y_3) = 0$ for all $Y_3 \in \Gamma_1$, then Ψ_q^{-1} is Γ_1-minimal.
- If $\varphi'(Y_3) = 0$ for all $Y_3 \in \Gamma_2$, then Ψ_q^{-1} is Γ_2-minimal.

Remark 4.5. *It is worth noting that if the fiber is minimal, then the fiber is Γ_3, Γ_1, and Γ_2 minimal. In addition, if the fiber is Γ_3-minimal, then it is also Γ_1-minimal and Γ_2-minimal, as $\Gamma_1 \subset \Gamma_3$ and $\Gamma_2 \subset \Gamma_3$.*

Now, we are ready to state our next Theorem.

Theorem 4.6. *If Ψ is a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ to a semi-Riemannian metric manifold (Σ_n, g_n), then*
(a) The fiber Ψ_q^{-1} is Γ_3-minimal iff the Legendre variation of σY_3 is zero for all Y_3 in Γ_3.
(b) The fiber $\Psi^{-1}q$ is Γ_1-minimal iff the Hamiltonian variation of σY_3 is zero for all Y_3 in Γ_1.
(c) The fiber $\Psi^{-1}q$ is Γ_2-minimal iff σ_{Γ_2} can be expressed as the sum of an exact and a co-exact 1-form, where σ_{Γ_2} denotes the harmonic variation of σY_3 for all Y_3 in Γ_2.

Proof. (a)⇒: Let Ψ be a *PSSSRM* that maps *LPS* manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian metric manifold (Σ_n, g_n) and let $\Psi^{-1}q$ be Γ_3-minimal. Then, for any $Y_3 \in \Gamma_3$, we have $g_m(\Gamma_2, Y_3) = 0$ from (**??**). From the definition of the Hodge star operator, we have for $V_1, V_2, ..., V_k \in \amalg(\ker \Psi*)$. From the definition of the global scalar product (.|.) on the module of all forms on the fiber, we get

$$(\sigma_{Y_3}|\sigma_{\Gamma_2}) = \int_{\Psi^{-1}q} \sigma Y_3 \Lambda * \sigma_{\Gamma_2} = 0. \tag{63}$$

Denote by δ the co-differential operator on the fiber $\Psi^{-1}q$. Since σY_3 is closed, for any 2-form β on $\Psi^{-1}q$, we have

$$0 = (d\sigma Y_3|\beta) = (d\sigma_{Y_3}|\delta\beta). \tag{64}$$

Since Ψ_q^{-1} is compact, by (63) and (64) we conclude that σ_{Γ_2} is co-exact.
⇐: Suppose that σ_{Γ_2} is co-exact. We have $\sigma_{\Gamma_2} = \delta\psi$ for some 2-form ψ. Then, for any $Y_3 \in \Gamma_3$, (

$$(\sigma_{Y_3}|\sigma_{\Gamma_2}) = (\sigma_{Y_3}|\delta\psi) = (d\sigma_{Y_3}|\psi) = 0$$

and then

$$\varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(\Gamma_2, Y_3) * 1 = -k \int_{\Psi_q^{-1}} (\sigma_{Y_3} \Lambda * \sigma_{\Gamma_2}) = -k(\sigma_{Y_3}|\sigma_{\Gamma_2}) = 0,$$

i.e. Ψ_q^{-1} is Γ_3-minimal.
(b) ⇒: Let the fiber Ψ_q^{-1} be Γ_1 -minimal. Then, we have

$$0 = \varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(Y_3, \Gamma_2) * 1 = -k \int_{\Psi_q^{-1}} (\sigma_{Y_3} \Lambda * \sigma_{\Gamma_2}) = -k(\sigma_{Y_3}|\sigma_{\Gamma_2}),$$

that is, $(\sigma_{Y_3}|\sigma_{\Gamma_2}) = 0$. Since $Y_3 \in \Gamma_1$, $\sigma_{Y_3} = df$ for some function f on the fiber Ψ_q^{-1}. Thus,

$$(df|\sigma_{\Gamma_2}) = (f|\delta\sigma_{\Gamma_2}) = 0.$$

Hence it follows that $\delta\sigma_{\Gamma_2} = 0$, i.e. σ_{Γ_2} is co-closed.
⇐: Suppose that σ_{Γ_2} is co-closed. Let $Y_3 \in \Gamma_1$, then there exist a function $f \in \mathcal{F}(\Psi_q^{-1})$ such that $\sigma_{Y_3} = df$. Hence, we have

$$(\sigma_{Y_3}|\sigma_{\Gamma_2}) = (df|\sigma_{\Gamma_2}) = (f|\delta\sigma_{\Gamma_2}) = 0.$$

Therefore,

$$\varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(\Gamma_2, Y_3) * 1 = -k \int_{\Psi_q^{-1}} (\sigma_{Y_3} \Lambda * \sigma_{\Gamma_2}) = -k(\sigma_{Y_3}|\sigma_{\Gamma_2}) = 0,$$

that is $\varphi'(Y_3) = 0$ for $Y_3 \in \Gamma_1$, i.e. Ψ_q^{-1} is Γ_1-minimal.

(c) $\Rightarrow$: If the fiber Ψ_q^{-1} is Γ_2-minimal, then for $Y_3 \in \Gamma_2$, we have

$$0 = \varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(Y_3, \Gamma_2) * 1 = -k \int_{\Psi_q^{-1}} (\sigma_{Y_3} \Lambda * \sigma_{\Gamma_2}) = -k(\sigma_{Y_3} | \sigma_{\Gamma_2}).$$

It means that, σ_{Γ_2} is orthogonal to harmonic 1-forms on the fiber Ψ_q^{-1}. Thus, by the Hodge decomposition theorem, we conclude that σ_{Γ_2} is the sum of an exact and a co-exact 1-form.
$\Leftarrow$: Let σ_{Γ_2} be the sum of an exact 1-form $\omega_1 = df$ and a co-exact 1-form $\omega_2 = \delta\psi$. For $Y_3 \in \Gamma_2$, we have

$$(\sigma_{Y_3} | \sigma_{\Gamma_2}) = (\sigma_{Y_3} | df + \delta\psi) = (\sigma_{Y_3} | df) + (\sigma_{Y_3} | \delta\psi) = (\delta\sigma_{Y_3} | f) + (d\sigma_{Y_3} | \psi) = 0,$$

since $d\sigma_{Y_3} = \delta\sigma_{Y_3} = 0$. Thus,

$$\varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(Y_3, \Gamma_2) * 1 = -k \int_{\Psi_q^{-1}} (\sigma_{Y_3} \Lambda * \sigma_{\Gamma_2}) = -k(\sigma_{Y_3} | \sigma_{\Gamma_2}),$$

that is, the fiber is Γ_2-minimal. □

Theorem 4.7. *Let Ψ be a PSSSRM from LPS manifold $(\Sigma_m, \Omega, \xi, \wedge, g_m)$ onto a semi-Riemannian metric manifold (Σ_n, g_n). If $\Gamma_2 \in \Gamma_3$ Then,*
(a) Ψ_q^{-1} is Γ_3-minimal iff Ψ_q^{-1} is minimal.
(b) Ψ_q^{-1} is Γ_1-minimal iff σ_{Γ_2} is a harmonic variation.
(c) Ψ_q^{-1} is Γ_2-minimal iff σ_{Γ_2} is an exact 1-form.

Proof. (a) If the fiber Ψ_q^{-1} is Γ_3-minimal, then by Theorem **??**-(a) we have, σ_{Γ_2} is co-exact. Hence σ_{Γ_2} is co-closed. Taking into account the fact that $d\sigma_{\Gamma_2} = 0$, we deduce that σ_{Γ_2} is harmonic. But this is a contradiction because of Hodge decomposition theorem. So, σ_{Γ_2} must be zero. Hence we conclude that $\Gamma_2 = 0$. The converse is clear.
(b) $\Rightarrow$: If the fiber Ψ_q^{-1} is Γ_1-minimal, then we have $\delta\sigma_{\Gamma_2} = 0$ from Theorem **??**-(b). Since $d\sigma_{\Gamma_2} = 0$, σ_{Γ_2} is also harmonic, i.e. $\Delta\sigma_{\Gamma_2} = 0$.
$\Leftarrow$: If σ_{Γ_2} is harmonic, then σ_{Γ_2} is co-closed. By Theorem **??**-(b), the fiber Ψ_q^{-1} is Γ_1-minimal.
(c) $\Rightarrow$: Assume that Ψ_q^{-1} is Γ_2-minimal. then, from Theorem **??**-(c), σ_{Γ_2} is the sum of an exact 1-form and a co-exact 1-form. On the other hand, the condition $\Gamma_2 \in \Gamma_3$ implies that σ_{Γ_2} is orthogonal to every co-exact 1-form on Ψ_q^{-1}. Thus, σ_{Γ_2} must be exact.
$\Leftarrow$: Let σ_{Γ_2} be an exact 1-form. For $Y_3 \in \Gamma_2$, we obtain

$$\varphi'(Y_3) = -k \int_{\Psi_q^{-1}} g_m(Y_3, \Gamma_2) * 1 = -k \int_{\Psi_q^{-1}} (\sigma_{Y_3} \Lambda * \sigma_{\Gamma_2})$$
$$= -k(\sigma_{Y_3} | \sigma_{\Gamma_2}) = (\sigma_{Y_3} | df) = (\delta\sigma_{Y_3} | f) = 0,$$

that is, Ψ_q^{-1} is Γ_2-minimal. □

Remark 4.8. *It is well known that, the fibers of a submersion is minimal iff the submersion is harmonic.*

5. Examples

Now, we present some examples for *PSSSRM*s.

Example 5.1. *Every almost para-contact submersion from an almost para-contact manifold into a semi-Riemannian metric manifold is a P-wise SS Riemannian map with $\theta = 0$ and $(range\Psi_*)^{\perp} = 0$.*

Example 5.2. *Every anti-invariant Riemannian submersion from an almost para-contact manifold into a semi-Riemannian manifold is a P-wise SS Riemannian map with $\theta = \frac{\pi}{2}$ and $(range\Psi_*)^\perp = 0$.*

Example 5.3. *Every proper P-wise SS Riemannian submersion with the slant function θ is a P-wise SS Riemannian map with $(range\Psi_*)^\perp = 0$.*

Example 5.4. *Every proper slant Riemannian submersion with the slant angle θ is a P-wise SS Riemannian map with $(range\Psi_*)^\perp = 0$.*

Example 5.5. *Every proper semi-slant Riemannian map is a P-wise SS Riemannian map with a constant slant function.*

Example 5.6. *Let $R^{2m+1} = \{(x_1, x_2, ..., x_m, y_1, y_2, ..., y_m, z : x_i, y_i, z \in R, i = 1, 2, ..., m)\}$. Consider R^{2m+1} with the following structure:*

$$\Omega\left(\sum_{i=1}^{m}(X_i\frac{\partial}{\partial x_i} + Y_i\frac{\partial}{\partial y_i}) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{m} Y_i\frac{\partial}{\partial x_i} + \sum_{i=1}^{m} X_i\frac{\partial}{\partial y_i} + \sum_{i=1}^{m} X_i y_i\frac{\partial}{\partial z},$$

$$g_m = -\wedge\otimes\wedge + \frac{1}{4}\sum_{i=1}^{m}(dx_i\otimes dx_i + dy_i\otimes dy_i),$$

$$\wedge = -\frac{1}{2}(dz - \sum_{i=1}^{m} y_i dx_i),\ \ \xi = 2\frac{\partial}{\partial z}.$$

Then, $(R^{2m+1}, \Omega, \xi, \wedge, g_m)$ is LPS manifold. The vector fields $E_i = 2\frac{\partial}{\partial y_i}$, $E_{m+i} = 2(\frac{\partial}{\partial x_i} + y_i\frac{\partial}{\partial z})$ and ξ form a Ω-basis for the contact metric structure.

Example 5.7. *Considering Example* **6***, we define a map*

$$\Psi : R^9 \to R^7$$

by

$$\Psi(x_1, .., x_4, y_1, .., y_4, z) = (\cos y_1 x_1 - \sin y_1 x_3, \sin y_2 x_2 - \cos y_2 y_3, 0, 0, 0, 0, 0).$$

Then it follows that

$$(\ker\Psi_*) = span\{E_4, E_8, E_9, \sin y_1 E_5 + \cos y_1 E_7, \cos y_2 E_6 + \sin y_1 E_3\},$$

where

$$D = span\{E_4, E_8\},$$

$$D_\theta = span\{\sin y_1 E_5 + \cos y_1 E_7, \cos y_2 E_6 + \sin y_2 E_3\},$$

$$\xi = E_9,$$

and

$$(\ker\Psi_*)^\perp = span\{E_1, E_2, \cos y_1 E_5 - \sin y_1 E_7, \sin y_2 E_6 - \cos y_2 E_3\},$$

where $\theta = cos^{-1}\frac{\sin 2y_2}{2\sqrt{\sin^2 y_1 + \cos^2 y_2 - y_2^2\cos^2 y_2}}$; where $\sin^2 y_1 + \cos^2 y_2 - y_2^2\cos^2 y_2 > 0$ is a point wise slant angle and Ψ is a PSSSRM from LPS manifold into semi-Riemannian metric manifold.

6. Conclusion

The first variational formula is fundamental to differential geometry and is essential to understanding the complexities of geometric objects, especially those that fall under the domain of surfaces. Its importance extends to various applications, with a primary focus on the study of minimal surfaces and the calculus of variations in a geometric context. In addition, the first variational formula serves a substantial part in the analysis of geodesics in Riemannian geometry contributing to deduce geodesic equations and expanding comprehension of curved spaces. Furthermore, it is used in the derivation of the Jacobi equation, which provides insights into the curvature of a manifold along geodesics. As a result, variational problems pertaining to minimal surfaces, geodesics, and variational principles can be systematically investigated through the use of the first variational formula, which malleable tool in differential geometry that improves our understanding of the geometry of spaces and the behavior of curves and surfaces within them.

However, Harmonic maps are necessary for the study of geometric structures on manifolds. They have intimate relationships, for instance, with minimal surfaces, isometric embeddings, and conformal mappings. The harmonicity condition provides insights into the interplay of geometry and analysis by establishing the equilibrium between curvature and deformation.

Thus, we have extended the understanding of Pointwise semi-slant maps which are defined on LP-sasakian manifold to a semi-Riemannian manifold and studied its behavior and significance through using the first variational formula on the fibers to derive necessary and sufficient conditions for their harmonicity.

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References

[1] M. A. Akyol and Y. Gunduzalp, *Semi-invariant semi-Riemannian submersions*, Commun. Fac. Sci. Univ. Ank. Series A1 **67**(1) (2018), 32–42.

[2] M. A. Akyol and Y. Gunduzalp, *Pointwise slant Riemannian maps (PSRM) into almost Hermitian manifolds*, Mediterr. J. Math. **20**:116 (2023), https://doi.org/10.1007/s00009-023-02321-5.

[3] J. P. Bourguinon and H. B. Lowson, *Stability and isolation phenomena for Yang-Mills fields*, Comm. Math. Phys. **79**(2) (1981), 189–230.

[4] G. Baditoiu and S. Ianus, *Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces*, Diff. Geom. Appl. **16** (2002), 79–84.

[5] B. Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk. J. Math. **36** (2012), 630–640.

[6] C. Chinea, *Almost contact metric submersions*, Rend. Circ. Mat. Palermo **43**(1) (1985), 89–104.

[7] M. Faghfouria and S. Mashmouli, *On anti-invariant semi-Riemannian submersions from Lorentzian para-Sasakian manifolds*, Filomat **32**(10) (2018), 3465–3478.

[8] A. E. Fischer, *Riemannian maps between Riemannian manifolds*, Contemp. Math. **132** (1992), 331–366.

[9] G., *G-Riemannian submersions and nonlinear gauge field equations of general relativity*, Global analysis-analysis on manifolds, 324–349, Teubner-Texte Math. **57** Teubner, Leipzig, 1983.

[10] Y. Gunduzalp and B. Sahin, *Para-contact para-complex semi-Riemannian submersions*, Bull. Malays. Math. Sci. Soc. **37**(1) (2014), 139–152.

[11] A. Gray, *Pseudo-Riemannian almost product manifolds and submersions*, J. Math. Mech. **16** (1967), 715–737.

[12] Y. Gunduzalp and B. Sahin, *Paracontact semi-Riemannian submersions*, Turk. J. Math. **37** (2013), 114–128.

[13] A. Gray, *Pseudo-Riemannian almost product manifolds and submersion*, J. Math. Mech. **16** (1967), 715–737.

[14] E. Garcia-Rio and D. N. Küpeli, *Semi-Riemannian Maps and Their Applications*, Kluwer Academic, Dordrecht, 1999.

[15] Y. Gunduzalp, *Slant submersions in paracontact geometry*, Hacet. J. Math. Stat. **49** (2020), 822–834.

[16] Y. Gunduzalp and M. A. Akyol, *Pointwise slant Riemannian maps from Kaehler manifolds*, Journal of Geometry and Physics **179** (2022), 104589.

[17] S. Ianus and M. Visinescu, *Kaluza-Klein theory with scalar fields and generalized Hopf manifolds*, Classical Quantum Gravity **4** (1987), 1317–1325.

[18] S. Ianus and M. Visinescu, *Space-time compactification and Riemannian submersion*, The mathematical heritage of C. F. Gauss, 358–371, World Sci. Publ., River Edge, NY, 1991.

[19] J. P. Bourguinon and H. B. Lowson, *A mathematicians visit to Kaluza-Klein theory*, Rend. Sem. Mat. Univ. Politec. Torino (1989), Special Issue, 143–163.

[20] R. Kaushal, R. Kumar and R. K. Nagaich, *On geometry of pointwise slant lightlike submersions with totally umbilical fibers*, Afrika Matematika **33**(22) (2022).
[21] J. W. Lee and B. Sahin, *Pointwise slant submersions*, Bull. Korean Math. Soc. **51**(4) (2014), 1115–1126.
[22] H. B. Lawson, *Minimal varieties in real and complex geometry*, Univ. of Montreal, 1972.
[23] P. Libermann, *Sur les structures presque para-complexes*, C.R. Acad. Sci. Paris Ser. I Math. **234** (1992), 2517–2519.
[24] M. T. Mustafa, *Applications of harmonic morphisms to gravity*, J. Math. Phys. **41**(10) (2000), 6918–6929.
[25] K. Miura, *Construction of Harmonic Maps between Semi-Riemannian Spheres*, Tsukuba J. Math. **31**(2) (2007), 397–409.
[26] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 458–469.
[27] B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Academic Press, New York, 1983.
[28] K. S. Park and B. Sahin, *Semi-slant Riemannian maps into almost Hermitian manifolds*, Czechoslov. Math. J. **64**(4) (2014), 1045–1061.
[29] R. Prasad and S. Pandey, *Slant Riemannian maps from an almost contact manifold*, Filomat **31**(13) (2017), 3999–4007.
[30] R. Prasad and S. Pandey, *Semi-slant Riemannian maps from almost contact manifolds*, Analele Universitatii, Oradea Fasc. Matematica Tom **XXV**(2) (2018), 127–141.
[31] R. Prasad and S. Pandey, *Hemi-slant Riemannian maps from almost contact metric manifolds*, P.J.M. **9**(2) (2020), 811–823.
[32] R. Prasad and S. Pandey, *Semi-slant Submersion from an almost Para-cosymplectic manifold*, Novi Sad J. Math. **47**(2) (2017), 93–105.
[33] S. Pandey, R. Prasad, P. Gupta and S. Singh, *Semi-invariant conformal submersions with horizontal Reeb vector field*, Filomat **37**(12) (2023), 3819–3836.
[34] P. K. Rashevskij, *The scalar field in a stratified space*, Trudy Sem. Vektor. Renzor. Anal. **6** (1948), 225–248.
[35] B. Sahin, *Anti-invariant Riemannian submersions from almost Hermitian manifolds*, Cent. Eur. J. Math. **8**(3) (2010), 437–447.
[36] B. Sahin, *Semi-invariant submersions from almost Hermitian manifolds*, Canadian Math. Bull. **56**(1) (2013), 173–182.
[37] B. Sahin, *Slant Riemannian maps from almost Hermitian manifolds*, Quaest. Math. **36**(3) (2013), 449–461.
[38] B. Sahin, *Riemannian Submersions, Riemannian Maps in Hermitian Geometry, and Their Applications*, Elsevier, Academic Press, 2017.
[39] S. A. Sepet and M. Ergut, *Pointwise bi-slant submersions from cosymplectic manifolds*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **69**(2) (2020), 1310–1319.
[40] M. D. Siddiqi and M. Ahmad, *Anti-invariant semi-Riemannian submersions from indefinite almost contact metric manifolds*, Konuralp Journal of Mathematics **8**(1) (2020), 38–49.
[41] B. Watson, *Almost Hermitian submersions*, J. Differential Geometry **11**(1) (1976), 147–165.