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Cycloidal maximal surfaces in the 3-dimensional Lorentz-Minkowski space

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Abstract. We introduce new examples of maximal surfaces in Lorentz-Minkowski 3-dimensional space by solving the Björling problem for cycloidal curves and vector fields determined by their normal and the binormal vector fields. We also provide their Weierstrass representation and determine the associated family of such surfaces. We analyze their geometric properties and show that one family of parametric curves of surfaces from associated family are generalized helices lying on a non-degenerated quadric. Finally, we study a Lorentzian counterpart of the Henneberg surface and show that its adjoint is a maximal surface over a spacelike Lorentzian astroid.

1. Introduction

In differential geometry, a minimal surface is defined as a surface with zero mean curvature. The study of minimal surfaces constitutes a prominent area of differential geometry and has applications in other mathematical disciplines, including the calculus of variations, potential theory, and mathematical physics, see for example [9, 12, 20, 21, 35, 36]. The term *minimal* originates from a variational problem, as such a surface represents a solution to Plateau's problem, which seeks a surface of minimal area for given boundary conditions, [34].

Counterparts of minimal surfaces are also studied in Lorentz-Minkowski space, where a specific type is referred to as a maximal surface. These surfaces are spacelike, meaning that the induced metric is Riemannian, and the term *maximal* signifies that they maximize the surface area for given boundary conditions. Maximal surfaces play a significant role in various problems in physics, particularly in General relativity, [3, 6, 25, 33]. For example, in [6] they serve as a special kind of initial data surfaces for Einsteins equations, since the vanishing mean curvature condition simplifies the structure of equations or represent a useful tool in the proof of positivity of the gravitational mass, [33]. The geometric importance of maximal surfaces has also been recognized, and there are many works dealing with such surfaces and their properties,

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for example [5, 8, 17–19]. One of the important results related to maximal surfaces is the Björling formula, [2, 28], which, at least theoretically, provides an efficient tool for obtaining an explicit parametrization of such surfaces. In the Lorentz-Minkowski space, the Björling problem consists of finding a maximal surface containing a given regular spacelike analytic curve $\alpha\colon I \to \mathbb{R}^3_1$, called a core curve of a surface, and a unit timelike analytic vector field V along α, such that α' and V are orthogonal along I and V is the surface normal vector field. Since functions α and V are analytic, they have holomorphic extensions $\alpha(z)$ and $V(z)$ in a simply-connected domain $\Omega \subset \mathbb{C}$ that contains $I \times \{0\}$ and the maximal surface is given by the parametrization $\mathbf{x}\colon \Omega \to \mathbb{R}^3_1$

$$\mathbf{x}(u, v) = \mathrm{Re}\left(\alpha(z) + \mathrm{i}\int_{u_0}^{z} V(w) \times \alpha'(w)\,\mathrm{d}w\right) \tag{1}$$

for a fixed $u_0 \in I$ and $z = u + \mathrm{i}v \in \Omega$, where $\times$ is the Lorentzian cross product, [22]. Although the formula seems simple, there is not a large number of examples of maximal surfaces given by an explicit parametrization, due to challenging calculation of integrals. In [22, 29], authors use the formula to obtain maximal surfaces whose core curve is a circle or a helix. In this paper we consider maximal surfaces whose core curve is a cycloidal curve. The cycloidal curves are also well-known in mathematical physics. A particular class of them, straight (ordinary) cycloids appear as a solution of the brachistochrone problem, i.e. a problem of finding the smooth curve joining two points along which a particle will slide from the higher to the lower point in the least possible time, [10]. In this context, maximal surfaces over straight (ordinary) cycloids can be viewed as solutions to two combined variational problems. Generally, cycloidal curves are generated by a point on a circle that rolls without sliding-either inside or outside another circle (or, in the case of a straight cycloid, along a straight line). In Lorentz-Minkowski space they are studied in [7]. We will refer to the maximal surfaces generated over such curves as cycloidal maximal surfaces.

In [14] authors analyzed global geometric properties of maximal surfaces over Euclidean cycloidal curves, while we consider maximal surfaces over the so-called Lorentzian cycloidal curves [7] as well and present examples of such surfaces. Next, we establish their Weierstrass representation and obtain the adjoint surfaces and a family of the associated surfaces. Finally, we analyze some of their properties. First we prove that u-parametric curves of the associated family of cycloidal maximal surface are generalized helices that lie on a non-degenerate quadrics. Then, theory implies that their projections onto planes orthogonal to helical axes are cycloidal curves, [7]. Further, we provide an explicit parametrization of the Lorentzian counterpart of the well-known Henneberg surface and its adjoint, [13, 30], as well as of the Catalan minimal surface, [16, 31, 32]. This can be done by using the idea of duality between maximal surfaces in $\mathbb{R}^3_1$ and minimal surfaces in $\mathbb{R}^3$ (e.g. [22]), which means that every maximal surface in $\mathbb{R}^3_1$ can be constructed from a minimal surface in $\mathbb{R}^3$ and vice versa. This concept appear as very useful. For example, Calabi [4] proved Bernstein-type theorem for maximal surfaces by using a one-to-one correspondence between minimal and maximal surfaces. Finally, it is also noteworthy to mention that the minimal surface in the isotropic 3-space appears as the "intermediate surface" between minimal and maximal surface, [1].

The paper is organized as follows. In Section 2, we recall the basic concepts of Lorentz-Minkowski 3-dimensional space and revise properties of maximal surfaces. The examples of maximal surfaces over Lorentzian, respectively Euclidean cycloidal curves are presented and analyzed in Section 3.

2. Preliminaries

The Lorentz-Minkowski 3-space $\mathbb{R}^3_1$ is the real affine space $\mathbb{R}^3_1$ whose underlying vector space is endowed with the Lorentzian pseudo-scalar product

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The pseudo-norm of a vector x is defined as the real number

$$\|x\| = \sqrt{|\langle x, x\rangle|} \geq 0.$$

The Lorentzian cross product of vectors x, y is defined by

$$x \times y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

A vector x in $\mathbb{R}^3_1$ is called spacelike if $\langle x, x\rangle > 0$ or $x = 0$, timelike if $\langle x, x\rangle < 0$ and null (lightlike) if $\langle x, x\rangle = 0$ and $x \neq 0$. The causal character of a regular curve is determined by the causal character of its velocity vectors.

A surface in $\mathbb{R}^3_1$ is called spacelike (resp. timelike, lightlike) if the induced metric is positive definite (resp. indefinite, of rank 1). A spacelike surface with a vanishing mean curvature is called a maximal surface.

Maximal surfaces can be given by the Weierstrass representation, which is set within a domain of complex numbers with identification $\mathbb{R}^2 = \mathbb{C}$, $z = u + \mathrm{i}v$, and includes usage of holomorphic functions. The advantages of using such a representation in $\mathbb{R}^3_1$ are not as significant as in the Euclidean space, where it is used for the global study of complete minimal surfaces, since there is only one complete maximal surface in $\mathbb{R}^3_1$, a spacelike plane [4, 5]. The following theorem is the adaptation of the Euclidean Weierstrass representation theorem to spacelike surfaces, [11, 22, 26].

Theorem 2.1 (Weierstrass representation). *Let ϕ_1, ϕ_2, ϕ_3 be complex holomorphic functions on a simply connected domain $U \subset \mathbb{R}^2 = \mathbb{C}$ such that*

(*a*) $-\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$

(*b*) $-\|\phi_1\|^2 + \|\phi_2\|^2 + \|\phi_3\|^2 > 0$

(*c*) *functions ϕ_k, $k = 1, 2, 3$ have no real period on U.*

Then the mapping $\mathbf{x}\colon U \to \mathbb{R}^3_1$ defined by

$$\mathbf{x}(u, v) = \mathrm{Re} \int^z (\phi_1, \phi_2, \phi_3)\, \mathrm{d}z \tag{2}$$

is a regular maximal isothermal map.

The Weierstrass representation (2) can be directly obtained from the Björling formula which determines the function $\phi(z) = (\phi_1(z), \phi_2(z), \phi_3(z))$, $z = u + \mathrm{i}v$, as

$$\phi(z) = \alpha'(z) + \mathrm{i}\, V(z) \times \alpha'(z),$$

where α is a regular spacelike holomorphic curve $\alpha\colon I \to \mathbb{R}^3_1$ and V a unit timelike holomorphic vector field along α, such that α' and V are orthogonal along I and V is the surface normal vector field.

Next, a maximal surface can also be given by a holomorphic function w and a meromorphic function g on U such that fg^2 is holomorphic, in the following way, [22]

$$w(z) = \phi_3 - \mathrm{i}\,\phi_2, \qquad g(z) = \frac{\phi_1}{\phi_3 - \mathrm{i}\,\phi_2}. \tag{3}$$

From $-\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$ and formula (3) we easily obtain

$$\phi_1 = gw, \quad \phi_2 = \frac{\mathrm{i}}{2}(1 - g^2)w, \quad \phi_3 = \frac{1}{2}(1 + g^2)w,$$

and a maximal surface is given by

$$\mathbf{x}(u, v) = \mathrm{Re} \int^z (2gw, \mathrm{i}\,(1 - g^2)w, (1 + g^2)w)\, \mathrm{d}z. \tag{4}$$

For a maximal surface we can adopt the definitions of the adjoint (the conjugate) surface, respectively the associate family of a minimal surface, [26]. Let S, $S^{\perp}$ be maximal surfaces locally parametrized by isothermal maps $\mathbf{x}\colon U \to S$, $\mathbf{y}\colon U \to S^{\perp}$.

Definition 2.2. *Two maximal surfaces S, $S^{\perp}$ in $\mathbb{R}^3_1$ are said to be adjoint (conjugate) if the local coordinate functions of one of them are the harmonic conjugates of the local coordinate functions of the other one. In terms of the isothermal parametrizations $\mathbf{x}$, $\mathbf{y}$ it holds $\mathbf{x}_u = \mathbf{y}_v$, $\mathbf{x}_v = -\mathbf{y}_u$.*

Theorem 2.3. *Let S be a maximal surface locally parametrized by an isothermal map $\mathbf{x}\colon U \to S$ and let $S^{\perp}$ be its adjoint surface given by $\mathbf{y}\colon U \to S^{\perp}$. Then the one-parameter family $(S_\theta)_{\theta\in\mathbb{R}}$ given by*

$$\mathbf{x}_\theta(u,v) := \mathbf{x}(u,v)\cos\theta + \mathbf{y}(u,v)\sin\theta,\ \theta\in\mathbb{R},$$

is a family of locally isometric maximal surfaces. The family $(S_\theta)_{\theta\in\mathbb{R}}$ is called the associated family of a surface S.

3. Cycloidal maximal surfaces

The purpose of this paper is to serve as a useful tool for studying maximal surfaces by providing explicit parametrizations of new examples. These examples will be constructed using the Björling formula, with cycloidal curves as the core curves. Consequently, we will refer to the resulting maximal surfaces as cycloidal maximal surfaces.

The idea for the curve choice comes from [32] where the author gives an overview of different algebraic minimal surfaces in Euclidean space and presents a new class of algebraic minimal surfaces with rational parametrization, the so-called cycloidal minimal surfaces. In present paper we differ between two classes of cycloidal maximal surfaces, the one, generated by Lorentzian cycloidal curves and the other, generated by Euclidean cycloidal curves. The vector field V is the linear combination of a normal and binormal vector field of the core curve. The idea for the vector field choice was presented in [27], where minimal surfaces with a circle or a helix as core curve were constructed. Further, Kaya and López present the analogous idea in the Lorentz-Minkowski space, for maximal spacelike surfaces in [22] and for minimal timelike surfaces in [15].

3.1. Maximal surfaces with a Lorentzian cycloid as a core curve

A spacelike Lorentzian cycloidal curve lying in a xz-plane is given by, [7]

$$\alpha(t) = \Big((R+r)\sinh t - r\sinh\Big(\frac{R+r}{r}t\Big), 0, (R+r)\cosh t + r\cosh\Big(\frac{R+r}{r}t\Big)\Big), \quad t,\in\mathbb{R} \tag{5}$$

where $r \neq 0, R > 0$. Binormal vector field of a curve $\alpha(t)$ is

$$\mathbf{b}(t) = \frac{\alpha'(t)\times\alpha''(t)}{\|\alpha'(t)\times\alpha''(t)\|} = \pm(0,1,0).$$

In the following, without loss of generality, we will assume that $\mathbf{b}(t) = (0,1,0)$. A normal vector is given by

$$\mathbf{n}(t) = \mathbf{b}(t)\times\mathbf{t}(t) = \Big(-\cosh\Big(\frac{R}{2r}t\Big), 0, \sinh\Big(\frac{R}{2r}t\Big)\Big)$$

where the tangent field is given by $\mathbf{t}(t) = \dfrac{\alpha'(t)}{\|\alpha'(t)\|}$. Following the Björling formula (1), vector field $V(t)$ needs to be unit timelike vector field along curve $\alpha(t)$. Since $\{\mathbf{n}(t), \mathbf{b}(t)\}$ is an orthonormal basis of a timelike plane $[\{\alpha'(t)\}]^{\perp}$, a vector field $V(t)$ can be chosen as

$$V(t) = \cosh\varphi(t)\mathbf{n}(t) + \sinh\varphi(t)\mathbf{b}(t),$$

where $\varphi(t)$ is an arbitrary function. In order to obtain integral in (1) explicitly, our choices for $\varphi(t)$ will be: $\varphi = 0$, $\varphi(t) = a$, $a \in \mathbb{R}, a \neq 0$ and $\varphi(t) = a\,t + b$, $a, b \in \mathbb{R}, a \neq 0$. Also, to obtain an explicit parametrization, we will use the following identities

$$\begin{aligned} \cos(u + \mathrm{i}v) &= \cos u \cosh v - \mathrm{i} \sin u \sinh v \\ \sin(u + \mathrm{i}v) &= \sin u \cosh v + \mathrm{i} \cos u \sinh v \\ \cosh(u + \mathrm{i}v) &= \cosh u \cos v + \mathrm{i} \sinh u \sin v \\ \sinh(u + \mathrm{i}v) &= \sinh u \cos v + \mathrm{i} \cosh u \sin v. \end{aligned} \tag{6}$$

Case 1. For $\varphi = 0$, vector field $V(t)$ corresponds to $\mathbf{n}(t)$ and the following holds.

Theorem 3.1. *Cycloidal maximal surfaces with a core curve (5) and unit normal vector field*

$$V(t) = \Big(-\cosh\Big(\frac{R}{2r}t\Big), 0, \sinh\Big(\frac{R}{2r}t\Big)\Big)$$

are given by

$$f(u,v) = \begin{pmatrix} (R+r)\sinh u \cos v - r\sinh\Big(\frac{R+r}{r}u\Big)\cos\Big(\frac{R+r}{r}v\Big) \\ -\frac{4r(R+r)}{R+2r}\sinh\Big(\frac{R+2r}{2r}u\Big)\sin\Big(\frac{R+2r}{2r}v\Big) \\ (R+r)\cosh u \cos v + r\cosh\Big(\frac{R+r}{r}u\Big)\cos\Big(\frac{R+r}{r}v\Big) \end{pmatrix}. \tag{7}$$

These surfaces are algebraic and allow rational parametrization. The core curve is a geodesic curve of such surfaces.

Proof. In order to apply (1), first we calculate

$$V(t) \times \alpha'(t) = (0, 2(R+r)\sinh\Big(\frac{R+2r}{2r}t\Big), 0),$$

$$\int^t V(t) \times \alpha'(t)\,\mathrm{d}t = (0, \frac{4r(R+r)}{R+2r}\cosh\Big(\frac{R+2r}{2r}t\Big), 0). \tag{8}$$

By inserting (5) and (8) into (1), then replacing the parameter t with a complex number $z = u + \mathrm{i}v$, using formulas (6), and extracting the real part, we obtain the required parametrization.

Next, since the parametrization (7) includes only trigonometric and hyperbolic functions, we can substitute them with their rational expressions in order to obtain a rational parametrization. Now from the rational parametrization by the parameter elimination, we obtain the algebraic equation of the surface.

The curve c is a geodesic curve due to the choice of a normal vector field of a surface. □

Corollary 3.2. *Parametrization (7) is an isothermal parametrization.*

Proof. A parametrization is an isothermal if and only if for the coefficients of the first fundamental form holds $E = G$ and $F = 0$. By a simple computation, we obtain

$$E = G = -\frac{1}{2}(R+r)^2\Big(\cos(2v) + \cos\Big(2\frac{R+r}{r}v\Big) + 2\cos\Big(\frac{R+2r}{r}v\Big) - 2\Big(1 + \cos\Big(\frac{R}{r}v\Big)\cosh\Big(\frac{R+2r}{r}\Big)u\Big)$$

$$F = 0$$

and the statement follows. □

Let us now obtain the Weierstrass representation of the surface (7). The required holomorphic functions are:

$$\phi_1 = (R+r)\Big(\cosh z - \cosh\Big(\frac{R+r}{r}z\Big)\Big)\,\mathrm{d}z = \frac{1}{2}\Big(e^{-z} + e^{z} - e^{-\frac{R+r}{r}z} - e^{\frac{R+r}{r}z}\Big)(R+r)\,\mathrm{d}z,$$
$$\phi_2 = 2\,\mathrm{i}\,(R+r)\sinh\Big(\frac{R+2r}{2r}z\Big)\,\mathrm{d}z = \mathrm{i}\,e^{-\frac{R+2r}{2r}z}\Big(e^{\frac{R+2r}{r}z} - 1\Big)(R+r)\,\mathrm{d}z,$$
$$\phi_3 = (R+r)\Big(\sinh z + \sinh\Big(\frac{R+r}{r}z\Big)\Big)\,\mathrm{d}z = \frac{1}{2}\Big(-e^{-z} + e^{z} - e^{-\frac{R+r}{r}z} + e^{\frac{R+r}{r}z}\Big)(R+r)\,\mathrm{d}z.$$

The Weierstrass representation of the surface (7) is

$$\begin{aligned} w(z) &= 4(R+r)\sinh\Big(\frac{R+2r}{2r}z\Big)\cosh^2\Big(\frac{R}{4r}z\Big) \\ &= \frac{1}{2}(R+r)\Big(2e^{-\frac{R+2r}{2r}z}\Big(e^{\frac{R+2r}{r}z} - 1\Big) + e^{\frac{R+r}{r}z} - e^{-\frac{R+r}{r}z} - e^{-z} + e^{z}\Big)\,\mathrm{d}z \\ g(z) &= -\tanh\Big(\frac{R}{4r}z\Big) = -\frac{e^{\frac{R}{r}z} - 1}{\Big(e^{\frac{R}{2r}z} + 1\Big)^2}. \end{aligned}$$

Theorem 3.3. *Let $f^{\perp}(u,v)$ be the adjoint surface of a surface $f(u,v)$ given by (7) and let*

$$F(u,v,\varphi) = f(u,v)\cos\varphi + f^{\perp}(u,v)\sin\varphi, \tag{9}$$

be its associated family of maximal surfaces. The u-parametric curves ($v = 0$) of a surface (9) are generalized helices (curves of constant slope) that lie on non-degenerate surfaces of the second order

$$-x^2 - \frac{(R+2r)^2\cot^2\varphi}{4r(R+r)}y^2 + z^2 = R^2\cos^2\varphi. \tag{10}$$

Proof. The adjoint surface of the surface (7) has the parametrization

$$f^{\perp}(u,v) = \begin{pmatrix} (R+r)\cosh u\sin v - r\cosh\big(\frac{R+r}{r}u\big)\sin\big(\frac{R+r}{r}v\big) \\ \frac{4r(R+r)}{R+2r}\cosh\big(\frac{R+2r}{2r}u\big)\cos\big(\frac{R+2r}{2r}v\big) \\ (R+r)\sinh u\sin v + r\sinh\big(\frac{R+r}{r}u\big)\sin\big(\frac{R+r}{r}v\big) \end{pmatrix}. \tag{11}$$

Since the surface (7) is a maximal surface with an isothermal parametrization, its adjoint surface (11) can be simply determined from the conditions $f(u,v)_u = f^{\perp}(u,v)_v$ and $f(u,v)_v = -f^{\perp}(u,v)_v$ (Definition 2.2). Therefore, it is obtained from (7) simply by replacing function $\sin v$ with $-\cos v$, function $\cos v$ with $\sin v$, and interchanging functions $\sinh u$ and $\cosh u$, since in this way parametrizations f, $f^{\perp}$ are harmonic conjugates. By inserting (7) and (11) in (9), we obtain that a parametrization of the associated family whose u-parametric curve ($v = 0$) is given by

$$c(u) = \begin{pmatrix} \cos\varphi\Big((R+r)\sinh u - r\sinh\big(\frac{R+r}{r}u\big)\Big) \\ \frac{4r(R+r)}{R+2r}\cosh\big(\frac{R+2r}{2r}u\big)\sin\varphi \\ \cos\varphi\Big((R+r)\cosh u + r\cosh\big(\frac{R+r}{r}u\big)\Big) \end{pmatrix}. \tag{12}$$

A straightforward computation shows that curve $c(u)$ is a generalized helix with axis $q = (0,1,0)$

$$\frac{\langle c', q\rangle}{\|c'\|} = \pm\sin\varphi$$

that lie on non-degenerate surface of the second order

$$-x^2 - \frac{(R+2r)^2 \cot^2 \varphi}{4r(R+r)} y^2 + z^2 = R^2 \cos^2 \varphi.$$

□

The next corollary is a direct consequence of the previous theorem, as shown in [7]:

Corollary 3.4. *Orthogonal projection of the parametric curves* (12) *of the associated family of maximal surfaces onto a plane orthogonal to a helical axis of* $c(u)$ *is a cycloidal curve.*

See Figure 1 as an example.

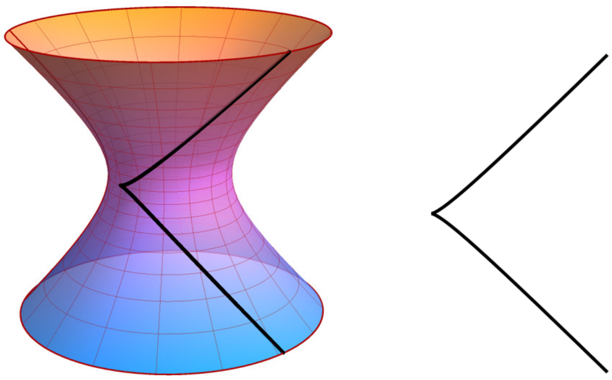

Figure 1: A special case of a general helix from the family (12) that lies on a pseudo-sphere with its projection, obtained for $R = 1$, $r = -\frac{1}{4}(2+\sqrt{2})$ and $\varphi = \pi/4$.

Case 2. For $\varphi(t) = a \in \mathbb{R}$, $a \neq 0$, we have $V(t) = \cosh(a)\,\mathbf{n}(t) + \sinh(a)\,\mathbf{b}(t)$ and obtain the following results.

Theorem 3.5. *Cycloidal maximal surfaces with a core curve (5) and unit normal vector field* $V(t) = \cosh(a)\,\mathbf{n}(t) + \sinh(a)\,\mathbf{b}(t)$ *where* $a \in \mathbb{R}\backslash\{0\}$, *are given by:*

$$f(u,v) = \begin{pmatrix} (R+r)\sinh u\, A + r\sinh\left(\frac{R+r}{r}u\right)B \\ -4\frac{r(R+r)}{R+2r}\cosh a \sin\left(\frac{R+2r}{2r}v\right)\sinh\left(\frac{R+2r}{2r}u\right) \\ (R+r)\cosh u\, A - r\cosh\left(\frac{R+r}{r}u\right)B \end{pmatrix} \tag{13}$$

where $A = \cos v + \sin v \sinh a$ *and* $B = -\cos\left(\frac{R+r}{r}v\right) + \sin\left(\frac{R+r}{r}v\right)\sinh a$. *These surfaces are algebraic and allow rational parametrization.*

The proof of this theorem, as well as proofs of the following theorems are analogous to the proof of Theorem (3.1), therefore it will all be omitted.

Holomorphic functions for the Weierstrass representation are:

$$\begin{aligned}
\phi_1 &= -2(R+r)\sinh\Big(\frac{R+2r}{2r}z\Big)\Big(\sinh\Big(\frac{R}{2r}z\Big) + \mathrm{i}\sinh a\cosh\Big(\frac{R}{2r}z\Big)\Big)\,\mathrm{d}z\\
&= -\frac{1}{4}\mathrm{i}\,A\,(R+r)e^{-a-\frac{R+r}{r}z}\Big(\big(e^a-\mathrm{i}\big)^2 e^{\frac{R}{r}z} + \big(e^a+\mathrm{i}\big)^2\Big)\,\mathrm{d}z,\\
\phi_2 &= 2\,\mathrm{i}\,(R+r)\cosh a\sinh\Big(\frac{R+2r}{2r}z\Big)\,\mathrm{d}z\\
&= \frac{1}{2}\,\mathrm{i}\,A\,(R+r)\big(e^{2a}+1\big)\,e^{-a-\frac{R+2r}{2r}z}\,\mathrm{d}z,\\
\phi_3 &= 2(R+r)\sinh\Big(\frac{R+2r}{2r}z\Big)\Big(\cosh\Big(\frac{R}{2r}z\Big) + \mathrm{i}\,\sinh a\sinh\Big(\frac{R}{2r}z\Big)\Big)\,\mathrm{d}z\\
&= \frac{1}{4}\mathrm{i}\,A\,(R+r)e^{-a-\frac{R+r}{r}z}\Big(\big(e^a-\mathrm{i}\big)^2 e^{\frac{R}{r}z} - \big(e^a+\mathrm{i}\big)^2\Big)\,\mathrm{d}z,
\end{aligned}$$

and the representation is

$$\begin{aligned}
w(z) &= A\,(R+r)e^{-\frac{R+2r}{2r}z}\Big(\cosh(a) + \cosh\Big(\frac{R}{2r}z\Big) + \,\mathrm{i}\,\sinh(a)\sinh\Big(\frac{R}{2r}z\Big)\Big)\,\mathrm{d}z\\
&= \frac{1}{4}\,\mathrm{i}\,A\,(R+r)e^{-a-\frac{R+r}{r}z}\Big(\big(e^a-\mathrm{i}\big)e^{\frac{R}{2r}z} - \mathrm{i}\,e^a + 1\Big)^2\,\mathrm{d}z,\\
g(z) &= \frac{\sinh\Big(\frac{a}{2}-\frac{R}{4r}z\Big) - \mathrm{i}\,\sinh\Big(\frac{1}{4}\Big(2a+\frac{R}{r}z\Big)\Big)}{\cosh\Big(\frac{a}{2}-\frac{R}{4r}z\Big) + \mathrm{i}\,\cosh\Big(\frac{1}{4}\Big(2a+\frac{R}{r}z\Big)\Big)}\\
&= -1 + \frac{2}{1+\frac{(1+\mathrm{i}\,e^a)e^{\frac{R}{2r}z}}{e^a+\mathrm{i}}}
\end{aligned}$$

where $A = e^{\frac{R+2r}{r}z} - 1$.

Case 3. Now we choose $\varphi(t) = a\,t + b$, $a \in \mathbb{R}\backslash\{0\}$ and without loss of generality, we can assume $b = 0$. For different choice of the parameter b we obtain the same surface, only rotated around x-axis.Then $V(t) = \cosh(a\,t)\,\mathbf{n}(t) + \sinh(a\,t)\,\mathbf{b}(t)$ and we obtain the following results.

Theorem 3.6. *Cycloidal maximal surfaces with a core curve (5) and unit normal vector field* $V(t) = \cosh(a\,t)\,\mathbf{n}(t) + \sinh(a\,t)\,\mathbf{b}(t)$ *where* $a \in \mathbb{R}\backslash\{0\}$, *are parametrized by*

$$f(u,v) = (x(u,v), y(u,v), z(u,v)) \tag{14}$$

where:

$$\begin{aligned}
x(u,v) = \frac{1}{2\,(a^2-1)\,((a-1)r-R)(ar+r+R)}&\Big((a^2-1)r(r+R)((a-1)r-R)\cosh\Big(\frac{ar+R+r}{r}u\Big)\sin\Big(\frac{ar+R+r}{r}u\Big)\\
&+ (ar+r+R)\Big(2((a-1)r-R)\Big((r+R)\sinh u\Big((a^2-1)\cos v + \sinh(au)(a\cos v\sin(av) - \sin v\cos(av))\Big)\\
&- \Big(a^2-1\Big)r\sinh\Big(\frac{R+r}{r}u\Big)\cos\Big(\frac{R+r}{r}v\Big)\Big) - (a^2-1)r(r+R)\cosh\Big(\frac{ar-r-R}{r}u\Big)\sin\Big(\frac{ar-r-R}{r}v\Big)\Big)\\
&+ 2(r+R)\cosh u((a-1)r-R)(ar+r+R)\cosh(au)(a\sin v\cos(av) - \cos v\sin(av))\Big)
\end{aligned}$$

$$y(u,v) = 2r(r+R)\Big(\frac{\sinh\big(\frac{2ar-R-2r}{2r}u\big)\sin\big(\frac{2ar-R-2r}{2r}v\big)}{2(a-1)r-R} - \frac{\sinh\big(\frac{2ar+R+2r}{2r}u\big)\sin\big(\frac{2ar+R+2r}{2r}v\big)}{2(a+1)r+R}\Big)$$

$$z(u,v) = \frac{1}{2(a^2-1)((a-1)r-R)(ar+r+R)}\Big(2(a^2-1)r((a-1)r-R)(ar+r+R)\cosh\Big(\frac{R+r}{r}u\Big)\cos\Big(\frac{R+r}{r}v\Big)$$
$$+ 2(r+R)\cosh u((a-1)r-R)(ar+r+R)\Big((a^2-1)\cos v + \sinh(au)(a\cos v\sin(av) - \sin v\cos(av))\Big)$$
$$+ (r+R)\Big((a^2-1)r\Big((-ar+r+R)\sinh(\frac{ar+R+r}{r}u)\sin(\frac{ar+R+r}{r}v) - (ar+r+R)\sinh(\frac{ar-r-R}{r}u)$$
$$\sin(\frac{ar-r-R}{r}v)\Big) + 2\sinh u((a-1)r-R)(ar+r+R)\cosh(au)(a\sin v\cos(av) - \cos v\sin(av))\Big)\Big).$$

These surfaces are algebraic and allow rational parametrization.

Holomorphic functions for the Weierstrass representation are:

$$\begin{aligned}
\phi_1 &= (R+r)\Big(\cosh(z) - \cosh\Big(\frac{R+r}{r}z\Big) - \mathrm{i}\,\sinh(az)\Big(\sinh(z) + \sinh\Big(\frac{R+r}{r}z\Big)\Big)\Big)\,\mathrm{d}z\\
&= \frac{1}{2}(R+r)\Big(B - \frac{1}{2}\,\mathrm{i}\left(e^{az} - e^{-az}\right)C\Big)\,\mathrm{d}z,\\
\phi_2 &= 2\,\mathrm{i}\,(R+r)\cosh(az)\sinh\Big(\frac{R+2r}{2r}z\Big)\,\mathrm{d}z\\
&= \frac{1}{2}\,\mathrm{i}\,e^{-\frac{2ar+2r+R}{2r}z}\left(1+e^{2az}\right)\left(e^{\frac{R+2r}{r}z} - 1\right)(R+r)\,\mathrm{d}z,\\
\phi_3 &= (R+r)\Big(\sinh(z) + \sinh\Big(\frac{R+r}{r}z\Big) - \mathrm{i}\,\sinh(az)\Big(\cosh(z) - \cosh\Big(\frac{R+r}{r}z\Big)\Big)\Big)\,\mathrm{d}z\\
&= \frac{1}{2}(R+r)\Big(C - \frac{1}{2}\,\mathrm{i}\left(e^{az} - e^{-az}\right)B\Big)\,\mathrm{d}z,
\end{aligned}$$

and the representation is

$$\begin{aligned}
w(z) &= 2\,\mathrm{i}\,(R+r)\sinh\Big(\frac{R+2r}{2r}z\Big)\Big(\cosh\Big(\frac{2ar+R}{4r}z\Big) - \mathrm{i}\,\cosh\Big(\frac{2ar-R}{4r}z\Big)\Big)^2\,\mathrm{d}z\\
&= \frac{1}{2}(R+r)\Big(-\frac{1}{2}\,\mathrm{i}\left(e^{az} - e^{-az}\right)B + \left(e^{2az}+1\right)\Big(e^{\frac{R+2r}{r}z} - 1\Big)e^{-\frac{2ar+2r+R}{2r}z} + C\Big)\,\mathrm{d}z,\\
g(z) &= \frac{\sinh(az)\cosh(\frac{R}{2r}z) - \mathrm{i}\,\sinh(\frac{R}{2r}z)}{\Big(\cosh\left(\frac{2ar-R}{4r}z\right) + \mathrm{i}\,\cosh\left(\frac{2ar+R}{4r}z\right)\Big)^2}\,\mathrm{d}z,\\
&= \frac{B - \frac{1}{2}\,\mathrm{i}\,(e^{az} - e^{-az})\,C}{C - \frac{1}{2}\,\mathrm{i}\,B\Big(e^{az} - e^{-az}\Big) + (e^{2az}+1)(e^{\frac{R+2r}{r}z} - 1)e^{-\frac{2ar+2r+R}{2r}z}},
\end{aligned}$$

where $A = e^{\frac{R+r}{r}z}$, $B = -A - \dfrac{1}{A} + e^{-z} + e^{z}$ and $C = A - \dfrac{1}{A} - e^{-z} + e^{z}$.

3.1.1. *Lorentzian Henneberg surface*

In the following we are going to analyze the spacelike counterpart of the Henneberg surface in Lorentz-Minkowski space. In Euclidean space, the Henneberg surface is a minimal surface obtained by a Björling formula with the Neile parabola (a semicubic parabola) as the core curve, therefore having it as a planar geodesic, [11, 13, 30]. It can be parametrized by

$$f(u,v) = \begin{pmatrix} 2\cos v\sinh u - \frac{2}{3}\cos 3v\sinh 3u \\ 2\cos 2v\cosh 2u \\ 2\sin v\sinh u + \frac{2}{3}\sin 3v\sinh 3u \end{pmatrix}. \tag{15}$$

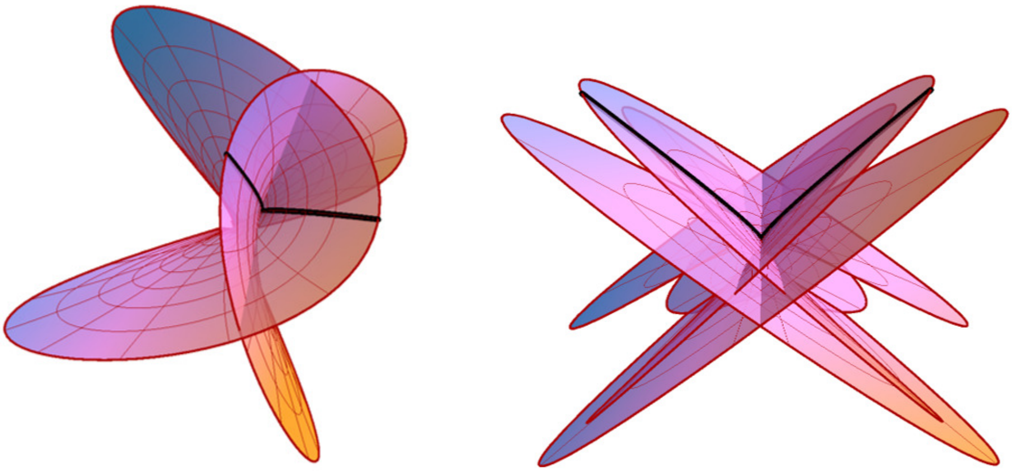

Figure 2: Cycloidal maximal surfaces. The left is the surface (13) for $R = 8$, $r = 4$ and $a = 0.1$ and the right is the surface (14) for $R = 4$, $r = 1$ and $a = 0.2$. The black curves are the core curves.

For $v = 0$ we have the Neile parabola, since

$$f(u, 0) = (2 \sinh u - \frac{2}{3} \sinh 3u, 2 \cosh 2u, 0)$$

satisfies the equation

$$9x^2 = (y-2)^3, \quad z = 0.$$

Its (Euclidean) adjoint surface is given by

$$f^{\perp}(u, v) = \begin{pmatrix} 2 \sin v \cosh u - \frac{2}{3} \sin 3v \cosh 3u \\ 2 \sin 2v \sinh 2u \\ -2 \cos v \cosh u - \frac{2}{3} \cos 3v \cosh 3u \end{pmatrix}. \tag{16}$$

In this paper we will call the surface (16) the adjoint Henneberg surface. We can notice that the adjoint Henneberg surface is a surface generated by a curve with $u = 0$

$$\begin{aligned} f^{\perp}(0, v) &= (2 \sin v - \frac{2}{3} \sin 3v, 0, -2 \cos v - \frac{2}{3} \cos 3v) \\ &= (\frac{8}{3} \sin^3 v, 0, -\frac{8}{3} \cos^3 v), \end{aligned}$$

which is a Euclidean asteroid in the $y = 0$ plane.

To obtain Lorentzian counterparts of these surfaces, we consider the well-known idea of duality between minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{R}^3_1$, [23, 24]. If (ϕ_1, ϕ_2, ϕ_3) are holomorphic function of a minimal surface in $\mathbb{R}^3$, then $(\psi_1, \psi_2, \psi_3) = (i\phi_1, \phi_2, \phi_3)$ are holomorphic functions of a maximal surface in $\mathbb{R}^3_1$.

Therefore, the Lorentzian dual surface of the Henneberg surfaces is parametrized by

$$f_L(u, v) = \begin{pmatrix} 2 \sin v \cosh u - \frac{2}{3} \sin 3v \cosh 3u \\ 2 \cos 2v \cosh 2u \\ 2 \sin v \sinh u + \frac{2}{3} \sin 3v \sinh 3u \end{pmatrix}.$$

In this paper we will call this surface Lorentzian Henneberg surface. Its first coordinate function matches that of surface (16), while the second and third coordinate functions correspond to those of surface (15).

Theorem 3.7. *Lorentzian Henneberg surface is a spacelike surface generated by the Björling formula over the Neile parabola* $(u = 0)$

$$f_L(0,v) = (2\sin v - \frac{2}{3}\sin 3v, 2\cos 2v, 0)$$

which satisfies the equation $9x^2 = -(y-2)^3,\ \ z = 0$.

The Lorentzian dual of the adjoint Henneberg surface is parametrized by

$$f_L^{\perp}(u,v) = \begin{pmatrix} -2\cos v\sinh u + \frac{2}{3}\cos 3v\sinh 3u \\ 2\sin 2v\sinh 2u \\ -2\cos v\cosh u - \frac{2}{3}\cos 3v\cosh 3u \end{pmatrix}. \tag{17}$$

This surface is the adjoint of the Lorentzian Henneberg surface, and we will refer to it as the adjoint Lorentzian Henneberg surface.

Theorem 3.8. *The adjoint Lorentzian Henneberg surface is a maximal surface with a Lorentzian astroid in* $y = 0$ *plane as a planar geodesic (Figure 4)*

$$f_L^{\perp}(u,0) = \begin{pmatrix} -2\sinh u + \frac{2}{3}\sinh 3u \\ 0 \\ -2\cosh u - \frac{2}{3}\cosh 3u \end{pmatrix} = \begin{pmatrix} \frac{8}{3}\sinh^3 u \\ 0 \\ -\frac{8}{3}\cosh^3 u \end{pmatrix}. \tag{18}$$

Proof. The parametric u-curve

$$\alpha(u) = f_L^{\perp}(u,0) = \Big(-2\sinh u + \frac{2}{3}\sinh 3u, 0, -2\cosh u - \frac{2}{3}\cosh 3u\Big) = (\frac{8}{3}\sinh^3 v, 0, -\frac{8}{3}\cosh^3 v),$$

is a spacelike curve in a Minkowski plane $y = 0$, with

$$\alpha'(u) = (8\cosh u\sinh^2 u, 0, -8\cosh u^2\sinh u) = 4\sinh(2u)(\sinh u, 0, -\cosh u)$$

and $\langle\alpha'(u), \alpha'(u)\rangle = 16\sinh^2 2u$. It represents a Lorentzian counterpart of a Euclidean astroid, [7]. Now, using the formula (1) we obtain the adjoint Lorentzian Henneberg surface (17).
The tangent, resp. binormal and normal vector field of a curve $\alpha(u)$ are

$$\mathbf{t} = \frac{\alpha'}{\|\alpha'\|} = (\sinh u, 0, -\cosh u), \quad \mathbf{b} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|} = (0, -1, 0), \quad \mathbf{n} = \mathbf{b} \times \mathbf{t} = (-\cosh u, 0, \sinh u).$$

In order to apply (1), we calculate

$$\mathbf{n} \times \alpha' = (0, -4\sinh(2u), 0),$$

$$\int^u \mathbf{n} \times \alpha'\, \mathrm{d}u = (0, -2\cosh(2u), 0). \tag{19}$$

Furthermore, by inserting (18) and (19) into (1), then by replacing the parameter u with a complex number $z = u + \mathrm{i}v$, and extracting the real part, we obtain the required parametrization (17). □

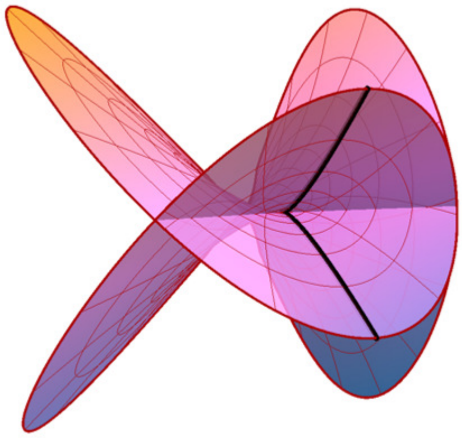

Figure 3: The adjoint Lorentzian Henneberg surface is a cycloidal maximal surface over the Lorentzian astroid.

Euclidean space $\mathbb{R}^3$		Lorentz-Minkowski space $\mathbb{R}^3_1$
Henneberg surface		Lorentzian Henneberg surface
$\begin{pmatrix} 2\cos v \sinh u - \frac{2}{3}\cos 3v \sinh 3u \\ 2\cos 2v \cosh 2u \\ 2\sin v \sinh u + \frac{2}{3}\sin 3v \sinh 3u \end{pmatrix}$	Dual $\rightleftarrows$	$\begin{pmatrix} 2\sin v \cosh u - \frac{2}{3}\sin 3v \cosh 3u \\ 2\cos 2v \cosh 2u \\ 2\sin v \sinh u + \frac{2}{3}\sin 3v \sinh 3u \end{pmatrix}$
$\updownarrow$ Adjoint		$\updownarrow$ Adjoint
$\begin{pmatrix} 2\sin v \cosh u - \frac{2}{3}\sin 3v \cosh 3u \\ 2\sin 2v \sinh 2u \\ -2\cos v \cosh u - \frac{2}{3}\cos 3v \cosh 3u \end{pmatrix}$	Dual $\rightleftarrows$	$\begin{pmatrix} -2\cos v \sinh u + \frac{2}{3}\cos 3v \sinh 3u \\ 2\sin 2v \sinh 2u \\ -2\cos v \cosh u - \frac{2}{3}\cos 3v \cosh 3u \end{pmatrix}$
Adjoint Henneberg surface		Adjoint Lorentzian Henneberg surface

Remark. Surface (17) is obtained from (7) for $R = -\frac{4}{3}$ and $r = -\frac{2}{3}$.

3.1.2. Lorentzian Catalan surface

Another well-known example of a minimal surface is the Catalan minimal surface [16, 31, 32], a minimal surface generated over a straight (ordinary) cycloid (a curve obtained by a rolling a curve along a line). In this work, we examine its Lorentzian counterpart, which we call the Lorentzian Catalan maximal surface.

Theorem 3.9. *The Lorentzian Catalan surface is a maximal surface with a straight cycloid in $y = 0$ plane as a planar geodesic*

$$\alpha(t) = (\sinh t - t, 0, 1 - \cosh t). \tag{20}$$

Proof. The straight cycloid (20) is a spacelike curve whose tangent, resp. binormal and normal vector fields are

$$\mathbf{t}(t) = \Big(\sinh\frac{t}{2}, 0, -\cosh\frac{t}{2}\Big), \quad \mathbf{b}(t) = (0,1,0), \quad \mathbf{n}(t) = \Big(\cosh\frac{t}{2}, 0, -\sinh\frac{t}{2}\Big).$$

In order to apply (1), we calculate

$$\mathbf{n}(t)\times\alpha'(t) = \Big(0, 2\sinh\frac{t}{2}, 0\Big),$$

$$\int^t \mathbf{n}(t)\times\alpha'(t)\,\mathrm{d}t = \Big(0, 4\cosh\frac{t}{2}, 0\Big). \tag{21}$$

Further, by inserting (20) and (21) into (1), then by replacing the parameter t with a $z = u + \mathrm{i}v$, and extracting the real part, we obtain the parametrization of Lorentzian Catalan maximal surface

$$f_{CL}(u,v) = \begin{pmatrix} \sinh u\cos v - u \\ -4\sinh\frac{u}{2}\sin\frac{v}{2} \\ 1-\cosh u\cos v \end{pmatrix}. \tag{22}$$

□

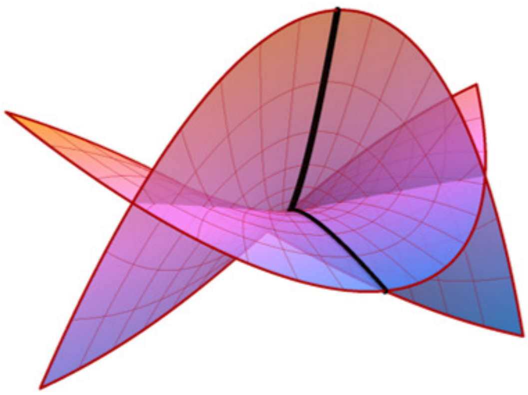

Figure 4: Lorentzian Catalan's surface is a maximal surface over a Lorentzian straight cycloid

Corollary 3.10. *Parametrization (22) is an isothermal parametrization.*

Proof. By a simple calculation, we obtain

$$E = G = 2\cos^2\Big(\frac{v}{2}\Big)(\cosh(u) - \cos(v)), \quad F = 0,$$

and the statement follows. □

Let us now obtain the Weierstrass representation of the surface (22). The required holomorphic functions are:

$$\phi_1 = (\cosh z - 1)\,\mathrm{d}z, \quad \phi_2 = 2\,\mathrm{i}\sinh\frac{z}{2}\,\mathrm{d}z, \quad \phi_3 = -\sinh z\,\mathrm{d}z.$$

The Weierstrass representation of the surface (7) is

$$w(z) = (2\sinh\frac{z}{2} - \sinh z)\,\mathrm{d}z, \quad g(z) = -\coth\frac{z}{4}.$$

Surface (22) is a dual surface of the Catalan minimal surface, [11].

Euclidean space $\mathbb{R}^3$	Lorentz-Minkowski space $\mathbb{R}^3_1$
Catalan surface	Lorentzian Catalan surface

$$\begin{pmatrix} u - \sin u \cosh v \\ -4 \sin \frac{u}{2} \sinh \frac{v}{2} \\ 1 - \cos u \cosh v \end{pmatrix} \overset{\text{Dual}}{\rightleftarrows} \begin{pmatrix} -v + \cos u \sinh v \\ -4 \sin \frac{u}{2} \sinh \frac{v}{2} \\ 1 - \cos u \cosh v \end{pmatrix}$$

$$\text{Adjoint} \updownarrow \qquad\qquad \updownarrow \text{Adjoint}$$

$$\begin{pmatrix} -v + \cos u \sinh v \\ 4 \cos \frac{u}{2} \cosh \frac{v}{2} \\ 1 - \sin u \sinh v \end{pmatrix} \overset{\text{Dual}}{\rightleftarrows} \begin{pmatrix} -u + \sin u \cosh v \\ 4 \cos \frac{u}{2} \cosh \frac{v}{2} \\ 1 - \sin u \sinh v \end{pmatrix}$$

Adjoint Catalan surface	Adjoint Lorentzian Catalan surface

3.2. *Maximal surfaces with a Euclidean cycloid as a core curve*

Euclidean cycloidal curves that lie in xz-plane are not a proper choice for a core curves since they are spacelike curves only locally. Therefore, we embed these curves in a spacelike yz-plane. A spacelike Euclidean cycloidal curve in yz-plane is given by (see [7])

$$\alpha(t) = \Big(0, (R+r)\sin t - r\sin\Big(\frac{R+r}{r}t\Big), (R+r)\cos t - r\cos\Big(\frac{R+r}{r}t\Big)\Big), \quad t \in \mathbb{R} \tag{23}$$

and normal, resp. binormal vector field of a curve $\alpha(t)$ are

$$\mathbf{n}(t) = \frac{1}{2\sin\big(\frac{R}{2r}t\big)}\Big(0, \sin t - \sin\big(\frac{R+r}{r}t\big), \cos t - \cos\big(\frac{R+r}{r}t\big)\Big),$$
$$\mathbf{b}(t) = (1, 0, 0).$$

Again the vector field $V(t)$ is chosen as $V(t) = \sinh\varphi(t)\,\mathbf{n}(t) + \cosh\varphi(t)\,\mathbf{b}(t)$, where $\varphi(t)$ is an arbitrary function and choices for $\varphi(t)$ are the same as for surfaces with Lorentzian cycloids.

Case 1. If $\varphi = 0$, then $V(t) = \mathbf{b}(t)$ and the obtained surface is yz-plane.

Case 2. If $\varphi(t) = a \in \mathbb{R}$, $a \neq 0$, then $V(t) = \sinh(a)\,\mathbf{n}(t) + \cosh(a)\,\mathbf{b}(t)$ and we obtain the following results.

Theorem 3.11. *Cycloidal maximal surfaces with a core curve (23) and normal unit vector* $V(t) = \sinh(a)\,\mathbf{n}(t) + \cosh(a)\,\mathbf{b}(t)$ *where* $a \in \mathbb{R}\backslash\{0\}$, *are parametrized by*

$$f(u,v) = \begin{pmatrix} -\frac{4r(R+r)}{R}\sin\left(\frac{R}{2r}u\right)\sinh a \sinh\left(\frac{R}{2r}v\right) \\ (R+r)\cosh v \sin u - r\cosh\left(\frac{R+r}{r}v\right)\sin\left(\frac{R+r}{r}u\right) - \cosh a\, A \\ (R+r)\cos u \cosh v - r\cos\left(\frac{R+r}{r}u\right)\cosh\left(\frac{R+r}{r}v\right) - \cosh a\, B \end{pmatrix} \tag{24}$$

where A and B are given by

$$A = (R + r)\sin u \sinh v - r \sin\left(\frac{R + r}{r}u\right)\sinh\left(\frac{R + r}{r}v\right)$$
$$B = (R + r)\cos u \sinh v - r \cos\left(\frac{R + r}{r}u\right)\sinh\left(\frac{R + r}{r}v\right).$$

These surfaces are algebraic and allow rational parametrization.

Holomorphic functions for the Weierstrass representation are:

$$\begin{aligned}
\phi_1 &= 2\,\mathrm{i}\,(R + r)\sin\left(\frac{R}{2r}z\right)\sinh a\,\mathrm{d}z \\
&= \frac{1}{2}e^{-a-\frac{\mathrm{i}Rz}{2r}}\left(-1 + e^{2a}\right)\left(-1 + e^{\frac{\mathrm{i}Rz}{r}}\right)(R + r)\,\mathrm{d}z, \\
\phi_2 &= (R + r)\Big(\cos z - \cos\left(\frac{R + r}{r}z\right) + \mathrm{i}\cosh a\Big(\sin z - \sin\left(\frac{R + r}{r}z\right)\Big)\Big)\,\mathrm{d}z \\
&= -\frac{1}{4}e^{-a-\frac{\mathrm{i}(R+r)z}{r}}\left(-1 + e^{\frac{\mathrm{i}Rz}{r}}\right)\left(C + e^{\frac{\mathrm{i}(2r+R)z}{r}} + 2e^{a+\frac{\mathrm{i}(2r+R)z}{r}} + e^{2a+\frac{\mathrm{i}(2r+R)z}{r}}\right)(R + r)\,\mathrm{d}z, \\
\phi_3 &= (R + r)\Big(\mathrm{i}\Big(\cos z - \cos\left(\frac{R + r}{r}z\right)\Big)\cosh a - \sin z + \sin\left(\frac{R + r}{r}z\right)\Big)\,\mathrm{d}z \\
&= -\frac{1}{4}\mathrm{i}\,e^{-a-\frac{\mathrm{i}(R+r)z}{r}}\left(-1 + e^{\frac{\mathrm{i}Rz}{r}}\right)\left(-C + e^{\frac{\mathrm{i}(2r+R)z}{r}} + 2e^{a+\frac{\mathrm{i}(2r+R)z}{r}} + e^{2a+\frac{\mathrm{i}(2r+R)z}{r}}\right)(R + r)\,\mathrm{d}z
\end{aligned}$$

and the representation is

$$w(z) = \frac{1}{2}\mathrm{i}\,e^{-a-\frac{\mathrm{i}(R+r)z}{r}}(-1 + e^{a})^2\left(-1 + e^{\frac{\mathrm{i}Rz}{r}}\right)(R + r)\,\mathrm{d}z$$
$$g(z) = -\frac{\mathrm{i}\,e^{\frac{\mathrm{i}(2r+R)z}{2r}}(1 + e^{a})}{-1 + e^{a}}$$

where $C = 1 - 2e^{a} + e^{2a}$.

Case 3. Next, we choose $\varphi(t) = a\,t + b$, $a \in \mathbb{R}\backslash\{0\}$ and again, without loss of generality, we assume $b = 0$. Then $V(t) = \cosh(a\,t)\,\mathbf{n}(t) + \sinh(a\,t)\,\mathbf{b}(t)$ and we obtain the following results.

Theorem 3.12. *Cycloidal maximal surfaces with a core curve (23) and normal unit vector field* $V(t) = \sinh(a\,t)\,\mathbf{n}(t) + \cosh(a\,t)\,\mathbf{b}(t)$ *where* $a \in \mathbb{R}\backslash\{0\}$*, are parametrized by*

$$f(u, v) = (x(u, v), y(u, v), z(u, v)) \tag{25}$$

where:

$$\begin{aligned}
x(u, v) = &-\frac{4r(r+R)}{4a^2r^2+R^2}\Big(\cos\left(\frac{R}{2r}u\right)\cosh au\Big(-R\cosh\left(\frac{R}{2r}v\right)\sin av + 2ar\cos av\sinh\left(\frac{R}{2r}v\right)\Big) \\
&+ \sin\left(\frac{R}{2r}u\right)\sinh au(2ar\cosh\left(\frac{R}{2r}v\right)\sin av + R\cos av\sinh\left(\frac{R}{2r}v\right)\Big),
\end{aligned}$$

$$\begin{aligned}
y(u, v) = &\frac{1}{(1 + a^2)\,A}\Big((R + r)A\cosh v\Big(\sin u(1 + a^2 - a\cosh au\sin av) + \cos u\sin av\sinh au\Big) \\
&+ (1 + a^2)r\cosh\Big(\frac{R+r}{r}v\Big)\Big(\sin\left(\frac{R+r}{r}u\right)(-A + ar(r + R)\cosh au\sin av) \\
&- (r + R)^2\cos\Big(\frac{R+r}{r}u\Big)\sin av\sinh au\Big) + (r + R)\cos av\Big(-A\,(\cosh au\sin u + a\cos u\sinh au)\sinh v \\
&+ (1 + a^2)r\Big((r + R)\cosh au\sin\Big(\frac{R+r}{r}u\Big) + ar\cos\Big(\frac{R+r}{r}u\Big)\sinh au\Big)\sinh\Big(\frac{R+r}{r}v\Big)\Big)\Big),
\end{aligned}$$

$$\begin{aligned} z(u,v) = & \frac{1}{(1+a^2)A}\Big((1+a^2)r\cosh\Big(\frac{R+r}{r}v\Big)\Big(\cos\Big(\frac{R+r}{r}u\Big)\Big(-A+ar(R+r)\cosh au\sin av\Big) \\ & + (R+r)^2\sin\Big(\frac{R+r}{r}u\Big)\sin av\sinh au\Big) + (R+r)A\cosh v\Big((1+a^2)\cos u - \sin av(a\cos u\cosh au \\ & + \sin u\sinh au)\Big) + (R+r)\cos av\Big(A(-\cos u\cosh au + a\sin u\sinh au)\sinh v \\ & + (1+a^2)r\Big((R+r)\cos\Big(\frac{R+r}{r}u\Big)\cosh au - ar\sin\Big(\frac{R+r}{r}u\Big)\sinh au\Big)\sinh\Big(\frac{R+r}{r}v\Big)\Big)\Big) \end{aligned}$$

where A is given by $A = r^2 + a^2r^2 + 2rR + R^2$. These surfaces are algebraic and allow rational parametrization.

Holomorphic functions for the Weierstrass representation are

$$\begin{aligned} \phi_1 &= 2\,\mathrm{i}\,(R+r)\sin\Big(\frac{R}{2r}z\Big)\sinh az\,\mathrm{d}z \\ &= \frac{1}{2}e^{-az-\frac{\mathrm{i}Rz}{2r}}\left(-1+e^{2az}\right)\left(-1+e^{\frac{\mathrm{i}Rz}{r}}\right)(R+r)\,\mathrm{d}z, \\ \phi_2 &= (R+r)\Big(\cos z - \cos\Big(\frac{R+r}{r}z\Big) + \mathrm{i}\,\cosh az\Big(\sin z - \sin\Big(\frac{R+r}{r}z\Big)\Big)\Big)\,\mathrm{d}z \\ &= -\frac{1}{4}e^{-\left(a+\frac{\mathrm{i}(R+r)}{r}\right)z}\Big(-1+e^{\frac{\mathrm{i}Rz}{r}}\Big)\Big(B + e^{\frac{\mathrm{i}(2r+R)z}{r}} + 2e^{\left(a+\mathrm{i}\left(2+\frac{R}{r}\right)\right)z} + e^{\left(2a+\mathrm{i}\left(2+\frac{R}{r}\right)\right)z}\Big)(r+R)\mathrm{d}z, \\ \phi_3 &= (R+r)\Big(\mathrm{i}\Big(\cos z - \cos\Big(\frac{R+r}{r}z\Big)\Big)\cosh az - \sin z + \sin\Big(\frac{R+r}{r}z\Big)\Big)\,\mathrm{d}z \\ &= \frac{1}{4}\mathrm{i}\,e^{-\left(a+\frac{\mathrm{i}(R+r)}{r}\right)z}\Big(-1+e^{\frac{\mathrm{i}Rz}{r}}\Big)\Big(B - e^{\frac{\mathrm{i}(2r+R)z}{r}} - 2e^{\left(a+\mathrm{i}\left(2+\frac{R}{r}\right)\right)z} - e^{\left(2a+\mathrm{i}\left(2+\frac{R}{r}\right)\right)z}\Big)(r+R)\mathrm{d}z \end{aligned}$$

and the representation is

$$w(z) = \frac{1}{2}\mathrm{i}\,e^{-\left(a+\frac{\mathrm{i}(R+r)}{r}\right)z}\left(-1+e^{az}\right)^2\left(-1+e^{\frac{\mathrm{i}Rz}{r}}\right)(R+r)\,\mathrm{d}z$$

$$g(z) = -\frac{\mathrm{i}e^{\frac{\mathrm{i}(2r+R)z}{2r}}\left(1+e^{az}\right)}{-1+e^{az}}$$

where $B = 1 - 2e^{az} + e^{2az}$.

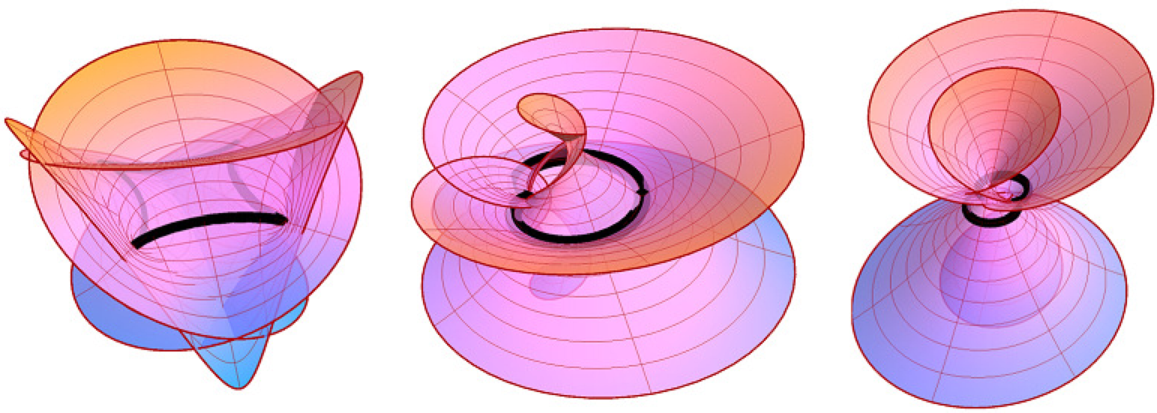

Figure 5: Cycloidal maximal surfaces (24) where the core curve is euclidean astroid ($R = -4$, $r = 1$, $a = 1$), cardioid ($R = 1$, $r = 1$, $a = 1$) and nodoid ($R = 2$, $r = 1$, $a = 2$).

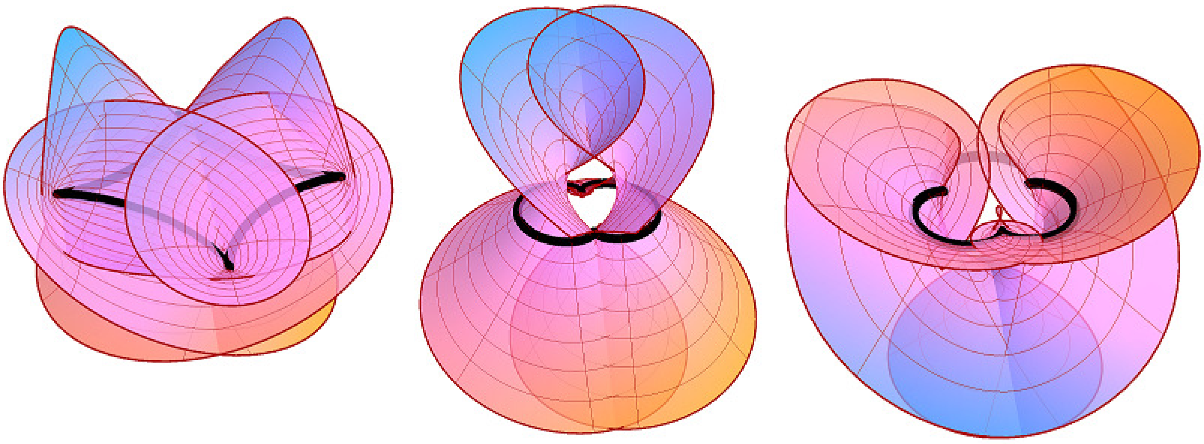

Figure 6: Cycloidal maximal surfaces(25) where the core curve is euclidean astroid ($R = -4$, $r = 1$, $a = 0.5$), nodoid ($R = 2$, $r = 1$, $a = 1$) and epicycloid ($R = 3$, $r = 1$, $a = 1$)

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