Filomat 40:1 (2026), 211–220
https://doi.org/10.2298/FIL2601211J

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized hybrid (b, c)-inverses

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Abstract. We study the hybrid (b, c)-inverse in a more general setting. The new concept of the right (m, n)-hybrid (b, c)-inverse is defined and studied. In particular, if $m = n = 1$, then the right (m, n)-hybrid (b, c)-inverse is precisely the general right hybrid (b, c)-inverse. Some examples and counter-examples to illustrate the concepts and results are presented. Moreover, the relationship between right (m, n)-hybrid (b, c)-inverses, right hybrid (b, c)-inverses and (b, c)-inverses is studied. Various properties of right (m, n)-hybrid (b, c)-inverses are investigated. Some well-known results on right hybrid (b, c)-inverses are unified and extended.

1. Introduction

Throughout this paper, R is an associative ring with 1 and $\mathbb{N}^+$ is the set of positive integers. An involution $*: R \to R$ is an anti-isomorphism which satisfies $(a^*)^* = a$, $(ab)^* = b^*a^*$, $(a+b)^* = a^* + b^*$ for all $a, b \in R$. For any $a \in R$, we use $lann(a) = \{x \in R : xa = 0\}$ and $rann(a) = \{x \in R : ax = 0\}$ to denote the left annihilator and right annihilator of a, respectively. For any element $a \in R$, the commutant and the double commutant of a, respectively, are defined by $comm(a) = \{x \in R \mid xa = ax\}$ and $comm^2(a) = \{x \in R \mid xy = yx \ for \ all \ y \in comm(a)\}$. An element $a \in R$ is called regular if there exists $x \in R$ such that $a = axa$. Such an $x = a^-$ is called an inner inverse of a. According to [1], $a \in R$ is said to be strongly regular if $a \in a^2R \cap Ra^2$, while a is said to be right (resp., left) regular if there is x such that $a^2x = a$ (resp., $xa^2 = a$). It is well known that an element a is group invertible if and only if it is strongly regular. Further results related to the group inverse can be found in [2] and [12]. If $a, x \in R$ and $k \in \mathbb{N}^+$, as recalled from [9] that x is the pseudo core inverse of a if it satisfies $xa^{k+1} = a^k, ax^2 = x$ and $(ax)^* = ax$. According to [6], y is the Bott-Duffin (e, f)-inverse of a if $y = ey = yf$, $yae = e$ and $fay = f$, where e and f are idempotent elements.

In 2012, Drazin defined three new classes of outer generalized inverses over a ring with identity, which are called (b, c)-inverses, hybrid (b, c)-inverses and annihilator (b, c)-inverses, respectively. Given any ring R with identity and any $a, b, c, y \in R$, recall from [6] that y is the (b, c)-inverse of a if $yay = y, yR = bR$ and $Ry = Rc$. And y is the annihilator (b, c)-inverse of a if $yay = y, lann(y) = lann(b)$ and $rann(y) = rann(c)$.

2020 *Mathematics Subject Classification*. Primary 16U90; Secondary 15A10, 16E50.
Keywords. right (m, n)-hybrid (b, c)-inverse, (b, c)-inverse, hybrid (b, c)-inverse, strongly regular ring.
Received: 28 April 2024; Revised: 03 December 2024; Accepted: 11 February 2025
Communicated by Dijana Mosić
Research supported by the National Science Foundation of China (No. 12161049) and the Natural Science Research Fund for Colleges and Universities in Anhui Province (2023AH040148).
The authors wish to express their sincere thanks to the referees for the valuable comments and suggestions which led to the improvements of this paper.
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Moreover, y is the hybrid (b,c)-inverse of a if $yay = y, yR = bR$ and $rann(y) = rann(c)$. Some characterizations of these generalized inverses were also given in [6]. The reverse order law for (b,c)-inverses and hybrid (b,c)-inverses was investigated in [3] and [11], respectively.

More generally, Drazin introduced left and right (b,c)-inverses for a semigroup S. Recall that x is a left (resp., right) (b,c)-inverse of a if it satisfies [5] $xab = b, x \in Sc$ (resp., $cax = c, x \in bS$) with $a,b,c,x \in S$. One-sided annihilator (b,c)-inverses for associative rings were studied in [13]. Let R be any associative ring with $a,b,c,x \in R$. Then x is a left annihilator (b,c)-inverse of a, if x satisfies $xab = b, rann(c) \subseteq rann(x)$. Dually, a is called right annihilator (b,c)-invertible if there exists $y \in R$ such that $cay = c, lann(b) \subseteq lann(y)$. Furthermore, right and left hybrid (b,c)-inverses for associative rings with identity were studied in [8]. An element a is right hybrid (b,c)-invertible if there exists $y \in R$ such that $yay = y$, $yR = bR$ and $rann(y) = rann(c)$. Note that right hybrid (b,c)-inverses are precisely hybrid (b,c)-inverses defined in [6]. The left hybrid (b,c)-invertibility can be defined dually.

In this paper, we investigate a more general case of right hybrid (b,c)-inverses in associative rings, which is called the right (m,n)-hybrid (b,c)-inverse. In particular, if $m = n = 1$, then right and left (m,n)-hybrid (b,c)-inverses are precisely the general right and left hybrid (b,c)-inverses, respectively. We shall give an example to show that a right (m,n)-hybrid (b,c)-invertible element need not be right hybrid (b,c)-invertible, and a right hybrid (b,c)-invertible element need not be right (m,n)-hybrid (b,c)-invertible. The relationship between right (m,n)-hybrid (b,c)-inverses, right hybrid (b,c)-inverses and (b,c)-inverses is discussed. Various properties of right (m,n)-hybrid (b,c)-inverses are investigated. As an application, we study the properties of Bott-Duffin (e,f)-inverse by using the reverse order law of right (m,n)-hybrid (b,c)-inverses. Some well-known results on right hybrid (b,c)-inverses are unified and extended.

This paper is organized as follows:

In Section 2, we define and study the concept of the right (m,n)-hybrid (b,c)-inverse. In particular, we give a new characterization of Drazin inverses and pseudo core inverses from the point of view of right (m,n)-hybrid (b,c)-inverses (Corollary 2.8). If R is a strongly regular ring, we prove that a is right (m,n)-hybrid (b,c)-invertible if and only if a is right hybrid (b,c)-invertible if and only if a is (b,c)-invertible (Proposition 2.14).

Section 3 is a study of the intertwining properties and Cline's formula for right (m,n)-hybrid (b,c)-inverses. Let $a_1, a_2, b, c, x \in R$ and $b, c \in comm(a_2a_1)$ for $m,n \in \mathbb{N}^+$. If x is the right (m,n)-hybrid (b,c)-inverse of a_2a_1, then we show $a_1x^2a_2$ is the right hybrid $((a_1ba_2)^m, (a_1ca_2)^n)$-inverse of a_1a_2 (Theorem 3.4).

Section 4 is concerned with the reverse order law and the triple reverse order law of right (m,n)-hybrid (b,c)-inverses. The relationship between Bott-Duffin (e,e)-inverses, Bott-Duffin (f,f)-inverses and Bott-Duffin (e,f)-inverses is investigated, which can be regarded as an application of the reverse order law of right (m,n)-hybrid (b,c)-inverses (Proposition 4.3).

2. Right (m,n)-hybrid (b,c)-inverses

In this section, we define and study the concept of right (m,n)-hybrid (b,c)-inverses, which is a more general case of hybrid (b,c)-inverses.

We begin with the following definition.

Definition 2.1. *Let $a,b,c \in R$ and $m,n \in \mathbb{N}^+$. We say that a is right (m,n)-hybrid (b,c)-invertible if there exists $y \in R$ such that*

$$yay = y, yR = b^mR \text{ and } rann(y) = rann(c^n).$$

If such y exists, then y is called the right (m,n)-hybrid (b,c)-inverse of a. Dually, we say that $z \in R$ is the left (m,n)-hybrid (b,c)-inverse of a if

$$zaz = z, Rz = Rc^n \text{ and } lann(z) = lann(b^m).$$

Clearly, if $m = n = 1$, then right and left (m,n)-hybrid (b,c)-inverses are precisely the general right and left hybrid (b,c)-inverses, respectively. In what follows, we just discuss the case of the right (m,n)-hybrid (b,c)-inverse. The case of left (m,n)-hybrid (b,c)-inverses can be discussed dually.

Theorem 2.2. *Let $a, b, c \in R$ and $m, n \in \mathbb{N}^+$. Then a has at most one right (m, n)-hybrid (b, c)-inverse.*

Proof. The proof is similar to that of [6, Theorem 6.4]. □

We give the following auxiliary proposition that will be used later.

Proposition 2.3. *Let $a, b, c, y \in R$ and $m, n \in \mathbb{N}^+$. Then the following two statements are equivalent:*
(1) y is the right (m, n)-hybrid (b, c)-inverse of a;
(2) $yab^m = b^m, c^n ay = c^n, yR \subseteq b^m R$ and $rann(c^n) \subseteq rann(y)$.

Proof. (1) ⇒ (2) Since y is the right (m, n)-hybrid (b, c)-inverse of a, we have $yay = y$ and $rann(y) = rann(c^n)$. It follows that $1 - ay \in rann(y) = rann(c^n)$, thus $c^n = c^n ay$. Since $b^m \in yR$, there is $t \in R$ such that $b^m = yt$. This implies that $yab^m = yayt = yt = b^m$.

(2) ⇒ (1) It is straightforward. □

It was shown in [8, Theorem 2.2] that an element a is right hybrid (b, c)-invertible if and only if $c \in cabR$ and $rann(cab) \subseteq rann(b)$. Accordingly, we give the following characterization for a right (m, n)-hybrid (b, c)-invertible element.

Proposition 2.4. *Let $a, b, c \in R$ and $m, n \in \mathbb{N}^+$. Then the following statements are equivalent:*
(1) a is right (m, n)-hybrid (b, c)-invertible;
(2) $c^n \in c^n ab^m R, rann(c^n ab^m) \subseteq rann(b^m)$;
(3) $R = ab^m R \oplus rann(c^n), b^m \in Rab^m$;
(4) $R = b^m R \oplus rann(c^n a), c^n \in c^n aR$.

Proof. (1) ⇒ (2) If $y \in R$ is the right (m, n)-hybrid (b, c)-inverse of a, then we have $c^n ay = c^n$ and $yR \subseteq b^m R$ by Proposition 2.3. It follows that $c^n \in c^n ab^m R$. It suffices to show $rann(c^n ab^m) \subseteq rann(b^m)$. Choose $s \in rann(c^n ab^m)$, then $c^n ab^m s = 0$. Therefore, we get $ab^m s \in rann(c^n) \subseteq rann(y)$, that is, $yab^m s = b^m s = 0$ since $yab^m = b^m$. This shows that $s \in rann(b^m)$, as desired.

(2) ⇒ (3) Since $c^n \in c^n ab^m R$ and $rann(c^n ab^m) \subseteq rann(b^m)$, there is $t \in R$ such that $c^n = c^n ab^m t$ and $rann(c^n ab^m) \subseteq rann(ab^m)$. It follows that $c^n ab^m = c^n ab^m tab^m$ and $R = ab^m R \oplus rann(c^n)$ by [8, Corollary 6.4]. Consequently, we have $(1 - tab^m) \in rann(c^n ab^m) \subseteq rann(b^m)$, which implies $b^m = b^m tab^m \in Rab^m$.

(3) ⇒ (4) If $R = ab^m R \oplus rann(c^n)$, then $rann(c^n ab^m) \subseteq rann(ab^m)$ and $c^n \in c^n ab^m R \subseteq c^n aR$ by [8, Corollary 6.4]. Then there is $t \in R$ such that $c^n a = c^n ab^m ta$. This implies that $c^n a(1 - b^m ta) = 0$. Let $u = 1 - b^m ta$. Then $u \in rann(c^n a)$, and thus

$$1 = b^m ta + u \in b^m R + rann(c^n a).$$

Therefore, $R = b^m R + rann(c^n a)$. Since $b^m \in Rab^m$, we have $rann(ab^m) \subseteq rann(b^m)$. It follows that $rann(c^n ab^m) \subseteq rann(ab^m) \subseteq rann(b^m)$. Therefore, $rann(c^n a) \cap b^m R = \{0\}$ by [8, Lemma 6.3], which implies $R = b^m R \oplus rann(c^n a)$.

(4) ⇒ (1) If $R = b^m R \oplus rann(c^n a)$, then we have $c^n aR \subseteq c^n ab^m R$ by [8, Lemma 6.3]. Since $c^n \in c^n aR \subseteq c^n ab^m R$, there is $w \in R$ such that $c^n = c^n ab^m w$. Let $x = b^m w$. Then $xR \subseteq b^m R$, and $c^n = c^n ax$. This implies that $rann(x) \subseteq rann(c^n)$. Choose $r \in rann(c^n)$, then $c^n r = c^n ab^m wr = 0$. It follows that $b^m wr \in rann(c^n a) \cap b^m R = \{0\}$, and thus $b^m wr = xr = 0$, which gives $r \in rann(x)$. Therefore, we have $rann(x) = rann(c^n)$. Moreover, since $c^n ab^m = c^n ab^m wab^m$, we conclude that

$$(b^m - b^m wab^m) \in rann(c^n a) \cap b^m R = \{0\}.$$

Then $b^m = b^m wab^m = xab^m$. This implies that $xR = b^m R$, $x = b^m w = (b^m wab^m)w = xax$. Therefore, a is right (m, n)-hybrid (b, c)-invertible with the right (m, n)-hybrid (b, c)-inverse x. □

We next give an example to show the class of right (m, n)-hybrid (b, c)-inverses is quite different from that of right hybrid (b, c)-inverses.

Example 2.5. *Let $R = M_2(\mathbb{F})$ be the ring of all 2 by 2 matrices over a field $\mathbb{F}$. On the one hand, let*

$$a = I_2,\ b = c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in R.$$

It is clear $b^m = c^n = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *for integers* $m, n \geq 2$. *This implies that* a *is right* (m, n)*-hybrid* (b, c)*-invertible. However, we have*

$$c \notin cabR,\ rann(cab) \nsubseteq rann(b)$$

since $cab = cb = 0$*, that is,* a *is not right hybrid* (b, c)*-invertible.*

On the other hand, let

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{and } c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in R.$$

Then we have $cab = c$, $b \in Rcab$. *This implies that* $c \in cabR$, $rann(cab) \subseteq rann(b)$. *Therefore,* a *is right hybrid* (b, c)*-invertible. However, it is easy to see that* $b^m = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *and* $c^n = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *for integers* $m, n \geq 2$*, and it is clear* $c^n \notin c^n ab^m R$. *Thus* a *is not right* (m, n)*-hybrid* (b, c)*-invertible by Proposition 2.4.*

In particular, if $m = n$, then the following proposition shows the relationship between the right (m, m)-hybrid (b, b)-inverse and the (b^m, b^m)-inverse.

Proposition 2.6. *Let* $a, b \in R$ *and* $m \in \mathbb{N}^+$. *Then* a *is right* (m, m)*-hybrid* (b, b)*-invertible if and only if* a *is* (b^m, b^m)*-invertible.*

Proof. Assume that a is right (m, m)-hybrid (b, b)-invertible. Then there exists $y \in R$ such that $yay = y$, $yR = b^m R$. Let $b^m = ys, y = b^m t$ for some $s, t \in R$. Then we have $b^m tab^m = b^m$, that is, b^m is regular. It follows that $[(b^m)^- b^m - 1] \in rann(b^m) = rann(y)$, and thus $y = y(b^m)^- b^m \in Rb^m$. Moreover, since $(1 - ay) \in rann(y) = rann(b^m)$, we have $b^m = b^m ay \in Ry$. This implies that $Ry = Rb^m$. Therefore, a is (b^m, b^m)-invertible. The converse is clear. □

If R is a ring with an involution, then we can get the similar result as follows.

Theorem 2.7. *Let* $a, b \in R$ *and* $m \in \mathbb{N}^+$. *Then* a *is right (m, m)-hybrid* (b, b^*)*-invertible if and only if* a *is* $(b^m, (b^*)^m)$*-invertible.*

Proof. Since a is right (m, m)-hybrid (b, b^*)-invertible, there exists $y \in R$ such that $yay = y, yR = b^m R$ and $rann(y) = rann((b^*)^m)$. To complete the proof, it suffices to show $Ry = R(b^*)^m$. Because b^m is regular, by the proof of Proposition 2.6, we have $b^m = b^m (b^m)^- b^m$. It follows that $(b^m)^* = (b^*)^m = (b^*)^m [(b^m)^-]^* (b^*)^m$, that is, $(b^*)^m$ is regular. Since $[((b^m)^-)^* (b^*)^m - 1] \in rann((b^*)^m) = rann(y)$, we have $y = y[(b^m)^-]^* (b^*)^m \in R(b^*)^m$. Also, combining with $(ay - 1) \in rann(y) = rann((b^*)^m)$, we have $(b^*)^m ay = (b^*)^m$. Thus, $Ry = R(b^*)^m$. The converse is clear. □

The next corollary gives a new characterization of Drazin inverses and pseudo core inverses from the point of view of the right (m, n)-hybrid (b, c)-inverse.

Corollary 2.8. *(1) An element* $a \in R$ *is Drazin invertible if and only if* a *is right* (m, m)*-hybrid (a, a)-invertible for some positive integer m.*

(2) An element $a \in R$ *is pseudo core invertible if and only if* a *is right* (m, m)*-hybrid* (a, a^*)*-invertible for some positive integer m.*

The next proposition shows the condition under which x being the right (m, n)-hybrid (b, c)-inverse of an element a implies $x \in comm(a)$ (resp., $x \in comm^2(a)$).

Proposition 2.9. *Let* $a, b, c, x \in R$ *and* $m, n \in \mathbb{N}^+$. *If* x *is the right* (m, n)*-hybrid* (b, c)*-inverse of* a*, then we have the following implications:*

(1) $b^m, c^n \in comm(a)$ *imply* $x \in comm(a)$*;*

(2) $b^m, c^n \in comm^2(a)$ *imply* $x \in comm^2(a)$.

Proof. (1) Since $b^m, c^n \in comm(a)$, we have $ab^m = b^m a = xab^m a = xa^2 b^m$, $c^n a = ac^n = ac^n ax = c^n a^2 x$. By Theorem 3.1, $x \in comm(a)$.

(2) If $b^m, c^n \in comm^2(a)$, then $b^m k = kb^m$ and $c^n k = kc^n$ for any $k \in comm(a)$. It suffices to show $xk = kx$. Since $ak = ka$, we have $c^n kab^m = c^n akb^m$. Also, since $kb^m = b^m k$ and $c^n k = kc^n$, we conclude that

$$kb^m = b^m k = xab^m k = xakb^m,$$
$$c^n k = kc^n = kc^n ax = c^n kax.$$

This implies that $xk = kx$ by Theorem 3.1, as desired. □

Corollary 2.10. *Let $a, b, c, z \in R$. If z is the right hybrid (b, c)-inverse of a, then*
(1) $b, c \in comm(a)$ imply that $z \in comm(a)$;
(2) $b, c \in comm^2(a)$ imply that $z \in comm^2(a)$.

Based on Proposition 2.9, one may suspect that if $x \in comm(a)$, then $b^m, c^n \in comm(a)$. However, the following example eliminates the possibility.

Example 2.11. *Let $R = M_2(\mathbb{F})$ be the ring of all 2 by 2 matrices over a field $\mathbb{F}$. Let*

$$y = a = c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in R.$$

It can be easily checked that $b^m = b, c^n = c$ and $ay = ya$ for some $m, n \in \mathbb{N}^+$. This implies that y is the right (m, n)-hybrid (b, c)-inverse of a. However, we have $ab^m \neq b^m a$.

Note that if $b^m, c^n \in comm(a)$ and x is the right (m, n)-hybrid (b, c)-inverse of a, then $b^m c^n \in comm(x)$. In fact, if $b^m, c^n \in comm(a)$, then we get $xb^m a = b^m$ and $ac^n x = c^n$ by Proposition 2.3. Thus, $b^m c^n x = xb^m ac^n x = xb^m(ac^n x) = xb^m c^n$. This implies that $b^m c^n \in comm(x)$. However, the following example shows that in general we can not conclude $b^m, c^n \in comm(x)$ from b^m and $c^n \in comm(a)$.

Example 2.12. *Let $R = M_2(\mathbb{F})$ be the ring of all 2 by 2 matrices over a field $\mathbb{F}$. Taking*

$$a = I_2, x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in R.$$

Then we have

$$b^m = b, c^n = c \text{ and } x = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} c^n.$$

It is easy to see that $xR = b^m R, Rc^n = Rx$ and $xax = x$ for some $m, n \in \mathbb{N}^+$. Therefore, $rann(c^n) = rann(x)$. This implies that x is the right (m, n)-hybrid (b, c)-inverse of a. However, we have $xb^m \neq b^m x$ and $c^n x \neq xc^n$.

Lemma 2.13. *Let R be a strongly regular ring. Then for any $a \in R$, $aR = a^m R$ and $Ra = Ra^m$ hold for any positive integer m.*

Proof. If R is a strongly regular ring, then $a \in Ra^2 \cap a^2 R$ for any element $a \in R$. This implies that $a = sa^2 = a^2 t$ for some $s, t \in R$, and thus we have $a = sa^2 = s^2 a^3 = \ldots = s^{(m-1)} a^m \in Ra^m$ for $m \geq 2$. Since $a \in Ra$ and $Ra^m \subseteq Ra$, we deduce that $Ra = Ra^m$ for any positive integer m. Similarly, we have $aR = a^m R$. □

As shown by Example 2.5, a right (m, n)-hybrid (b, c)-invertible element need not be right hybrid (b, c)-invertible, and a right hybrid (b, c)-invertible element need not be right (m, n)-hybrid (b, c)-invertible. However, for a strongly regular ring, the next proposition shows the equivalences of the right (m, n)-hybrid (b, c)-invertibility, the right hybrid (b, c)-invertibility and the (b, c)-invertibility.

Proposition 2.14. *Let $a, b, c \in R$ and $m, n \in \mathbb{N}^+$. If R is a strongly regular ring, then the following statements are equivalent:*
(1) a is right (m, n)-hybrid (b, c)-invertible;
(2) a is right hybrid (b, c)-invertible;
(3) a is (b, c)-invertible.

Proof. (1) ⇒ (2) If a is right (m, n)-hybrid (b, c)-invertible, then there is $y \in R$ such that $yay = y, yR = b^mR$ and $rann(y) = rann(c^n)$. By Lemma 2.13, we have $yR = b^mR = bR$ and $Rc^n = Rc$. This implies that $rann(c) = rann(c^n) = rann(y)$, and thus a is right hybrid (b, c)-invertible.

(2) ⇒ (3) Since a is right hybrid (b, c)-invertible, there is $x \in R$ such that $xax = x, xR = bR$ and $rann(x) = rann(c)$. Then $x(1 - ax) = 0$, and thus $(1 - ax) \in rann(x) = rann(c)$. Therefore, $cax = c$ and hence $Rc \subseteq Rx$. Since R is strongly regular, c is group invertible. Suppose that $c' \in R$ is the group inverse of c, then we have $(c'c - 1) \in rann(c) = rann(x)$. Then $xc'c = x$, and so we have $Rx \subseteq Rc$. Therefore, we obtain $Rx = Rc$.

(3) ⇒ (1) If a is (b, c)-invertible, then $b \in Rcab$ and $c \in cabR$. By Lemma 2.13, we have $c^n \in c^nabR = c^nab^mR$, and $b^m \in Rcab^m = Rc^nab^m$. This implies that $rann(c^nab^m) \subseteq rann(b^m)$. Therefore, a is right (m, n)-hybrid (b, c)-invertible by Proposition 2.4. □

Corollary 2.15. *Let $a, b \in R$ and $m, n \in \mathbb{N}^+$. If b is left regular and $m \le n$, then a is right (m, n)-hybrid (b, b)-invertible if and only if a is (b^m, b^n)-invertible.*

Proof. Since b is left regular, there is $x \in R$ such that $b = xb^2 = x^2b^3 = \cdots = x^{n-1}b^n \in Rb^n$. This shows that $b^m = x^{n-1}b^{m-1}b^n \in Rb^n$. If a is right (m, n)-hybrid (b, b)-invertible, then there is $y \in R$ such that $yay = y, yR = b^mR$ and $rann(b^n) = rann(y)$. Moreover, it is clear that b^m is regular and $b^n = b^nay$. Therefore, we have

$$(1 - (b^m)^- b^m) \in rann(b^m) \subseteq rann(b^n) = rann(y).$$

Then $y = y(b^m)^- b^m$. This implies that $Rb^n \subseteq Ry \subseteq Rb^m \subseteq Rb^n$, and hence $Ry = Rb^n$. The converse is obvious. □

3. Intertwining property and Cline's formula for right (m, n)-hybrid (b, c)-inverses

It was proved in [4] that if ab is Drazin invertible, then so is ba, and we have $(ba)^D = b[(ab)^D]^2a$. This equality is called Cline's formula. It plays an important role in connecting the Drazin inverse of a sum of two elements with the Drazin inverse of a matrix (see [10]). Moreover, Drazin studied the intertwining property for (b, c)-inverse in [7]. It was shown in [7, Theorem 2.3] that if S is a semigroup and $a_i, b_i, c_i, y_i \in S$ $(i = 1, 2)$ such that each a_i is (b_i, c_i)-invertible with (b_i, c_i)-inverse y_i, then for any $d \in S$, $da_1 = a_2d, db_1 = b_2d$ and $dc_1 = c_2d$ imply $dy_1 = y_2d$. Motivated by these results, in this section we further study the intertwining property and Cline's formula for right (m, n)-hybrid (b, c)-inverses.

Theorem 3.1. *Let $a_i, b_i, c_i, x_i, y \in R$ $(i = 1, 2)$ and $m, n \in \mathbb{N}^+$. If each a_i is right (m, n)-hybrid (b_i, c_i)-invertible with the right (m, n)-hybrid (b_i, c_i)-inverse x_i, then $x_2y = yx_1$ if and only if $c_2^nya_1b_1^m = c_2^na_2yb_1^m$, $yb_1^m = x_2a_2yb_1^m$ and $c_2^ny = c_2^nya_1x_1$.*

Proof. Assume that for any $y \in R$, the implication $x_2y = yx_1$ holds. Since x_i is the right (m, n)-hybrid (b_i, c_i)-inverse of a_i, we have

$$x_1a_1b_1^m = b_1^m, c_2^na_2x_2 = c_2^n, x_2a_2x_2 = x_2 \text{ and } x_1a_1x_1 = x_1.$$

Then we have the following implications:

$$c_2^na_2yb_1^m = c_2^na_2yx_1a_1b_1^m = c_2^na_2x_2ya_1b_1^m = c_2^nya_1b_1^m,$$
$$yb_1^m = yx_1a_1b_1^m = x_2ya_1b_1^m = x_2a_2x_2ya_1b_1^m = x_2a_2yx_1a_1b_1^m = x_2a_2yb_1^m,$$
$$c_2^ny = c_2^na_2x_2y = c_2^na_2yx_1 = c_2^na_2yx_1a_1x_1 = c_2^na_2x_2ya_1x_1 = c_2^nya_1x_1.$$

Conversely, if $c_2^n y = c_2^n y a_1 x_1$, then $c_2^n(y - ya_1x_1) = 0$. Since x_i is the right (m,n)-hybrid (b_i, c_i)-inverse of a_i, we have $x_iR = b_i^m R$. Combining with $(y - ya_1x_1) \in rann(c_2^n) = rann(x_2)$, we get $x_2y = x_2ya_1x_1$. Let $x_1 = b_1^m t$ with $t \in R$. Then we have

$$yx_1 = yb_1^m t = x_2a_2yb_1^m t = x_2a_2yx_1.$$

Since $c_2^n ya_1b_1^m = c_2^n a_2 y b_1^m$, we also have

$$(ya_1b_1^m - a_2yb_1^m) \in rann(c_2^n) = rann(x_2).$$

Thus, $x_2ya_1b_1^m = x_2a_2yb_1^m$. It follows that $x_2ya_1b_1^m t = x_2a_2yb_1^m t$, that is, $x_2ya_1x_1 = x_2a_2yx_1$. Therefore, $x_2y = yx_1$. □

Corollary 3.2. *Let $a_i, b_i, c_i, x_i \in R$ $(i = 1, 2)$ and $m, n \in \mathbb{N}^+$. If x_i is the right (m,n)-hybrid (b_i, c_i)-inverse of a_i, then for any $y \in R$, $ya_1 = a_2y$, $yb_1^m = b_2^m y$ and $yc_1^n = c_2^n y$ imply $x_2y = yx_1$.*

Proof. Since $ya_1 = a_2y$, $yb_1^m = b_2^m y$ and $yc_1^n = c_2^n y$, we conclude that

$$c_2^n ya_1b_1^m = c_2^n a_2yb_1^m,\ yb_1^m = b_2^m y = x_2a_2b_2^m y = x_2a_2yb_1^m,$$
$$c_2^n y = yc_1^n = yc_1^n a_1x_1 = c_2^n ya_1x_1.$$

By Theorem 3.1, we have $yx_1 = x_2y$. □

More generally, we can get the following theorem.

Theorem 3.3. *Let $a_i, b_i, c_i, x_i, d \in R$ $(i = 1, 2)$ and $m, n \in \mathbb{N}^+$. If x_i is the right (m,n)-hybrid (b_i, c_i)-inverse of a_i, then $x_2d = dx_1$ if and only if $x_2da_1x_1 = x_2a_2dx_1$, $lann(b_2^m) \subseteq lann(db_1^m)$ and $rann(c_1^n) \subseteq rann(c_2^n d)$.*

Proof. If x_i is the right (m,n)-hybrid (b_i, c_i)-inverse of a_i and $x_2d = dx_1$, then we have

$$x_2a_2dx_1 = x_2a_2x_2d = x_2d = dx_1 = dx_1a_1x_1 = x_2da_1x_1.$$

Let $k \in lann(b_2^m) = lann(x_2)$. Then $kx_2 = 0$. This shows that

$$kx_2a_2dx_1a_1b_1^m = kx_2a_2x_2da_1b_1^m = kx_2da_1b_1^m = kdx_1a_1b_1^m = kdb_1^m = 0.$$

Combining with $k \in lann(b_2^m)$, we get $lann(b_2^m) \subseteq lann(db_1^m)$. Let $l \in rann(c_1^n) = rann(x_1)$. Then $x_1l = 0$, and thus $dx_1l = 0$. Since $dx_1l = dx_1a_1x_1l = x_2da_1x_1l = 0$, we have $c_2^n a_2x_2da_1x_1l = 0$. It follows that

$$c_2^n a_2x_2da_1x_1l = c_2^n a_2dx_1a_1x_1l = c_2^n a_2dx_1l = c_2^n dl = 0.$$

Therefore, $rann(c_1^n) \subseteq rann(c_2^n d)$.

Conversely, since x_i is the right (m,n)-hybrid (b_i, c_i)-inverse of a_i, we have

$$rann(c_i^n) = rann(x_i),\ x_iR = b_i^m R \text{ and } x_ia_ib_i^m = b_i^m.$$

Then $lann(b_i^m) = lann(x_i)$, and thus $(x_2a_2 - 1)b_2^m = 0$. Also because $(x_2a_2 - 1) \in lann(b_2^m) \subseteq lann(db_1^m)$, we get $x_2a_2db_1^m = db_1^m$. Since $c_1^n = c_1^n a_1x_1$, it follows that

$$(1 - a_1x_1) \in rann(c_1^n) \subseteq rann(c_2^n d).$$

Hence, $c_2^n d = c_2^n da_1x_1$. Next since $x_2da_1x_1 = x_2a_2dx_1$, we have $x_2(da_1 - a_2d)x_1 = 0$. Combining with $(da_1 - a_2d)x_1 \in rann(x_2) = rann(c_2^n)$, we get $c_2^n(da_1 - a_2d)x_1 = 0$. Since $c_2^n(da_1 - a_2d) \in lann(x_1) = lann(b_1^m)$, we have $c_2^n da_1b_1^m = c_2^n a_2db_1^m$. This shows that $x_2d = dx_1$ by Theorem 3.1. □

We next consider Cline's formula for the right (m,n)-hybrid (b,c)-inverse.

Theorem 3.4. *Let $a_1, a_2, b, c, x \in R$ and $b, c \in comm(a_2a_1)$ for $m, n \in \mathbb{N}^+$. If x is the right (m,n)-hybrid (b,c)-inverse of a_2a_1, then $a_1x^2a_2$ is the right hybrid $((a_1ba_2)^m, (a_1ca_2)^n)$-inverse of a_1a_2.*

Proof. If $b, c \in comm(a_2a_1)$, then $b^m, c^n \in comm(a_2a_1)$. Thus, $xa_2a_1 = a_2a_1x$ by Proposition 2.9. Since x is the right (m, n)-hybrid (b, c)-inverse of a_2a_1, we deduce that

$$(a_1x^2a_2)a_1a_2(a_1x^2a_2) = a_1xa_2a_1xa_2a_1x^2a_2 = a_1x^2a_2,$$
$$(a_1x^2a_2)a_1a_2(a_1ba_2)^m = a_1xa_2a_1b^m(a_2a_1)^{m-1}a_2 = a_1b^m(a_2a_1)^{m-1}a_2 = (a_1ba_2)^m,$$
$$(a_1ca_2)^na_1a_2(a_1x^2a_2) = a_1(a_2a_1)^{n-1}c^na_2a_1xa_2 = a_1(a_2a_1)^{n-1}c^na_2 = (a_1ca_2)^n.$$

Let $x = b^ms$ for some $s \in R$ since $xR = b^mR$. It follows that

$$a_1x^2a_2 = a_1a_2a_1x^3a_2 = \cdots = a_1(a_2a_1)^mx^{m+2}a_2 = a_1(a_2a_1)^mb^msx^{m+1}a_2$$
$$= (a_1ba_2)^ma_1sx^{m+1}a_2 \in (a_1ba_2)^mR.$$

Therefore, we have $a_1x^2a_2R = (a_1ba_2)^mR$. If $t \in rann[(a_1ca_2)^n]$, then $(a_1ca_2)^nt = 0$, and thus $a_2(a_1ca_2)^nt = c^n(a_2a_1)^na_2t = 0$. It follows that

$$(a_2a_1)^na_2t \in rann(c^n) \subseteq rann(x) \subseteq \cdots \subseteq rann(x^{n+1}).$$

This implies that $x^{n+1}(a_2a_1)^na_2t = 0$. Then $xa_2t = 0$, and hence $a_1x^2a_2t = 0$. Therefore, $rann[(a_1ca_2)^n] \subseteq rann(a_1x^2a_2)$. □

By Theorem 3.4, we have the following corollary immediately.

Corollary 3.5. *Let $a, b, c, x \in R$ such that $b^m, c^n \in comm(a)$ for some $m, n \in \mathbb{N}^+$. If x is the right (m, n)-hybrid (b, c)-inverse of a, then*

(1) a^2x is the right hybrid (b^ma, c^na)-inverse of x;

(2) x is the right hybrid (b^ma, c^na)-inverse of a^2x.

Proof. (1) Since $b^m, c^n \in comm(a)$, we have $xa = ax$ by Proposition 2.9. Since x is the right (m, n)-hybrid (b, c)-inverse of a, it follows that

$$(a^2x)x(a^2x) = a^2x^2a^2x = a^2x,$$
$$(a^2x)xb^ma = a(xax)ab^m = b^ma,$$
$$(c^na)xa^2x = c^na^2x = c^naxa = c^na.$$

Since $x = b^mt$ for some $t \in R$, we get $a^2x = a^2b^mt = b^maat \in b^maR$. Let $k \in rann(c^na)$. Then $ak \in rann(c^n) = rann(x)$. This implies that $k \in rann(a^2x)$ since $a^2xk = axak = 0$. Therefore, we have $rann(c^na) \subseteq rann(a^2x)$.

(2) Since x is the right (m, n)-hybrid (b, c)-inverse of a and $b^m, c^n \in comm(a)$, it can be easily checked that $x(a^2x)x = x$, $xa^2xb^ma = b^ma$ and $(c^na)a^2xx = c^na$. Let $x = b^mg$ for some $g \in R$. Then we have

$$x = xax = ax^2 = ab^mgx \in b^maR.$$

Let $h \in rann(c^na)$. Then we get $ah \in rann(c^n) = rann(x)$, and thus $xh = x^2ah = 0$. This implies that $h \in rann(x)$, and hence $rann(c^na) \subseteq rann(x)$. □

The following theorem can be regarded as a generalization of Cline's formula for right hybrid (b, c)-inverses, which is closely related to right (m, n)-hybrid (b, c)-inverses.

Theorem 3.6. *Let $a_1, a_2, b, c, x \in R$ and $m, n \in \mathbb{N}^+$. If x is the right (m, n)-hybrid (b, c)-inverse of a_2a_1 such that $a_2a_1b^m \in b^msR$, $rann(tc^n) \subseteq rann(c^na_2a_1)$ for some $s, t \in R$, then $a_1x^2a_2$ is the right hybrid (a_1b^ms, tc^na_2)-inverse of a_1a_2.*

Proof. If $a_2a_1b^m \in b^msR$, then $lann(b^m) \subseteq lann(b^ms) \subseteq lann(a_2a_1b^m)$. Since $rann(c^n) \subseteq rann(tc^n) \subseteq rann(c^na_2a_1)$ and $xa_2a_1a_2a_1x = xa_2a_1a_2a_1x$, it follows from Theorem 3.3 that $xa_2a_1 = a_2a_1x$. Since x is the right (m, n)-hybrid (b, c)-inverse of a_2a_1, we deduce that $(a_1x^2a_2)a_1a_2(a_1x^2a_2) = a_1x^2a_2$. Moreover, since $(xa_2a_1 - 1) \in lann(b^m) \subseteq lann(a_2a_1b^m)$, we get $xa_2a_1a_2a_1b^m = a_2a_1b^m$. This implies that

$$(a_1x^2a_2)a_1a_2(a_1b^ms) = a_1xa_2a_1b^ms = a_1b^ms.$$

Also since $(1 - a_2a_1x) \in rann(c^n) \subseteq rann(c^na_2a_1)$, we get $c^na_2a_1 = c^na_2a_1a_2a_1x$. Then $(tc^na_2)a_1a_2(a_1x^2a_2) = tc^na_2a_1xa_2 = tc^na_2$.

To complete the proof, it remains to show $a_1x^2a_2 \in a_1b^msR$, $rann(tc^na_2) \subseteq rann(a_1x^2a_2)$. In fact, since $a_2a_1b^m \in b^msR$, there is $k \in R$ such that $a_2a_1b^m = b^msk$. Let $x \in b^mg$ for some $g \in R$. Then we get

$$\begin{aligned} a_1x^2a_2 = a_1xa_2a_1x^2a_2 = a_1a_2a_1xx^2a_2 &= a_1(a_2a_1b^m)gx^2a_2 \\ &= a_1b^mskgx^2a_2 \in a_1b^msR. \end{aligned}$$

If $e \in rann(tc^na_2)$, then $tc^na_2e = 0$. Thus $a_2e \in rann(tc^n) \subseteq rann(c^na_2a_1)$. This shows that $c^na_2a_1a_2e = 0$. As $a_2a_1a_2e \in rann(c^n) = rann(x)$, we deduce that $xa_2a_1a_2e = 0$. Therefore, $x^2a_2a_1a_2e = xa_2e = 0$. It follows that $e \in rann(a_1x^2a_2)$ since $a_1x^2a_2e = 0$. □

4. The reverse order law for right (m, n)-hybrid (b, c)-inverses

In this section, we discuss the reverse order law for right (m,n)-hybrid (b,c)-inverses. As an application, the relationships between Bott-Duffin (e,e)-inverses, Bott-Duffin (f,f)-inverses and Bott-Duffin (e,f)-inverses are studied.

Theorem 4.1. *Let $a_i, b_i, c_i, x_i \in R$ and let x_i be the right (m,n)-hybrid (b_i,c_i)-inverse of a_i for each i $(i = 1,2)$. If $lann(b_1^m) \subseteq lann(a_2)$ and $rann(c_2^n) \subseteq rann(a_1)$ for some $m,n \in \mathbb{N}^+$, then x_2x_1 is the right (m,n)-hybrid (b_2,c_1)-inverse of a_1a_2.*

Proof. Since x_i is the right (m,n)-hybrid (b_i,c_i)-inverse of a_i and $lann(b_1^m) \subseteq lann(a_2)$, we have $(x_1a_1 - 1) \in lann(b_1^m) \subseteq lann(a_2)$, and hence $x_1a_1a_2 = a_2$. This implies that

$$\begin{aligned} x_2x_1a_1a_2x_2x_1 = x_2a_2x_2x_1 = x_2x_1, \\ (x_2x_1)a_1a_2b_2^m = x_2a_2b_2^m = b_2^m. \end{aligned}$$

Since $x_2 \in b_2^mR$, we get $x_2x_1 \in b_2^mR$. As $rann(c_2^n) \subseteq rann(a_1)$, we get $(1 - a_2x_2) \in rann(c_2^n) \subseteq rann(a_1)$. It yields that $c_1^na_1a_2x_2x_1 = c_1^na_1x_1 = c_1^n$ since $a_1 = a_1a_2x_2$. Combining with $rann(c_1^n) \subseteq rann(x_1) \subseteq rann(x_2x_1)$, we conclude that x_2x_1 is the right (m,n)-hybrid (b_2,c_1)-inverse of a_1a_2. □

We have the following corollary for right hybrid (b_i,c_i)-inverses immediately.

Corollary 4.2. *Let $a_i, b_i, c_i, x_i \in R$ $(i = 1,2)$. If x_i is the right hybrid (b_i,c_i)-inverse of a_i such that $lann(b_1) \subseteq lann(a_2)$ and $rann(c_2) \subseteq rann(a_1)$, then x_2x_1 is the right hybrid (b_2,c_1)-inverse of a_1a_2.*

Next, we explore the relationship between Bott-Duffin (e,e)-inverses, Bott-Duffin (f,f)-inverses and Bott-Duffin (e,f)-inverses, which can be regarded as an application of the reverse order law of right (m,n)-hybrid (b,c)-inverses.

Proposition 4.3. *Let $a_i, e, f, x_i \in R$ $(i = 1,2)$ and let e, f be two idempotent elements of R. If the following three conditions are satisfied:*

(1) x_1 is the Bott-Duffin (f,f)-inverse of a_1,

(2) x_2 is the Bott-Duffin (e,e)-inverse of a_2,

(3) $lann(f) \subseteq lann(a_2)$, $rann(e) \subseteq rann(a_1)$,

then x_2x_1 is the Bott-Duffin (e,f)-inverse of a_1a_2.

Proof. If x_1 is the Bott-Duffin (f,f)-inverse of a_1, then $x_1a_1f^m = x_1a_1f = f = f^m$ and $f^na_1x_1 = fa_1x_1 = f = f^n$. Since $x_1 = f^mx_1 = x_1f^n$, we have $x_1R \subseteq f^mR$ and $rann(f^n) \subseteq rann(x_1)$. Then x_1 is the right (m,n)-hybrid (f,f)-inverse of a_1 by Proposition 2.3. Similarly, we can show that x_2 is the right (m,n)-hybrid (e,e)-inverse of a_2. Since f, e are idempotent elements with $lann(f) \subseteq lann(a_2)$ and $rann(e) \subseteq rann(a_1)$, it follows that x_2x_1 is the right (m,n)-hybrid (e,f)-inverse of a_1a_2 by Theorem 4.1, that is, x_2x_1 is the right hybrid (e,f)-inverse of a_1a_2. Therefore, x_2x_1 is the Bott-Duffin (e,f)-inverse of a_1a_2 by [14, Corollary 3.7]. □

We conclude this section by giving the triple reverse order law of the right (m,n)-hybrid (b,c)-inverse.

Theorem 4.4. *Let $a_i, b_i, c_i, x_i \in R$ and $m, n \in \mathbb{N}^+$ $(i = 1, 2, 3)$. If x_i is the right (m,n)-hybrid (b_i, c_i)-inverse of a_i such that*

$$lann(b_1^m) \subseteq lann(a_2), lann(b_2^m) \subseteq lann(a_3),$$
$$rann(c_2^n) \subseteq rann(a_1), rann(c_3^n) \subseteq rann(a_2),$$

then $x_3x_2x_1$ is the right (m,n)-hybrid (b_3, c_1)-inverse of $a_1a_2a_3$.

Proof. By the assumption, it is clear that $(x_1a_1 - 1) \in lann(b_1^m) \subseteq lann(a_2)$ and $(x_2a_2 - 1) \in lann(b_2^m) \subseteq lann(a_3)$, thus we have $x_1a_1a_2 = a_2$ and $x_2a_2a_3 = a_3$. Similarly, since $(1 - a_2x_2) \in rann(c_2^n) \subseteq rann(a_1)$ and $(1 - a_3x_3) \in rann(c_3^n) \subseteq rann(a_2)$, we also get $a_1 = a_1a_2x_2$ and $a_2 = a_2a_3x_3$. This implies that

$$\begin{aligned}(x_3x_2x_1)(a_1a_2a_3)(x_3x_2x_1) &= x_3x_2(x_1a_1a_2)a_3x_3x_2x_1 = x_3x_2(a_2a_3x_3)x_2x_1\\ &= x_3x_2a_2x_2x_1 = x_3x_2x_1.\end{aligned}$$

Since $x_3 \in b_3^m R$, we have $x_3x_2x_1 \in b_3^m R$. Also because $rann(c_1^n) = rann(x_1) \subseteq rann(x_3x_2x_1)$, we have $rann(c_1^n) \subseteq rann(x_3x_2x_1)$. Since $x_3a_3b_3^m = b_3^m$ and $c_1^n a_1x_1 = c_1^n$, we deduce that

$$x_3x_2(x_1a_1a_2)a_3b_3^m = x_3(x_2a_2a_3)b_3^m = x_3a_3b_3^m = b_3^m,$$
$$c_1^n a_1(a_2a_3x_3)x_2x_1 = c_1^n(a_1a_2x_2)x_1 = c_1^n a_1x_1 = c_1^n.$$

Therefore, $x_3x_2x_1$ is the right (m,n)-hybrid (b_3, c_1)-inverse of $a_1a_2a_3$. □

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