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Impact of Schouten-van Kampen connection on LP-Sasakian manifolds

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Abstract. In this paper, we explore the geometric structure of LP-Sasakian manifolds in the context of the Schouten-van Kampen connection. We investigate the interplay between this connection and various curvature-related properties of the manifold. It is established that an LP-Sasakian manifold is locally ϕ-symmetric with respect to the Schouten-van Kampen connection if and only if the same holds for the Levi-Civita connection. Furthermore, we demonstrate that if an LP-Sasakian manifold is ϕ-recurrent under the Schouten-van Kampen connection, then it necessarily satisfies the η-Einstein condition with respect to the Levi-Civita connection. We also prove that quasi-conharmonically flat, conharmonically flat and ϕ-conharmonically flat LP-Sasakian manifolds admitting the Schouten-van Kampen connection are likewise η-Einstein manifolds.

1. Introduction

In differential geometry, the exploration of manifolds with distinct geometric structure is essential for understanding complex spaces, particularly those that emerge in physics, such as general relativity and cosmology. Among these structures, the para-Sasakian manifold stand out as a valuable framework-much like the Sasakian manifold for examining geometries exhibiting certain symmetries. The concept of LP-Sasakian manifolds was first introduced by K. Matsumoto in 1989 [10]. Independently, I. Mihai and R. Rosca [12] also developed the notion and established several foundational results within this framework. Subsequent investigations into LP-Sasakian manifolds have been carried out by various researchers, including K. Mastumoto and I. Mihai [11], as well as U.C. De, K. Matsumoto and A.A. Shaikh [6] and many others such as ([4], [7], [18]).

The Schouten-van Kampen connection, first introduced by Van Kampen in 1930 for the analyzing non-holomorphic manifolds [17]. It is regarded as one of the most intrinsic connections on differentiable manifolds endowed with an affine connection [1]. In 2006, Bejancu [2] examined various properties of the Schouten-van Kampen connection of foliated manifolds. Subsequently, Olszak [13] focused on the Schouten-van Kampen connection in the framework of almost contact metric structure and showed some

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significant results. Recently, Zeren et al. Characterize LP-Sasakian manifolds with respect to the Schouten-van Kampen connection and showed some interesting results.

Schouten-van Kampen connection $\check{\nabla}$ on Riemannian manifold (M^n, g) is given by

$$\check{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\nabla_{X_1}\xi - (\nabla_{X_1}\eta)(X_2)\xi, \tag{1}$$

for all $X_1,\ X_2 \in \chi(M)$.

A function is termed harmonic if it satisfies the condition that its Laplacian vanishes. In general, applying a transformation to a harmonic function does not guarantee that the resulting function remains harmonic. The specific conditions under which harmonicity is preserved have been investigated by Ishii [9], who introduced a specialized subclass of conformal transformations known as conharmonic transformations. These transformations maintain the harmonic nature of functions and are defined by a conformal change of the metric

$$\bar{g}_{ij} = e^{2\sigma} g_{ij},$$

where σ is a smooth scalar field. For the transformation to be conharmonic, the conformal factor σ must satisfy the following differential condition:

$$\sigma^{i}{}_{,i} + \sigma^{\prime i}\sigma_{,i} = 0$$

in which commas denotes covariant derivatives with respect to the original metric g. This condition ensures that the Laplace operator transforms in a way that preserves its vanishing property, thereby keeping harmonic functions invariant under the transformation.

Conharmonic curvature tensor of LP-Sasakian manifold is defined by [9]

$$K(X_1, X_2)X_3 = R(X_1, X_2)X_3 - \frac{1}{(2n-1)}[S(X_2, X_3)X_1 - S(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2], \tag{2}$$

where $X_1,\ X_2,\ X_3 \in \chi(M)$ R, S and Q are the Riemannian curvature tensor, Ricci tensor and Ricci operator respectively.

Subsequent to Ishii's work, several researchers have made substantial contributions to the theory and applications of conharmonic geometry. Yano and Bochner studied various curvature invariants under conformal and projective transformations, laying foundational concepts for understanding how curvature tensors behave under different geometric mappings [22]. Building on these ideas, Ryszard explored manifolds admitting conharmonic transformations and investigated the properties that remain invariant under such transformations [20]. Prvanovi'c analyzed manifolds with recurrent conharmonic curvature tensors, contributing to the classification of special types of manifolds where the conharmonic tensor exhibits particular symmetries or recurrence relations [14]. Later, Chaki extended the study of conharmonic curvature by introducing the notion of conharmonic flatness and its implications in the context of quasi-Einstein and semi-symmetric manifolds [3]. In more recent developments, De and De [5] give various geometric structures satisfying specific curvature conditions involving the conharmonic tensor, thereby broadening the scope of its applicability.

The conharmonic curvature tensor $\check{C}$ of LP-Sasakian manifold with respect to Schouten-van Kampen connection $\check{\nabla}$ is given by

$$\check{K}(X_1, X_2)X_3 = \check{R}(X_1, X_2)X_3 - \frac{1}{(2n-1)}[\check{S}(X_2, X_3)X_1 - \check{S}(X_1, X_3)X_2 + g(X_2, X_3)\check{Q}X_1 - g(X_1, X_3)\check{Q}X_2], \tag{3}$$

where $\check{R}$, $\check{S}$ and $\check{S}$ are Riemannian curvature tensor, Ricci tensor and scalar curvature tensor with respect to Schouten-van Kampen connection $\tilde{\nabla}$ respectively.

Definition 1.1. *An n-dimensional LP-Sasakian manifold M^n is said to be η-Einstein manifold if the Ricci tensor S is of the form $S(X_1, X_2) = ag(X_1, X_2) + b\eta(X_1)\eta(X_2)$, for all $X_1,\ X_2 \in \chi(M)$, where a and b are scalars.*

2. Lorentzian Para-Sasakian Manifolds

Let M^n be an n-dimensional differentiable manifolds equipped with the structure (ϕ, ξ, η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field, η is a 1-form on M^n such that

$$\eta(\xi) = -1, \tag{4}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{5}$$

which implies

$$\phi\xi = 0, \tag{6}$$

$$\eta(\phi) = 0, \tag{7}$$

$$rank(\phi) = n - 1. \tag{8}$$

Then the manifold M^n admit a Lorentzian metric g, such that

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{9}$$

and M^n is said to admit a Lorentzian almost paracontact structure (ϕ, ξ, η, g). In this case, we have

$$g(X_1, \xi) = \eta(X_1), \quad \nabla_{X_1}\xi = \phi X_1, \tag{10}$$

$$\Omega(X_1, X_2) = g(X_1, \phi X_2) = g(\phi X_1, X_2) = \Omega(X_2, X_1), \tag{11}$$

where Ω is another 2-form on M^n.

If we replace ξ by $-\xi$ in equations (4) and (5), the the structure (ϕ, ξ, η) becomes an almost paracontact structure on M^n defined by Sato ([15], [16]). The Lorentzian metric given by equation (10) stands analogues to the almost paracontact Riemannian metric for any almost paracontact manifold [10].

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian paracontact manifold [10] if

$$\Omega(X_1, X_2) = \frac{1}{2}((\nabla_{X_1}\eta)X_2 + (\nabla_{X_2}\eta)X_1). \tag{12}$$

A Lorentzian almost paracontact manifold M^n equipped with the structure (ϕ, ξ, η, g) is called Lorentzian para-Sasakian manifold [10] if

$$(\nabla_{X_1}\phi)X_2 = g(\phi X_1, \phi X_2)\xi + \eta(X_2)\phi^2 X_1. \tag{13}$$

In Lorentzian para-Sasakian manifolds the 1-form η is closed. Also in [10], it is provided that if an n-dimensional Lorentzian para-Sasakian manifold (M^n, g) admits a timelike unit vector field ξ such that the 1-form η associated to ξ is closed and satisfies

$$(\nabla_{X_1}\nabla_{X_2})X_3 = g(X_1, X_2)\eta(X_3) + g(X_1, X_3)\eta(X_2) + 2\eta(X_1)\eta(X_2)\eta(X_3), \tag{14}$$

then M^n admits a Lorentzian para-Sasakian structure. Also n-dimensional Lorentzian para-Sasakian manifold M^n satisfies the following conditions:

$$(\nabla_{X_1}\eta)X_2 = -g(X_1, X_2) - \eta(X_1)\eta(X_2), \tag{15}$$

$$R(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{16}$$

$$R(\xi, X_1)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \tag{17}$$

$$\eta(R(X_1, X_2)X_3) = g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2), \tag{18}$$

for all $X_1,\ X_2,\ X_3 \in \chi(M)$.

On using equations (1), (10) and (15) Schouten-van Kampen connection of LP-Sasakian manifold is given by

$$\breve{\nabla}_{X_1} X_2 = \nabla_{X_1} X_2 + \eta(X_2)\phi X_1 - g(\phi X_1, X_2)\xi. \tag{19}$$

Example: Consider a 3-dimensional Lorentzian manifold M^3 with the coordinate system (x, y, t) and the Lorentzian metric

$$g = dx^2 + e^{2x} dy^2 - dt^2.$$

This metric has Lorentzian signature $(+, +, -)$. Define LP-Sasakian structure as almost contact structure

$$\phi(\partial_x) = e^x \partial_y, \quad \phi(\partial_y) = -e^{-x}\partial_x, \quad \phi(\partial_t) = 0,$$

characteristic vector field $\xi = \partial_t$ and 1-form $\eta = dt$. Ricci tensor $S(X_1, X_2)$ is given by:

$$S(X_1, X_2) = \sum_{i=1}^{3} g(R(X_1, E_i)X_2, E_i),$$

where $E_1,\ E_2,\ E_3 = \partial_x,\ \partial_y, \partial_t$ is an orthogonal basis.

Computing the Christoffel symbols using the Koszul formula

$$2g(\nabla_{X_1} X_2, X_3) = X_1 g(X_2, X_3) + X_2 g(X_3, X_1) - X_3 g(X_1, X_2) + g([X_1, X_2], X_3) + g([X_3, X_1], X_2) - g([X_2, X_3], X_1),$$

we get

$$\begin{aligned} &\nabla_{\partial_x}\partial_x = 0, && \nabla_{\partial_x}\partial_y = e^x \partial_y, && \nabla_{\partial_x}\partial_t = 0, \\ &\nabla_{\partial_y}\partial_y = -e^{-x}\partial_x, && \nabla_{\partial_y}\partial_t = 0, && \nabla_{\partial_t}\partial_t = 0. \end{aligned}$$

The components of the Riemann curvature tensor are as follows:

$R(\partial_x, \partial_y)\partial_x = \partial_y, \quad R(\partial_x, \partial_y)\partial_y = -2e^{-x}\partial_x, \quad R(\partial_x, \partial_y)\partial_t = 0,$

$R(\partial_x, \partial_t)\partial_x = 0, \quad R(\partial_x, \partial_t)\partial_y = 0, \quad R(\partial_x, \partial_t)\partial_t = 0,$

$R(\partial_y, \partial_t)\partial_x = 0, \quad R(\partial_y, \partial_t)\partial_y = 0, \quad R(\partial_y, \partial_t)\partial_t = 0.$

The Ricci curvature components as

$$S(\partial_x, \partial_x) = -1, \quad S(\partial_y, \partial_y) = e^{-2x}, \quad S(\partial_t, \partial_t) = 0,$$

for off-diagonal terms

$$S(\partial_x, \partial_y) = S(\partial_x, \partial_t) = S(\partial_y, \partial_t) = 0.$$

Thus, the Ricci curvature matrix is

$$S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{-2x} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A manifold is an Einstein manifold if $S = \lambda g$, but we have matrix

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

comparing S with g, we see that no constant λ satisfies $S = \lambda g$, hence the manifold is not Einstein.

A manifold is η-Einstein if

$$S = \alpha g + \beta \eta \otimes \eta.$$

Now,

$$\eta \otimes \eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

substituting g and $\eta \otimes \eta$, and matching the Ricci tensor, we get

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{-2x} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Solving for $\alpha,\ \beta$, we get $\alpha = -1$ and $\beta = 1$.

Thus, the manifold is an η-Einstein manifold with $S = -g + \eta \otimes \eta$.

3. Curvature Tensor of LP-Sasakian Manifolds with respect to Schouten-van Kampen Connection

Riemannian curvature tensor $\check{R}$ of LP-Sasakian manifold M^n with respect to Schouten-van Kampen connection is given by

$$\check{R}(X_1, X_2)X_3 = \check{\nabla}_{X_1}\check{\nabla}_{X_2}X_3 - \check{\nabla}_{X_2}\check{\nabla}_{X_1}X_3 - \check{\nabla}_{[X_1,X_2]}X_3, \tag{20}$$

which on using equation (19), above equation reduces to

$$\begin{aligned} \check{R}(X_1, X_2)X_3 &= R(X_1, X_2)X_3 + g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1 \\ &+ g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi + \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2. \end{aligned} \tag{21}$$

which is relation between curvature tensor of connections ∇ and $\tilde{\nabla}$, where

$$R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3.$$

From equation (21), we have

$$\begin{aligned} {}'\check{R}(X_1, X_2, X_3, X_4) &= R(X_1, X_2, X_3, X_4) + g(X_3, \phi X_1)g(\phi X_2, X_4) - g(X_3, \phi X_2)g(\phi X_1, X_4) \\ &+ g(X_2, X_3)\eta(X_1)\eta(X_4) - g(X_1, X_3)\eta(X_2)\eta(X_4) + g(X_1, X_4)\eta(X_2)\eta(X_3) - g(X_2, X_4)\eta(X_1)\eta(X_3). \end{aligned} \tag{22}$$

where

$${}'\check{R}(X_1, X_2, X_3, X_4) = g(\check{R}(X_1, X_2)X_3, X_4)$$

and

$${}'R(X_1, X_2, X_3, X_4) = g(R(X_1, X_2)X_3, X_4).$$

Putting $X_1 = X_4 = e_i$ in equation 21 and taking summation over $i,\ 1 \le i \le n$, we get

$$\check{S}(X_2, X_3) = S(X_2, X_3) + (n-1)\eta(X_2)\eta(X_3), \tag{23}$$

where, $\tilde{S}$ and S are the Ricci tensor of the connections $\check{\nabla}$ and ∇ respectively. Again putting $X_2 = X_3 = e_i$ in equation (23) and taking summation over $i,\ 1 \le i \le n$, we get

$$\check{r} = r - (n-1), \tag{24}$$

where, $\breve{r}$ and r are the scalar curvature tensor of the connections $\breve{\nabla}$ and ∇ respectively.
From equation (21), we have

$$\breve{R}(X_1, \xi)X_3 = \breve{R}(\xi, X_2)X_3 = \breve{R}(X_1, X_3)\xi = 0. \tag{25}$$

Again from equation (23), we have

$$\breve{S}(X_2, \xi) = \breve{S}(\xi, X_2) = 0 \tag{26}$$

and

$$\breve{Q}X_1 = QX_1 + (n-1)X_1, \tag{27}$$

where $\breve{Q}$ and Q are the Ricci operators with respect to the connection $\breve{\nabla}$ and ∇ respectively.

4. Locally ϕ-Symmetric LP-Sasakian Manifolds with respect to Schouten-van Kampen Connection

An LP-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_{X_5}R)(X_1, X_2)X_3) = 0 \tag{28}$$

for all vector fields $X_1,\ X_2,\ X_3,\ X_5$ orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds [21]. An LP-Sasakian manifold M^n is said to be ϕ-symmetric if

$$\phi^2((\nabla_{X_5}R)(X_1, X_2)X_3) = 0 \tag{29}$$

for arbitrary vector fields $X_1,\ X_2,\ X_3,\ X_5$.

Analogous to the definition of locally ϕ-symmetric LP-Sasakian manifold with respect to Levi-Civita connection, we define a locally ϕ-symmetric LP-Sasakian manifold with respect to the Schouten-van Kampen connection by

$$\phi^2((\breve{\nabla}_{X_5}\breve{R})(X_1, X_2)X_3) = 0, \tag{30}$$

for all vector fields $X_1,\ X_2,\ X_3,\ X_5$ orthogonal to ξ. In the view of equation (19), we have

$$(\breve{\nabla}_{X_5}\breve{R})(X_1, X_2)X_3 = (\nabla_{X_5}\breve{R})(X_1, X_2)X_3 + \eta(\breve{R}(X_1, X_2)X_3)\phi X_5 + g(\phi X_5, \breve{R}(X_1, X_2)X_3)\xi. \tag{31}$$

Now differentiating equation (21) covariantly with respect to X_5, we get

$$\begin{aligned}(\nabla_{X_5}\breve{R})(X_1, X_2)X_3 &= (\nabla_{X_5}R)(X_1, X_2)X_3 + g(X_1, (\nabla_{X_5}\phi)X_3)\phi X_2 + g(X_1, \phi X_3)(\nabla_{X_5}\phi)X_2 + g(X_2, X_3)(\nabla_{X_5}\eta)(X_1)\xi \\ &+ g(X_2, X_3)\eta(X_1)\nabla_{X_5}\xi - g(X_1, X_3)(\nabla_{X_5}\eta)(X_2)\xi - g(X_1, X_3)\eta(X_2)\nabla_{X_5}\xi - \{(\nabla_{X_5}\eta)(X_2)X_1 - (\nabla_{X_5}\eta)(X_1)X_2\}\eta(X_3) \\ &+ \{\eta(X_2)X_1 - \eta(X_1)X_2\}(\nabla_{X_5}\eta)(X_3).\end{aligned} \tag{32}$$

Using equations (10), (13) and (15) in equation (32), we have

$$\begin{aligned}(\nabla_{X_5}\breve{R})(X_1, X_2)X_3 &= (\nabla_{X_5}R)(X_1, X_2)X_3 + \{g(X_5, X_3)\eta(X_1) + g(X_1, X_5)\eta(X_3) + 2\eta(X_1)\eta(X_3)\eta(X_5)\}\phi X_2 \\ &+ \{g(X_1, \phi X_3)g(X_2, X_5) + 2g(X_1, \phi X_3)\eta(X_2)\eta(X_5) + g(X_2, X_3)g(\phi X_5, X_1) - g(X_1, X_3)g(\phi X_5, X_2)\}\xi \\ &+ \{g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)\}\phi X_5\{g(\phi X_5, X_2)X_1 - g(\phi X_5, X_1)X_2\}\eta(X_3) + \{\eta(X_2)X_1 - \eta(X_1)X_2\}g(\phi X_5, X_3).\end{aligned} \tag{33}$$

Now taking the inner product of the equation (21) with ξ, we get

$$\eta(\breve{R}(X_1, X_2)X_3) = 0. \tag{34}$$

Also from equation (21), we have

$$g(X_5,\phi(\check{R}(X_1,X_2)X_3))\xi = g(X_5,\phi X_1)g(X_2,X_3) + g(X_1,X_3)g(X_5,\phi X_2)\xi - g(X_1,X_3)g(X_5,\phi X_2)\xi + g(X_1,\phi X_3)g(X_5,X_2)\xi + g(X_1,\phi X_3)\eta(X_2)\eta(X_5)\xi - g(X_2,\phi X_3)g(X_1,X_5)\xi - g(X_2,\phi X_3)\eta(X_1)\eta(X_5)\xi + g(X_5,\phi X_1)\eta(X_2)\eta(X_3)\xi - g(X_5,\phi X_2)\eta(X_1)\eta(X_3)\xi. \tag{35}$$

By the virtue of equations (30), (31), (32) and (33), we get

$$\phi^2((\check{\nabla}_{X_5}\check{R})(X_1,X_2)X_3) = \phi^2(\nabla_{X_5}R)(X_1,X_2)X_3) + \{g(X_3,X_5)\eta(X_1) + g(X_1,X_5)\eta(X_3) + 2\eta(X_1)\eta(X_3)\eta(X_5)\}\phi^2(\phi X_2) + \{g(X_1,\phi X_3)g(X_2,X_5) + 2g(X_1,\phi X_3)\eta(X_2)\eta(X_5) + g(X_2,X_3)g(X_1,\phi X_5) - g(X_1,X_3)g(X_2,\phi X_5)\}\phi^2\xi + \{g(X_2,X_3)\eta(X_1) - g(X_1,X_3)\eta(X_2)\}\phi^2(\phi X_5) + \{g(\phi X_5,X_2)\phi^2X_1 - g(\phi X_5,X_1)\phi^2X_2\}\eta(X_3) + \{\eta(X_2)\phi^2(X_1) - \eta(X_1)\phi^2(X_2)\}g(\phi X_5,X_3). \tag{36}$$

Consider $X_1,\ X_2,\ X_3$ and X_5 are the orthogonal to ξ, then equation (35) yields

$$\phi^2((\check{\nabla}_{X_5}\check{R})(X_1,X_2)X_3) = \phi^2((\nabla_{X_5}R)(X_1,X_2)X_3). \tag{37}$$

Thus, we can state as follows:

Theorem 4.1. *In an LP-Sasakian manifolds the Schouten-van Kampen connection $\check{\nabla}$ is locally ϕ-symmetric iff the Levi-Civita is so.*

5. ϕ-Recurrent LP-Sasakian Manifolds with respect to Schouten-van Kampen Connection

An n-dimensional LP-Sasakian manifold M^n is said to be ϕ-recurrent if there exist a non-zero 1-form A such that

$$\phi^2((\nabla_{X_5}R)(X_1,X_2)X_3) = A(X_5)R(X_1,X_2)X_3. \tag{38}$$

If $X_1,\ X_2,\ X_3,\ X_5$ are orthogonal to ξ then the manifold is called locally ϕ-recurrent manifold.
If the 1-form A vanishes, then the manifold is reduced to ϕ-symmetric manifold ([8], [19]).
An n-dimensional LP-Sasakian manifold M^n is said to be ϕ-recurrent with respect to Schouten-van Kampen connection if there exist a non-zero 1-form A such that

$$\phi^2((\check{\nabla}_{X_5}\check{R})(X_1,X_2)X_3) = A(X_5)\check{R}(X_1,X_2)X_3, \tag{39}$$

for arbitrary vector fields $X_1,\ X_2,\ X_3$ and X_5.

Suppose M^n is ϕ-recurrent with respect to Schouten-van Kampen connection, then in view of equations (5) and (39), we can write

$$g((\check{\nabla}_{X_5}\check{R})(X_1,X_2)X_3,X_4) - \eta((\check{\nabla}_{X_5}\check{R})(X_1,X_2)X_3)\eta(X_4) = A(X_5)g(\check{R}(X_1,X_2)X_3,X_4). \tag{40}$$

By the virtue of equations (31) and (34) above equation reduced to

$$g((\nabla_{X_5}\check{R})(X_1,X_2)X_3,X_4) + \eta(\check{R}(X_1,X_2)X_3)g(X_4,\phi X_5) + \eta((\nabla_{X_5}\check{R})(X_1,X_2)X_3)\eta(X_4) = A(X_5)g(\check{R}(X_1,X_2)X_3,X_4), \tag{41}$$

which on using equations (21) and (33), above equation takes the form

$$g((\nabla_{X_5}R)(X_1,X_2)X_3,X_4) + \{g(X_5,X_3)\eta(X_1) + g(X_1,X_5)\eta(X_3) + 2\eta(X_1)\eta(X_3)\eta(X_5)\}g(\phi X_2,X_4) + \{g(X_1,\phi X_3)g(X_2,X_5) + 2g(X_1,\phi X_3)\eta(X_2)\eta(X_5)\}\eta(X_4) + \{g(X_2,X_3)\eta(X_1) - g(X_1,X_3)\eta(X_2)\}g(\phi X_5,X_4) + \eta(\nabla_{X_5}R)(X_1,X_2)X_3\eta(X_4) + \{g(X_1,X_4)\eta(X_2) - g(X_2,X_4)\eta(X_1)\}g(\phi X_5,X_3) - \{g(X_2,X_3)\eta(X_1) + g(X_1,X_3)\eta(X_2)\}g(X_4,\phi X_5) + g(X_4,\phi X_5)\eta(R((X_1,X_2)X_3) - \{g(X_2,X_5) + 2\eta(X_3)\eta(X_5)\}g(X_1,\phi X_3)\eta(X_4) + \{g(X_2,\phi X_5)\eta(X_1) - g(X_1,\phi X_5)\eta(X_2)\}\eta(X_3)\eta(X_4) = A(X_5)g(R((X_1,X_2)X_3,X_4) + A(X_5)g(X_1,\phi X_3)g(X_2,\phi X_4) - A(X_5)g(X_2,\phi X_3)g(X_1,\phi X_4) + A(X_5)g(X_2,X_3)\eta(X_1)\eta(X_4) - A(X_5)g(X_1,X_3)\eta(X_2)\eta(X_4) + A(X_5)g(X_1,X_4)\eta(X_2)\eta(X_3) - A(X_5)g(X_2,X_4)\eta(X_1)\eta(X_3).$$

(42)

Putting $X_3 = \xi$ in above equation and using equations (4), (6) and (10), we get

$$\begin{aligned} & g((\nabla_{X_5}R)(X_1, X_2)\xi, X_4) - \{g(X_1, X_5) + \eta(X_1)\eta(X_5)\}g(\phi X_2, X_4) + \{\eta(X_2)X_1 - \eta(X_1)X_2\}g(X_1, \phi X_5) \\ & + \eta((\nabla_{X_5}R)(X_1, X_2)\xi)\eta(X_4) - \{g(X_2, \phi X_5)\eta(X_1) - g(X_1, \phi X_5)\eta(X_2)\}\eta(X_4) \\ & = A(X_5)g(R(X_1, X_2)\xi, X_4) - g(\phi X_1, X_4)\eta(X_2)A(X_5) - g(X_2, X_4)\eta(X_1)A(X_5). \end{aligned} \tag{43}$$

Now, putting $X_1 = X_4 = e_i$ in above equation and taking summation over i, $1 \leq i \leq n$, we get

$$(\nabla_{X_5}S)(X_2, \xi) - g(X_2, \phi X_5) - \sum_{i=1}^{n} \eta((\nabla_{X_5}R)(e_i, X_2)\xi)g(e_i, \xi) = A(X_5)S(X_2, \xi) + (n-1)\eta(X_2)\eta(X_4). \tag{44}$$

Let us denote the third term of left hand side of equation (44) by E. In this case E vanishes Namely, we have

$$g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) = g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) - g(R(\nabla_{X_5}e_i, X_2)\xi, \xi) - g(R(e_i, \nabla_{X_5}X_2)\xi, \xi) - g(R(e_i, X_2)\nabla_{X_5}\xi, \xi) \tag{45}$$

at $p \in M^n$. In local coordinates $\nabla_{X_5}e_i = X_5^j \Gamma_{ji}^h e_h$, where Γ_{ji}^h are the Christoffel symbols. Since $\{e_i\}$ is an orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, δ_{ij} is the Kronecker delta and hence the Christoffel symbols are zero. Therefore $\nabla_{X_5}e_i = 0$. Since R is skew-symmetric, we have

$$g(R(e_i, \nabla_{X_5}X_2)\xi, \xi) = 0. \tag{46}$$

Using equation (46) in equation (45), we get

$$g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) = g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) - g(R(e_i, X_2)\nabla_{X_5}\xi, \xi). \tag{47}$$

In view of $g(R(e_i, X_2)\xi, \xi) = -g(R(\xi, \xi)e_i, X_2) = 0$ and $(\nabla_{X_5}g) = 0$, we have

$$g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) - g(R(e_i, X_2)\xi, \nabla_{X_5}\xi) = 0, \tag{48}$$

which implies

$$g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) = -g(R(e_i, X_2)\xi, \nabla_{X_5}\xi) - g(R(e_i, X_2)\nabla_{X_5}\xi, \xi).$$

Since R is skew-symmetric, we have

$$g((\nabla_{X_5}R)(e_i, X_2)\xi, \xi) = 0. \tag{49}$$

Using equation (49) in equation (44), we have

$$(\nabla_{X_5}S)(X_2, \xi) - g(\phi X_5, X_2) = A(X_5)S(X_2, \xi) + (n-1)\eta(X_2)\eta(X_5). \tag{50}$$

Now, we have

$$(\nabla_{X_5}S)(X_2, \xi) = \nabla_{X_5}S(X_2, \xi) - S(\nabla_{X_5}X_2, \xi) - S(X_2, \nabla_{X_5}\xi). \tag{51}$$

Using equations (10) and (15) in above equation, we have

$$(\nabla_{X_5}S)(X_2, \xi) = -(n-1)g(X_2, \phi X_5) - S(X_2, \phi X_5). \tag{52}$$

Using equation (52) in equation (50), we get

$$S(X_2, \phi X_5) = -ng(X_2 \phi X_5) - 2(n-1)\eta(X_2)A(X_5). \tag{53}$$

Now replacing X_5 by ϕX_5 in above equation, we get

$$S(X_2, X_5) = -ng(X_2, X_5) - (2n-1)\eta(X_2)\eta(X_5). \tag{54}$$

Theorem 5.1. *If a LP-Sasakian manifold is ϕ-recurrent with respect to the Schouten-van Kampen connection, then the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

6. Quasi-Conharmonically flat LP-Sasakian Manifolds with respect to Schouten-van Kampen Connection

An LP-Sasakian manifold M^n is said to be quasi-conharmonically flat with respect to Schouten-van Kampen connection if

$$g(\check{K}(\phi X_1, X_2)X_3, \phi X_4) = 0, \tag{55}$$

where $\check{K}$ is the conharmonic curvature tensor with respect to Schouten-van Kampen connection $\check{\nabla}$. In the view of equation (3), we have

$$\begin{aligned} g(\check{K}(X_1, X_2)X_3, X_4) &= g(\check{R}(X_1, X_2)X_3, X_4) - \frac{1}{(2n-1)}\{\check{S}(X_2, X_3)g(X_1, X_4) \\ &- \check{S}(X_1, X_3)g(X_2, X_4) + g(X_2, X_3)\check{S}(X_1, X_4) - g(X_1, X_3)\check{S}(X_2, X_4)\}. \end{aligned} \tag{56}$$

Replacing X_1 by ϕX_1 and X_4 by ϕX_4 in above equation, we get

$$\begin{aligned} g(\check{K}(\phi X_1, X_2)X_3, \phi X_4) &= g(\check{R}(\phi X_1, X_2)X_3, \phi X_4) - \frac{1}{(2n-1)}\{\check{S}(X_2, X_3)g(\phi X_1, \phi X_4) \\ &- \check{S}(\phi X_1, X_3)g(\phi X_2, \phi X_4) + g(X_2, X_3)\check{S}(\phi X_1, \phi X_4) - g(\phi X_1, X_3)\check{S}(X_2, \phi X_4)\}. \end{aligned} \tag{57}$$

Now, suppose that M^n is quasi-conharmonically flat with respect to Schouten-van Kampen connection. Then from equations (55) and (56), we have

$$\begin{aligned} g(\check{R}(\phi X_1, X_2)X_3, \phi X_4) &= \frac{1}{(2n-1)}\{\check{S}(X_2, X_3)g(\phi X_1, \phi X_4) - \check{S}(\phi X_1, X_3)g(\phi X_2, \phi X_4) \\ &+ g(X_2, X_3)\check{S}(\phi X_1, \phi X_4) - g(\phi X_1, X_3)\check{S}(X_2, \phi X_4)\}. \end{aligned} \tag{58}$$

Using equations (21) and (23) in above equation, we have

$$\begin{aligned} g(R(\phi X_1, X_2)X_3, \phi X_4) &= -g(\phi X_1, \phi X_3)g(\phi X_2, X_4) - g(X_2, \phi X_3)g(\phi X_1, X_4) + g(\phi X_1, \phi X_4)\eta(X_2)\eta(X_3) \\ &+ \frac{1}{(2n-1)}\{(n-1)g(\phi X_1, \phi X_4)\eta(X_2)\eta(X_3) + S(X_2, X_3)g(\phi X_1, \phi X_4) - S(\phi X_1, X_3)g(X_2, \phi X_4) \\ &+ S(\phi X_1, \phi X_4)g(X_2, X_3) - S(X_2, \phi X_4)g(\phi X_1, X_3)\}. \end{aligned} \tag{59}$$

Let $\{e_1, e_2, ...e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n. Then $\{\phi e_1, \phi e_2, ...\phi e_{n-1}, \xi\}$ is also local orthonormal basis of M^n. Putting $X_1 = X_4 = e_i$ in equation (59) and taking summation over i, $1 \leq i \leq n-1$ and using the fact that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, X_2)X_3, \phi e_i) = S(X_2, X_3) + (n-2)g(X_2, X_3), \tag{60}$$

we get

$$S(X_2, X_3) = ag(X_2, X_3) + b\eta(X_2)\eta(X_3), \tag{61}$$

where $a = \frac{-2n^2}{(n+2)}$ and $b = \frac{(3n^2-5n)}{(n+2)}$.
Thus, we can state as follows:

Theorem 6.1. *If an LP-Sasakian manifold M^n is quasi-conharmonically flat with respect to Schouten-van Kampen connection then the manifold is an η-Einstein manifold with respect to Levi-Civita connection.*

7. Conharmonically flat LP-Sasakian Manifolds with respect to Schouten-van Kampen Connection $\check{\nabla}$

An n-dimensional LP-Sasakian manifold M^n is said to be conharmonically flat if the conharmonic curvature tensor vanishes.

In this section, we assume that $\check{K}(X_1, X_2)X_3 = 0$, where $\check{K}$ denotes the conharmonic curvature tensor with respect to the Schouten-van Kampen connection $\check{\nabla}$.

Let M^n be an n-dimensional conharmonically flat LP-Sasakian manifold with respect to the Schouten-van Kampen connection, *i.e.* $\check{K} = 0$, then from equation (3), we have

$$\check{R}(X_1, X_2)X_3 = \frac{1}{(2n-1)}[\check{S}(X_2, X_3)X_1 - \check{S}(X_1, X_3)X_2 + g(X_2, X_3)\check{Q}X_1 - g(X_1, X_3)\check{Q}X_2]. \quad (62)$$

Transvection of X_4 in equation (62), gives

$$g(\check{R}(X_1, X_2)X_3, X_4) = \frac{1}{(2n-1)}[\check{S}(X_2, X_3)g(X_1, X_4) - \check{S}(X_1, X_3)g(X_2, X_4) + g(X_2, X_3)\check{S}(X_1, X_4) - g(X_1, X_3)\check{S}(X_2, X_4)]. \quad (63)$$

Let $e_i, (1 \leq i \leq n)$ be an orthonormal basis. Taking summation over $X_1 = X_4 = e_i$ $(1 \leq i \leq n)$ in above equation, we get

$$\check{S}(X_2, X_3) = \frac{\check{r}}{(n+1)}g(X_2, X_3). \quad (64)$$

Using equations (23) and (24) in equation (64), we get

$$S(X_2, X_3) = ag(X_2, X_3) + b\eta(X_2)\eta(X_3), \quad (65)$$

where $a = \frac{r-(n-1)}{(n+1))}$ and $b = -(n-1)$.
Thus, we can state as follows:

Theorem 7.1. *A conharmonically flat LP-Sasakian manifold M^n admitting Schouten-van Kampen connection $\check{\nabla}$ is an η-Einstein manifold.*

Now, from equations (2), (3), (21), (23), and (24), we have

$$\begin{aligned}\check{K}(X_1, X_2)X_3 &= K(X_1, X_2)X_3 + g(X_1, \phi X_3)\phi X_2 - g(X_2, \phi X_3)\phi X_1 + g(X_2, X_3)\eta(X_1)\xi - g(X_1, X_3)\eta(X_2)\xi \\ &+ \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2 - \frac{1}{(2n-1)}[(n-1)\eta(X_2)\eta(X_3)X_1 - (n-1)\eta(X_1)\eta(X_3)X_2].\end{aligned} \quad (66)$$

Substitute $X_3 = \xi$ in above equation (66), we get

$$\check{K}(X_1, X_2)\xi = K(X_1, X_2)\xi - \frac{(3n-2)}{(2n-1)}[\eta(X_2)X_1 - \eta(X_1)X_2]. \quad (67)$$

If X_1 and X_2 are horizontal vector fields then from equation (67), it follows that

$$\check{K}(X_1, X_2)\xi = K(X_1, X_2)\xi.$$

Thus, we can state as follows:

Theorem 7.2. *On an n-dimensional LP-Sasakian manifold M^n, ξ-conharmonic curvature tensor of Schouten-van Kampen connection and Levi-Civita connection are identical provided that the vector fields on M^n are horizontal vector fields.*

8. ϕ-Conharmonically flat LP-Sasakian Manifolds with respect to Schouten-van Kampen Connection

An n-dimensional differentiable manifold (M^n, g) satisfying the equation

$$\phi^2(K(\phi X_1, \phi X_2)\phi X_3) = 0, \tag{68}$$

is called ϕ-conharmonically flat. Analogous to the equation (68) an n-dimensional LP-Sasakian manifold is said to be ϕ-conharmonically flat with respect to Schouten-van Kampen connection if it satisfies

$$\phi^2(\breve{K}(\phi X_1, \phi X_2)\phi X_3) = 0, \tag{69}$$

where $\breve{K}$ is the conharmonic curvature tensor of the manifold with respect to Schouten-van Kampen connection.

Suppose M^n is ϕ-conharmonically flat LP-Sasakian manifold with respect to Schouten-van Kampen connection. It is easy to see that

$$\phi^2(\breve{K}(\phi X_1, \phi X_2)\phi X_3) = 0$$

holds if and only if

$$g(\phi(\breve{K}(\phi X_1, \phi X_2)\phi X_3), \phi X_4) = 0, \tag{70}$$

for $X_1,\ X_2\ X_3,\ X_4 \in \chi(M)$. So by the virtue of equation (58) ϕ-concircularly flat means

$$\begin{aligned} g(\breve{R}(\phi X_1, \phi X_2)\phi X_3, \phi X_4) &= \frac{1}{(2n-1)}\{\breve{S}(\phi X_2, \phi X_3)g(\phi X_1, \phi X_4) - \breve{S}(\phi X_1, \phi X_3)g(\phi X_2, \phi X_4) \\ &+ \breve{S}(\phi X_1, \phi X_4)g(\phi X_2, \phi X_3) - \breve{S}(\phi X_2, \phi X_4)g(\phi X_1, \phi X_3)\}, \end{aligned} \tag{71}$$

which on using equation (21) and (23), the above equation reduced to

$$\begin{aligned} g(R(\phi X_1, \phi X_2)\phi X_3, \phi X_4) &= -g(\phi X_3, X_1)g(X_2, \phi X_4) + g(\phi X_3, X_2)g(X_1, \phi X_4) \\ &+ \frac{1}{(2n-1)}\{-(n-1)S(X_1, X_3)\eta(X_2)\eta(X_4) + (n-1)S(X_2, X_3)\eta(X_2)\eta(X_4) + (n-1)g(X_1, X_4)\eta(X_2)\eta(X_3) \\ &- S(X_1, X_3)g(X_2, X_4) + S(X_2, X_3)g(X_1, X_4) - (n-1)g(X_2, X_4)\eta(X_1)\eta(X_3) + S(X_1, X_4)g(X_2, X_3) \\ &- S(X_2, X_4)g(X_1, X_3) + (n-1)S(X_1, X_4)\eta(X_2)\eta(X_3) + (n-1)g(X_2, X_3)\eta(X_1)\eta(X_4) \\ &- (n-1)S(X_2, X_4)\eta(X_1)\eta(X_3) - (n-1)g(X_1, X_3)\eta(X_2)\eta(X_4). \end{aligned} \tag{72}$$

Let $\{e_1, e_2, ..., e_{n-1}, \xi\}$ are the local orthonormal basis of the vector field in M^n. Using the fact that $\{\phi e_1, \phi e_2, ..., \phi e_{n-1}, \xi\}$ is also local orthonormal basis. Putting $X_1 = X_4 = e_i$ in equation (72) and summing over i; $1 \leq i \leq n$ and using the fact that

$$\sum_{i=1}^{n-1} g(R(\phi e_i, \phi X_2)\phi X_3, \phi e_i) = S(X_2, X_3) + (n-1)\eta(X_2)\eta(X_4), \tag{73}$$

we get

$$S(X_2, X_3) = ag(X_2, X_3) + b\eta(X_2)\eta(X_3), \tag{74}$$

where $a = \frac{(r-n+2)}{(2n+1)}$ and $b = \frac{-(3n^2+4n-rn+r)}{(2n+1)}$, which shows that M^n is an η-Einstein manifold.
Thus, we can state the following:

Theorem 8.1. *An n-dimensional ϕ-conharmonically flat LP-Sasakian manifold admitting Schouten-van Kampen connection is an η-Einstein manifold with respect to Levi-Civita connection.*

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