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Semi-symmetric ∗-Ricci tensor on generalized Sasakian space forms

Junxia Zhu[a], Rongsheng Ma[a,*]

[a] *School of Science, Yanshan University, Qinhuangdao, 066004, China*

Abstract. We study the ∗-Ricci tensor on generalized Sasakian space forms in both Riemannian and semi-Riemannian settings. We investigate the ∗-Ricci semi-symmetric property of generalized Sasakian space forms and its application: this provides a new approach to quantify how close a Sasakian space form is to being a constant curvature space. At the end of this paper, we present an example of a Sasakian space form with semi-symmetric ∗-Ricci tensor and provide its classification.

1. Introduction

In Riemannian geometry, a simple connected Riemannian manifold (M, g) with constant sectional curvature c is a real space form and the curvature tensor is

$$R(X, Y)Z = c(g(Y, Z)X - g(X, Z)Y),$$

according to different values of c, there are three models of real space form: Euclidean space $\mathbb{R}^n(c = 0)$, sphere $S^n(c > 0)$ and hyperbolic space $\mathbb{H}^n(c < 0)$. In complex geometry, if a Kǎhler manifold (M, J, g) has constant holomorphic sectional curvature c then it ia a complex space form and the curvature is

$$R(X, Y)Z = \frac{c}{4}(g(Y, Z)X - g(X, Z)Y + g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ),$$

and the three models are $\mathbb{C}^n(c = 0)$, $\mathbb{CP}^n(c > 0)$ and $\mathbb{CH}^n(c < 0)$. In [32] (or [33]), F.Tricerri and L.Vanhecke gave a generalization of complex space form. For an almost Hermitian manifold (M, J, g), if the curvature tensor R satisfies:

$$R(X, Y)Z = F_1(g(Y, Z)X - g(X, Z)Y) + F_2(g(X, JZ)JY - g(Y, JZ)JX + 2g(X, JY)JZ),$$

where F_1, F_2 are differentiable functions on M, then M is a generalized complex space form. Moreover, they showed that for a connected generalized complex space form of dimension at least six, if F_2 is non-zero, then it was necessarily a complex space form.

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* Corresponding author: Rongsheng Ma
Email addresses: zhujx318@ysu.edu.cn (Junxia Zhu), mars@ysu.edu.cn (Rongsheng Ma)
ORCID iDs: https://orcid.org/0000-0002-8787-6149 (Junxia Zhu), https://orcid.org/0000-0001-6380-4864 (Rongsheng Ma)

In contact geometry, the Sasakian space form corresponds to the complex space form. If the ϕ-sectional curvature of a Sasakian manifold is constant c, then the curvature tensor of the manifold has the following form:

$$\begin{aligned} R(X,W)Z =& \frac{c+3}{4}(g(W,Z)X - g(X,Z)W) + \frac{c-1}{4}(g(X,\phi Z)\phi W - g(W,\phi Z)\phi X \\ &+ 2g(X,\phi W)\phi Z)) + \frac{c-1}{4}(\eta(X)\eta(Z)W - \eta(W)\eta(Z)X + g(X,Z)\eta(W)\zeta - g(W,Z)\eta(X)\zeta), \end{aligned} \tag{1}$$

and M is called Sasakian-space-form. The notion of generalized Sasakian space form was introduced by P.Alegre, D.E.Blair and A.Carriazo in [2]. The semi-Riemannian setting of generalized Sasakian space forms was given in [4]. In [20] and [21], the authors studied the Ricci solitons on Lorentzian generalized Sasakian space forms and get beautiful and significant results. For further key results in Riemannian geometry, see the works [18, 19].

Although generalized Sasakian space form is inspired by Sasakian space form, it is not only a simple generalization, but contains a large class of almost contact metric manifolds. For example, when trans-Sasakian three-manifold satisfies some certain condition, it is a generalized Sasakian space form and contact generalized Sasakian space form is a generalized (κ, μ)-space (see [3]). Generally speaking, the study of generalized Sasakian space forms can be approached from three perspectives. One is the study of various curvature tensors on the generalized Sasakian space form and their relationships. U.C.De and A.Sarkar gave the the necessary and sufficient condition that the generalized Sasakian space form being projective flat (see [8]), and they also studied the quasi-conformal curvature tensor (see [28]). U.K.Kim studied the conformal flat and locally symmetric generalized Sasakian space form (see [17]). The corresponding author of this paper also obtained some good results (see [22]). The second aspect is the study of various solitons on generalized Sasakian space form. For example, people have studied the conformal Ricci soliton and quasi-Yamabe soliton on generalized Sasakian space form (see [11]) and there are good results about Ricci soliton on three-dimensional generalized Sasakian space form (see [14, 27]). The third is the study of submanifolds of generalized Sasakian space form (see [1, 5, 16]).

In contact geometry, $*$-Ricci tensor is a tensor similar to Ricci tensor but it has different geometric properties. In 1959, S.Tachibana gave the notion of $*$-Ricci tensor in complex geometric (see [29, 30]). In 2002, T.Hamada extended the notion of $*$-Ricci tensor to almost contact manifolds in [13]. Here we introduce an intuitive way to understand $*$- Ricci tensor. Let (M, J, h) be a Kӑhler manifold, its Ricci tensor has expression

$$\mathrm{Ric}(X,Y) = \frac{1}{2}\,\mathrm{trace}\{J \circ R(X, JY)\}.$$

In contact geometry, the almost contact structure includes a $(1,1)$-tensor field ϕ, which corresponds to the almost complex structure J. However, replacing J by ϕ in above formula is not the Ricci tensor of the almost contact manifold. It is $*$-Ricci tensor in contact geometry:

$$\mathrm{Ric}^*(X,Y) = \frac{1}{2}\,\mathrm{trace}\{\phi \circ R(X, \phi Y)\}.$$

After the $*$-Ricci tensor of contact manifold being proposed, it attracts great interest of mathematicians. The corresponding author of this paper investigated the $*$-Ricci tensor on trans-Sasakian three-manifolds (see [23]). After G.Kaimakamis and K.Panagiotidou defined $*$-Ricci solitons on real hypersurfaces in complex space form (see [15]), many mathematicians had studied $*$-Ricci solitons on different almost contact manifolds in many aspects, and obtained important and meaningful results. For example, Venkatesha, D.M.Naik and H.A.Kumara proved that if a three-dimensional Kenmotsu manifold was a $*$-Ricci soliton, then it was of constant curvature -1 (see [34]). P.Majhi, U.C.De and Y.J.Suh proved that if a three-dimensional Sasakian manifold was a $*$-Ricci soliton, then it was a manifold with constant curvature and the potential vector field was a Killing vector field (see [26]). A.Ghosh and D.S.Patra gave a complete classification of $*$-Ricci soliton of non-Sasakian (κ, μ)-contact manifolds (see [12]). The corresponding author of this paper investigated the $*$-Ricci tensor on (κ, μ)-contact manifolds and the Hopf real hypersurfaces in the complex quadric (see [24, 25]).

In the present paper, we study ∗-Ricci tensor and ∗-Ricci operator on generalized Sasakian space forms. The ∗-Ricci semi-symmetric property provides a deeper insight than the projective curvature tensor P. While P might only determine if a Sasakian space form has constant curvature, the ∗-Ricci property quantifies its degree of resemblance to such a space. We derive necessary and sufficient conditions for the ∗-Ricci tensor to be semi-symmetric on both generalized and generalized indefinite Sasakian space forms. We also determine when the ∗-Ricci operator is Reeb flow invariant and Reeb-parallel, and explore the relationship between ∗-Ricci and Ricci semi-symmetry.

We use $U, W, V, X, Y,$ and Z to denote the smooth tangent vector fields on the manifold, and all manifolds and functions mentioned in our paper are smooth.

2. Generalized Sasakian space forms

If a Riemannian manifold M admits a vector field ζ(we call it Reeb vector field or characteristic vector field), a 1-form η and a $(1,1)$-tensor field ϕ that satisfy:

$$\phi\zeta = 0, \quad \eta \circ \phi = 0,$$

$$\phi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1,$$

$$\eta(X) = g(\zeta, X),$$

$$g(X, W) = g(\phi X, \phi W) + \eta(X)\eta(W),$$

then the manifold is almost contact metric manifold and the triple (ϕ, ζ, η) is almost contact structure on the manifold. If the 2-form $d\eta$ and the metric g satisfy:

$$d\eta(X, W) = g(X, \phi W),$$

then M is a contact metric manifold and the triple (ϕ, ζ, η) is a contact structure on the manifold.

We can define a vector field on the product $\mathbb{R} \times M^{2n+1}$ by $(h\frac{d}{dx}, W)$, in which x is the coordinate on $\mathbb{R}$, and h is a C^∞ function on $\mathbb{R} \times M^{2n+1}$. Define an almost complex structure J on $\mathbb{R} \times M^{2n+1}$ by

$$J(h\frac{d}{dx}, W) = (\eta(W)\frac{d}{dx}, \phi W - h\zeta),$$

and it is easy to check $J^2 = -\mathrm{I}$. If J is integrable then the almost contact structure (ϕ, ζ, η) is normal. A normal contact metric manifold is a Sasakian manifold.

For a vector field W orthogonal to ζ, the plane spanned by W and ϕW is called a ϕ-section. The curvature $K(W, \phi W)$ of this plane is known as the ϕ-sectional curvature. The curvature of a Sasakian manifold is determined by ϕ-sectional curvatures entirely (see [6]).

A Sasakian manifold with constant ϕ-sectional curvature c is a Sasakian space form and the curvature tensor is given by Equation (1). In [2], the author replaced the constants in Equation (1) by three smooth functions defining on the manifold. For an almost contact metric manifold M, if the curvature tensor is given by

$$\begin{aligned} R(X, W)Z =& f_1(g(W, Z)X - g(X, Z)W) + f_2(g(X, \phi Z)\phi W - g(W, \phi Z)\phi X + 2g(X, \phi W)\phi Z) \\ &+ f_3(\eta(X)\eta(Z)W - \eta(W)\eta(Z)X + g(X, Z)\eta(W)\zeta - g(W, Z)\eta(X)\zeta), \end{aligned} \tag{2}$$

where $f_1, f_2, f_3 \in C^\infty(M)$, then M is generalized Sasakian space form. The ϕ-sectional curvature of a generalized Sasakian space form is $f_1 + 3f_2$.

Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form, then $f_1 - f_3$ is constant on M. Moreover if the dimension of M^{2n+1} is greater than three, that is $n > 1$, then M^{2n+1} is Sasakian (see [3, Theorem 3.5]) and it is a Sasakian space form (see [2, Corollary 3.16]).

For a generalized Sasakian space form $(M^{2n+1}, f_1, f_2, f_3)$, we have the following useful equations from Equation (2):

$$\mathrm{Ric}(X, W) =(2nf_1 + 3f_2 - f_3)g(X, W) - (3f_2 + (2n - 1)f_3)\eta(X)\eta(W), \tag{3}$$
$$QX =(2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n - 1)f_3)\eta(X)\zeta, \tag{4}$$
$$r =2n(2n + 1)f_1 + 6nf_2 - 4nf_3, \tag{5}$$

where Ric is the Ricci tensor, Q is the Ricci operator and r is the scalar curvature.

For an almost contact metric manifold, the $*$-Ricci tensor Ric^* is (see [13]):

$$\mathrm{Ric}^*(X, Y) = \frac{1}{2}\, trace\{Z \to R(X, \phi Y)\phi Z\},$$

Let $\{e_1, \dots, e_{2n+1}\}$ be the local orthonormal basis of an almost contact metric manifold, then we have

$$\mathrm{Ric}^*(X, Y) =\frac{1}{2}\sum_{i=1}^{2n+1} g(R(X, \phi Y)\phi e_i, e_i) = -\frac{1}{2}\sum_{i=1}^{2n+1} g(R(X, \phi Y)e_i, \phi e_i) = \frac{1}{2}\sum_{i=1}^{2n+1} g(\phi R(X, \phi Y)e_i, e_i),$$

Thus we know that $*$-Ricci tensor is also half of the trace of $\phi \circ R(X, \phi Y)$:

$$\mathrm{Ric}^*(X, Y) = \frac{1}{2}\, \mathrm{trace}\{\phi \circ R(X, \phi Y)\}.$$

Theorem 2.1. *Let $(M^{2n+1}, \phi, \zeta, \eta, g, f_1, f_2, f_3)$ be a generalized Sasakian space form. Then the $*$-Ricci tensor of M^{2n+1} is*

$$\mathrm{Ric}^*(W, Z) = (f_1 + (2n + 1)f_2)g(\phi W, \phi Z), \tag{6}$$

*thus the $*$-*Ricci *operator Q^* and the $*$-scale curvature r^* are:*

$$Q^*W = - (f_1 + (2n + 1)f_2)\phi^2 W, \tag{7}$$
$$r^* =2n(f_1 + (2n + 1)f_2). \tag{8}$$

Proof. Let $\{e_1, \dots, e_{2n+1}\}$ be the local orthonormal basis of M. From the definition of $*$-Ricci tensor, we have:

$$\begin{aligned}\mathrm{Ric}^*(W, Z) =&\frac{1}{2}\sum_{i=1}^{2n+1} g(R(W, \phi Z)\phi e_i, e_i)\\ =&\frac{1}{2}(f_1(-g(\phi^2 Z, W) + g(\phi W, \phi Z)) + f_2(g(\phi^2 W, \phi^2 Z) + g(\phi W, \phi Z) - 4ng(W, \phi^2 Z)))\\ =&(f_1 + (2n + 1)f_2)g(\phi W, \phi Z).\end{aligned}$$

From the above equation and $\mathrm{Ric}^*(W, Z) = g(Q^*W, Z)$, we have

$$Q^*W = -(f_1 + (2n + 1)f_2)\phi^2 W,$$

and

$$r^* = \sum_{i=1}^{2n+1} \mathrm{Ric}^*(e_i, e_i) = (f_1 + (2n + 1)f_2)\sum_{i=1}^{2n+1} g(\phi e_i, \phi e_i) = 2n(f_1 + (2n + 1)f_2).$$

□

If a semi-Riemannian manifold M admits a vector field ζ, a 1-form η and a $(1,1)$ tensor field ϕ satisfying:

$$\phi\zeta = 0, \quad \eta \circ \phi = 0,$$

$$\phi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1,$$

$$\eta(X) = \varepsilon g(\zeta, X),$$

$$g(X, W) = g(\phi X, \phi W) + \varepsilon \eta(X)\eta(W),$$

where $\varepsilon = g(\zeta, \zeta) = \pm 1$ is the significant of ζ, then such manifold is ε-almost contact metric manifold (see [9]) or almost contact pseudo-metric manifold (see [7]).

For a ε-almost contact metric manifold M, if the curvature tensor is given by

$$\begin{aligned} R(U, W)X = & f_1(g(W, X)U - g(U, X)W) + f_2(g(U, \phi X)\phi W - g(W, \phi X)\phi U + 2g(U, \phi W)\phi X) \\ & + f_3(\eta(U)\eta(X)W - \eta(W)\eta(X)U + \varepsilon g(U, X)\eta(W)\zeta - \varepsilon g(W, X)\eta(U)\zeta), \end{aligned} \tag{9}$$

where $f_1, f_2, f_3 \in C^\infty(M)$, then M is a generalized indefinite Sasakian space form (see [4]). It is the correspondence of generalized Sasakian space form in semi-Riemannian settings.

From Equation (9), the Ricci tensor of generalized indefinite Sasakian space form M_ε^{2n+1} is:

$$\mathrm{Ric}(X, W) = (2nf_1 + 3f_2 - \varepsilon f_3)g(X, W) - (3\varepsilon f_2 + (2n-1)f_3)\eta(X)\eta(W). \tag{10}$$

Suppose $\{e_1, \dots, e_{2n}, e_{2n+1} = \zeta\}$ is the local orthonormal basis of generalized indefinite Sasakian space form M_ε^{2n+1} and $\{\varepsilon_1, \dots, \varepsilon_{2n}, \varepsilon_{2n+1}\}$ is its signature. Then the $*$-Ricci tensor Ric^* of M_ε^{2n+1} is

$$\mathrm{Ric}^*(X, Y) = \frac{1}{2}\, trace\{Z \to R(X, \phi Y)\phi Z\} = \frac{1}{2}\sum_{i=1}^{2n+1} \varepsilon_i g(R(X, \phi Y)\phi e_i, e_i).$$

Theorem 2.2. *Let $(M_\varepsilon^{2n+1}, \phi, \zeta, \eta, g, f_1, f_2, f_3)$ be a generalized indefinite Sasakian space form. Then the $*$-Ricci tensor of M_ε^{2n+1} is*

$$\mathrm{Ric}^*(W, Z) = (f_1 + (2n+1)f_2)g(\phi W, \phi Z).$$

Proof. The proof is similar to Theorem 2.1, we omit it here. □

3. Semi-symmetric ∗-Ricci tensor

In this section we study the semi-symmetric $*$-Ricci tensor on generalized Sasakian space form.

Theorem 3.1. *Let $(M^{2n+1}, \phi, \zeta, \eta, g, f_1, f_2, f_3)$ be a generalized Sasakian space form. Then M^{2n+1} is $*$-Ricci semi-symmetric if and only if $f_1 = f_3$ or $f_1 + (2n+1)f_2 = 0$. When $f_1 + (2n+1)f_2 = 0$, it is $*$-Ricci flat.*

Proof. Firstly let us calculate $(R(X, Y)\mathrm{Ric}^*)(Z, W)$:

$$\begin{aligned} & (R(X, Y)\mathrm{Ric}^*)(Z, W) \\ = & -\mathrm{Ric}^*(R(X, Y)Z, W) - \mathrm{Ric}^*(Z, R(X, Y)W) \\ = & -(f_1 + (2n+1)f_2)(g(\phi R(X, Y)Z, \phi W) + g(\phi Z, \phi R(X, Y)W)) \\ = & -(f_1 + (2n+1)f_2)(f_1(-g(Y, Z)\eta(X)\eta(W) + g(X, Z)\eta(Y)\eta(W) - g(Y, W)\eta(X)\eta(Z) + g(X, W)\eta(Y)\eta(Z)) \\ & + f_3(g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z) + g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W))) \\ = & -(f_1 + (2n+1)f_2)(f_3 - f_1)(g(Y, Z)\eta(X)\eta(W) - g(X, Z)\eta(Y)\eta(W) + g(Y, W)\eta(X)\eta(Z) - g(X, W)\eta(Y)\eta(Z)). \end{aligned}$$

If M is $*$-Ricci semi-symmetric, that is $(R(X,Y)\mathrm{Ric}^*)(Z,W)=0$, putting $X=Z=\zeta$, we have

$$\begin{aligned}0 =&(R(\zeta,Y)\mathrm{Ric}^*)(\zeta,W)\\ =&-\mathrm{Ric}^*(R(\zeta,Y)\zeta,W)-\mathrm{Ric}^*(\zeta,R(\zeta,Y)W)\\ =&-(f_1+(2n+1)f_2)(f_3-f_1)(g(Y,W)-\eta(Y)\eta(W))\\ =&-(f_1+(2n+1)f_2)(f_3-f_1)g(\phi Y,\phi W),\end{aligned}$$

from the arbitrariness of vector field Y,W, we have $(f_1+(2n+1)f_2)(f_3-f_1)=0$. Thus $f_1+(2n+1)f_2=0$ or $f_3-f_1=0$.

Conversely, if $f_1+(2n+1)f_2=0$ or $f_3-f_1=0$, then

$$\begin{aligned}&(R(X,Y)\mathrm{Ric}^*)(Z,W)\\ =&-\mathrm{Ric}^*(R(X,Y)Z,W)-\mathrm{Ric}^*(Z,R(X,Y)W)\\ =&-(f_1+(2n+1)f_2)(f_3-f_1)(g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)+g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z))\\ =&0,\end{aligned}$$

so M is $*$-Ricci semi-symmetric, thus we complete the proof. □

Remark 3.2. *In [10], it has been proved that if a generalized Sasakian space form is $*$-Ricci semi-symmetric, then it is $*$-Ricci flat or $f_3-f_1=0$. We prove that this condition is both necessary and sufficient.*

In [28, Theorem 6.1], it has been proved that a $(2n+1)$-dimensional $(n>1)$ generalized Sasakian space form (M^{2n+1},f_1,f_2,f_3) is Ricci semi-symmetric if and only if $f_1=f_3$ or $3f_2+(2n-1)f_3=0$. Actually, this condition is also true in dimension 3. We give a new proof here.

Theorem 3.3. *Let $(M^{2n+1},\phi,\zeta,\eta,g,f_1,f_2,f_3)$ be a generalized Sasakian space form. Then M^{2n+1} is Ricci semi-symmetric if and only if $f_1=f_3$ or $3f_2+(2n-1)f_3=0$.*

Proof. From Equation (3), we have:

$$\begin{aligned}&(R(X,Y)\mathrm{Ric})(Z,W)\\ =&-\mathrm{Ric}(R(X,Y)Z,W)-\mathrm{Ric}(Z,R(X,Y)W)\\ =&-(2nf_1+3f_2-f_3)g(R(X,Y)Z,W)+(3f_2+(2n-1)f_3)\eta(R(X,Y)Z)\eta(W)\\ &-(2nf_1+3f_2-f_3)g(R(X,Y)W,Z)+(3f_2+(2n-1)f_3)\eta(R(X,Y)W)\eta(Z)\\ =&(3f_2+(2n-1)f_3)(f_1-f_3)(g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)+g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z)).\end{aligned}$$

If M is Ricci semi-symmetric, that is $(R(X,Y)\mathrm{Ric})(Z,W)=0$, putting $X=W=\zeta$ in above equation, we have

$$\begin{aligned}0 =&(R(\zeta,Y)\mathrm{Ric}^*)(Z,\zeta)\\ =&-\mathrm{Ric}(R(\zeta,Y)Z,\zeta)-\mathrm{Ric}(Z,R(\zeta,Y)\zeta)\\ =&(3f_2+(2n-1)f_3)(f_1-f_3)(g(Y,W)-\eta(Y)\eta(W))\\ =&(3f_2+(2n-1)f_3)(f_1-f_3)g(\phi Y,\phi W),\end{aligned}$$

from the arbitrariness of vector field Y,W, we have $(3f_2+(2n-1)f_3)(f_1-f_3)=0$. Thus we have $3f_2+(2n-1)f_3=0$ or $f_3-f_1=0$.

Conversely, if $3f_2+(2n-1)f_3=0$ or $f_3-f_1=0$, then

$$\begin{aligned}&(R(X,Y)\mathrm{Ric})(Z,W)\\ =&-\mathrm{Ric}(R(X,Y)Z,W)-\mathrm{Ric}(Z,R(X,Y)W)\\ =&(3f_2+(2n-1)f_3)(f_1-f_3)(g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)+g(Y,W)\eta(X)\eta(Z)-g(X,W)\eta(Y)\eta(Z))\\ =&0,\end{aligned}$$

so M is Ricci semi-symmetric, thus we complete the proof. □

For generalized indefinite Sasakian space form $(M_\varepsilon^{2n+1}, f_1, f_2, f_3)$, we have the following two theorems and we omit the proof of these two theorems because they are basically the same as the proof of Theorem 3.1 and Theorem 3.3.

Theorem 3.4. *Let $(M_\varepsilon^{2n+1}, f_1, f_2, f_3)$ be a generalized indefinite Sasakian space form. Then M_ε^{2n+1} is $*$-Ricci semi-symmetric if and only if $f_1 = f_3$ or $f_1 + (2n+1)f_2 = 0$.*

Theorem 3.5. *Let $(M_\varepsilon^{2n+1}, f_1, f_2, f_3)$ be a generalized indefinite Sasakian space form. Then M_ε^{2n+1} is Ricci semi-symmetric if and only if $f_1 = \varepsilon f_3$ or $3\varepsilon f_2 + (2n-1)f_3 = 0$.*

From Equation (3), Equation (10), Theorem 2.1,Theorem 2.2,Theorem 3.1,Theorem 3.3,Theorem 3.4 and Theorem 3.5, we have the following table in which M^{2n+1} is generalized Sasakian space form and M_ε^{2n+1} is generalized indefinite Sasakian space form.

Table 1: **Difference between $*$-Ricci tensor and Ricci tensor**

	M^{2n+1}	M_ε^{2n+1}
Ricci tensor	$(2nf_1 + 3f_2 - f_3)g(X,W)$ $-(3f_2 + (2n-1)f_3)\eta(X)\eta(W)$	$(2nf_1 + 3f_2 - \varepsilon f_3)g(X,W)$ $-(3\varepsilon f_2 + (2n-1)f_3)\eta(X)\eta(W)$
$*$-Ricci tensor	$(f_1 + (2n+1)f_2)g(\phi X, \phi W)$	
Ricci semi-symmetric	$f_1 = f_3$ or $3f_2 + (2n-1)f_3 = 0$	$f_1 = \varepsilon f_3$ or $3\varepsilon f_2 + (2n-1)f_3 = 0$
$*$-Ricci semi-symmetric	$f_1 = f_3$ or $f_1 + (2n+1)f_2 = 0$	

As evidenced by the table above, the generalized indefinite Sasakian space form and the generalized Sasakian space form exhibit distinct properties with respect to the Ricci tensor, while sharing identical characteristics in terms of the $*$-Ricci tensor. More specifically, the signature of the Reeb vector field has no influence on the properties of the $*$-Ricci tensor, yet it significantly affects those of the Ricci tensor. This indicates that the $*$-Ricci tensor is primarily associated with the structure of the tangent directions in the almost contact manifold other than that of the Reeb vector field.

Lemma 3.6. *Let (M^3, f_1, f_2, f_3) be a three-dimensional non-Sasakian contact generalized Sasakian space form. Then $f_1 = f_3$ if and only if $f_1 + 3f_2 = 0$.*

Proof. In [3, Proposition 3.7], it has been proved that for three-dimensional non-Sasakian contact generalized Sasakian space form, there exists $2f_1 + 3f_2 - f_3 = 0$. So $f_1 = f_3$ if and only if $f_1 + 3f_2 = 0$. □

From above lemma, we have the following corollary:

Corollary 3.7. *Let (M^3, f_1, f_2, f_3) be a three-dimensional non-Sasakian contact generalized Sasakian space form, then the following conditions are equivalent:*
(1) M is $*$-Ricci semi-symmetric;
(2) $f_1 = f_3$;
(3) $f_1 + 3f_2 = 0$.

Corollary 3.8. *Let (M^3, f_1, f_2, f_3) be a three-dimensional contact generalized Sasakian space form. If M is Sasakian manifold, then it is $*$-Ricci semi-symmetric if and only if $f_1 = \frac{3}{4}, f_2 = f_3 = -\frac{1}{4}$.*

Proof. Since M^3 is Sasakian manifold, from [2, Theorem 3.15], we have $f_2 = f_3 = f_1 - 1$. If M^3 is $*$-Ricci semi-symmetric, from Theorem 3.1, there must be $f_1 + 3f_2 = 0$. Thus we can get $f_1 = \frac{3}{4}, f_2 = f_3 = -\frac{1}{4}$. Conversely, if $f_1 = \frac{3}{4}, f_2 = f_3 = -\frac{1}{4}$, then $f_1 + 3f_2 = 0$, from Theorem 3.1, M^3 is $*$-Ricci semi-symmetric. □

Corollary 3.9. *Let (M^3, f_1, f_2, f_3) be a three-dimensional contact generalized Sasakian space form. If M is Sasakian, then it is $*$-Ricci semi-symmetric if and only if it is a Sasakian space form with constant ϕ-sectional curvature $c = 0$.*

Proof. Since M is Sasakian and $*$-Ricci semi-symmetric, from Corollary 3.8, we have $f_1 = \frac{3}{4}, f_2 = f_3 = -\frac{1}{4}$. From [2, Proposition 3.11], the ϕ-sectional curvature of M^3 is $c = f_1 + 3f_2 = 0$. □

Lemma 3.10. *Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form and $n > 1$. Then M is $*$-Ricci semi-symmetric if and only if $f_1 = \frac{2n+1}{2n+2}, f_2 = f_3 = -\frac{1}{2n+2}$.*

Proof. From [3, Theorem 3.5] and [2, Theorem 3.15], we know that M is Sasakian and $f_2 = f_3 = f_1 - 1$. So $f_1 \neq f_3$.

If M is $*$-Ricci semi-symmetric, from Theorem 3.1, there must be $f_1 + (2n+1)f_2 = 0$. We can get $f_1 = \frac{2n+1}{2n+2}, f_2 = f_3 = -\frac{1}{2n+2}$.

Conversely, if $f_1 = \frac{2n+1}{2n+2}, f_2 = f_3 = -\frac{1}{2n+2}$, then f_1 and f_2 satisfies $f_1 + (2n+1)f_2 = 0$, from Theorem 3.1, M is $*$-Ricci semi-symmetric. We complete the proof. □

From Corollary 3.8, we find that a Sasakian three-dimensional generalized Sasakian space form also satisfies above lemma, so we have:

Theorem 3.11. *Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form. If M is Sasakian, then M is $*$-Ricci semi-symmetric if and only if $f_1 = \frac{2n+1}{2n+2}$ and $f_2 = f_3 = -\frac{1}{2n+2}$.*

When $(M^{2n+1}, f_1, f_2, f_3)$ is a contact generalized Sasakian space form such that $f_1 = \frac{2n+1}{2n+2}, f_2 = f_3 = -\frac{1}{2n+2}$, from [2, Corollary 3.16], we know the ϕ-sectional curvature of M is $c = f_1 + 3f_2 = \frac{n-1}{n+1}$, thus we have

Theorem 3.12. *Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form. If M is Sasakian, then M is $*$-Ricci semi-symmetric if and only if it is Sasakian space form with constant ϕ-sectional curvature $c = f_1 + 3f_2 = \frac{n-1}{n+1}$.*

An equivalent statement of the theorem is as follows:

Theorem 3.13. *Let M^{2n+1} be a Sasakian space form with constant ϕ-sectional curvature c. Then M is $*$-Ricci semi-symmetric if and only if $c = \frac{n-1}{n+1}$.*

Remark 3.14. *For a Sasakian space form, we can use the property of $*$-Ricci semi-symmetric to know how closed it is to be a constant curvature space. We know that if the projective curvature tensor P of a manifold is equal to zero, then the manifold has constant curvature. So the projective curvature tensor P is the measure of whether a manifold to be a constant curvature space. But we can not know that how close the manifold to be the constant curvature space from projective curvature tensor P. We can denote the curvature of Sasakian space form by*

$$R(X, W)Z = \frac{c+3}{4}R_1(X, W)Z + \frac{c-1}{4}R_2(X, W)Z + \frac{c-1}{4}R_3(X, W)Z.$$

If the ϕ-sectional curvature $c = 1$, then the curvature is

$$R(X, W)Z = R_1(X, W)Z = g(W, Z)X - g(X, Z)W,$$

it is a constant curvature space. So we know for the three components R_1, R_2, R_3 of R, the less of R_2, R_3, the more the Sasakian space form close to the constant curvature space. That is the more the ϕ-sectional curvature c close to one, the more the Sasakian space form close to the constant curvature space. When the Sasakian space form M^{2n+1} is $*$-Ricci semi-symmetric, we know that it is not a constant curvature space and the ϕ-sectional curvature of M is $0 \leq c = \frac{n-1}{n+1} < 1$, so with the increasing of the dimension of M, it is more and more close to a manifold with constant curvature since c is more close to one.

Generally speaking, the $*$-Ricci flatness ($\mathrm{Ric}^* = 0$) of a generalized Sasakian space form is the sufficient condition for $*$-Ricci semi-symmetry, not a necessary condition. But if the generalized Sasakian space form is contact, then the $*$-Ricci flatness is the necessary and sufficient condition of $*$-Ricci semi-symmetry.

Theorem 3.15. *Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form, then it is $*$-Ricci semi-symmetric if and only if it is $*$-Ricci flat.*

Proof. Firstly we assume $(M^{2n+1}, f_1, f_2, f_3)$ is $*$-Ricci semi-symmetric.

Case **I**: If M^{2n+1} is three-dimensional non-Sasakian, then from Corollary 3.7, we have $f_1 + 3f_2 = 0$. It is $*$-Ricci flat from Theorem 2.1.

Case **II**: If M^{2n+1} is three-dimensional Sasakian, then from Corollary 3.8, we have $f_1 = \frac{3}{4}, f_2 = f_3 = -\frac{1}{4}$. Thus $f_1 + 3f_2 = 0$ and M^{2n+1} is $*$-Ricci flat from Theorem 2.1.

Case **III**: If the dimension of M^{2n+1} is greater than three, then from Lemma 3.10, we have $f_1 = \frac{2n+1}{2n+2}, f_2 = f_3 = -\frac{1}{2n+2}$. Thus $f_1 + (2n+1)f_2 = 0$ and M^{2n+1} is $*$-Ricci flat from Theorem 2.1.

Conversely if $(M^{2n+1}, f_1, f_2, f_3)$ is $*$-Ricci flat, then from Theorem 2.1, we have $f_1 + (2n+1)f_2 = 0$. Thus M^{2n+1} is $*$-Ricci semi-symmetric from Theorem 3.1. □

Theorem 3.16. *Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form. If M is Sasakian, then M is Ricci semi-symmetric if and only if $f_1 = 1, f_2 = f_3 = 0$.*

Proof. Since M is Sasakian, from [2, Theorem 3.15], we have $f_2 = f_3 = f_1 - 1$, thus $f_1 \neq f_3$. Firstly we assume that M is Ricci semi-symmetric, from Theorem 3.3, there must be $f_2 + (2n-1)f_3 = 0$, so $f_1 = 1, f_2 = f_3 = 0$.

Conversely if $f_1 = 1, f_2 = f_3 = 0$, they satisfy $f_2 + (2n-1)f_3 = 0$, from Theorem 3.3, M is Ricci semi-symmetric. □

Remark 3.17. *From Theorem 3.11 and Theorem 3.16 we know that, there is no generalized Sasakian space form which is Sasakian and both* $*$-Ricci *semi-symmetric and Ricci semi-symmetric.*

Actually, if a generalized Sasakian space form is Sasakian, then it is a Sasakian space form. So we have

Corollary 3.18. *There is no Sasakian space form that both $*$-Ricci semi-symmetric and Ricci semi-symmetric.*

4. Reeb flow invariant $*$-Ricci operator

In this section we study Reeb flow invariant $*$-Ricci operator on generalized Sasakian space form.

Theorem 4.1. *Let $(M^{2n+1}, \phi, \zeta, \eta, g, f_1, f_2, f_3)$ be a contact generalized Sasakian space form. Then*

$$\mathcal{L}_\zeta Q^* = \nabla_\zeta Q^*.$$

The $*$-Ricci operator Q^* of M is Reeb flow invariant (Reeb parallel) if and only if the $*$-scale curvature r^* is invariant along Reeb vector field, that is $\zeta r^* = 0$.

Proof. From Theorem 2.1, we know $Q^*W = -(f_1 + (2n+1)f_2)\phi^2 W$, then

$$\begin{aligned}\mathcal{L}_\zeta(Q^*W) =& (\mathcal{L}_\zeta Q^*)W + Q^*(\mathcal{L}_\zeta W)\\ =& -(f_1 + (2n+1)f_2)((\mathcal{L}_\zeta\phi)(\phi W) + \phi(\mathcal{L}_\zeta\phi)W) - \zeta(f_1 + (2n+1)f_2)\phi^2 W - (f_1 + (2n+1)f_2)\phi^2(\mathcal{L}_\zeta W).\end{aligned}$$

From [6, Lemma 6.2], we know $h\phi + \phi h = 0$, that is $(\mathcal{L}_\zeta\phi)(\phi W) + \phi(\mathcal{L}_\zeta\phi)W = 0$. Put it in above equation, we have:

$$(\mathcal{L}_\zeta Q^*)W = -\zeta(f_1 + (2n+1)f_2)\phi^2 W. \tag{11}$$

Since

$$\begin{aligned}\nabla_\zeta(Q^*W) =& (\nabla_\zeta Q^*)X + Q^*(\nabla_\zeta W)\\ =& -\zeta(f_1 + (2n+1)f_2)\phi^2 W - (f_1 + (2n+1)f_2)\phi^2(\nabla_\zeta W) - (f_1 + (2n+1)f_2)((\nabla_\zeta\phi)(\phi W) + \phi(\nabla_\zeta\phi)W),\end{aligned}$$

and M is contact, $\nabla_\zeta\phi = 0$, the above equation is:

$$(\nabla_\xi Q^*)W = -\zeta(f_1 + (2n+1)f_2)\phi^2 W. \tag{12}$$

From Equation (11) and (12), we have

$$(\mathcal{L}_\zeta Q^*)W = (\nabla_\zeta Q^*)W = -\zeta(f_1 + (2n+1)f_2)\phi^2 W.$$

Put $r^* = 2n(f_1 + (2n+1)f_2)$ in above equation,

$$(\mathcal{L}_\zeta Q^*)W = (\nabla_\zeta Q^*)W = -\frac{1}{2n}\zeta r^* \phi^2 W,$$

so we know $(\mathcal{L}_\zeta Q^*)W = 0$ (or $(\nabla_\zeta Q^*)W = 0$) if and only if $\zeta r^* = 0$. So Q^* is Reeb flow invariant (or Reeb parallel) if and only if the $*$-scale curvature r^* is invariant along Reeb vector field. □

Lemma 4.2. *Let (M^3, f_1, f_2, f_3) be a three-dimensional non-Sasakian contact generalized Sasakian space form. If M is $*$-Ricci semi-symmetric, then it is Ricci semi-symmetric and the $*$-Ricci operator on M is Reeb flow invariant and Reeb parallel.*

Proof. If M is $*$-Ricci semi-symmetric, from Corollary 3.7, $f_1 + 3f_2 = 0$ and $f_1 = f_3$. M is Ricci semi-symmetric from Theorem 3.3. From Lemma 3.6, the $*$-scale curvature $r^* = 2(f_1 + 3f_2) = 0$. From Theorem 3.1, the $*$-Ricci operator on M is Reeb flow invariant and Reeb parallel. □

Lemma 4.3. *Let (M^3, f_1, f_2, f_3) be a three-dimensional contact generalized Sasakian space form. If M is Sasakian and $*$-Ricci semi-symmetric, then the $*$-Ricci operator on M is Reeb flow invariant and Reeb parallel.*

Proof. Since M is Sasakian and $*$-Ricci semi-symmetric, from Corollary 3.8, we have $f_1 = \frac{3}{4}, f_2 = f_3 = -\frac{1}{4}$. From Throrem 2.1, the $*$-scale curvature $r^* = 2(f_1 + 3f_2) = 0$, it is invariant along Reeb vector field. From Theorem 4.1, the $*$-Ricci operator on M is Reeb flow invariant and Reeb parallel. □

From Lemma 4.2 and Lemma 4.3, we can conclude that:

Theorem 4.4. *Let (M^3, f_1, f_2, f_3) be a three-dimensional contact generalized Sasakian space form. If M is $*$-Ricci semi-symmetric, then the $*$-Ricci operator on M is Reeb flow invariant and Reeb parallel.*

Since the $*$-scale curvature r^* of a Sasakian space form is constant, so $\zeta r^* = 0$, thus we have the following corollary:

Corollary 4.5. *The $*$-Ricci operator of a Sasakian space form is Reeb flow invariant and Reeb parallel.*

Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form, if $n > 1$, from [3, Theorem 3.5], M is a Sasakian manifold, then from [2, Corollary 3.16], M is a Sasakian space form. Thus we have the following corollary:

Corollary 4.6. *Let $(M^{2n+1}, f_1, f_2, f_3)$ be a contact generalized Sasakian space form. If $n > 1$, then the $*$-Ricci operator of M is Reeb flow invariant and Reeb parallel.*

Actually, from Theorem 4.4 and Corollary 4.6, we have proved:

Theorem 4.7. *The $*$-Ricci operator on a $*$-Ricci semi-symmetric contact generalized Sasakian space form is Reeb flow invariant and Reeb parallel.*

5. Example

First we recall the notion of D-homothetic deformation of a contact metric structure. Suppose $(\phi_0, \zeta_0, \eta_0, g_0)$ is a contact metric structure on a manifold M^{2n+1}, then

$$\phi' = \phi_0, \ \ \zeta' = \frac{1}{a}\zeta_0, \ \ \eta' = a\eta_0, \ \ g' = ag_0 + a(a-1)\eta_0 \otimes \eta_0,$$

is also a contact metric structure on M^{2n+1} in which a is a positive constant. Moreover if $(\phi_0, \zeta_0, \eta_0, g_0)$ is a Sasakian structure with constant ϕ-sectional curvature c_0, then $(\phi', \zeta', \eta', g')$ is also a Sasakian structure but the constant ϕ-sectional curvature changes to $c' = \frac{c_0+3}{a} - 3$ (see [6]).

We know there is a standard Sasakian structure $(\phi_0, \zeta_0, \eta_0, g_0)$ on the sphere S^{2n+1} and $(S^{2n+1}, \phi_0, \zeta_0, \eta_0, g_0)$ is a Sasakian space form with constant ϕ-sectional curvature $c_0 = 1$. Setting $a = \frac{2n+2}{2n+1}$, the D-homothetic deformation of standard Sasakian structure $(\phi_0, \zeta_0, \eta_0, g_0)$ is

$$\phi' = \phi_0, \ \ \zeta' = \frac{2n+1}{2n+2}\zeta_0, \ \ \eta' = \frac{2n+2}{2n+1}\eta_0, \ \ g' = \frac{2n+2}{2n+1}g_0 + \frac{2n+2}{(2n+1)^2}\eta_0 \otimes \eta_0.$$

Thus $(S^{2n+1}, \phi', \zeta', \eta', g')$ is a Sasakian space form with constant ϕ-sectional curvature $c' = \frac{4}{a} - 3 = \frac{n-1}{n+1}$ and we denote it by $\widetilde{S}^{2n+1}(\frac{n-1}{n+1})$. From Theorem 3.13, we know $\widetilde{S}^{2n+1}(\frac{n-1}{n+1})$ is $*$-Ricci semi-symmetric.

Actually, from the classification of complete and simply connected Sasakian space form in [31], we can get

Theorem 5.1. *Complete and simply connected $*$-Ricci semi-symmetric Sasakian space form is isomorphic to* $\widetilde{S}^{2n+1}(\frac{n-1}{n+1})$.

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