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Topological indices, spectra and energies of prime ideal sum graphs of commutative rings

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Abstract. Let R be a commutative ring with the nonzero identity. The prime ideal sum graph of R, denoted by $PIS(R)$, is a graph whose vertex-set is the set of all nonzero proper ideals of R, and two distinct vertices I and J are adjacent if and only if $I + J$ is a prime ideal of R. In this study, our aim is to find out the topological indices, spectra and energies of the prime ideal sum graphs of $\mathbb{Z}_n$, where $n = p^\alpha, pq, p^2q, p^2q^2, pqr, p^3q, p^2qr, pqrs$; p, q, r, s being distinct prime integers, $\alpha \in \mathbb{Z}^+$ and $\mathbb{Z}_n$ is the ring of integers modulo n.

1. Introduction

We briefly review the fundamental concepts, notations and findings of graph theory to facilitate subsequent discussions. As for the terms or concepts not explained in this article, readers can refer to "Introduction to Graph Theory" by D.B. West.
A graph $G = (V, E)$, consists of a non-empty set V called the vertex set and a symmetric binary relation E on V known as the edge set, which may be empty. Two elements u and v in V are said to be adjacent if $(u, v) \in E$. A graph $H = (W, F)$ is termed as a subgraph of G if H itself is a graph and $\phi \neq W \subset V$ and $F \subset E$. If V is finite, then G is referred to as a finite graph, otherwise it is infinite. The order of G is defined as the number of vertices in V, while the size of G is the number of edges in G. A graph is called complete if every pair of distinct vertices is adjacent in G, and it is denoted by K_n, where n represents the number of vertices. The degree of a vertex $u \in V$ denoted by d_u is the number of edges incident to u.

During the last few years, researchers have shown interest in graph indices such as the First Zagreb index, the Second Zagreb index, the forgotten topological index, and the Sombor index. Therefore, many research articles have been written on different graph indices. Chemical graph theory is a sub-discipline of graph theory which embraces graph theory in solution of molecular problems. In this context, a graph is a molecule in which vertices correspond to atoms, and edges correspond to chemical bonds. Topological indices can be defined as mathematical functions that provide numerical characteristics of molecular graphs. These indices are important in estimating the physicochemical and biological characteristics of molecules, and therefore essential in the design of drugs.There are many topological indices in the literature. These

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indices are very useful tools for chemists and researchers that deal with chemical graph theory and other related fields. They enable the visualization of molecular shapes, estimate physical and chemical properties of a material, and help to design new material with desired properties, particularly in pharmaceutical and material chemistry. At the same time, their mathematical beauty also makes them an object of emphasis in pure researches in graph theory. Before we proceed, let us summarise some topological indices which will be also studied in this manuscript.

• The Sombor index

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2};$$

• The Randic index

$$RI(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u d_v}};$$

• The first Zagreb index

$$M_1(G) = \sum_{u \in V(G)} d_u^2;$$

• The second Zagreb index

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v;$$

• The Harmonic index

$$HI(G) = \sum_{uv \in E(G)} \frac{2}{d_u + d_v};$$

• The ABC index

$$ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}};$$

• The first Zagreb coindex

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} d_u + d_v;$$

• The second Zagreb coindex

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_u d_v;$$

where d_u and d_v represent the degrees of u and v respectively.

The concept of studying the graph-theoretical properties of a commutative ring associated with zero divisors was introduced in [3] and [6]. Anwar et al. and Gursoy et al. [5, 15] discuss the Forgotten and Sombor indices of zero-divisor graphs and co-zero divisor graphs of commutative rings. Rather et al. and Rehman et al. [23, 25] are among the most recent contributors to this field.

The adjacency matrix of a graph G is the $n \times n$ matrix $A(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1 & , \quad \text{if } v_i v_j \in E(G) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

In any case, the adjacency matrix is a symmetric $n \times n$ matrix, where every entry on the main diagonal is 0. The number of 1s in row i (or column i) is the degree of v_i. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of the adjacency matrix $A(G)$. The energy $E(G)$ of the graph G is defined as the sum of the absolute values of its eigenvalues i.e.,

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The Sombor matrix of G is defined as: $SO(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} \sqrt{d_{v_i}^2 + d_{v_j}^2} & , \quad \text{if } v_iv_j \in E(G) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

The Sombor matrix is a real, symmetric matrix and the eigenvalues of $SO(G)$, denoted by $\rho_1, \rho_2, \ldots, \rho_n$ are said to form the Sombor spectrum of the graph G. The Sombor Energy $E_{SO}(G)$ is defined as

$$E_{SO}(G) = \sum_{i=1}^{n} |\rho_i|.$$

The Laplacian and signless Laplacian matrices of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G. Their spectra are respectively the Laplacian spectrum and signless Laplacian spectrum of the graph G. The eigenvalues of $L(G)$ and $Q(G)$ will be denoted by $\mu_1, \mu_2, \ldots, \mu_n$ and $q_1, q_2, \ldots, q_n$, respectively. Then the Laplacian energy and signless Laplacian energy of G are defined as:

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| \quad \text{and} \quad QE(G) = \sum_{i=1}^{n} \left| q_i - \frac{2m}{n} \right|.$$

The Randic matrix of G is defined as: $R(G) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} \dfrac{1}{\sqrt{d_{v_i} d_{v_j}}} & , \quad \text{if } v_iv_j \in E(G) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

If $\xi_1, \xi_2, \ldots, \xi_n$ are the eigen values of $R(G)$, then the Randic Energy $RE(G)$ is defined as:

$$RE(G) = \sum_{i=1}^{n} |\xi_i|.$$

These matrices are real symmetric and positive semi-definite having real eigenvalues.The Sombor spectrum is an approach to envelope the structural information of graphs through vertex degrees to analyze the stability and properties of molecules in chemistry. The Laplacian spectrum plays an important role in analyzing connectivity, graph cut and network evolution. The signless Laplacian spectrum again contains useful information for understanding graph characteristics such as bipartite and clustering. The Randic spectrum is important for investigation of molecular branching and large complex objects, which has many applications in chemistry, biology and social networks. Altogether, these spectra are useful instruments in graph theory and in other branches of science as well. Details about Laplacian, signless Laplacian and Randic matrices can be found in [4, 22] and the references therein.

Based on the findings mentioned above, this article examines some degree-based topological indices, spectra and energies for the prime ideal sum graphs of $\mathbb{Z}_n$, a commutative ring with unity. The computation of approximate eigenvalues of all the matrices discussed in our article has been facilitated through the utilization of matlab software.

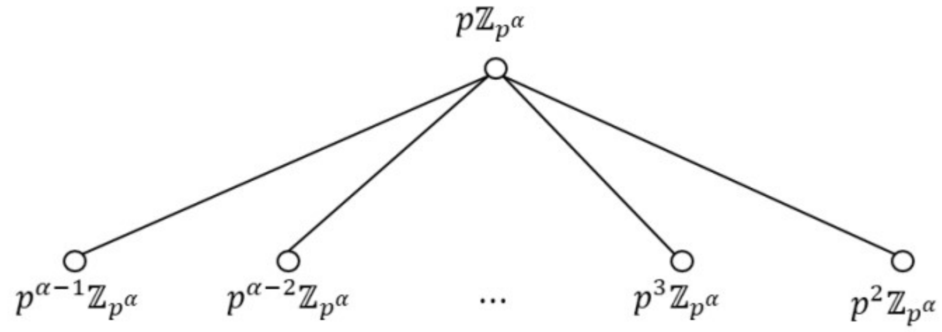

Figure 1: Prime Ideal Sum Graph of $\mathbb{Z}_{p^\alpha}$

Figure 2: Prime Ideal Sum Graph of $\mathbb{Z}_{pq}$

2. Main Results

Our first result deals with the estimation of Sombor indices of different prime ideal sum graphs of $\mathbb{Z}_n$, which is stated as below:

Theorem 2.1. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(i) *If $n = p^\alpha$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* $(\alpha - 2)\sqrt{\alpha^2 - 4\alpha + 5}$.
(ii) *If $n = pq$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 0.
(iii) *If $n = p^2q$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 13.199.
(iv) *If $n = p^2q^2$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 59.796.
(v) *If $n = pqr$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 43.8.
(vi) *If $n = p^3q$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 35.316.
(vii) *If $n = p^2qr$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 171.64.
($viii$) *If $n = pqrs$, then the Sombor index of $PIS(\mathbb{Z}_n)$ is* 491.692.

Proof. (i) First, we need all the non-zero proper ideals of $\mathbb{Z}_{p^\alpha}$ and they are: $u_1 = p\mathbb{Z}_{p^\alpha}, u_2 = p^2\mathbb{Z}_{p^\alpha}$, $u_3 = p^3\mathbb{Z}_{p^\alpha}, \ldots, u_{\alpha-2} = p^{\alpha-2}\mathbb{Z}_{p^\alpha}, u_{\alpha-1} = p^{\alpha-1}\mathbb{Z}_{p^\alpha}$. Now, as $p\mathbb{Z}_{p^\alpha}$ is a prime ideal of $\mathbb{Z}_{p^\alpha}$ and $p\mathbb{Z}_{p^\alpha} + p^r\mathbb{Z}_{p^\alpha} = (p, p^r)\mathbb{Z}_{p^\alpha} = p\mathbb{Z}_{p^\alpha}, \quad 2 \le r \le \alpha - 1$, which is a prime ideal. So, $p\mathbb{Z}_{p^\alpha}$ is adjacent to every other vertex. Next, consider the following graph of $\mathbb{Z}_{p^\alpha}$: Now we get,

$$\begin{aligned} SO(PIS(\mathbb{Z}_{p^\alpha})) &= \sqrt{d_{u_1}^2 + d_{u_2}^2} + \sqrt{d_{u_1}^2 + d_{u_3}^2} + \cdots + \sqrt{d_{u_1}^2 + d_{u_{\alpha-1}}^2} \\ &= \sqrt{(\alpha-2)^2 + 1} + \sqrt{(\alpha-2)^2 + 1} + \cdots + \sqrt{(\alpha-2)^2 + 1} \\ &= (\alpha - 2)\sqrt{\alpha^2 - 4\alpha + 5}. \end{aligned}$$

(ii) As $\mathbb{Z}_{pq}$ has only two proper ideals namely, $p\mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq}$. From the graph given in Fig.2, we observe that both $p\mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq}$ are isolated vertices. So the vertices $p\mathbb{Z}_{pq}$ and $q\mathbb{Z}_{pq}$ have degree zero. Thus,

$$SO(PIS(\mathbb{Z}_{pq})) = 0.$$

(iii) The ideals of $\mathbb{Z}_{p^2q}$ are: $u_1 = p\mathbb{Z}_{p^2q}$, $u_2 = q\mathbb{Z}_{p^2q}$, $u_3 = pq\mathbb{Z}_{p^2q}$, $u_4 = p^2\mathbb{Z}_{p^2q}$, where the vertices u_1 and u_2 are the prime ideals. The graph of $PIS(\mathbb{Z}_{p^2q})$ is represented in Fig.3. It can be seen from the graph that

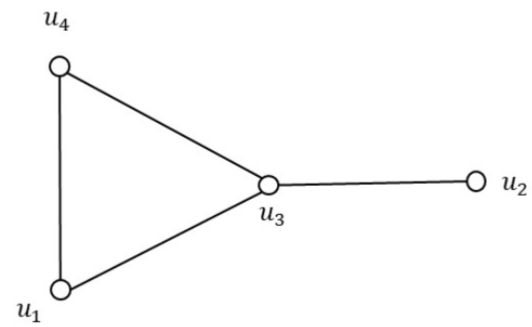

Figure 3: Prime Ideal Sum Graph of $\mathbb{Z}_{p^2q}$

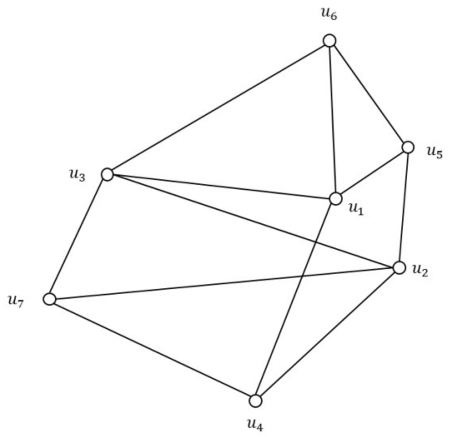

Figure 4: Prime Ideal Sum Graph of $\mathbb{Z}_{p^2q^2}$

$d_{u_1} = 2 = d_{u_4}$, $d_{u_3} = 3$, $d_{u_2} = 1$. So,

$$\begin{aligned} SO(PIS(\mathbb{Z}_{p^2q})) &= \sum_{uv\in E(PIS(\mathbb{Z}_{p^2q}))} \sqrt{d_u^2 + d_v^2} \\ &= \sqrt{d_{u_1}^2 + d_{u_3}^2} + \sqrt{d_{u_1}^2 + d_{u_4}^2} + \sqrt{d_{u_2}^2 + d_{u_3}^2} + \sqrt{d_{u_3}^2 + d_{u_4}^2} \\ &= 13.199. \end{aligned}$$

(iv) Consider the ideals, $u_1 = p\mathbb{Z}_{p^2q^2}$, $u_2 = q\mathbb{Z}_{p^2q^2}$, $u_3 = pq\mathbb{Z}_{p^2q^2}$, $u_4 = p^2q\mathbb{Z}_{p^2q^2}$, $u_5 = pq^2\mathbb{Z}_{p^2q^2}$, $u_6 = p^2\mathbb{Z}_{p^2q^2}$, $u_7 = q^2\mathbb{Z}_{p^2q^2}$ of $\mathbb{Z}_{p^2q^2}$. The graph can be seen in Fig.4. Number of vertices of degree $4 = 3(u_1, u_2, u_3)$ and number of vertices of degree $3 = 4(u_5, u_6, u_7, u_4)$. So, we get

$$\begin{aligned} SO(PIS(\mathbb{Z}_{p^2q^2})) &= \sqrt{d_{u_1}^2 + d_{u_3}^2} + \sqrt{d_{u_1}^2 + d_{u_4}^2} + \sqrt{d_{u_1}^2 + d_{u_5}^2} + \sqrt{d_{u_1}^2 + d_{u_6}^2} + \sqrt{d_{u_2}^2 + d_{u_3}^2} + \sqrt{d_{u_2}^2 + d_{u_4}^2} \\ &+ \sqrt{d_{u_2}^2 + d_{u_5}^2} + \sqrt{d_{u_2}^2 + d_{u_7}^2} + \sqrt{d_{u_3}^2 + d_{u_6}^2} + \sqrt{d_{u_3}^2 + d_{u_7}^2} + \sqrt{d_{u_4}^2 + d_{u_7}^2} + \sqrt{d_{u_5}^2 + d_{u_6}^2} \\ &= 59.796. \end{aligned}$$

(v) The proper ideals of $\mathbb{Z}_{pqr}$ are: $p\mathbb{Z}_{pqr}$, $q\mathbb{Z}_{pqr}$, $r\mathbb{Z}_{pqr}$, $pq\mathbb{Z}_{pqr}$, $qr\mathbb{Z}_{pqr}$, $pr\mathbb{Z}_{pqr}$. Let $u_1 = p\mathbb{Z}_{pqr}$, $u_2 = q\mathbb{Z}_{pqr}$, $u_3 = r\mathbb{Z}_{pqr}$, $u_4 = pq\mathbb{Z}_{pqr}$, $u_5 = pr\mathbb{Z}_{pqr}$, $u_6 = qr\mathbb{Z}_{pqr}$, where the ideals generated by u_1, u_2 and u_3 are the prime ideals. The graph of $PIS(\mathbb{Z}_{pqr})$ can be seen in Fig.5. Number of vertices of degree $2 = 3(u_1, u_2, u_3)$ and number of vertices of degree $4 = 3(u_4, u_5, u_6)$. So,

$$SO(PIS(\mathbb{Z}_{pqr})) = \sqrt{d_{u_1}^2 + d_{u_4}^2} + \sqrt{d_{u_1}^2 + d_{u_5}^2} + \sqrt{d_{u_2}^2 + d_{u_4}^2} + \sqrt{d_{u_2}^2 + d_{u_6}^2} + \sqrt{d_{u_3}^2 + d_{u_5}^2} + \sqrt{d_{u_3}^2 + d_{u_6}^2}$$

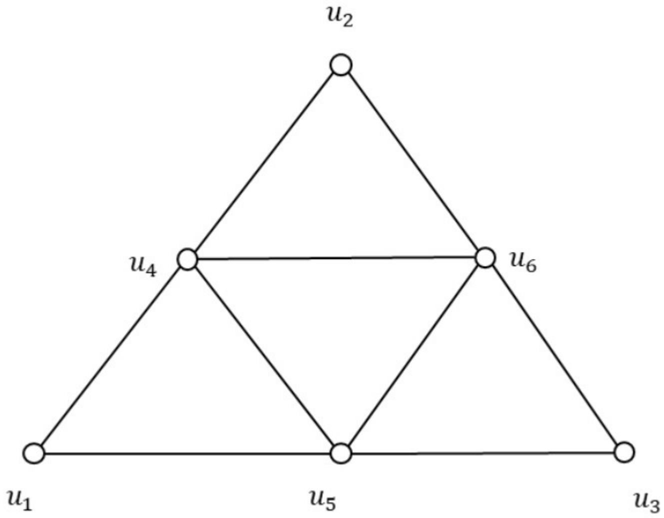

Figure 5: Prime Ideal Sum Graph of $\mathbb{Z}_{pqr}$

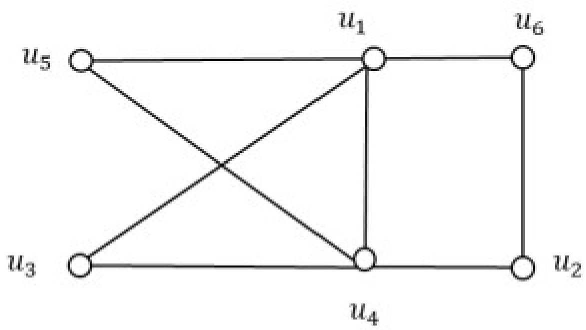

Figure 6: Prime Ideal Sum Graph of $\mathbb{Z}_{p^3q}$

$$\begin{aligned} &+ \sqrt{d_{u_4}^2 + d_{u_5}^2} + \sqrt{d_{u_4}^2 + d_{u_6}^2} + \sqrt{d_{u_5}^2 + d_{u_6}^2} \\ &= 43.8. \end{aligned}$$

(vi) The proper ideals of $\mathbb{Z}_{p^3q}$ are: $p\mathbb{Z}_{p^3q}$, $q\mathbb{Z}_{p^3q}$, $p^2\mathbb{Z}_{p^3q}$, $pq\mathbb{Z}_{p^3q}$, $p^3\mathbb{Z}_{p^3q}$, $p^2q\mathbb{Z}_{p^3q}$. Let $u_1 = p\mathbb{Z}_{p^3q}$, $u_2 = q\mathbb{Z}_{p^3q}$, $u_3 = p^2\mathbb{Z}_{p^3q}$, $u_4 = pq\mathbb{Z}_{p^3q}$, $u_5 = p^3\mathbb{Z}_{p^3q}$, $u_6 = p^2q\mathbb{Z}_{p^3q}$. The graph of $PIS(\mathbb{Z}_{p^3q})$ can be seen in Fig.6. Number of vertices of degree 2 = 4 and number of vertices of degree 4 = 2. So,

$$\begin{aligned} SO(PIS(\mathbb{Z}_{p^3q})) &= \sqrt{d_{u_1}^2 + d_{u_3}^2} + \sqrt{d_{u_1}^2 + d_{u_4}^2} + \sqrt{d_{u_1}^2 + d_{u_5}^2} + \sqrt{d_{u_1}^2 + d_{u_6}^2} \\ &+ \sqrt{d_{u_2}^2 + d_{u_4}^2} + \sqrt{d_{u_2}^2 + d_{u_6}^2} + \sqrt{d_{u_3}^2 + d_{u_4}^2} + \sqrt{d_{u_4}^2 + d_{u_5}^2} \\ &= 35.316. \end{aligned}$$

(vii) Consider the proper ideals of $\mathbb{Z}_{p^2qr}$: $u_1 = p\mathbb{Z}_{p^2qr}$, $u_2 = q\mathbb{Z}_{p^2qr}$, $u_3 = r\mathbb{Z}_{p^2qr}$, $u_4 = p^2\mathbb{Z}_{p^2qr}$, $u_5 = pq\mathbb{Z}_{p^2qr}$, $u_6 = pr\mathbb{Z}_{p^2qr}$, $u_7 = p^2q\mathbb{Z}_{p^2qr}$, $u_8 = p^2r\mathbb{Z}_{p^2qr}$, $u_9 = qr\mathbb{Z}_{p^2qr}$, $u_{10} = pqr\mathbb{Z}_{p^2qr}$, where u_1, u_2 and u_3 are prime ideals. The graph of $PIS(\mathbb{Z}_{p^2qr})$ can be seen in Fig.7. Number of vertices of degree 4 = 6 and number of vertices of degree 6 = 4. We have,

$$SO(PIS(\mathbb{Z}_{p^2qr})) = \sqrt{d_{u_1}^2 + d_{u_4}^2} + \sqrt{d_{u_1}^2 + d_{u_5}^2} + \sqrt{d_{u_1}^2 + d_{u_6}^2} + \sqrt{d_{u_1}^2 + d_{u_7}^2} + \sqrt{d_{u_1}^2 + d_{u_8}^2}$$

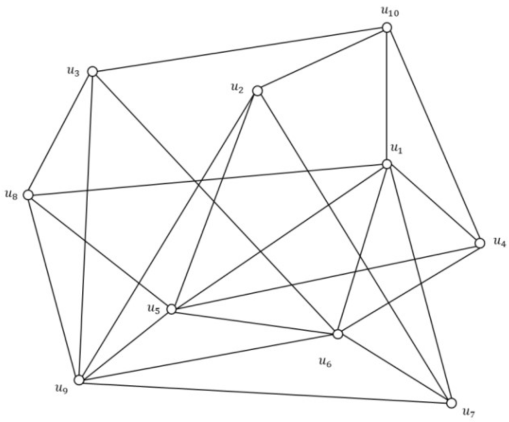

Figure 7: Prime Ideal Sum Graph of $\mathbb{Z}_{p^2qr}$

$$
\begin{aligned}
&+ \sqrt{d_{u_1}^2 + d_{u_{10}}^2} + \sqrt{d_{u_2}^2 + d_{u_5}^2} + \sqrt{d_{u_2}^2 + d_{u_7}^2} + \sqrt{d_{u_2}^2 + d_{u_9}^2} + \sqrt{d_{u_2}^2 + d_{u_{10}}^2} \\
&+ \sqrt{d_{u_3}^2 + d_{u_6}^2} + \sqrt{d_{u_3}^2 + d_{u_8}^2} + \sqrt{d_{u_3}^2 + d_{u_9}^2} + \sqrt{d_{u_3}^2 + d_{u_{10}}^2} + \sqrt{d_{u_4}^2 + d_{u_5}^2} \\
&+ \sqrt{d_{u_4}^2 + d_{u_6}^2} + \sqrt{d_{u_4}^2 + d_{u_{10}}^2} + \sqrt{d_{u_5}^2 + d_{u_6}^2} + \sqrt{d_{u_5}^2 + d_{u_8}^2} + \sqrt{d_{u_5}^2 + d_{u_9}^2} \\
&+ \sqrt{d_{u_6}^2 + d_{u_7}^2} + \sqrt{d_{u_6}^2 + d_{u_9}^2} + \sqrt{d_{u_7}^2 + d_{u_9}^2} + \sqrt{d_{u_8}^2 + d_{u_9}^2} \\
&= 171.64.
\end{aligned}
$$

(*viii*) The proper ideals of $\mathbb{Z}_{pqrs}$ are: $u_1 = p\mathbb{Z}_{pqrs}$, $u_2 = q\mathbb{Z}_{pqrs}$, $u_3 = r\mathbb{Z}_{pqrs}$, $u_4 = pq\mathbb{Z}_{pqrs}$, $u_5 = s\mathbb{Z}_{pqrs}$, $u_6 = pr\mathbb{Z}_{pqrs}$, $u_7 = ps\mathbb{Z}_{pqrs}$, $u_8 = qr\mathbb{Z}_{pqrs}$, $u_9 = qs\mathbb{Z}_{pqrs}$, $u_{10} = pqr\mathbb{Z}_{pqrs}$, $u_{11} = rs\mathbb{Z}_{pqrs}$, $u_{12} = pqs\mathbb{Z}_{pqrs}$, $u_{13} = prs\mathbb{Z}_{pqrs}$, $u_{14} = qrs\mathbb{Z}_{pqrs}$. The graph of $PIS(\mathbb{Z}_{pqrs})$ is as follows : Number of vertices of degree eight = 6 and number of vertices of degree six = 8. So we get,

$$
\begin{aligned}
SO(PIS(\mathbb{Z}_{pqrs})) &= \sqrt{d_{u_1}^2 + d_{u_4}^2} + \sqrt{d_{u_1}^2 + d_{u_6}^2} + \sqrt{d_{u_1}^2 + d_{u_7}^2} + \sqrt{d_{u_1}^2 + d_{u_{10}}^2} + \sqrt{d_{u_1}^2 + d_{u_{12}}^2} \\
&+ \sqrt{d_{u_1}^2 + d_{u_{13}}^2} + \sqrt{d_{u_2}^2 + d_{u_4}^2} + \sqrt{d_{u_2}^2 + d_{u_8}^2} + \sqrt{d_{u_2}^2 + d_{u_9}^2} + \sqrt{d_{u_2}^2 + d_{u_{10}}^2} \\
&+ \sqrt{d_{u_2}^2 + d_{u_{12}}^2} + \sqrt{d_{u_2}^2 + d_{u_{14}}^2} + \sqrt{d_{u_3}^2 + d_{u_6}^2} + \sqrt{d_{u_3}^2 + d_{u_8}^2} + \sqrt{d_{u_3}^2 + d_{u_{10}}^2} \\
&+ \sqrt{d_{u_3}^2 + d_{u_{11}}^2} + \sqrt{d_{u_3}^2 + d_{u_{13}}^2} + \sqrt{d_{u_3}^2 + d_{u_{14}}^2} + \sqrt{d_{u_4}^2 + d_{u_6}^2} + \sqrt{d_{u_4}^2 + d_{u_7}^2} \\
&+ \sqrt{d_{u_4}^2 + d_{u_8}^2} + \sqrt{d_{u_4}^2 + d_{u_9}^2} + \sqrt{d_{u_4}^2 + d_{u_{13}}^2} + \sqrt{d_{u_4}^2 + d_{u_{14}}^2} + \sqrt{d_{u_5}^2 + d_{u_7}^2} \\
&+ \sqrt{d_{u_5}^2 + d_{u_9}^2} + \sqrt{d_{u_5}^2 + d_{u_{11}}^2} + \sqrt{d_{u_5}^2 + d_{u_{12}}^2} + \sqrt{d_{u_5}^2 + d_{u_{13}}^2} + \sqrt{d_{u_5}^2 + d_{u_{14}}^2} \\
&+ \sqrt{d_{u_6}^2 + d_{u_7}^2} + \sqrt{d_{u_6}^2 + d_{u_8}^2} + \sqrt{d_{u_6}^2 + d_{u_{11}}^2} + \sqrt{d_{u_6}^2 + d_{u_{12}}^2} + \sqrt{d_{u_6}^2 + d_{u_{14}}^2} \\
&+ \sqrt{d_{u_7}^2 + d_{u_9}^2} + \sqrt{d_{u_7}^2 + d_{u_{10}}^2} + \sqrt{d_{u_7}^2 + d_{u_{11}}^2} + \sqrt{d_{u_7}^2 + d_{u_{14}}^2} + \sqrt{d_{u_8}^2 + d_{u_9}^2} \\
&+ \sqrt{d_{u_8}^2 + d_{u_{11}}^2} + \sqrt{d_{u_8}^2 + d_{u_{12}}^2} + \sqrt{d_{u_8}^2 + d_{u_{13}}^2} + \sqrt{d_{u_9}^2 + d_{u_{10}}^2} + \sqrt{d_{u_9}^2 + d_{u_{11}}^2} \\
&+ \sqrt{d_{u_9}^2 + d_{u_{13}}^2} + \sqrt{d_{u_{10}}^2 + d_{u_{11}}^2} + \sqrt{d_{u_{11}}^2 + d_{u_{12}}^2} \\
&= 491.692.
\end{aligned}
$$

□

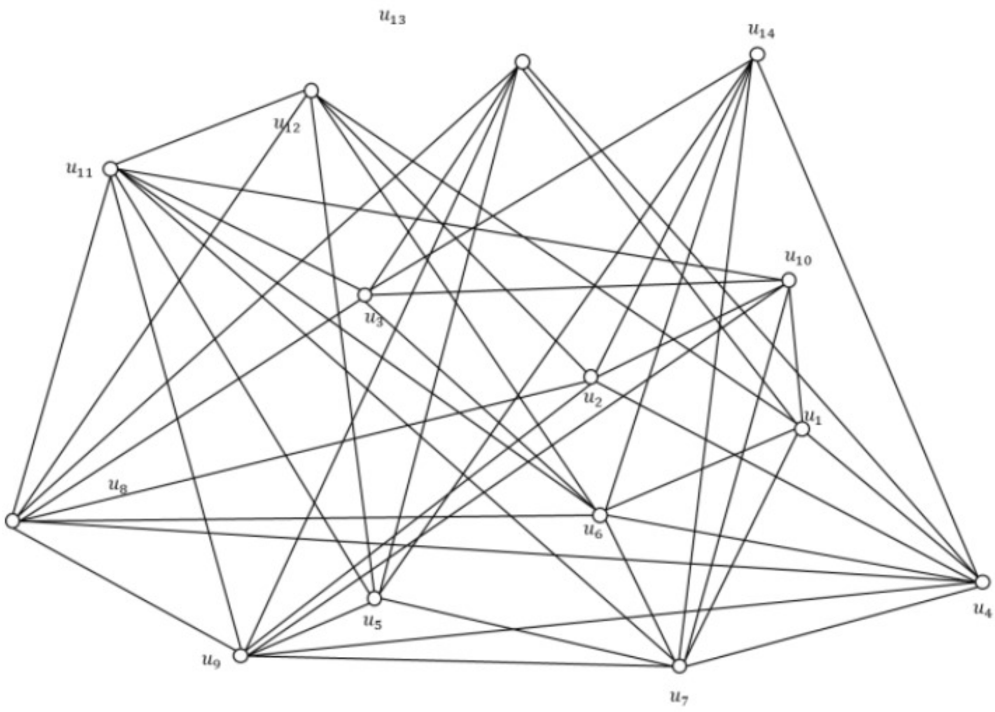

Figure 8: Prime Ideal Sum Graph of $\mathbb{Z}_{pqrs}$

The following result deals with the evaluation of Randic indices of different prime ideal sum graphs of $\mathbb{Z}_n$.

Theorem 2.2. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = p^\alpha$, then the Randic index of $PIS(\mathbb{Z}_n)$ is $\sqrt{\alpha - 2}$.*
(*ii*) *If $n = pq$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 0.*
(*iii*) *If $n = p^2q$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 1.893.*
(*iv*) *If $n = p^2q^2$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 3.809.*
(*v*) *If $n = pqr$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 2.871.*
(*vi*) *If $n = p^3q$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 2.724.*
(*vii*) *If $n = p^2qr$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 4.485.*
(*viii*) *If $n = pqrs$, then the Randic index of $PIS(\mathbb{Z}_n)$ is 6.964.*

Proof. (*i*) Using Fig.1, we have

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_2}}} + \frac{1}{\sqrt{d_{u_1}d_{u_3}}} + \cdots + \frac{1}{\sqrt{d_{u_1}d_{u_{\alpha-1}}}} \\ &= \sqrt{\alpha - 2}. \end{aligned}$$

(*ii*) Using Fig.2, we get

$$R(PIS(\mathbb{Z}_n)) = 0.$$

(*iii*) Using Fig.3, we obtain

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_3}}} + \frac{1}{\sqrt{d_{u_1}d_{u_4}}} + \frac{1}{\sqrt{d_{u_2}d_{u_3}}} + \frac{1}{\sqrt{d_{u_3}d_{u_4}}} \\ &= 1.893. \end{aligned}$$

(iv) Using Fig.4, we find that

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_3}}} + \frac{1}{\sqrt{d_{u_1}d_{u_4}}} + \frac{1}{\sqrt{d_{u_1}d_{u_5}}} + \frac{1}{\sqrt{d_{u_1}d_{u_6}}} + \frac{1}{\sqrt{d_{u_2}d_{u_3}}} + \frac{1}{\sqrt{d_{u_2}d_{u_4}}} + \frac{1}{\sqrt{d_{u_2}d_{u_5}}} \\ &+ \frac{1}{\sqrt{d_{u_2}d_{u_7}}} + \frac{1}{\sqrt{d_{u_3}d_{u_6}}} + \frac{1}{\sqrt{d_{u_3}d_{u_7}}} + \frac{1}{\sqrt{d_{u_4}d_{u_7}}} + \frac{1}{\sqrt{d_{u_5}d_{u_6}}} \\ &= 3.809. \end{aligned}$$

(v) Using Fig.5, we have

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_4}}} + \frac{1}{\sqrt{d_{u_1}d_{u_5}}} + \frac{1}{\sqrt{d_{u_2}d_{u_4}}} + \frac{1}{\sqrt{d_{u_2}d_{u_6}}} + \frac{1}{\sqrt{d_{u_3}d_{u_5}}} + \frac{1}{\sqrt{d_{u_3}d_{u_6}}} + \frac{1}{\sqrt{d_{u_4}d_{u_5}}} \\ &+ \frac{1}{\sqrt{d_{u_4}d_{u_6}}} + \frac{1}{\sqrt{d_{u_5}d_{u_6}}} \\ &= 2.871. \end{aligned}$$

(vi) Using Fig.6, we find that

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_3}}} + \frac{1}{\sqrt{d_{u_1}d_{u_4}}} + \frac{1}{\sqrt{d_{u_1}d_{u_5}}} + \frac{1}{\sqrt{d_{u_1}d_{u_6}}} + \frac{1}{\sqrt{d_{u_2}d_{u_4}}} + \frac{1}{\sqrt{d_{u_2}d_{u_6}}} + \frac{1}{\sqrt{d_{u_3}d_{u_5}}} \\ &+ \frac{1}{\sqrt{d_{u_4}d_{u_5}}} \\ &= 2.724. \end{aligned}$$

(vii) Using Fig.7, we get

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_4}}} + \frac{1}{\sqrt{d_{u_1}d_{u_5}}} + \frac{1}{\sqrt{d_{u_1}d_{u_6}}} + \frac{1}{\sqrt{d_{u_1}d_{u_7}}} + \frac{1}{\sqrt{d_{u_1}d_{u_8}}} + \frac{1}{\sqrt{d_{u_1}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_2}d_{u_5}}} \\ &+ \frac{1}{\sqrt{d_{u_2}d_{u_7}}} + \frac{1}{\sqrt{d_{u_2}d_{u_9}}} + \frac{1}{\sqrt{d_{u_2}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_3}d_{u_6}}} + \frac{1}{\sqrt{d_{u_3}d_{u_8}}} + \frac{1}{\sqrt{d_{u_3}d_{u_9}}} + \frac{1}{\sqrt{d_{u_3}d_{u_{10}}}} \\ &+ \frac{1}{\sqrt{d_{u_4}d_{u_5}}} + \frac{1}{\sqrt{d_{u_4}d_{u_6}}} + \frac{1}{\sqrt{d_{u_4}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_5}d_{u_6}}} + \frac{1}{\sqrt{d_{u_5}d_{u_8}}} + \frac{1}{\sqrt{d_{u_5}d_{u_9}}} + \frac{1}{\sqrt{d_{u_6}d_{u_7}}} \\ &+ \frac{1}{\sqrt{d_{u_6}d_{u_9}}} + \frac{1}{\sqrt{d_{u_7}d_{u_9}}} + \frac{1}{\sqrt{d_{u_8}d_{u_9}}} \\ &= 4.485. \end{aligned}$$

$(viii)$ Using Fig.8, we obtain

$$\begin{aligned} R(PIS(\mathbb{Z}_n)) &= \frac{1}{\sqrt{d_{u_1}d_{u_4}}} + \frac{1}{\sqrt{d_{u_1}d_{u_6}}} + \frac{1}{\sqrt{d_{u_1}d_{u_7}}} + \frac{1}{\sqrt{d_{u_1}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_1}d_{u_{12}}}} + \frac{1}{\sqrt{d_{u_1}d_{u_{13}}}} + \frac{1}{\sqrt{d_{u_2}d_{u_4}}} \\ &+ \frac{1}{\sqrt{d_{u_2}d_{u_8}}} + \frac{1}{\sqrt{d_{u_2}d_{u_9}}} + \frac{1}{\sqrt{d_{u_2}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_2}d_{u_{12}}}} + \frac{1}{\sqrt{d_{u_2}d_{u_{14}}}} + \frac{1}{\sqrt{d_{u_3}d_{u_6}}} + \frac{1}{\sqrt{d_{u_3}d_{u_8}}} \\ &+ \frac{1}{\sqrt{d_{u_3}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_3}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_3}d_{u_{13}}}} + \frac{1}{\sqrt{d_{u_3}d_{u_{14}}}} + \frac{1}{\sqrt{d_{u_4}d_{u_6}}} + \frac{1}{\sqrt{d_{u_4}d_{u_7}}} + \frac{1}{\sqrt{d_{u_4}d_{u_8}}} \\ &+ \frac{1}{\sqrt{d_{u_4}d_{u_9}}} + \frac{1}{\sqrt{d_{u_4}d_{u_{13}}}} + \frac{1}{\sqrt{d_{u_4}d_{u_{14}}}} + \frac{1}{\sqrt{d_{u_5}d_{u_7}}} + \frac{1}{\sqrt{d_{u_5}d_{u_9}}} + \frac{1}{\sqrt{d_{u_5}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_5}d_{u_{12}}}} \\ &+ \frac{1}{\sqrt{d_{u_5}d_{u_{13}}}} + \frac{1}{\sqrt{d_{u_5}d_{u_{14}}}} + \frac{1}{\sqrt{d_{u_6}d_{u_7}}} + \frac{1}{\sqrt{d_{u_6}d_{u_8}}} + \frac{1}{\sqrt{d_{u_6}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_6}d_{u_{12}}}} + \frac{1}{\sqrt{d_{u_6}d_{u_{14}}}} \\ &+ \frac{1}{\sqrt{d_{u_7}d_{u_9}}} + \frac{1}{\sqrt{d_{u_7}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_7}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_7}d_{u_{14}}}} + \frac{1}{\sqrt{d_{u_8}d_{u_9}}} + \frac{1}{\sqrt{d_{u_8}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_8}d_{u_{12}}}} \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\sqrt{d_{u_8}d_{u_{13}}}} + \frac{1}{\sqrt{d_{u_9}d_{u_{10}}}} + \frac{1}{\sqrt{d_{u_9}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_9}d_{u_{13}}}} + \frac{1}{\sqrt{d_{u_{10}}d_{u_{11}}}} + \frac{1}{\sqrt{d_{u_{11}}d_{u_{12}}}} \\
&= \quad 6.964.
\end{aligned}
$$

□

Following theorem, is confined to the evaluation of Zagreb indices of different prime ideal sum graphs of $\mathbb{Z}_n$.

Theorem 2.3. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(i) *If $n = p^\alpha$, then both the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are $\alpha^2 - 3\alpha + 2$ and $\alpha^2 - 3\alpha + 2$ respectively.*
(ii) *If $n = pq$, then both the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 0.
(iii) *If $n = p^2q$, then the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 18 *and* 19 *respectively.*
(iv) *If $n = p^2q^2$, then the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 84 *and* 146 *respectively.*
(v) *If $n = pqr$, then the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 60 *and* 96 *respectively.*
(vi) *If $n = p^3q$, then the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 48 *and* 68 *respectively.*
(vii) *If $n = p^2qr$, then the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 240 *and* 588 *respectively.*
($viii$) *If $n = pqrs$, then the First and Second Zagreb indices of $PIS(\mathbb{Z}_n)$ are* 672 *and* 2352 *respectively.*

Proof. (i) With the help of Fig.1, we get

$$
\begin{aligned}
M_1(PIS(\mathbb{Z}_n)) &= d_{u_1}^2 + d_{u_2}^2 + \cdots + d_{u_{\alpha-1}}^2 \\
&= 1^2 + 1^2 + \cdots + (\alpha-2)^2 \\
&= \alpha^2 - 3\alpha + 2,
\end{aligned}
$$

$$
\begin{aligned}
M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_2} + d_{u_1}d_{u_3} + \cdots + d_{u_1}d_{u_{\alpha-1}} \\
&= (\alpha-1)\cdot 1 + (\alpha-1)\cdot 1 + \cdots + (\alpha-1)\cdot 1 \\
&= \alpha^2 - 3\alpha + 2.
\end{aligned}
$$

(ii) With the help of Fig.2, we have

$$M_1(PIS(\mathbb{Z}_n)) = 0,$$

$$M_2(PIS(\mathbb{Z}_n)) = 0.$$

(iii) With the help of Fig.3, we obtain

$$
\begin{aligned}
M_1(PIS(\mathbb{Z}_n)) &= \sum_{i=1}^{4} d_{u_i}^2 \\
&= 18,
\end{aligned}
$$

$$
\begin{aligned}
M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_3} + d_{u_1}d_{u_4} + d_{u_2}d_{u_3} + d_{u_3}d_{u_4} \\
&= 19.
\end{aligned}
$$

(iv) With the help of Fig.4, we get

$$
\begin{aligned}
M_1(PIS(\mathbb{Z}_n)) &= \sum_{i=1}^{7} d_{u_i}^2 \\
&= 3\cdot 4^2 + 4\cdot 3^2 \\
&= 84,
\end{aligned}
$$

$$
\begin{aligned}
M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_3} + d_{u_1}d_{u_4} + d_{u_1}d_{u_5} + d_{u_1}d_{u_6} + d_{u_2}d_{u_3} + d_{u_2}d_{u_4} + d_{u_2}d_{u_5} + d_{u_2}d_{u_7} + d_{u_3}d_{u_6} \\
&+ d_{u_3}d_{u_7} + d_{u_4}d_{u_7} + d_{u_5}d_{u_6} \\
&= 146.
\end{aligned}
$$

(v) With the help of Fig.5, we arrived at

$$\begin{aligned} M_1(PIS(\mathbb{Z}_n)) &= \sum_{i=1}^{6} d_{u_i}^2 \\ &= 3 \cdot 2^2 + 3 \cdot 4^2 \\ &= 60, \end{aligned}$$

$$\begin{aligned} M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_4} + d_{u_1}d_{u_5} + d_{u_2}d_{u_4} + d_{u_2}d_{u_6} + d_{u_3}d_{u_5} + d_{u_3}d_{u_6} + d_{u_4}d_{u_5} \\ &+ d_{u_4}d_{u_6} + d_{u_5}d_{u_6} \\ &= 96. \end{aligned}$$

(vi) With the help of Fig.6, we get

$$\begin{aligned} M_1(PIS(\mathbb{Z}_n)) &= \sum_{i=1}^{6} d_{u_i}^2 \\ &= 2 \cdot 4^2 + 4 \cdot 2^2 \\ &= 48, \end{aligned}$$

$$\begin{aligned} M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_3} + d_{u_1}d_{u_4} + d_{u_1}d_{u_5} + d_{u_1}d_{u_6} + d_{u_2}d_{u_4} + d_{u_2}d_{u_6} + d_{u_3}d_{u_4} + d_{u_4}d_{u_5} \\ &= 68. \end{aligned}$$

(vii) With the help of Fig.7, we have

$$\begin{aligned} M_1(PIS(\mathbb{Z}_n)) &= \sum_{i=1}^{10} d_{u_i}^2 \\ &= 4 \cdot 6^2 + 6 \cdot 4^2 \\ &= 240, \end{aligned}$$

$$\begin{aligned} M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_4} + d_{u_1}d_{u_5} + d_{u_1}d_{u_6} + d_{u_1}d_{u_7} + d_{u_1}d_{u_8} + d_{u_1}d_{u_{10}} + d_{u_2}d_{u_5} + d_{u_2}d_{u_7} + d_{u_2}d_{u_9} \\ &+ d_{u_2}d_{u_{10}} + d_{u_3}d_{u_6} + d_{u_3}d_{u_8} + d_{u_3}d_{u_9} + d_{u_3}d_{u_{10}} + d_{u_4}d_{u_5} + d_{u_4}d_{u_6} + d_{u_4}d_{u_{10}} + d_{u_5}d_{u_6} \\ &+ d_{u_5}d_{u_8} + d_{u_5}d_{u_9} + d_{u_6}d_{u_7} + d_{u_6}d_{u_9} + d_{u_7}d_{u_9} + d_{u_8}d_{u_9} \\ &= 588. \end{aligned}$$

($viii$) With the help of Fig.8, we obtain

$$\begin{aligned} M_1(PIS(\mathbb{Z}_n)) &= \sum_{i=1}^{14} d_{u_i}^2 \\ &= 6 \cdot 8^2 + 8 \cdot 6^2 \\ &= 672, \end{aligned}$$

$$\begin{aligned} M_2(PIS(\mathbb{Z}_n)) &= d_{u_1}d_{u_4} + d_{u_1}d_{u_6} + d_{u_1}d_{u_7} + d_{u_1}d_{u_{10}} + d_{u_1}d_{u_{12}} + d_{u_1}d_{u_{13}} + d_{u_2}d_{u_4} + d_{u_2}d_{u_8} + d_{u_2}d_{u_9} \\ &+ d_{u_2}d_{u_{10}} + d_{u_2}d_{u_{12}} + d_{u_2}d_{u_{14}} + d_{u_3}d_{u_6} + d_{u_3}d_{u_8} + d_{u_3}d_{u_{10}} + d_{u_3}d_{u_{11}} + d_{u_3}d_{u_{13}} + d_{u_3}d_{u_{14}} \\ &+ d_{u_4}d_{u_6} + d_{u_4}d_{u_7} + d_{u_4}d_{u_8} + d_{u_4}d_{u_9} + d_{u_4}d_{u_{13}} + d_{u_4}d_{u_{14}} + d_{u_5}d_{u_7} + d_{u_5}d_{u_9} + d_{u_5}d_{u_{11}} \\ &+ d_{u_5}d_{u_{12}} + d_{u_5}d_{u_{13}} + +d_{u_5}d_{u_{14}} + d_{u_6}d_{u_7} + d_{u_6}d_{u_8} + d_{u_6}d_{u_{11}} + d_{u_6}d_{u_{12}} + d_{u_6}d_{u_{14}} + d_{u_7}d_{u_9} \\ &+ d_{u_7}d_{u_{10}} + d_{u_7}d_{u_{11}} + d_{u_7}d_{u_{14}} + d_{u_8}d_{u_9} + d_{u_8}d_{u_{11}} + d_{u_8}d_{u_{12}} + d_{u_8}d_{u_{13}} + d_{u_9}d_{u_{10}} + d_{u_9}d_{u_{11}} \\ &+ d_{u_9}d_{u_{13}} + d_{u_{10}}d_{u_{11}} + d_{u_{11}}d_{u_{12}} \\ &= 2352. \end{aligned}$$

□

Now we determine the Harmonic indices of different prime ideal sum graph of $\mathbb{Z}_n$.The findings have been confined in Theorem 2.4.

Theorem 2.4. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = p^\alpha$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* $\dfrac{2(\alpha-2)}{(\alpha-1)}$.
(*ii*) *If $n = pq$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 0.
(*iii*) *If $n = p^2q$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 1.8.
(*iv*) *If $n = p^2q^2$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 3.451.
(*v*) *If $n = pqr$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 2.75.
(*vi*) *If $n = p^3q$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 2.75.
(*vii*) *If $n = p^2qr$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 4.933.
(*viii*) *If $n = pqrs$, then the Harmonic index of $PIS(\mathbb{Z}_n)$ is* 6.928.

Proof. (*i*) From Fig.1, we get

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_2}} + \frac{2}{d_{u_1}+d_{u_3}} + \cdots + \frac{2}{d_{u_1}+d_{u_{\alpha-1}}} \\ &= \frac{2(\alpha-2)}{(\alpha-1)}. \end{aligned}$$

(*ii*) From Fig.2, we have

$$HI(PIS(\mathbb{Z}_n)) = 0.$$

(*iii*) From Fig.3, we obtain

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_3}} + \frac{2}{d_{u_1}+d_{u_4}} + \frac{2}{d_{u_2}+d_{u_3}} + \frac{2}{d_{u_3}+d_{u_4}} \\ &= 1.8. \end{aligned}$$

(*iv*) From Fig.4, we get

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_3}} + \frac{2}{d_{u_1}+d_{u_4}} + \frac{2}{d_{u_1}+d_{u_5}} + \frac{2}{d_{u_1}+d_{u_6}} + \frac{2}{d_{u_2}+d_{u_3}} + \frac{2}{d_{u_2}+d_{u_4}} + \frac{2}{d_{u_2}+d_{u_5}} \\ &+ \frac{2}{d_{u_2}+d_{u_7}} + \frac{2}{d_{u_3}+d_{u_6}} + \frac{2}{d_{u_3}+d_{u_7}} + \frac{2}{d_{u_4}+d_{u_7}} + \frac{2}{d_{u_5}+d_{u_6}} \\ &= 3.451. \end{aligned}$$

(*v*) From Fig.5, we conclude that

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_4}} + \frac{2}{d_{u_1}+d_{u_5}} + \frac{2}{d_{u_2}+d_{u_4}} + \frac{2}{d_{u_2}+d_{u_6}} + \frac{2}{d_{u_3}+d_{u_5}} + \frac{2}{d_{u_3}+d_{u_6}} + \frac{2}{d_{u_4}+d_{u_5}} \\ &+ \frac{2}{d_{u_4}+d_{u_6}} + \frac{2}{d_{u_5}+d_{u_6}} \\ &= 2.75. \end{aligned}$$

(*vi*) From Fig.6, we observe that

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_3}} + \frac{2}{d_{u_1}+d_{u_4}} + \frac{2}{d_{u_1}+d_{u_5}} + \frac{2}{d_{u_1}+d_{u_6}} + \frac{2}{d_{u_2}+d_{u_4}} + \frac{2}{d_{u_2}+d_{u_6}} + \frac{2}{d_{u_3}+d_{u_4}} \\ &+ \frac{2}{d_{u_4}+d_{u_5}} \\ &= 2.75. \end{aligned}$$

(*vii*) From Fig.7, it is obvious that

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_4}}+\frac{2}{d_{u_1}+d_{u_5}}+\frac{2}{d_{u_1}+d_{u_6}}+\frac{2}{d_{u_1}+d_{u_7}}+\frac{2}{d_{u_1}+d_{u_8}}+\frac{2}{d_{u_1}+d_{u_{10}}}+\frac{2}{d_{u_2}+d_{u_5}} \\ &+ \frac{2}{d_{u_2}+d_{u_7}}+\frac{2}{d_{u_2}+d_{u_9}}+\frac{2}{d_{u_2}+d_{u_{10}}}+\frac{2}{d_{u_3}+d_{u_6}}+\frac{2}{d_{u_3}+d_{u_8}}+\frac{2}{d_{u_3}+d_{u_9}}+\frac{2}{d_{u_3}+d_{u_{10}}} \\ &+ \frac{2}{d_{u_4}+d_{u_5}}+\frac{2}{d_{u_4}+d_{u_6}}+\frac{2}{d_{u_4}+d_{u_{10}}}+\frac{2}{d_{u_5}+d_{u_6}}+\frac{2}{d_{u_5}+d_{u_8}}+\frac{2}{d_{u_5}+d_{u_9}}+\frac{2}{d_{u_6}+d_{u_7}} \\ &+ \frac{2}{d_{u_6}+d_{u_9}}+\frac{2}{d_{u_7}+d_{u_9}}+\frac{2}{d_{u_8}+d_{u_9}} \\ &= 4.933. \end{aligned}$$

(*viii*) From Fig.8, we get

$$\begin{aligned} HI(PIS(\mathbb{Z}_n)) &= \frac{2}{d_{u_1}+d_{u_4}}+\frac{2}{d_{u_1}+d_{u_6}}+\frac{2}{d_{u_1}+d_{u_7}}+\frac{2}{d_{u_1}+d_{u_{10}}}+\frac{2}{d_{u_1}+d_{u_{12}}}+\frac{2}{d_{u_1}+d_{u_{13}}}+\frac{2}{d_{u_2}+d_{u_4}} \\ &+ \frac{2}{d_{u_2}+d_{u_8}}+\frac{2}{d_{u_2}+d_{u_9}}+\frac{2}{d_{u_2}+d_{u_{10}}}+\frac{2}{d_{u_2}+d_{u_{12}}}+\frac{2}{d_{u_2}+d_{u_{14}}}+\frac{2}{d_{u_3}+d_{u_6}}+\frac{2}{d_{u_3}+d_{u_8}} \\ &+ \frac{2}{d_{u_3}+d_{u_{10}}}+\frac{2}{d_{u_3}+d_{u_{11}}}+\frac{2}{d_{u_3}+d_{u_{13}}}+\frac{2}{d_{u_3}+d_{u_{14}}}+\frac{2}{d_{u_4}+d_{u_6}}+\frac{2}{d_{u_4}+d_{u_7}}+\frac{2}{d_{u_4}+d_{u_8}} \\ &+ \frac{2}{d_{u_4}+d_{u_9}}+\frac{2}{d_{u_4}+d_{u_{13}}}+\frac{2}{d_{u_4}+d_{u_{14}}}+\frac{2}{d_{u_5}+d_{u_7}}+\frac{2}{d_{u_5}+d_{u_9}}+\frac{2}{d_{u_5}+d_{u_{11}}}+\frac{2}{d_{u_5}+d_{u_{12}}} \\ &+ \frac{2}{d_{u_5}+d_{u_{13}}}+\frac{2}{d_{u_5}+d_{u_{14}}}+\frac{2}{d_{u_6}+d_{u_7}}+\frac{2}{d_{u_6}+d_{u_8}}+\frac{2}{d_{u_6}+d_{u_{11}}}+\frac{2}{d_{u_6}+d_{u_{12}}}+\frac{2}{d_{u_6}+d_{u_{14}}} \\ &+ \frac{2}{d_{u_7}+d_{u_9}}+\frac{2}{d_{u_7}+d_{u_{10}}}+\frac{2}{d_{u_7}+d_{u_{11}}}+\frac{2}{d_{u_7}+d_{u_{14}}}+\frac{2}{d_{u_8}+d_{u_9}}+\frac{2}{d_{u_8}+d_{u_{11}}}+\frac{2}{d_{u_8}+d_{u_{12}}} \\ &+ \frac{2}{d_{u_8}+d_{u_{13}}}+\frac{2}{d_{u_9}+d_{u_{10}}}+\frac{2}{d_{u_9}+d_{u_{11}}}+\frac{2}{d_{u_9}+d_{u_{13}}}+\frac{2}{d_{u_{10}}+d_{u_{11}}}+\frac{2}{d_{u_{11}}+d_{u_{12}}} \\ &= 6.928. \end{aligned}$$

□

The next theorem deals with the estimation of ABC indices of different prime ideal sum graphs of $\mathbb{Z}_n$.

Theorem 2.5. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = p^\alpha$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* $\dfrac{(\alpha-2)(\sqrt{\alpha-2})}{\sqrt{\alpha-1}}$.
(*ii*) *If $n = pq$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 0.
(*iii*) *If $n = p^2q$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 2.937.
(*iv*) *If $n = p^2q^2$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 7.72.
(*v*) *If $n = pqr$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 4.855.
(*vi*) *If $n = p^3q$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 5.562.
(*vii*) *If $n = p^2qr$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 13.779.
(*viii*) *If $n = pqrs$, then the ABC index of $PIS(\mathbb{Z}_n)$ is* 17.935.

Proof. (*i*) Observing Fig.1, we obtain

$$\begin{aligned} ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_2}-2}{d_{u_1}d_{u_2}}}+\sqrt{\frac{d_{u_1}+d_{u_3}-2}{d_{u_1}d_{u_3}}}+\cdots+\sqrt{\frac{d_{u_1}+d_{u_{\alpha-1}}-2}{d_{u_1}d_{u_{\alpha-1}}}} \\ &= \frac{(\alpha-2)(\sqrt{\alpha-2})}{\sqrt{\alpha-1}}. \end{aligned}$$

(*ii*) Observing Fig.2, we get

$$ABC(PIS(\mathbb{Z}_n)) = 0.$$

(*iii*) Fig.3, gives us

$$\begin{aligned} ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_3}-2}{d_{u_1}d_{u_3}}}+\sqrt{\frac{d_{u_1}+d_{u_4}-2}{d_{u_1}d_{u_4}}}+\sqrt{\frac{d_{u_2}+d_{u_3}-2}{d_{u_2}d_{u_3}}}+\sqrt{\frac{d_{u_3}+d_{u_4}-2}{d_{u_3}d_{u_4}}} \\ &= 2.937. \end{aligned}$$

(*iv*) From Fig.4, it is obvious that

$$\begin{aligned} ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_3}-2}{d_{u_1}d_{u_3}}}+\sqrt{\frac{d_{u_1}+d_{u_4}-2}{d_{u_1}d_{u_4}}}+\sqrt{\frac{d_{u_1}+d_{u_5}-2}{d_{u_1}d_{u_5}}}+\sqrt{\frac{d_{u_1}+d_{u_6}-2}{d_{u_1}d_{u_6}}} \\ &+ \sqrt{\frac{d_{u_2}+d_{u_3}-2}{d_{u_2}d_{u_3}}}+\sqrt{\frac{d_{u_2}+d_{u_4}-2}{d_{u_2}d_{u_4}}}+\sqrt{\frac{d_{u_2}+d_{u_5}-2}{d_{u_2}d_{u_5}}}+\sqrt{\frac{d_{u_2}+d_{u_7}-2}{d_{u_2}d_{u_7}}} \\ &+ \sqrt{\frac{d_{u_3}+d_{u_6}-2}{d_{u_3}d_{u_6}}}+\sqrt{\frac{d_{u_3}+d_{u_7}-2}{d_{u_3}d_{u_7}}}+\sqrt{\frac{d_{u_4}+d_{u_7}-2}{d_{u_4}d_{u_7}}}+\sqrt{\frac{d_{u_5}+d_{u_6}-2}{d_{u_5}d_{u_6}}} \\ &= 7.72. \end{aligned}$$

(*v*) Observing Fig.5, we obtain

$$\begin{aligned} ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_4}-2}{d_{u_1}d_{u_4}}}+\sqrt{\frac{d_{u_1}+d_{u_5}-2}{d_{u_1}d_{u_5}}}+\sqrt{\frac{d_{u_2}+d_{u_4}-2}{d_{u_2}d_{u_4}}}+\sqrt{\frac{d_{u_2}+d_{u_6}-2}{d_{u_2}d_{u_6}}} \\ &+ \sqrt{\frac{d_{u_3}+d_{u_5}-2}{d_{u_3}d_{u_5}}}+\sqrt{\frac{d_{u_3}+d_{u_6}-2}{d_{u_3}d_{u_6}}}+\sqrt{\frac{d_{u_4}+d_{u_5}-2}{d_{u_4}d_{u_5}}}+\sqrt{\frac{d_{u_4}+d_{u_6}-2}{d_{u_4}d_{u_6}}} \\ &+ \sqrt{\frac{d_{u_5}+d_{u_6}-2}{d_{u_5}d_{u_6}}} \\ &= 4.855. \end{aligned}$$

(*vi*) From Fig.6, we conclude that

$$\begin{aligned} ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_3}-2}{d_{u_1}d_{u_3}}}+\sqrt{\frac{d_{u_1}+d_{u_4}-2}{d_{u_1}d_{u_4}}}+\sqrt{\frac{d_{u_1}+d_{u_5}-2}{d_{u_1}d_{u_5}}}+\sqrt{\frac{d_{u_1}+d_{u_6}-2}{d_{u_1}d_{u_6}}} \\ &+ \sqrt{\frac{d_{u_2}+d_{u_4}-2}{d_{u_2}d_{u_4}}}+\sqrt{\frac{d_{u_2}+d_{u_6}-2}{d_{u_2}d_{u_6}}}+\sqrt{\frac{d_{u_3}+d_{u_4}-2}{d_{u_3}d_{u_4}}}+\sqrt{\frac{d_{u_4}+d_{u_5}-2}{d_{u_4}d_{u_5}}} \\ &= 5.562. \end{aligned}$$

(*vii*) Observing Fig.7, we arrive at

$$\begin{aligned} ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_4}-2}{d_{u_1}d_{u_4}}}+\sqrt{\frac{d_{u_1}+d_{u_5}-2}{d_{u_1}d_{u_5}}}+\sqrt{\frac{d_{u_1}+d_{u_6}-2}{d_{u_1}d_{u_6}}}+\sqrt{\frac{d_{u_1}+d_{u_7}-2}{d_{u_1}d_{u_7}}} \\ &+ \sqrt{\frac{d_{u_1}+d_{u_8}-2}{d_{u_1}d_{u_8}}}+\sqrt{\frac{d_{u_1}+d_{u_{10}}-2}{d_{u_1}d_{u_{10}}}}+\sqrt{\frac{d_{u_2}+d_{u_5}-2}{d_{u_2}d_{u_5}}}+\sqrt{\frac{d_{u_2}+d_{u_7}-2}{d_{u_2}d_{u_7}}} \\ &+ \sqrt{\frac{d_{u_2}+d_{u_9}-2}{d_{u_2}d_{u_9}}}+\sqrt{\frac{d_{u_2}+d_{u_{10}}-2}{d_{u_2}d_{u_{10}}}}+\sqrt{\frac{d_{u_3}+d_{u_6}-2}{d_{u_3}d_{u_6}}}+\sqrt{\frac{d_{u_3}+d_{u_8}-2}{d_{u_3}d_{u_8}}} \end{aligned}$$

$$
\begin{aligned}
&+ \sqrt{\frac{d_{u_3}+d_{u_9}-2}{d_{u_3}d_{u_9}}} + \sqrt{\frac{d_{u_3}+d_{u_{10}}-2}{d_{u_3}d_{u_{10}}}}\\
&= 13.779.
\end{aligned}
$$

$(viii)$ Fig.8, gives us

$$
\begin{aligned}
ABC(PIS(\mathbb{Z}_n)) &= \sqrt{\frac{d_{u_1}+d_{u_4}-2}{d_{u_1}d_{u_4}}} + \sqrt{\frac{d_{u_1}+d_{u_6}-2}{d_{u_1}d_{u_6}}} + \sqrt{\frac{d_{u_1}+d_{u_7}-2}{d_{u_1}d_{u_7}}} + \sqrt{\frac{d_{u_1}+d_{u_{10}}-2}{d_{u_1}d_{u_{10}}}}\\
&+ \sqrt{\frac{d_{u_1}+d_{u_{12}}-2}{d_{u_1}d_{u_{12}}}} + \sqrt{\frac{d_{u_1}+d_{u_{13}}-2}{d_{u_1}d_{u_{13}}}} + \sqrt{\frac{d_{u_2}+d_{u_4}-2}{d_{u_2}d_{u_4}}} + \sqrt{\frac{d_{u_2}+d_{u_8}-2}{d_{u_2}d_{u_8}}}\\
&+ \sqrt{\frac{d_{u_2}+d_{u_9}-2}{d_{u_2}d_{u_9}}} + \sqrt{\frac{d_{u_2}+d_{u_{10}}-2}{d_{u_2}d_{u_{10}}}} + \sqrt{\frac{d_{u_2}+d_{u_{12}}-2}{d_{u_2}d_{u_{12}}}} + \sqrt{\frac{d_{u_2}+d_{u_{14}}-2}{d_{u_2}d_{u_{14}}}}\\
&+ \sqrt{\frac{d_{u_3}+d_{u_6}-2}{d_{u_3}d_{u_6}}} + \sqrt{\frac{d_{u_3}+d_{u_8}-2}{d_{u_3}d_{u_8}}} + \sqrt{\frac{d_{u_3}+d_{u_{10}}-2}{d_{u_3}d_{u_{10}}}} + \sqrt{\frac{d_{u_3}+d_{u_{11}}-2}{d_{u_3}d_{u_{11}}}}\\
&+ \sqrt{\frac{d_{u_3}+d_{u_{13}}-2}{d_{u_3}d_{u_{13}}}} + \sqrt{\frac{d_{u_3}+d_{u_{14}}-2}{d_{u_3}d_{u_{14}}}} + \sqrt{\frac{d_{u_4}+d_{u_6}-2}{d_{u_4}d_{u_6}}} + \sqrt{\frac{d_{u_4}+d_{u_7}-2}{d_{u_4}d_{u_7}}}\\
&+ \sqrt{\frac{d_{u_4}+d_{u_8}-2}{d_{u_4}d_{u_8}}} + \sqrt{\frac{d_{u_4}+d_{u_9}-2}{d_{u_4}d_{u_9}}} + \sqrt{\frac{d_{u_4}+d_{u_{13}}-2}{d_{u_4}d_{u_{13}}}} + \sqrt{\frac{d_{u_4}+d_{u_{14}}-2}{d_{u_4}d_{u_{14}}}}\\
&+ \sqrt{\frac{d_{u_5}+d_{u_7}-2}{d_{u_5}d_{u_7}}} + \sqrt{\frac{d_{u_5}+d_{u_9}-2}{d_{u_5}d_{u_9}}} + \sqrt{\frac{d_{u_5}+d_{u_{11}}-2}{d_{u_5}d_{u_{11}}}} + \sqrt{\frac{d_{u_5}+d_{u_{12}}-2}{d_{u_5}d_{u_{12}}}}\\
&+ \sqrt{\frac{d_{u_5}+d_{u_{13}}-2}{d_{u_5}d_{u_{13}}}} + \sqrt{\frac{d_{u_5}+d_{u_{14}}-2}{d_{u_5}d_{u_{14}}}} + \sqrt{\frac{d_{u_6}+d_{u_7}-2}{d_{u_6}d_{u_7}}} + \sqrt{\frac{d_{u_6}+d_{u_8}-2}{d_{u_6}d_{u_8}}}\\
&+ \sqrt{\frac{d_{u_6}+d_{u_{11}}-2}{d_{u_6}d_{u_{11}}}} + \sqrt{\frac{d_{u_6}+d_{u_{12}}-2}{d_{u_6}d_{u_{12}}}} + \sqrt{\frac{d_{u_6}+d_{u_{14}}-2}{d_{u_6}d_{u_{14}}}} + \sqrt{\frac{d_{u_7}+d_{u_9}-2}{d_{u_7}d_{u_9}}}\\
&= 17.935.
\end{aligned}
$$

□

Now we determine the Zagreb coindices of different prime ideal sum graphs of $\mathbb{Z}_n$. Before, stating our main result, it is necessary to mention following facts, which are proved in [20].

Fact 1. $\overline{M_1}(G) = 2m(n-1) - M_1(G)$.

Fact 2. $\overline{M_2}(G) = 2m^2 - \dfrac{M_1(G)}{2} - M_2(G)$.

Theorem 2.6. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(i) *If $n = p^\alpha$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are $\alpha^2-3\alpha+2$ and $\dfrac{(\alpha-2)(\alpha-5)}{2}$ respectively.*

(ii) *If $n = pq$, then both the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 0.

(iii) *If $n = p^2q$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 6 *and* 4 *respectively.*

(iv) *If $n = p^2q^2$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 60 *and* 100 *respectively.*

(v) *If $n = pqr$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 30 *and* 36 *respectively.*

(vi) *If $n = p^3q$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 32 *and* 36 *respectively.*

(vii) *If $n = p^2qr$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 192 *and* 444 *respectively.*

$(viii)$ *If $n = pqrs$, then the First and Second Zagreb coindices of $PIS(\mathbb{Z}_n)$ are* 576 *and* 2352 *respectively.*

Proof. Follows from Fact 1, Fact 2 and Theorem 2.3. □

Following theorem is confined to the estimation of Sombor spectra and energies of different prime ideal sum graphs of $\mathbb{Z}_n$.

Theorem 2.7. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = p^\alpha$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{0^{[\alpha-2]}, \sqrt{(\alpha-1)(\alpha^2-2\alpha+2)}, -\sqrt{(\alpha-1)(\alpha^2-2\alpha+2)}\}$ and the Sombor Energy is $2\sqrt{(\alpha-1)(\alpha^2-2\alpha+2)}$.*

(*ii*) *If $n = pq$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{0^{[2]}\}$ and the Sombor Energy is 0.*

(*iii*) *If $n = p^2q$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{-5.178, -2.828, 0.752, 7.253\}$ and the Sombor Energy is 16.011.*

(*iv*) *If $n = p^2q^2$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{-13.2, -6.557, -5.446, 0, 1.07, 6.557, 17.577\}$ and the Sombor Energy is 50.407.*

(*v*) *If $n = pqr$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{-8.118^{[2]}, -4.924, 2.462^{[2]}, 16.236\}$ and the Sombor Energy is 42.32.*

(*vi*) *If $n = p^3q$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{-8.932, -7.291, 0, 0.448, 2.303, 13.473\}$ and the Sombor Energy is 32.447.*

(*vii*) *If $n = p^2qr$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{-17.427, -17.366, -13.844, -11.385, 0.894, 3.224, 4.535, 5.656, 10.193, 35.519\}$ and the Sombor Energy is 120.103.*

(*viii*) *If $n = pqrs$, then the Sombor spectrum of $PIS(\mathbb{Z}_n)$ is $\{-25.455, -24.687^{[3]}, -22.627^{[2]}, -0.627, 8.485^{[3]}, 16.202^{[3]}, 71.383\}$ and the Sombor Energy is 290.841.*

Proof. (*i*) Since the PIS graph of $\mathbb{Z}_n$ for $n = p^\alpha$ is a star graph on α vertices and from [12, Theorem 2.6], the Sombor characteristic polynomial of the star graph $S_n = K_{1,n-1}$ is $c(\lambda) = \lambda^{n-2}(\lambda^2-(n-1)(n^2-2n+2))$, so the characteristic polynomial in our case is $c(\lambda) = \lambda^{\alpha-2}(\lambda^2-(\alpha-1)(\alpha^2-2\alpha+2))$. Thus the eigen values are 0, $(\alpha-2)$ times and the rest two eigen values are $\sqrt{(\alpha-1)(\alpha^2-2\alpha+2)}$ and $-\sqrt{(\alpha-1)(\alpha^2-2\alpha+2)}$. From [12, Theorem 2.6], it follows that $E_{SO}(PIS(\mathbb{Z}_n)) = 2\sqrt{(\alpha-1)(\alpha^2-2\alpha+2)}$.

(*ii*) It is obvious.

(*iii*) The Sombor matrix is

$$SO(G) = \begin{bmatrix} 0 & 0 & \sqrt{d_{u_1}^2 + d_{u_3}^2} & \sqrt{d_{u_1}^2 + d_{u_4}^2} \\ 0 & 0 & \sqrt{d_{u_2}^2 + d_{u_3}^2} & 0 \\ \sqrt{d_{u_3}^2 + d_{u_1}^2} & \sqrt{d_{u_3}^2 + d_{u_2}^2} & 0 & \sqrt{d_{u_3}^2 + d_{u_4}^2} \\ \sqrt{d_{u_4}^2 + d_{u_1}^2} & 0 & \sqrt{d_{u_4}^2 + d_{u_3}^2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \sqrt{13} & \sqrt{8} \\ 0 & 0 & \sqrt{10} & 0 \\ \sqrt{13} & \sqrt{10} & 0 & \sqrt{13} \\ \sqrt{8} & 0 & \sqrt{13} & 0 \end{bmatrix}.$$

The eigen values are $\{-5.178, -2.828, 0.752, 7.253\}$ and $E_{SO}(PIS)(\mathbb{Z}_n) = 16.011$.

(iv) The Sombor matrix is

$$SO(G)=\begin{bmatrix} 0 & 0 & \sqrt{d_{u_1}^2+d_{u_3}^2} & \sqrt{d_{u_1}^2+d_{u_4}^2} & \sqrt{d_{u_1}^2+d_{u_5}^2} & \sqrt{d_{u_1}^2+d_{u_6}^2} & 0 \\ 0 & 0 & \sqrt{d_{u_2}^2+d_{u_3}^2} & \sqrt{d_{u_2}^2+d_{u_4}^2} & \sqrt{d_{u_2}^2+d_{u_5}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_7}^2} \\ \sqrt{d_{u_3}^2+d_{u_1}^2} & \sqrt{d_{u_3}^2+d_{u_2}^2} & 0 & 0 & 0 & \sqrt{d_{u_3}^2+d_{u_6}^2} & \sqrt{d_{u_3}^2+d_{u_7}^2} \\ \sqrt{d_{u_4}^2+d_{u_1}^2} & \sqrt{d_{u_4}^2+d_{u_2}^2} & 0 & 0 & 0 & 0 & \sqrt{d_{u_4}^2+d_{u_7}^2} \\ \sqrt{d_{u_5}^2+d_{u_1}^2} & \sqrt{d_{u_5}^2+d_{u_2}^2} & 0 & 0 & 0 & \sqrt{d_{u_5}^2+d_{u_6}^2} & 0 \\ \sqrt{d_{u_6}^2+d_{u_1}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_3}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_5}^2} & 0 & 0 \\ 0 & \sqrt{d_{u_7}^2+d_{u_2}^2} & \sqrt{d_{u_7}^2+d_{u_3}^2} & \sqrt{d_{u_7}^2+d_{u_4}^2} & 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & \sqrt{32} & 5 & 5 & 5 & 0 \\ 0 & 0 & \sqrt{32} & 5 & 5 & 0 & 5 \\ \sqrt{32} & \sqrt{32} & 0 & 0 & 0 & 5 & 5 \\ 5 & 5 & 0 & 0 & 0 & 0 & \sqrt{18} \\ 5 & 5 & 0 & 0 & 0 & \sqrt{18} & 0 \\ 5 & 0 & 5 & 0 & \sqrt{18} & 0 & 0 \\ 0 & 5 & 5 & \sqrt{18} & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\{-13.2, -6.557, -5.446, 0, 1.07, 6.557, 17.577\}$ and $E_{SO}(PIS)(\mathbb{Z}_n) = 50.407$.

(v) The Sombor matrix is

$$SO(G)=\begin{bmatrix} 0 & 0 & 0 & \sqrt{d_{u_1}^2+d_{u_4}^2} & \sqrt{d_{u_1}^2+d_{u_5}^2} & 0 \\ 0 & 0 & 0 & \sqrt{d_{u_2}^2+d_{u_4}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_6}^2} \\ 0 & 0 & 0 & 0 & \sqrt{d_{u_3}^2+d_{u_5}^2} & \sqrt{d_{u_3}^2+d_{u_6}^2} \\ \sqrt{d_{u_4}^2+d_{u_1}^2} & \sqrt{d_{u_4}^2+d_{u_2}^2} & 0 & 0 & \sqrt{d_{u_4}^2+d_{u_5}^2} & \sqrt{d_{u_4}^2+d_{u_6}^2} \\ \sqrt{d_{u_5}^2+d_{u_1}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_3}^2} & \sqrt{d_{u_5}^2+d_{u_4}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_6}^2} \\ 0 & \sqrt{d_{u_6}^2+d_{u_2}^2} & \sqrt{d_{u_6}^2+d_{u_3}^2} & \sqrt{d_{u_6}^2+d_{u_4}^2} & \sqrt{d_{u_6}^2+d_{u_5}^2} & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 & \sqrt{20} & \sqrt{20} & 0 \\ 0 & 0 & 0 & \sqrt{20} & 0 & \sqrt{20} \\ 0 & 0 & 0 & 0 & \sqrt{20} & \sqrt{20} \\ \sqrt{20} & \sqrt{20} & 0 & 0 & \sqrt{32} & \sqrt{32} \\ \sqrt{20} & 0 & \sqrt{20} & \sqrt{32} & 0 & \sqrt{32} \\ 0 & \sqrt{20} & \sqrt{20} & \sqrt{32} & \sqrt{32} & 0 \end{bmatrix}.$$

The eigen values are $\{-8.118^{[2]}, -4.924, 2.462^{[2]}, 16.236\}$ and $E_{SO}(PIS)(\mathbb{Z}_n) = 42.32$.

(vi) The Sombor matrix is

$$SO(G)=\begin{bmatrix} 0 & 0 & \sqrt{d_{u_1}^2+d_{u_3}^2} & \sqrt{d_{u_1}^2+d_{u_4}^2} & \sqrt{d_{u_1}^2+d_{u_5}^2} & \sqrt{d_{u_1}^2+d_{u_6}^2} \\ 0 & 0 & 0 & \sqrt{d_{u_2}^2+d_{u_4}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_6}^2} \\ \sqrt{d_{u_3}^2+d_{u_1}^2} & 0 & 0 & \sqrt{d_{u_3}^2+d_{u_4}^2} & 0 & 0 \\ \sqrt{d_{u_4}^2+d_{u_1}^2} & \sqrt{d_{u_4}^2+d_{u_2}^2} & \sqrt{d_{u_4}^2+d_{u_3}^2} & 0 & \sqrt{d_{u_4}^2+d_{u_5}^2} & 0 \\ \sqrt{d_{u_5}^2+d_{u_1}^2} & 0 & 0 & \sqrt{d_{u_5}^2+d_{u_4}^2} & 0 & 0 \\ \sqrt{d_{u_6}^2+d_{u_1}^2} & \sqrt{d_{u_6}^2+d_{u_2}^2} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & \sqrt{20} & \sqrt{32} & \sqrt{20} & \sqrt{20} \\ 0 & 0 & 0 & \sqrt{20} & 0 & \sqrt{8} \\ \sqrt{20} & 0 & 0 & \sqrt{20} & 0 & 0 \\ \sqrt{32} & \sqrt{20} & \sqrt{20} & 0 & \sqrt{20} & 0 \\ \sqrt{20} & 0 & 0 & \sqrt{20} & 0 & 0 \\ \sqrt{20} & \sqrt{8} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\{-8.932, -7.291, 0, 0.448, 2.303, 13.473\}$ and $E_{SO}(PIS)(\mathbb{Z}_n) = 32.447$.

(vii) The Sombor matrix is

$$SO(G)=\begin{bmatrix} 0 & 0 & 0 & \sqrt{d_{u_1}^2+d_{u_4}^2} & \sqrt{d_{u_1}^2+d_{u_5}^2} & \sqrt{d_{u_1}^2+d_{u_6}^2} & \sqrt{d_{u_1}^2+d_{u_7}^2} & \sqrt{d_{u_1}^2+d_{u_8}^2} & 0 & \sqrt{d_{u_1}^2+d_{u_{10}}^2} \\ 0 & 0 & 0 & 0 & \sqrt{d_{u_2}^2+d_{u_5}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_7}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_9}^2} & \sqrt{d_{u_2}^2+d_{u_{10}}^2} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{d_{u_3}^2+d_{u_6}^2} & 0 & \sqrt{d_{u_3}^2+d_{u_8}^2} & \sqrt{d_{u_3}^2+d_{u_9}^2} & \sqrt{d_{u_3}^2+d_{u_{10}}^2} \\ \sqrt{d_{u_4}^2+d_{u_1}^2} & 0 & 0 & 0 & \sqrt{d_{u_4}^2+d_{u_5}^2} & \sqrt{d_{u_4}^2+d_{u_6}^2} & 0 & 0 & 0 & \sqrt{d_{u_4}^2+d_{u_{10}}^2} \\ \sqrt{d_{u_5}^2+d_{u_1}^2} & \sqrt{d_{u_5}^2+d_{u_2}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_4}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_6}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_8}^2} & \sqrt{d_{u_5}^2+d_{u_9}^2} & 0 \\ \sqrt{d_{u_6}^2+d_{u_1}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_3}^2} & \sqrt{d_{u_6}^2+d_{u_4}^2} & \sqrt{d_{u_6}^2+d_{u_5}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_7}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_9}^2} & 0 \\ \sqrt{d_{u_7}^2+d_{u_1}^2} & \sqrt{d_{u_7}^2+d_{u_2}^2} & 0 & 0 & 0 & \sqrt{d_{u_7}^2+d_{u_6}^2} & 0 & 0 & \sqrt{d_{u_7}^2+d_{u_9}^2} & 0 \\ \sqrt{d_{u_8}^2+d_{u_1}^2} & 0 & \sqrt{d_{u_8}^2+d_{u_3}^2} & 0 & \sqrt{d_{u_8}^2+d_{u_5}^2} & 0 & 0 & 0 & \sqrt{d_{u_8}^2+d_{u_9}^2} & 0 \\ 0 & \sqrt{d_{u_9}^2+d_{u_2}^2} & \sqrt{d_{u_9}^2+d_{u_3}^2} & 0 & \sqrt{d_{u_9}^2+d_{u_5}^2} & \sqrt{d_{u_9}^2+d_{u_6}^2} & \sqrt{d_{u_9}^2+d_{u_7}^2} & \sqrt{d_{u_9}^2+d_{u_8}^2} & 0 & 0 \\ \sqrt{d_{u_{10}}^2+d_{u_1}^2} & \sqrt{d_{u_{10}}^2+d_{u_2}^2} & \sqrt{d_{u_{10}}^2+d_{u_3}^2} & \sqrt{d_{u_{10}}^2+d_{u_4}^2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 0 & \sqrt{52} & 6 & 6 & \sqrt{52} & \sqrt{52} & 0 & \sqrt{52} \\ 0 & 0 & 0 & 0 & \sqrt{52} & 0 & \sqrt{32} & 0 & \sqrt{52} & \sqrt{32} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{52} & 0 & \sqrt{32} & \sqrt{52} & \sqrt{32} \\ \sqrt{52} & 0 & 0 & 0 & \sqrt{52} & \sqrt{52} & 0 & 0 & 0 & \sqrt{32} \\ 6 & \sqrt{52} & 0 & \sqrt{52} & 0 & \sqrt{72} & 0 & \sqrt{52} & \sqrt{72} & 0 \\ 6 & 0 & \sqrt{52} & \sqrt{52} & \sqrt{72} & 0 & \sqrt{52} & 0 & \sqrt{72} & 0 \\ \sqrt{52} & \sqrt{32} & 0 & 0 & 0 & \sqrt{52} & 0 & 0 & \sqrt{52} & 0 \\ \sqrt{52} & 0 & \sqrt{32} & 0 & \sqrt{52} & 0 & 0 & 0 & \sqrt{52} & 0 \\ 0 & \sqrt{52} & \sqrt{52} & 0 & \sqrt{72} & \sqrt{72} & \sqrt{52} & \sqrt{52} & 0 & 0 \\ \sqrt{52} & 6 & \sqrt{32} & \sqrt{32} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\{-17.427, -17.366, -13.844, -11.385, 0.894, 3.224, 4.535, 5.656, 10.193, 35.519\}$ and $E_{SO}(PIS)(\mathbb{Z}_n) = 120.103$.

(*viii*) The Sombor matrix is

$$
= \begin{bmatrix}
0 & 0 & 0 & \sqrt{d_{u_1}^2+d_{u_4}^2} & 0 & \sqrt{d_{u_1}^2+d_{u_6}^2} & \sqrt{d_{u_1}^2+d_{u_7}^2} & 0 & 0 & \sqrt{d_{u_1}^2+d_{u_{10}}^2} & 0 & \sqrt{d_{u_1}^2+d_{u_{12}}^2} & \sqrt{d_{u_1}^2+d_{u_{13}}^2} & 0 \\
0 & 0 & 0 & \sqrt{d_{u_2}^2+d_{u_4}^2} & 0 & 0 & 0 & \sqrt{d_{u_2}^2+d_{u_8}^2} & \sqrt{d_{u_2}^2+d_{u_9}^2} & \sqrt{d_{u_2}^2+d_{u_{10}}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_{12}}^2} & 0 & \sqrt{d_{u_2}^2+d_{u_{14}}^2} \\
0 & 0 & 0 & 0 & 0 & \sqrt{d_{u_3}^2+d_{u_6}^2} & 0 & \sqrt{d_{u_3}^2+d_{u_8}^2} & 0 & \sqrt{d_{u_3}^2+d_{u_{10}}^2} & \sqrt{d_{u_3}^2+d_{u_{11}}^2} & 0 & \sqrt{d_{u_3}^2+d_{u_{13}}^2} & \sqrt{d_{u_3}^2+d_{u_{14}}^2} \\
\sqrt{d_{u_4}^2+d_{u_1}^2} & \sqrt{d_{u_4}^2+d_{u_2}^2} & 0 & 0 & 0 & \sqrt{d_{u_4}^2+d_{u_6}^2} & \sqrt{d_{u_4}^2+d_{u_7}^2} & \sqrt{d_{u_4}^2+d_{u_8}^2} & \sqrt{d_{u_4}^2+d_{u_9}^2} & 0 & 0 & 0 & \sqrt{d_{u_4}^2+d_{u_{13}}^2} & \sqrt{d_{u_4}^2+d_{u_{14}}^2} \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{d_{u_5}^2+d_{u_7}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_9}^2} & 0 & \sqrt{d_{u_5}^2+d_{u_{11}}^2} & \sqrt{d_{u_5}^2+d_{u_{12}}^2} & \sqrt{d_{u_5}^2+d_{u_{13}}^2} & \sqrt{d_{u_5}^2+d_{u_{14}}^2} \\
\sqrt{d_{u_6}^2+d_{u_1}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_3}^2} & \sqrt{d_{u_6}^2+d_{u_4}^2} & 0 & 0 & \sqrt{d_{u_6}^2+d_{u_7}^2} & \sqrt{d_{u_6}^2+d_{u_8}^2} & 0 & 0 & \sqrt{d_{u_6}^2+d_{u_{11}}^2} & \sqrt{d_{u_6}^2+d_{u_{12}}^2} & 0 & \sqrt{d_{u_6}^2+d_{u_{14}}^2} \\
\sqrt{d_{u_7}^2+d_{u_1}^2} & 0 & 0 & \sqrt{d_{u_7}^2+d_{u_4}^2} & \sqrt{d_{u_7}^2+d_{u_5}^2} & \sqrt{d_{u_7}^2+d_{u_6}^2} & 0 & 0 & \sqrt{d_{u_7}^2+d_{u_9}^2} & \sqrt{d_{u_7}^2+d_{u_{10}}^2} & \sqrt{d_{u_7}^2+d_{u_{11}}^2} & 0 & 0 & \sqrt{d_{u_7}^2+d_{u_{14}}^2} \\
0 & \sqrt{d_{u_8}^2+d_{u_2}^2} & \sqrt{d_{u_8}^2+d_{u_3}^2} & \sqrt{d_{u_8}^2+d_{u_4}^2} & 0 & \sqrt{d_{u_8}^2+d_{u_6}^2} & 0 & 0 & \sqrt{d_{u_8}^2+d_{u_9}^2} & 0 & \sqrt{d_{u_8}^2+d_{u_{11}}^2} & \sqrt{d_{u_8}^2+d_{u_{12}}^2} & \sqrt{d_{u_8}^2+d_{u_{13}}^2} & 0 \\
0 & \sqrt{d_{u_9}^2+d_{u_2}^2} & 0 & \sqrt{d_{u_9}^2+d_{u_4}^2} & \sqrt{d_{u_9}^2+d_{u_5}^2} & 0 & \sqrt{d_{u_9}^2+d_{u_7}^2} & \sqrt{d_{u_9}^2+d_{u_8}^2} & 0 & \sqrt{d_{u_9}^2+d_{u_{10}}^2} & \sqrt{d_{u_9}^2+d_{u_{11}}^2} & 0 & \sqrt{d_{u_9}^2+d_{u_{13}}^2} & 0 \\
\sqrt{d_{u_{10}}^2+d_{u_1}^2} & \sqrt{d_{u_{10}}^2+d_{u_2}^2} & \sqrt{d_{u_{10}}^2+d_{u_3}^2} & 0 & 0 & 0 & \sqrt{d_{u_{10}}^2+d_{u_7}^2} & 0 & \sqrt{d_{u_{10}}^2+d_{u_9}^2} & 0 & \sqrt{d_{u_{10}}^2+d_{u_{11}}^2} & 0 & 0 & 0 \\
0 & 0 & \sqrt{d_{u_{11}}^2+d_{u_3}^2} & 0 & \sqrt{d_{u_{11}}^2+d_{u_5}^2} & \sqrt{d_{u_{11}}^2+d_{u_6}^2} & \sqrt{d_{u_{11}}^2+d_{u_7}^2} & \sqrt{d_{u_{11}}^2+d_{u_8}^2} & \sqrt{d_{u_{11}}^2+d_{u_9}^2} & \sqrt{d_{u_{11}}^2+d_{u_{10}}^2} & 0 & \sqrt{d_{u_{11}}^2+d_{u_{12}}^2} & 0 & 0 \\
\sqrt{d_{u_{12}}^2+d_{u_1}^2} & \sqrt{d_{u_{12}}^2+d_{u_2}^2} & 0 & 0 & \sqrt{d_{u_{12}}^2+d_{u_5}^2} & \sqrt{d_{u_{12}}^2+d_{u_6}^2} & 0 & \sqrt{d_{u_{12}}^2+d_{u_8}^2} & 0 & 0 & \sqrt{d_{u_{12}}^2+d_{u_{11}}^2} & 0 & 0 & 0 \\
\sqrt{d_{u_{13}}^2+d_{u_1}^2} & 0 & \sqrt{d_{u_{13}}^2+d_{u_3}^2} & \sqrt{d_{u_{13}}^2+d_{u_4}^2} & \sqrt{d_{u_{13}}^2+d_{u_5}^2} & 0 & 0 & \sqrt{d_{u_{13}}^2+d_{u_8}^2} & \sqrt{d_{u_{13}}^2+d_{u_9}^2} & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{d_{u_{14}}^2+d_{u_2}^2} & \sqrt{d_{u_{14}}^2+d_{u_3}^2} & \sqrt{d_{u_{14}}^2+d_{u_4}^2} & \sqrt{d_{u_{14}}^2+d_{u_5}^2} & \sqrt{d_{u_{14}}^2+d_{u_6}^2} & \sqrt{d_{u_{14}}^2+d_{u_7}^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

$$
= \begin{bmatrix}
0 & 0 & 0 & 10 & 0 & 10 & 10 & 0 & 0 & \sqrt{72} & 0 & \sqrt{72} & \sqrt{72} & 0 \\
0 & 0 & 0 & 10 & 0 & 0 & 0 & 10 & 10 & \sqrt{72} & 0 & \sqrt{72} & 0 & \sqrt{72} \\
0 & 0 & 0 & 0 & 0 & 10 & 0 & 10 & 0 & \sqrt{72} & 10 & 0 & \sqrt{72} & \sqrt{72} \\
10 & 10 & 0 & 0 & 0 & \sqrt{128} & \sqrt{128} & \sqrt{128} & \sqrt{128} & 0 & 0 & 0 & 10 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 10 & 0 & 10 & \sqrt{72} & \sqrt{72} & \sqrt{72} \\
10 & 0 & 10 & \sqrt{128} & 0 & 0 & \sqrt{128} & \sqrt{128} & 0 & 0 & \sqrt{128} & 10 & 0 & 10 \\
10 & 0 & 0 & \sqrt{128} & 10 & \sqrt{128} & 0 & 0 & \sqrt{128} & 10 & \sqrt{128} & 0 & 0 & 10 \\
0 & 10 & 10 & \sqrt{128} & 0 & \sqrt{128} & 0 & 0 & \sqrt{128} & 0 & \sqrt{128} & 10 & 10 & 0 \\
0 & 10 & 0 & \sqrt{128} & 10 & 0 & \sqrt{128} & \sqrt{128} & 0 & 10 & \sqrt{128} & 0 & 10 & 0 \\
\sqrt{72} & \sqrt{72} & \sqrt{72} & 0 & 0 & 0 & 10 & 0 & 10 & 0 & 10 & 0 & 0 & 0 \\
0 & 0 & 10 & 0 & 10 & \sqrt{128} & \sqrt{128} & \sqrt{128} & \sqrt{128} & 10 & 0 & 10 & 0 & 0 \\
\sqrt{72} & \sqrt{72} & 0 & 0 & \sqrt{72} & 10 & 0 & 10 & 0 & 0 & 10 & 0 & 0 & 0 \\
\sqrt{72} & 0 & \sqrt{72} & 10 & \sqrt{72} & 0 & 0 & 10 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{72} & \sqrt{72} & 10 & \sqrt{72} & 10 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

The eigen values are $\{-25.455, -24.687^{[3]}, -22.627^{[2]}, -0.627, 8.485^{[3]}, 16.202^{[3]}, 71.383\}$ and $E_{SO}(PIS)(\mathbb{Z}_n) = 290.841$.

□

Following theorem deals with the evaluation of Laplacian spectra and energies of different prime ideal sum graphs of $\mathbb{Z}_n$.

Theorem 2.8. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = p^\alpha$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 1^{[\alpha-3]}, \alpha-1\}$ and the Laplacian Energy is $\dfrac{4(\alpha-2)}{\alpha-1}$.*

(*ii*) *If $n = pq$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0^{[2]}\}$ and the Laplacian Energy is 0.*

(*iii*) *If $n = p^2q$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 1, 3, 4\}$ and the Laplacian Energy is 6.*

(*iv*) *If $n = p^2q^2$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 1.753, 3.198, 3.445, 4.554, 4.801, 6.246\}$ and the Laplacian Energy is 10.667.*

(*v*) *If $n = pqr$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 1.697^{[2]}, 4, 5.302^{[2]}\}$ and the Laplacian Energy is 8.002.*

(*vi*) *If $n = p^3q$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 1.268, 2, 2.586, 4.732, 5.414\}$ and the Laplacian Energy is 9.624.*

(*vii*) *If $n = p^2qr$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 3^{[2]}, 3.586, 4.268, 5, 6.414, 7, 7.732, 8\}$ and the Laplacian Energy is 20.292.*

(*viii*) *If $n = pqrs$, then the Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 5^{[6]}, 7, 9, 9.561^{[3]}, 10^{[2]}\}$ and the Laplacian Energy is 34.683.*

Proof. (i) The Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of degrees of the vertices and $A(G)$ is the adjacency matrix, so

$$L(G) = \begin{bmatrix} \alpha-2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha-2 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The eigen values are $\{0, 1^{[\alpha-3]}, \alpha-1\}$ and $LE(G) = \dfrac{4(\alpha-2)}{\alpha-1}$.

(ii) Follows trivially.

(iii) The Laplacian matrix is

$$L(G) = \begin{bmatrix} 2 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

The eigen values are $\{0, 1, 3, 4\}$ and $LE(G) = 6$.

(iv) The Laplacian matrix is

$$L(G) = \begin{bmatrix} 4 & 0 & -1 & -1 & -1 & -1 & 0 \\ 0 & 4 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & 4 & 0 & 0 & -1 & -1 \\ -1 & -1 & 0 & 3 & 0 & 0 & -1 \\ -1 & -1 & 0 & 0 & 3 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 3 & 0 \\ 0 & -1 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}.$$

The eigen values are $\{0, 1.753, 3.198, 3.445, 4.554, 4.801, 6.246\}$ and $LE(G) = 10.667$.

(v) The Laplacian matrix is

$$L(G) = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{bmatrix}.$$

The eigen values are $\{0, 1.697^{[2]}, 4, 5.302^{[2]}\}$ and $LE(G) = 8.002$.

(vi) The Laplacian matrix is

$$L(G) = \begin{bmatrix} 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The eigen values are $\{0, 1.268, 2, 2.586, 4.732, 5.414\}$ and $LE(G) = 9.624$.

(*vii*) The Laplacian matrix is

$$L(G) = \begin{bmatrix} 6 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & 4 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 4 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \\ -1 & 0 & 0 & 4 & -1 & -1 & 0 & 0 & 0 & -1 \\ -1 & -1 & 0 & -1 & 6 & -1 & 0 & -1 & -1 & 0 \\ -1 & 0 & -1 & -1 & -1 & 6 & -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 & 0 & -1 & 4 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & 6 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

The eigen values are $\{0, 3^{[2]}, 3.586, 4.268, 5, 6.414, 7, 7.732, 8\}$ and $LE(G) = 20.292$.

(*viii*) The Laplacian matrix is

$$= \begin{bmatrix} 6 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 0 & 6 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 6 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & 8 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 6 & 0 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & 0 & 8 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & -1 & -1 & 8 & 0 & -1 & -1 & -1 & 0 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 & -1 & 0 & 8 & -1 & 0 & -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 8 & -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 6 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 8 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 6 & 0 & 0 \\ -1 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 6 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

The eigen values are $\{0, 5^{[6]}, 7, 9, 9.561^{[3]}, 10^{[2]}\}$ and $LE(G) = 34.683$.

□

The next theorem is related with the study of signless Laplacian spectra and energies of different prime ideal sum graphs of $\mathbb{Z}_n$.

Theorem 2.9. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = p^\alpha$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0, 1^{[\alpha-3]}, \alpha - 1\}$ and the signless Laplacian Energy is $\dfrac{4(\alpha-2)}{\alpha-1}$.*

(*ii*) *If $n = pq$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0^{[2]}\}$ and the signless Laplacian Energy is 0.*

(*iii*) *If $n = p^2q$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0.438, 1, 2, 4.561\}$ and the signless Laplacian Energy is 5.123.*

(*iv*) *If $n = p^2q^2$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0.950, 1.753, 2.213, 3.445, 3.761, 4.01, 7.076\}$ and the signless Laplacian Energy is 10.739.*

(*v*) *If $n = pqr$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{1.171, 1.382^{[2]}, 3.618^{[2]}, 6.828\}$ and the signless Laplacian Energy is 10.129.*

(*vi*) *If $n = p^3q$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{0.586, 0.897, 2, 2.854, 3.414, 6.249\}$ and the signless Laplacian Energy is 9.034.*

(*vii*) *If $n = p^2qr$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{1.711, 2.268, 2.645, 3.452, 4.803, 5, 5.374, 5.732, 6.710, 10.305\}$ and the signless Laplacian Energy is 18.248.*

(*viii*) *If $n = pqrs$, then the signless Laplacian spectrum of $PIS(\mathbb{Z}_n)$ is $\{3.143, 3.882, 4^{[2]}, 6^{[2]}, 6.844^{[2]}, 7^{[2]}, 8.976, 9^{[2]}, 14.312\}$ and the signless Laplacian Energy is 28.315.*

Proof. (*i*) The signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of degrees of the vertices and $A(G)$ is the adjacency matrix, so

$$Q(G) = \begin{bmatrix} \alpha-2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$Q(G) = \begin{bmatrix} \alpha-2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

The eigen values are $\{0, 1^{[\alpha-3]}, \alpha-1\}$ and $QE(G) = \dfrac{4(\alpha-2)}{\alpha-1}$.

(*ii*) It is obvious.

(*iii*) The signless Laplacian matrix is

$$Q(G) = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}.$$

The eigen values are $\{0.438, 1, 2, 4.561\}$ and $QE(G) = 5.123$.

(*iv*) The signless Laplacian matrix is

$$Q(G) = \begin{bmatrix} 4 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 4 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 4 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 3 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 3 \end{bmatrix}.$$

The eigen values are $\{0.586, 0.897, 2, 2.854, 3.414, 6.249\}$ and $QE(G) = 9.034$.

(*v*) The signless Laplacian matrix is

$$Q(G) = \begin{bmatrix} 2 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 & 1 \\ 0 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}.$$

The eigen values are $\{1.171, 1.382^{[2]}, 3.618^{[2]}, 6.828\}$ and $QE(G) = 10.129$.

(*vi*) The signless Laplacian matrix is

$$Q(G) = \begin{bmatrix} 4 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 4 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

The eigen values are $\{0, 1.268, 2, 2.586, 4.732, 5.414\}$ and $QE(G) = 9.624$.

(*vii*) The signless Laplacian matrix is

$$Q(G) = \begin{bmatrix} 6 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 4 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 4 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 6 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 6 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 4 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 4 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 6 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

The eigen values are $\{1.711, 2.268, 2.645, 3.452, 4.803, 5, 5.374, 5.732, 6.710, 10.305\}$ and $QE(G) = 18.248$.

(*viii*) The signless Laplacian matrix is

$$Q(G) = \begin{bmatrix} 6 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 6 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 6 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 8 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 6 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 8 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 8 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 8 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 8 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 6 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 8 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 6 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 6 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

The eigen values are $\{3.143, 3.882, 4^{[2]}, 6^{[2]}, 6.844^{[2]}, 7^{[2]}, 8.976, 9^{[2]}, 14.312\}$ and $QE(G) = 28.315$.

□

Finally, we estimate Randic spectra and energies of different prime ideal sum graphs of $\mathbb{Z}_n$. This estimation is discussed in Theorem 2.10.

Theorem 2.10. *Let $PIS(\mathbb{Z}_n)$ be the prime ideal sum graph of $\mathbb{Z}_n$. Then the followings hold:*

(*i*) *If $n = pq$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{0^{[2]}\}$ and the Randic Energy is* 0.

(*ii*) *If $n = p^2q$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{-0.728, -0.5, 0.228, 1\}$ and the Randic Energy is* 2.456.

(*iii*) *If $n = p^2q^2$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{-0.724, -0.441, -0.333, 0, 0.057, 0.441, 1\}$ and the Randic Energy is* 2.996.

(*iv*) *If $n = pqr$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{-0.5^{[3]}, 0.25^{[2]}, 1\}$ and the Randic Energy is* 3.

(*v*) *If $n = p^3q$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{-0.75, -0.64, 0^{[2]}, 0.390, 1\}$ and the Randic Energy is* 2.78.

(*vi*) *If $n = p^2qr$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{-0.628, -0.5, -0.403, -0.283, -0.099, 0.083, 0.203, 0.250, 0.320, 1.058\}$ and the Randic Energy is* 3.827.

(*vii*) *If $n = pqrs$, then the Randic spectrum of $PIS(\mathbb{Z}_n)$ is $\{-0.5, -0.392, -0.383, -0.377, -0.270, -0.243, 0.006, 0.166^{[3]}, 0.204, 0.217, 0.231, 1.007\}$ and the Randic Energy is* 4.328.

Proof. (*i*) Follows trivially.

(*ii*) The Randic matrix is

$$R(G)=\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_3}}} & \frac{1}{\sqrt{d_{u_1}d_{u_4}}} \\ 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_3}}} & 0 \\ \frac{1}{\sqrt{d_{u_3}d_{u_1}}} & \frac{1}{\sqrt{d_{u_3}d_{u_2}}} & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_4}}} \\ \frac{1}{\sqrt{d_{u_4}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_3}}} & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{4}} \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & 0 & \frac{1}{\sqrt{6}} & 0 \end{bmatrix}.$$

The eigen values are $\{-0.728, -0.5, 0.228, 1\}$ and $RE(G) = 2.456$.

(*iii*) The Randic Matrix is

$$R(G)=\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_3}}} & \frac{1}{\sqrt{d_{u_1}d_{u_4}}} & \frac{1}{\sqrt{d_{u_1}d_{u_5}}} & \frac{1}{\sqrt{d_{u_1}d_{u_6}}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_3}}} & \frac{1}{\sqrt{d_{u_2}d_{u_4}}} & \frac{1}{\sqrt{d_{u_2}d_{u_5}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_7}}} \\ \frac{1}{\sqrt{d_{u_3}d_{u_1}}} & \frac{1}{\sqrt{d_{u_3}d_{u_2}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_6}}} & \frac{1}{\sqrt{d_{u_3}d_{u_7}}} \\ \frac{1}{\sqrt{d_{u_4}d_{u_1}}} & \frac{1}{\sqrt{d_{u_4}d_{u_2}}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_7}}} \\ \frac{1}{\sqrt{d_{u_5}d_{u_1}}} & \frac{1}{\sqrt{d_{u_5}d_{u_2}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_6}}} & 0 \\ \frac{1}{\sqrt{d_{u_6}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_3}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_5}}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{d_{u_7}d_{u_2}}} & \frac{1}{\sqrt{d_{u_7}d_{u_3}}} & \frac{1}{\sqrt{d_{u_7}d_{u_4}}} & 0 & 0 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & 0 & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} & 0 & 0 & 0 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{9}} \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & 0 & 0 & 0 & \frac{1}{\sqrt{9}} & 0 \\ \frac{1}{\sqrt{12}} & 0 & \frac{1}{\sqrt{12}} & 0 & \frac{1}{\sqrt{9}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{9}} & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\{-0.724, -0.441, -0.333, 0, 0.057, 0.441, 1\}$ and $RE(G) = 2.996$.

(iv) The Randic matrix is

$$R(G) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_4}}} & \frac{1}{\sqrt{d_{u_1}d_{u_5}}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_6}}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_5}}} & \frac{1}{\sqrt{d_{u_3}d_{u_6}}} \\ \frac{1}{\sqrt{d_{u_4}d_{u_1}}} & \frac{1}{\sqrt{d_{u_4}d_{u_2}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_5}}} & \frac{1}{\sqrt{d_{u_4}d_{u_6}}} \\ \frac{1}{\sqrt{d_{u_5}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_3}}} & \frac{1}{\sqrt{d_{u_5}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_6}}} \\ 0 & \frac{1}{\sqrt{d_{u_6}d_{u_2}}} & \frac{1}{\sqrt{d_{u_6}d_{u_3}}} & \frac{1}{\sqrt{d_{u_6}d_{u_4}}} & \frac{1}{\sqrt{d_{u_6}d_{u_5}}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & 0 & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} \\ \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{16}} & 0 & \frac{1}{\sqrt{16}} \\ 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} & 0 \end{bmatrix}.$$

The eigen values are $\{-0.5^{[3]}, 0.25^{[2]}, 1\}$ and $RE(G) = 3$.

(v) The Randic Matrix is

$$R(G) = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_3}}} & \frac{1}{\sqrt{d_{u_1}d_{u_4}}} & \frac{1}{\sqrt{d_{u_1}d_{u_5}}} & \frac{1}{\sqrt{d_{u_1}d_{u_6}}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_6}}} \\ \frac{1}{\sqrt{d_{u_3}d_{u_1}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_6}}} \\ \frac{1}{\sqrt{d_{u_4}d_{u_1}}} & \frac{1}{\sqrt{d_{u_4}d_{u_2}}} & \frac{1}{\sqrt{d_{u_4}d_{u_3}}} & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_5}}} & 0 \\ \frac{1}{\sqrt{d_{u_5}d_{u_1}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_4}}} & 0 & 0 \\ \frac{1}{\sqrt{d_{u_6}d_{u_1}}} & \frac{1}{\sqrt{d_{u_6}d_{u_2}}} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{8}} & 0 & 0 & \frac{1}{\sqrt{8}} & 0 & 0 \\ \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 & \frac{1}{\sqrt{8}} & 0 \\ \frac{1}{\sqrt{8}} & 0 & 0 & \frac{1}{\sqrt{8}} & 0 & 0 \\ \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{4}} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\{-0.75, -0.64, 0^{[2]}, 0.390, 1\}$ and $RE(G) = 2.78$.

(vi) The Randic Matrix is

$$R(G) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_4}}} & \frac{1}{\sqrt{d_{u_1}d_{u_5}}} & \frac{1}{\sqrt{d_{u_1}d_{u_6}}} & \frac{1}{\sqrt{d_{u_1}d_{u_7}}} & \frac{1}{\sqrt{d_{u_1}d_{u_8}}} & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_{10}}}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_5}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_7}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_9}}} & \frac{1}{\sqrt{d_{u_2}d_{u_{10}}}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_6}}} & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_8}}} & \frac{1}{\sqrt{d_{u_3}d_{u_9}}} & \frac{1}{\sqrt{d_{u_3}d_{u_{10}}}} \\ \frac{1}{\sqrt{d_{u_4}d_{u_1}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_5}}} & \frac{1}{\sqrt{d_{u_4}d_{u_6}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_{10}}}} \\ \frac{1}{\sqrt{d_{u_5}d_{u_1}}} & \frac{1}{\sqrt{d_{u_5}d_{u_2}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_6}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_8}}} & \frac{1}{\sqrt{d_{u_5}d_{u_9}}} & 0 \\ \frac{1}{\sqrt{d_{u_6}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_3}}} & \frac{1}{\sqrt{d_{u_6}d_{u_4}}} & \frac{1}{\sqrt{d_{u_6}d_{u_5}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_7}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_9}}} & 0 \\ \frac{1}{\sqrt{d_{u_7}d_{u_1}}} & \frac{1}{\sqrt{d_{u_7}d_{u_2}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_7}d_{u_6}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_7}d_{u_9}}} & 0 \\ \frac{1}{\sqrt{d_{u_8}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_8}d_{u_3}}} & 0 & \frac{1}{\sqrt{d_{u_8}d_{u_5}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_8}d_{u_9}}} & 0 \\ 0 & \frac{1}{\sqrt{d_{u_9}d_{u_2}}} & \frac{1}{\sqrt{d_{u_9}d_{u_3}}} & 0 & \frac{1}{\sqrt{d_{u_9}d_{u_5}}} & \frac{1}{\sqrt{d_{u_9}d_{u_6}}} & \frac{1}{\sqrt{d_{u_9}d_{u_7}}} & \frac{1}{\sqrt{d_{u_9}d_{u_8}}} & 0 & 0 \\ \frac{1}{\sqrt{d_{u_{10}}d_{u_1}}} & \frac{1}{\sqrt{d_{u_{10}}d_{u_2}}} & \frac{1}{\sqrt{d_{u_{10}}d_{u_3}}} & \frac{1}{\sqrt{d_{u_{10}}d_{u_4}}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{24}} \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{16}} & 0 & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{16}} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{16}} \\ \frac{1}{\sqrt{24}} & 0 & 0 & 0 & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{24}} & 0 & 0 & 0 & \frac{1}{\sqrt{16}} \\ \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{36}} & 0 \\ \frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{36}} & 0 \\ \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{16}} & 0 & 0 & 0 & \frac{1}{\sqrt{24}} & 0 & 0 & \frac{1}{\sqrt{24}} & 0 \\ \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{16}} & 0 & \frac{1}{\sqrt{24}} & 0 & 0 & 0 & \frac{1}{\sqrt{24}} & 0 \\ 0 & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{24}} & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{24}} & 0 & 0 \\ \frac{1}{\sqrt{24}} & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} & \frac{1}{\sqrt{16}} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigen values are $\{-0.628, -0.5, -0.403, -0.283, -0.099, 0.083, 0.203, 0.250, 0.320, 1.058\}$ and $RE(G) = 3.827$.

(*vii*) The Randic Matrix is

$$= \begin{bmatrix}
0 & 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_6}}} & \frac{1}{\sqrt{d_{u_1}d_{u_7}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_{10}}}} & 0 & \frac{1}{\sqrt{d_{u_1}d_{u_{12}}}} & \frac{1}{\sqrt{d_{u_1}d_{u_{13}}}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_4}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_8}}} & \frac{1}{\sqrt{d_{u_2}d_{u_9}}} & \frac{1}{\sqrt{d_{u_2}d_{u_{10}}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_{12}}}} & 0 & \frac{1}{\sqrt{d_{u_2}d_{u_{14}}}} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_6}}} & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_8}}} & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_{10}}}} & \frac{1}{\sqrt{d_{u_3}d_{u_{11}}}} & 0 & \frac{1}{\sqrt{d_{u_3}d_{u_{13}}}} & \frac{1}{\sqrt{d_{u_3}d_{u_{14}}}} \\
\frac{1}{\sqrt{d_{u_4}d_{u_1}}} & \frac{1}{\sqrt{d_{u_4}d_{u_2}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_6}}} & \frac{1}{\sqrt{d_{u_4}d_{u_7}}} & \frac{1}{\sqrt{d_{u_4}d_{u_8}}} & \frac{1}{\sqrt{d_{u_4}d_{u_9}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_4}d_{u_{13}}}} & \frac{1}{\sqrt{d_{u_4}d_{u_{14}}}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_7}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_9}}} & 0 & \frac{1}{\sqrt{d_{u_5}d_{u_{11}}}} & \frac{1}{\sqrt{d_{u_5}d_{u_{12}}}} & \frac{1}{\sqrt{d_{u_5}d_{u_{13}}}} & \frac{1}{\sqrt{d_{u_5}d_{u_{14}}}} \\
\frac{1}{\sqrt{d_{u_6}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_3}}} & \frac{1}{\sqrt{d_{u_6}d_{u_4}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_7}}} & \frac{1}{\sqrt{d_{u_6}d_{u_8}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_{11}}}} & \frac{1}{\sqrt{d_{u_6}d_{u_{12}}}} & 0 & \frac{1}{\sqrt{d_{u_6}d_{u_{14}}}} \\
\frac{1}{\sqrt{d_{u_7}d_{u_1}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_7}d_{u_4}}} & \frac{1}{\sqrt{d_{u_7}d_{u_5}}} & \frac{1}{\sqrt{d_{u_7}d_{u_6}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_7}d_{u_9}}} & \frac{1}{\sqrt{d_{u_7}d_{u_{10}}}} & \frac{1}{\sqrt{d_{u_7}d_{u_{11}}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_7}d_{u_{14}}}} \\
0 & \frac{1}{\sqrt{d_{u_8}d_{u_2}}} & \frac{1}{\sqrt{d_{u_8}d_{u_3}}} & \frac{1}{\sqrt{d_{u_8}d_{u_4}}} & 0 & \frac{1}{\sqrt{d_{u_8}d_{u_6}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_8}d_{u_9}}} & 0 & \frac{1}{\sqrt{d_{u_8}d_{u_{11}}}} & \frac{1}{\sqrt{d_{u_8}d_{u_{12}}}} & \frac{1}{\sqrt{d_{u_8}d_{u_{13}}}} & 0 \\
0 & \frac{1}{\sqrt{d_{u_9}d_{u_2}}} & 0 & \frac{1}{\sqrt{d_{u_9}d_{u_4}}} & \frac{1}{\sqrt{d_{u_9}d_{u_5}}} & 0 & \frac{1}{\sqrt{d_{u_9}d_{u_7}}} & \frac{1}{\sqrt{d_{u_9}d_{u_8}}} & 0 & \frac{1}{\sqrt{d_{u_9}d_{u_{10}}}} & \frac{1}{\sqrt{d_{u_9}d_{u_{11}}}} & 0 & \frac{1}{\sqrt{d_{u_9}d_{u_{13}}}} & 0 \\
\frac{1}{\sqrt{d_{u_{10}}d_{u_1}}} & \frac{1}{\sqrt{d_{u_{10}}d_{u_2}}} & \frac{1}{\sqrt{d_{u_{10}}d_{u_3}}} & 0 & 0 & 0 & \frac{1}{\sqrt{d_{u_{10}}d_{u_7}}} & 0 & \frac{1}{\sqrt{d_{u_{10}}d_{u_9}}} & 0 & \frac{1}{\sqrt{d_{u_{10}}d_{u_{11}}}} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{d_{u_{11}}d_{u_3}}} & 0 & \frac{1}{\sqrt{d_{u_{11}}d_{u_5}}} & \frac{1}{\sqrt{d_{u_{11}}d_{u_6}}} & \frac{1}{\sqrt{d_{u_{11}}d_{u_7}}} & \frac{1}{\sqrt{d_{u_{11}}d_{u_8}}} & \frac{1}{\sqrt{d_{u_{11}}d_{u_9}}} & \frac{1}{\sqrt{d_{u_{11}}d_{u_{10}}}} & 0 & \frac{1}{\sqrt{d_{u_{11}}d_{u_{12}}}} & 0 & 0 \\
\frac{1}{\sqrt{d_{u_{12}}d_{u_1}}} & \frac{1}{\sqrt{d_{u_{12}}d_{u_2}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_{12}}d_{u_5}}} & \frac{1}{\sqrt{d_{u_{12}}d_{u_6}}} & 0 & \frac{1}{\sqrt{d_{u_{12}}d_{u_8}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_{12}}d_{u_{11}}}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{d_{u_{13}}d_{u_1}}} & 0 & \frac{1}{\sqrt{d_{u_{13}}d_{u_3}}} & \frac{1}{\sqrt{d_{u_{13}}d_{u_4}}} & \frac{1}{\sqrt{d_{u_{13}}d_{u_5}}} & 0 & 0 & \frac{1}{\sqrt{d_{u_{13}}d_{u_8}}} & \frac{1}{\sqrt{d_{u_{13}}d_{u_9}}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{d_{u_{14}}d_{u_2}}} & \frac{1}{\sqrt{d_{u_{14}}d_{u_3}}} & \frac{1}{\sqrt{d_{u_{14}}d_{u_4}}} & \frac{1}{\sqrt{d_{u_{14}}d_{u_5}}} & \frac{1}{\sqrt{d_{u_{14}}d_{u_6}}} & \frac{1}{\sqrt{d_{u_{14}}d_{u_7}}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

$$= \begin{bmatrix}
0 & 0 & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & 0 & 0 & \frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{48}} & 0 & 0 & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{36}} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} \\
\frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & 0 & 0 & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & 0 & 0 & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} \\
\frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{64}} & 0 & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & 0 & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} \\
\frac{1}{\sqrt{48}} & 0 & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{64}} & 0 & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{64}} & 0 & 0 & \frac{1}{\sqrt{48}} \\
0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{64}} & 0 & \frac{1}{\sqrt{64}} & 0 & 0 & \frac{1}{\sqrt{64}} & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & 0 \\
0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{64}} & 0 & \frac{1}{\sqrt{48}} & 0 \\
\frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & 0 & 0 & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{64}} & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & 0 & 0 \\
\frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & 0 & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{48}} & 0 & \frac{1}{\sqrt{48}} & 0 & 0 & \frac{1}{\sqrt{48}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{36}} & 0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{36}} & 0 & 0 & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{36}} & \frac{1}{\sqrt{48}} & \frac{1}{\sqrt{48}} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

The eigen values are $\{-0.5, -0.392, -0.383, -0.377, -0.270, -0.243, 0.006, 0.166^{[3]}, 0.204, 0.217, 0.231, 1.007\}$ and $RE(G) = 4.328$.

□

3. CONCLUSION

A more thorough examination of the fundamental characteristics of commutative rings is made possible by the prime ideal sum graph PIS(R), which offers a strong foundation for connecting their abstract algebraic structure to concrete, real-world systems. Based on their ideal lattice structure, invariants like the graph energy and a variety of topological indices obtained from PIS(R) may be used as distinctive fingerprints to categorize and identify commutative rings.

Sombor indices for prime ideal sum graphs have been discussed in Theorem 2.1. Redzepovic has shown that the Sombor index can be effectively utilized to predict the thermodynamic characteristics of compounds[24]. Alikhani et al. examine the Sombor index of polymer graphs and demonstrate that Sombor index of some graphs is calculated using their monomer units[2]. The Sombor indices may also be used to compute the Backbone DNA networks[21]. We have derived Randic indices of prime ideal sum graphs in Theorem 2.2, which are used to calculate the overall power of a networked graphical transmission system. Moreover the Zagreb indices for prime ideal sum graph have been obtained in Theorem 2.3, which determines the overall π-electron energy of molecules with high accuracy[18, 28]. ABC indices for prime ideal sum graphs have been studied in Theorem 2.5. The link between linear and branched alkane stability is ascertained by computing strain energy for cyclic alkanes using these indices[11].

Energies of prime ideal sum graphs are evaluated in Theorem 2.7 and 2.10. Graph energies are used in network analysis to quantify the robustness and stability of intricate networks, which help direct security and network design initiatives. Analysis and investigation of the graph's characteristics are made possible by the adjacency matrix, which offers a succinct depiction of its structure. Additionally, the spectrum of the adjacency matrix is used in network analysis to examine a number of characteristics of networked systems, including resilience, synchronization behavior, and centrality measurements.

We have calculated Laplacian spectrum of prime ideal sum graphs in Theorem 2.8. Applications for the Laplacian spectrum may be found in a variety of fields, such as machine learning, network analysis, and the study of graph properties such as connectivity, community structure and graph partitioning. Moreover, it is used in network research to examine a number of topics, including synchronization behavior, resilience, and network centrality. The eigenvalues of the Laplacian matrix shed light on the network's resilience and stability, making it easier to spot important nodes and comprehend how networked systems function. Furthermore, the Laplacian spectrum has been extensively utilized in machine learning methods for organizing and analyzing high-dimensional data in a graph-based form, including spectral clustering and dimensionality reduction approaches[7, 26]. The signless Laplacian matrix, is a key tool in algebraic graph theory for describing the connectivity and structure of graphs. The literature mentions several findings and applications[1, 8, 9].

Therefore, prime ideal sum graphs offer a rigorous algebraic framework for understanding chemical compound stability and designing reliable and effective communication and computing networks. Their promise as a flexible tool across several disciplines is highlighted by their capacity to reconcile the practical requirements of physical and virtual systems with the abstract qualities of ideals.

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