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New Simpson type inequalities for convex functions via conformable fractional integrals

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Abstract. This study proves equality for differentiable functions involving the conformable fractional integrals. Using the established identity, we offer new Simpson type inequalities for convex functions via conformable fractional integrals. We also consider some special cases which can be deduced from the main results.

1. Introduction and Preliminaries

The convexity of functions is a very important and fundamental concept in both areas of pure and applied mathematics. This function has attracted considerable attention and has been applied to various inequalities by many researchers. A convex function is defined as follows:

A function $f:[a,b]\to\mathbb{R}$ is convex if it satisfies an inequality:

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y),$$

for all $x,y\in[a,b]$ and $t\in[0,1]$.

The most famous inequality which has been used with convex functions is Simpson's inequality. This inequality, a well-known technique of numerical integration and approximations for definite integrals, was discovered by Thomas Simpson (1710-1761). Simpson's inequality is the following inequality

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$. Then, the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(b-a)^4.$$

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In 2010, Sarikaya et al. [1] introduced Simpson-type inequality for differentiable convex function, and they used the following lemma to prove the main inequalities.

Lemma 1.2. [1] Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on (a, b) such that $f' \in L_1([a, b])$ with $a < b$, then the following equality holds:

$$\begin{aligned}
&\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{b-a}{2}\int_0^1 \left[\left(\frac{t}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right) + \left(\frac{1}{3} - \frac{t}{2}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right] dt.
\end{aligned}$$

Using Lemma 1.2, Sarikaya et al. [1] established the inequalities as follows

Theorem 1.3. [1] Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L_1([a, b])$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{5(b-a)}{12}\left[|f'(a)| + |f'(b)|\right].
\end{aligned}$$

Theorem 1.4. [1] Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L_1([a, b])$ with $a < b$. If $|f'|^q$ is convex on $[a, b], q > 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left\{\left(\frac{3|f'(b)|^q + |f'(a)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.5. [1] Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L_1([a, b])$ with $a < b$. If $|f'|^q$ is convex on $[a, b], q \geq 1$, then the following inequality holds:

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leq \frac{b-a}{72}(5)^{1-\frac{1}{q}} \left[\left(\frac{61|f'(b)|^q + 29|f'(a)|^q}{18}\right)^{\frac{1}{q}} + \left(\frac{61|f'(a)|^q + 29|f'(b)|^q}{18}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

On the other hand, some researchers have studied the Simpson-type inequality via fractional calculus. Fractional calculus is an area of mathematics that expands the traditional derivative and integral ideas to non-integer orders. In recent decades, it has piqued the curiosity of mathematicians, physicists, and engineers [2–4]. In a fluid-dynamic traffic model, fractional derivatives can be utilized to simulate the irregular oscillation of earthquakes and to compensate for the inadequacies induced by the assumption of continuous traffic flow. Fractional derivatives are also used to model a wide range of chemical processes, as well as mathematical biology and other physics and engineering problems [5–9]. Further, it is demonstrated that several fractional systems produce results that are more appropriate than those produced by corresponding systems having integer derivatives [10, 11].

New studies have concentrated on developing a class of fractional integral operators and their applicability in a variety of scientific disciplines. Using only the derivative's fundamental limit formulation, a newly well-behaved straightforward fractional derivative known as the conformable derivative was developed in [12]. Some significant requirements that cannot be fulfilled by the Riemann-Liouville and Caputo definitions are fulfilled by the conformable derivative. Nevertheless, in [13] the author demonstrated that the conformable approach in [12] cannot yield good results when compared to the Caputo definition for specific functions. This flaw in the conformable definition was avoided by some extensions of the conformable approach [14, 15]. In addition, employing exponential and Mittag-Leffler functions in the kernels, several scholars created novel expanded fractional operators [16–20] for more details.

In 2006, kilbas et al. [18] defined fractional integrals, also called Riemann-Liouville integrals as follows:

Definition 1.6. *[18] For $f \in L^1[a,b]$, the Riemann-Liouville integrals $J^{\beta}_{a+}f(x)$ and $J^{\beta}_{b-}f(x)$ of order $\beta > 0$ are respectively given by*

$$J^{\beta}_{a+}f(x) = \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)dt, \quad x > a \tag{1}$$

and

$$J^{\beta}_{b-}f(x) = \frac{1}{\Gamma(\beta)} \int_x^{\varrho} (t-x)^{\beta-1} \phi(\varsigma)d\varsigma, \quad x < b, \tag{2}$$

where Γ denotes the Gamma function and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In the case $\beta = 1$, Riemann-Liouville integrals reduces to the classical integrals.

In 2015, Matloka [21] introduced Simpson-type inequality for h-convex function. He used Definition 1.6 to prove the following lemma:

Lemma 1.7. [21] Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on (a,b) such that $f' \in L_1([a,b])$

with $a < b$, then the following equality holds:

$$\begin{aligned} &\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[{}^{\beta}_{a}Jf\left(\frac{a+b}{2}\right) + {}^{\beta}J_b f\left(\frac{a+b}{2}\right)\right] \\ = \ &\frac{b-a}{2}\left[\int_0^1 \left(\frac{t^{\beta}}{2} - \frac{1}{3}\right) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 \left(\frac{1}{3} - \frac{t^{\beta}}{2}\right) f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt\right]. \end{aligned}$$

In 2017, Chen and Huang [22] presented Simpson type inequality for s-convex functions via fractional integrals using Lemma 1.7 to prove their main equalities. For $s = 1$, Chen and Huang [22] obtained the following Simpson type inequality for convex functions.

Theorem 1.8. [22] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1([a,b])$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[{}^{\beta}_{a}Jf\left(\frac{a+b}{2}\right) + {}^{\beta}J_b f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq \ &\frac{b-a}{2}\int_0^1 \left|\frac{t^{\beta}}{2} - \frac{1}{3}\right| dt \left[|f'(a)| + |f'(b)|\right]. \end{aligned}$$

Theorem 1.9. [22] Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1([a,b])$ with $a < b$. If $|f'|^q$ is convex on $[a,b], q > 1$, then the following inequality holds:

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^{\beta}}\left[{}^{\beta}_{a}Jf\left(\frac{a+b}{2}\right) + {}^{\beta}J_b f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left[\left(\frac{3\left|f'(b)\right|^q+\left|f'(a)\right|^q}{4}\right)^{1/q}+\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{1/q}\right],$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.10. [22] Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L_1([a,b])$ with $a<b$. If $|f'|^q$ is convex on $[a,b], q\geq 1$, then the following inequality holds:

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[{}_a^\beta Jf\left(\frac{a+b}{2}\right)+{}^\beta J_b f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|dt\right)^{1-\frac{1}{q}}\left[\left(\int_0^1\left|\frac{t^\beta}{2}-\frac{1}{3}\right|\left(\left(\frac{1+t}{2}\right)|f'(b)|^q+\left(\frac{1-t}{2}\right)|f'(a)|^q\right)dt\right)\right.$$
$$\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{t^\beta}{2}\right|\left(\left(\frac{1+t}{2}\right)|f'(a)|^q+\left(\frac{1-t}{2}\right)|f'(b)|^q\right)dt\right)\right].$$

Remark 1.11. *For classical integrals,*

(i) if we put $\beta=1$, then Lemma 1.7 leads to Lemma 1.2.

(ii) By setting $\beta=1$, then Theorem 1.8 leads to Theorem 1.3.

(ii) If we take $\beta=1$, then Theorem 1.9 leads to Theorem 1.4.

(iv) Taking $\beta=1$, then Theorem 1.10 leads to Theorem 1.5.

In 2017, Jarad et al. [19] introduced the following fractional conformable integral operators. They also provided certain characteristics and relationships between these operators and several other fractional operators in the literature. The fractional conformable integral operators are defined by

Definition 1.12. [19] For $f\in L^1[a,b]$, the fractional conformable integral operator ${}_a^\beta J^\alpha f(x)$ and ${}^\beta J_b^\alpha f(x)$ of order $\beta\in\mathbb{C}$, $\mathrm{Re}(\beta)>0$ and $\alpha\in(0,1]$ are respectively given by

$${}_a^\beta J^\alpha f(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt,\quad t>a, \tag{3}$$

and

$${}^\beta J_b^\alpha f(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt,\quad t<b. \tag{4}$$

Note that, the fractional integral in (3) coincides with the Riemann-Liouville fractional integral in (1) when $a=0$ and $\alpha=1$. Moreover, the fractional integral in (4) coincides with the Riemann-Liouville fractional integral in (2) when $b=0$ and $\alpha=1$. Some recent results connected with fractional integral inequalities, see [23–32] and the references cited therein.

The aim of this paper is to establish some new Simpson type inequalities associated with convex function via conformable fractional integrals. We also prove that the newly established equalities are the generalization of the existing Simpson type inequalities. The ideas and strategies for our results concerning Simpson type inequalities via conformable fractional integrals may open new avenues for further research in this area.

2. Main Results

To prove our main results, we consider the following lemma.

Lemma 2.1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on (a,b) such that $f'\in L_1([a,b])$ with $a<b$, then the following equality holds:*

$$\begin{aligned}
&\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\
=\ &\frac{(b-a)\alpha^{\beta}}{2}\left[\int_0^1\left(\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right.\\
&\left.+\int_0^1\left(\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right].
\end{aligned}\tag{5}$$

Proof. Let

$$\begin{aligned}
&\frac{(b-a)\alpha^{\beta}}{2}\left[\int_0^1\left(\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\right.\\
&\left.+\int_0^1\left(\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt\right]\\
=\ &\frac{(b-a)\alpha^{\beta}}{2}\left[I_1+I_2\right].
\end{aligned}\tag{6}$$

Integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1\left(\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt\\
&= -\frac{2}{b-a}\left(\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{3\alpha^{\beta}}\right)f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\Bigg|_0^1\\
&\quad+\frac{2}{b-a}\int_0^1\frac{\beta}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}(1-t)^{\alpha-1}f\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt.
\end{aligned}$$

Considering $x=\frac{1+t}{2}a+\frac{1-t}{2}b$, we obtain

$$\begin{aligned}
I_1 &= \frac{-2}{6\alpha^{\beta}(b-a)}f(a)-\frac{2}{3\alpha^{\beta}(b-a)}f\left(\frac{a+b}{2}\right)\\
&\quad+\frac{\beta}{2}\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(x-a)^{\alpha}}{\alpha}\right)(x-a)^{\alpha-1}dx\\
&= \frac{-2}{(b-a)\alpha^{\beta}}\left[\frac{f(a)}{6}+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]+\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{2}\,{}_{a}^{\beta}J^{\alpha}f(x).
\end{aligned}\tag{7}$$

Similarly, using the argument outlined above, we get

$$I_2 = \int_0^1\left(\frac{1}{3\alpha^{\beta}}-\frac{1}{2}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta}\right)f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

$$= \frac{-2}{(b-a)\,\alpha^{\beta}}\left[\frac{f(b)}{6}+\frac{1}{3}f\left(\frac{a+b}{2}\right)\right]+\left(\frac{2}{b-a}\right)^{\alpha\beta+1}\frac{\Gamma(\beta+1)}{2}\,{}^{\beta}J_{b}^{\alpha}f(x). \tag{8}$$

Substituting equalities (7) and (8) in the equality (6), we can write

$$\begin{aligned}
&\frac{(b-a)\,\alpha^{\beta}}{2}\left[I_1+I_2\right]\\
=\ &\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right].
\end{aligned}$$

Thus, the proof is completed. □

Remark 2.2. *In Lemma 2.1, we have the equalities as follows:*

(i) if we set $\alpha=1$ in (5)*, then Lemma 2.1 leads to Lemma 1.7.*

(ii) if we take $\alpha=1$ and $\beta=1$ in (5)*, then Lemma 2.1 leads to Lemma 1.2.*

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L_1([a,b])$. If $\left|f'\right|$ is convex on $[a,b]$, then the following equality holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
\leq\ &\frac{b-a}{2}\left[\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right),
\end{aligned} \tag{9}$$

where $\mathcal{B}$ denotes the beta function and

$$c=1-\left(1-\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}.$$

Proof. Taking the absolute value of both sides of (5), we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
\leq\ &\frac{b-a}{2}\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|\left[\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|+\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|\right]dt\right)
\end{aligned} \tag{10}$$

Since $\left|f'\right|$ is convex on $[a,b]$, we get

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
\leq\ &\frac{b-a}{2}\left[\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|\right.\\
&\left.\times\left[\left(\frac{1+t}{2}\left|f'(a)\right|+\frac{1-t}{2}\left|f'(b)\right|\right)+\left(\frac{1-t}{2}\left|f'(a)\right|+\frac{1+t}{2}\left|f'(b)\right|\right)\right]dt\right]\\
=\ &\frac{b-a}{2}\left[\int_0^c\left(\frac{1}{3}-\frac{(1-(1-t)^{\alpha})^{\beta}}{2}\right)dt+\int_c^1\left(\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right)dt\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right)
\end{aligned}$$

$$= \frac{b-a}{2}\left[\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right).$$

Thus, the proof is completed. □

Remark 2.4. *In Theorem 2.3, we have the inequalities as follows:*

(i) if we set $\alpha = 1$ in (9)*, then Theorem 2.3 leads to Theorem 1.8.*

(ii) if we take $\alpha = 1$ and $\beta = 1$ in (9)*, then Theorem 2.3 leads to Theorem 1.3.*

Theorem 2.5. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1([a,b])$. If $\left|f'\right|^q$ is convex on $[a,b]$ with $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
&\leq \frac{b-a}{2}\Theta_{\alpha}^{\beta}(p)\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{1/q}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{1/q}\right]\\
&\leq \frac{b-a}{2^{2/q-1}}\Theta_{\alpha}^{\beta}(p)\left[\left|f'(a)\right|+\left|f'(b)\right|\right] \qquad (11)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and

$$\Theta_{\alpha}^{\beta}(p)=\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}.$$

Proof. Using Hölder inequality in (10), we have

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]dt.
\end{aligned}$$

Since $\left|f'\right|^q$ is convex on $[a,b]$, we get

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\\
&\times\left[\left(\int_0^1\left(\frac{1+t}{2}\left|f'(a)\right|^q+\frac{1-t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}+\left(\int_0^1\left(\frac{1-t}{2}\left|f'(a)\right|^q+\frac{1+t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{2}\left(\int_0^1 \left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|^p dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{1/q}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{1/q}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Remark 2.6. *In Theorem 2.5, we have the inequalities as follows:*

(i) if we set $\alpha = 1$ in (11)*, then Theorem 2.5 leads to Theorem 1.9.*

(ii) if we take $\alpha = 1$ and $\beta = 1$ in (11)*, then Theorem 2.5 leads to Theorem 1.4.*

Theorem 2.7. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L_1([a,b])$. If $\left|f'\right|^q$ is convex on $[a,b]$ with $q \geq 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{b-a}{2^{1+\frac{1}{q}}}\left(\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\Psi_2\left|f'(a)\right|^q+\Psi_1\left|f'(b)\right|^q\right)^{1/q}+\left(\Psi_1\left|f'(a)\right|^q+\Psi_2\left|f'(b)\right|^q\right)^{1/q}\right], \qquad (12)
\end{aligned}
$$

where $\mathcal{B}$ denotes the beta function, $c = 1-\left(1-\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$,

$$
\Psi_1 = \frac{1-(1-c)^2}{6}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{2}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)
$$

and

$$
\begin{aligned}
\Psi_2 &= \frac{8c-(1-c)^2}{6}+\frac{1}{\alpha}\left(\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right. \\
&\left.-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)+\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{2}{\alpha}\right)\right).
\end{aligned}
$$

Proof. Applying power-mean inequality in (10), we have

$$
\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{(1-(1-t)^{\alpha})^{\beta}}{2}\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}} dt\right].
\end{aligned}
$$

Since $|f'|^q$ is convex on $[a,b]$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}_{a}^{\beta}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J_{b}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq\ & \frac{b-a}{2}\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|\right)^{1-\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|\left(\frac{1-t}{2}\left|f'(a)\right|^q+\frac{1+t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{1}{3}-\frac{(1-(1-t)^{\alpha})^{\beta}}{2}\right|\left(\frac{1+t}{2}\left|f'(a)\right|^q+\frac{1-t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}.
\end{aligned}$$

It is clearly seen that

$$\begin{aligned}
\Psi_1 &= \int_0^1(1-t)\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|dt \\
&= \frac{1-(1-c)^2}{6}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{2}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\Psi_2 &= \int_0^1(1+t)\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|dt \\
&= \int_0^1(2-(1-t))\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|dt \\
&= 2\int_0^1\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|dt+\int_0^1(1-t)\left|\frac{(1-(1-t)^{\alpha})^{\beta}}{2}-\frac{1}{3}\right|dt \\
&= \frac{8c-(1-c)^2}{6}+\frac{1}{\alpha}\left(\mathcal{B}\left(\beta+1,\frac{2}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)-2\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)+\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{2}{\alpha}\right)\right).
\end{aligned}$$

Thus, the proof is completed. □

Remark 2.8. *In Theorem 2.7, we have the inequalities as follows:*

(i) if we set $\alpha = 1$ *in* (12), *then Theorem 2.7 leads to Theorem 1.10.*
(ii) if we take $\alpha = 1$ *and* $\beta = 1$ *in* (12), *then Theorem 2.7 leads to Theorem 1.5.*

3. Examples

In this section, we give examples to support the our main results in the last section.

Example 3.1. *Let $f:[0,2]\to\mathbb{R}$ be defined by $f(x)=x^2$. From Theorem 2.3 with $\beta\in(0,10)$ and $\alpha\in(0,1)$, the left side of* (9) *becomes*

$$\begin{aligned}&\left|\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a}J^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}J^{\alpha}_{b}f\left(\frac{a+b}{2}\right)\right]-\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\ =\;&\left|\beta\left(\mathcal{B}\left(\beta+1,\frac{2}{\alpha}+1\right)-2\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)\right)+\frac{2}{3}\right| \qquad (13)\end{aligned}$$

and the right side of (9) *becomes*

$$\begin{aligned}&\frac{b-a}{2}\left[\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\\ =\;&4\left[\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right]. \qquad (14)\end{aligned}$$

where $\mathcal{B}$ denotes the beta function and

$$c=1-\left(1-\left(\frac{2}{3}\right)^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}.$$

After calculating (13) *and* (14), *the graph is shown as in Figure 1 using the MATLAB software.*

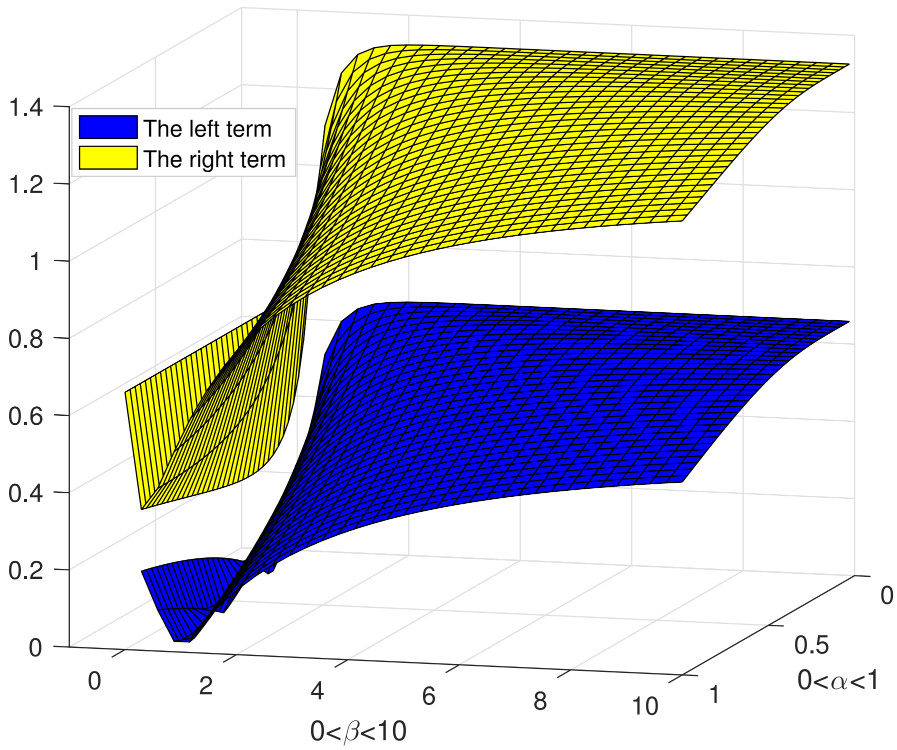

Figure 1: Plot illustration for Theorem 2.3.

Hence in Figure 1, we can see that the inequality (9) is valid.

Example 3.2. *Let $f : [0,2] \to \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 2.7 with $\beta \in (0,10), \alpha \in (0,1), q = 2$ and $p = 2$, the left side of the equality (11) becomes the equality (13) and the right side of the equality (11) becomes*

$$\begin{aligned} & \frac{b-a}{2}\Theta_{\alpha}^{\beta}(p)\left[\left(\frac{3\left|f'(a)\right|^{q}+\left|f'(b)\right|^{q}}{4}\right)^{1/q}+\left(\frac{\left|f'(a)\right|^{q}+3\left|f'(b)\right|^{q}}{4}\right)^{1/q}\right] \\ = & \left(2+2\sqrt{3}\right)\left(\frac{1}{\alpha}\left(\frac{\mathcal{B}\left(2\beta+1,\frac{1}{\alpha}\right)}{4}+\frac{\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)}{3}\right)+\frac{1}{9}\right)^{\frac{1}{2}}. \end{aligned} \tag{15}$$

We can see the graph of (13) and (15) from MATLAB software as in Figure 2.

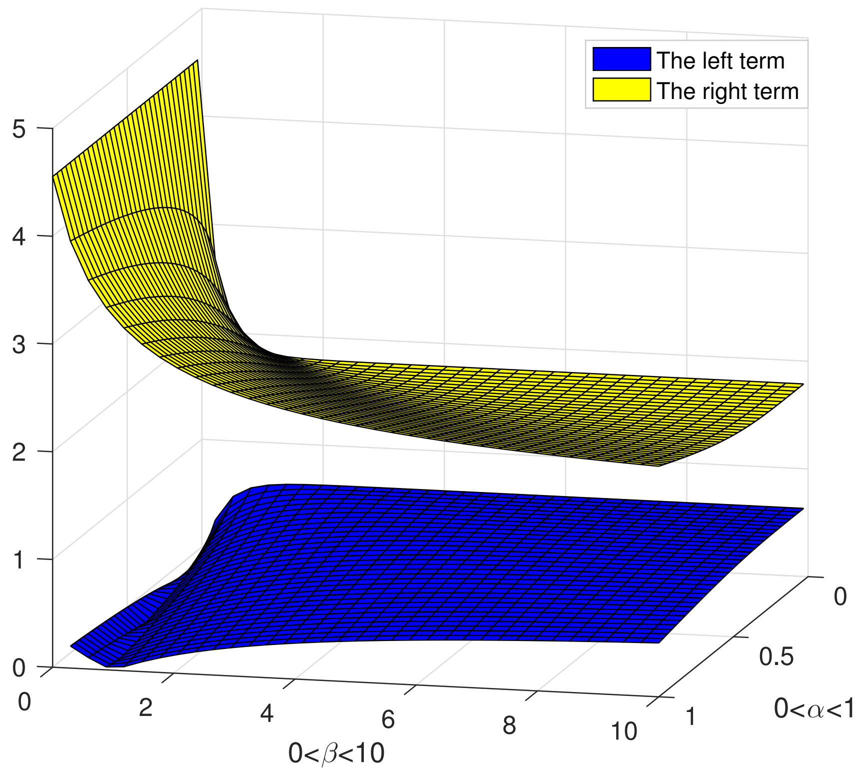

Figure 2: Plot illustration for Theorem 2.3.

Hence in Figure 2, we can see that the inequality (11) is valid.

Example 3.3. *Let $f : [0,2] \to \mathbb{R}$ be defined by $f(x) = x^2$. From Theorem 2.7 with $\beta \in (0,10), \alpha = (0,1]$ and $q = 2$, the left side of the equality (12) becomes the equality (13) and the right side of the equality (12) becomes*

$$\begin{aligned} & \frac{b-a}{2^{1+\frac{1}{q}}}\left(\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right)^{1-\frac{1}{q}} \\ & \times\left[\left(\Psi_{2}\left|f'(a)\right|^{q}+\Psi_{1}\left|f'(b)\right|^{q}\right)^{1/q}+\left(\Psi_{1}\left|f'(a)\right|^{q}+\Psi_{2}\left|f'(b)\right|^{q}\right)^{1/q}\right] \end{aligned}$$

$$= \frac{1}{2}\left(\frac{2c-1}{3}+\frac{1}{\alpha}\left(\frac{1}{2}\mathcal{B}\left(\beta+1,\frac{1}{\alpha}\right)-\mathcal{B}\left(\beta+1,\frac{1}{\alpha},\left(\frac{2}{3}\right)^{\frac{1}{\beta}}\right)\right)\right)^{1-\frac{1}{2}}$$
$$\times\left[\left(\Psi_2\left|f'(0)\right|^2+\Psi_1\left|f'(2)\right|^2\right)^{1/2}+\left(\Psi_1\left|f'(0)\right|^2+\Psi_2\left|f'(2)\right|^2\right)^{1/2}\right]. \quad (16)$$

The images of the (13) *and* (16) *expressions are drawn in Figure 3 in Matlab software.*

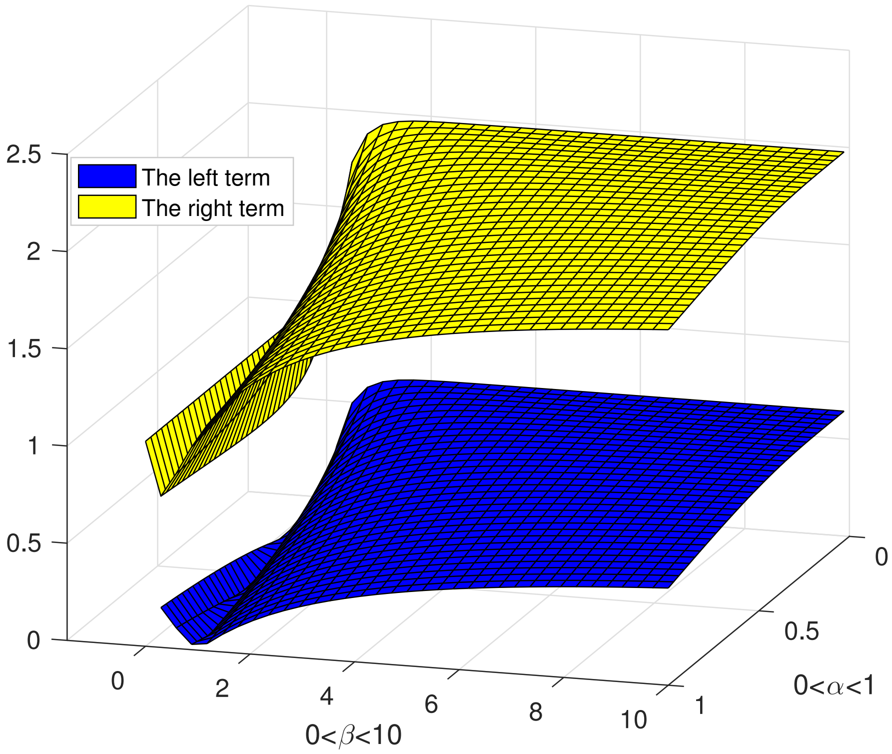

Figure 3: Plot illustration for Theorem 2.7.

Hence in Figure 3, we can see that the equality (12) *is valid.*

4. Conclusion

In this work, we established new estimates of Simpson type inequalities via conformable fractional integrals for convex functions. Our main results were proven to be generalizations of the Riemann-Liouville fractional integrals related to Simpson type inequalities. Examples were given to illustrate the investigated results. In future works, researchers can obtain similar inequalities of Simpson-type inequalities via conformable fractional integrals for convex functions by using quantum calculus.

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