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Explore the study of the fractional differential equation containing the right Hilfer derivative

Belqassim Azzouz[a]

[a]*Department of Mathematics, Ahmed Zabana University of Relizane, Algeria*
Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO),
University of Oran 1, Ahmed Ben Bella, Oran, Algeria

Abstract. Motivated by the Hilfer fractional derivative, we introduced a new problem involving the right Hilfer fractional operator. We present some important results through the use of lemmas. In this context, we discuss the existence and uniqueness of the solution using the Krasnoselskii fixed point theorem. Finally, we also demonstrate the application of the obtained results with the aid of an example.

1. Introduction

In recent years, the scientific community has focused more attention on fractional differential equations, as they are effective tools in modeling many phenomena in applied sciences and engineering applications such as acoustic control, rheology, polymer physics, porous media, medicine, electrochemistry, proteins, electromagnetics, economics, astrophysics, chemical engineering, signal processing, optics, chaotic dynamics, statistical physics, etc. (For details and examples, one may refer to papers [13, 14, 17, 19, 21] and references cited therein). Over the years, many researchers have been interested in discussing the qualitative analysis of fractional differential equations, including existence and uniqueness, as seen in [1, 6, 10, 11, 15]. Some authors have dedicated their efforts to further qualitative analysis of these kinds of equations. Many related articles on the existence and uniqueness of fractional differential equations under different types can be found, as seen in [2, 4, 5, 7, 9, 16]. Recently, the study of the existence and uniqueness of solutions of fractional integral equations through integral operators such as Caputo, Riemann-Liouville, Hadamard, Katugampola, and Hilfer has gained prominence in both analytical and functional contexts (Abbas et al. 2018). A significant amount of research has been completed so far on fractional differential equations involving Hilfer derivatives with initial and boundary conditions; therefore, it is worth further consideration. In [8], K. M. Furati et al. studied Hilfer fractional differential equations. In 2015, in the same context, J. Wang and Y. Zang investigated the existence of a solution to a nonlocal IVP for Hilfer fractional differential equations. For details, see [24]. In this regard, in [23], the authors considered implicit fractional equations with nonlocal conditions. In [12], S. Harikrishnan et al. studied the existence and stability results

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Communicated by Maria Alessandra Ragusa
Email address: belqassim.azzouz@univ-relizane.dz (Belqassim Azzouz)
ORCID iD: https://orcid.org/0009-0009-4132-7812 (Belqassim Azzouz)

for Langevin equations with Hilfer fractional derivatives.
Further, in [20], Suphawat et al. studied the nonlocal BVP.

$$ {}_H\mathfrak{D}^{p,\nu}_{a+}\vartheta(u) = R\left(u, \vartheta(u)\right), \quad 1 < p < 2, \quad 0 \le \nu \le 1 \qquad u \in [a, b], $$

with the integral boundary conditions

$$ \vartheta(a) = 0, \quad \vartheta(b) = \sum_{i=1}^{m} \lambda_i \mathfrak{T}^{\gamma_i}_{a+}\vartheta(\mu_i), \quad \gamma_i > 0 \qquad \lambda_i \in \mathbb{R} \qquad \mu_i \in [a, b]. $$

The Banach contraction mapping principle, Banach fixed point theorem with Hölder inequality, nonlinear contraction, Krasnoselskii's fixed point theorem, and the nonlinear Leray-Schauder alternative are employed to prove the existence of a solution to the integral boundary value problem (BVP).
Recently, Mohamed S. Abdo et al. [18] discussed the existence of a solution for Hilfer fractional differential equations with boundary value conditions

$$ \begin{cases} {}^H D^{p,\nu}_{a+} y(s) = R\left(s, y(s)\right), & s \in]a, b], \\ \mathfrak{I}^{1-\xi}_{a+}[c y(a^+) + d y(b^-)] = e_i, \quad c, d, e_i, \in \mathbb{R}, & \xi = p + \nu(1 - p), \end{cases} $$

where ${}^H D^{p,\nu}_{a+}(.)$ is the left Hilfer fractional derivative of order p with $0 < p < 1$ and type ν with $0 \le \nu \le 1$, $\mathfrak{I}^{1-\xi}_{a+}(.)$ is the left Riemann-Liouville fractional integral of order $1 - \xi$, $R :]a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. They obtained several existence results using Schauder's, Schaefer's, and Krasnoselskii's fixed-point theorems.
Motivated by the aforementioned works, this paper considers the following boundary value problem for a class of right Hilfer fractional differential equations

$$ (\mathbb{P}\tau) \begin{cases} {}_H\mathfrak{D}^{p,\nu}_{b-}\vartheta(u) = R\left(u, \vartheta(u), \vartheta(\tau u)\right), & 0 < \tau < 1, \qquad u \in [a, b[, \\ \mathfrak{T}^{1-\gamma}_{b-}\vartheta(b) = \sum_{i=1}^{m} \lambda_i \vartheta(\mu_i) + \vartheta_b, & \gamma = p + \nu - \nu p, \quad \mu_i \in [a, b[, \end{cases} $$

where ${}_H\mathfrak{D}^{p,\nu}_{b-}(.)$ represents the right Hilfer fractional derivative of order p with $0 < p < 1$ and type ν with $0 \le \nu \le 1$, $\mathfrak{T}^{1-\gamma}_{b-}(.)$ is the right Riemann-Liouville fractional integral of order $1 - \gamma$, $R : [a, b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, μ_i $(i = 0, 1, \cdots, m)$ are prefixed points satisfying $a < \mu_1 < \cdots < \mu_m < b$, λ_i is real numbers and ϑ_b is a constant.
The rest of the paper is organized as follows: In Section 2, essential definitions and useful lemmas are provided. In Section 3, we discuss the suitable conditions for the existence and uniqueness of the solution to (1.1) − (1.2). Section 4 focuses on an application to illustrate the results.

2. An auxiliary results

In this section, we present some background material for the forthcoming analysis. Interested readers can refer to [3, 14, 22].

Definition 2.1. *Let u, $w \in \mathbb{C}$, such that $Re(u) > 0$, $Re(w) > 0$. Then,*
a) The gamma function Γ is given as

$$ \Gamma(u) = \int_0^{\infty} t^{u-1} e^{-t} dt, \quad \textit{where } \Gamma(u + 1) = u\Gamma(u). $$

b) The beta function β is defined as follows

$$ \beta(u, w) = \int_0^1 t^{u-1}(1 - t)^{w-1} dt, \quad \textit{so that } \beta(u, w) = \frac{\Gamma(u)\Gamma(w)}{\Gamma(u + w)}. $$

Definition 2.2. *Let $\mathfrak{S} \in L^1[a,b]$. Then, the right Riemann-Liouville fractional integral of order $p>0$ is defined by*

$$\mathfrak{T}^{p}_{b-}\mathfrak{S}(u) = \frac{1}{\Gamma(p)}\int_u^b (t-u)^{p-1}\mathfrak{S}(t)dt.$$

Definition 2.3. *Let $u \in [a,b]$, $q \in \mathbb{N}$ such that $q = [p]$ and $\mathfrak{S} \in C[a,b]$.*
Then, the right Riemann-Liouville fractional derivative of order p is defined by

$${}_{RL}\mathfrak{D}^{p}_{b-}\mathfrak{S}(u) = \left(-\frac{d}{du}\right)^q \mathfrak{T}^{q-p}_{b-}\mathfrak{S}(u).$$

Definition 2.4. *Let $p>0$, $q \in \mathbb{N}$, $q-1<p\le q$, $\gamma = p+\nu(q-p)$, $\nu \in [0,1]$, $u \in [a,b]$ and $\mathfrak{S} \in C^q[a,b]$. Then, the right Hilfer fractional derivative of order p and type ν is determined as*

$${}_{H}\mathfrak{D}^{p,\nu}_{b-}\mathfrak{S}(u) = \mathfrak{T}^{\nu(q-p)}_{b-}\left(-\frac{d}{du}\right)^q \mathfrak{T}^{(1-\nu)(q-p)}_{b-}\mathfrak{S}(u).$$

Now, we consider the weighted space of continuous function

$$C_{1-\gamma}[a,b] = \left\{\mathfrak{S} :\ [a,b[\longrightarrow \mathbb{R},\ (b-u)^{1-\gamma}\mathfrak{S}(u) \in C[a,b]\right\} \text{where } 0<\gamma\le 1,$$

with the norm

$$\|\mathfrak{S}\|_{C_{1-\gamma}[a,b]} = \max_{u\in[a,b]}\left|(b-u)^{1-\gamma}\mathfrak{S}(u)\right|,$$

and

$$C^q_{1-\gamma}[a,b] = \left\{\mathfrak{S} :\ [a,b[\longrightarrow \mathbb{R},\ \left(-\frac{d}{du}\right)^{q-1}\mathfrak{S}(u) \in C[a,b] \text{ and} \left(-\frac{d}{du}\right)^{q}\mathfrak{S}(u) \in C_{1-\gamma}[a,b]\right\},$$

we also introduce the spaces

$$C^{p,\nu}_{1-\gamma}[a,b] = \left\{\mathfrak{S} :\ [a,b[\longrightarrow \mathbb{R},\ \mathfrak{S} \in C_{1-\gamma}[a,b] \text{ and } {}_{H}\mathfrak{D}^{p,\nu}_{b-}\mathfrak{S} \in C_{1-\gamma}[a,b]\right\},$$

and

$$C^{\gamma}_{1-\gamma}[a,b] = \left\{\mathfrak{S} :\ [a,b[\longrightarrow \mathbb{R},\ \mathfrak{S} \in C_{1-\gamma}[a,b] \text{ and } {}_{RL}\mathfrak{D}^{\gamma}_{b-}\mathfrak{S} \in C_{1-\gamma}[a,b]\right\}.$$

Lemma 2.5. *Let $\l_1,\ \l_2>0$. Then we have the following semigroup property*

$$\mathfrak{T}^{\l_1}_{b-}\mathfrak{T}^{\l_2}_{b-} = \mathfrak{T}^{\l_1+l_2}_{b-}.$$

Lemma 2.6. *Let $p>0$, $q\in\mathbb{N}$ such that $q=[p]$. Then,*

$$(1)\ {}_{RL}\mathfrak{D}^{p}_{b-}\mathfrak{T}^{p}_{b-}\mathfrak{S}(u) = \mathfrak{S}(u). \text{ If } \mathfrak{S} \in C_{q-\gamma}[a,b]$$

$$(2)\left(-\frac{d}{du}\right)^q \mathfrak{T}^{q}_{b-}\mathfrak{S}(u) = \mathfrak{S}(u). \text{ If } \mathfrak{S} \in C[a,b].$$

Lemma 2.7. *Let $p\in\mathbb{R}_+$ with $p<1$, $\nu \in [0,1]$ and $\gamma = p+\nu-\nu p$.*
If $\mathfrak{S} \in C^{\gamma}_{1-\gamma}[a,b]$, Then,

$${}_{RL}\mathfrak{D}^{\gamma}_{b-}\,{}^{\mathfrak{T}}\mathfrak{T}^{p}_{b-}\mathfrak{S}(u) = {}_{RL}\mathfrak{D}^{\nu-\nu p}_{b-}\mathfrak{S}(u).$$

and

$$\mathfrak{T}^{\gamma}_{b-}\,{}_{RL}\mathfrak{D}^{\gamma}_{b-}\mathfrak{S}(u) = \mathfrak{T}^{p}_{b-}\,{}_{H}\mathfrak{D}^{p,\nu}_{b-}\mathfrak{S}(u).$$

Proposition 2.8. *Let $\mathfrak{l} > 0$, and let $\gamma \in \mathbb{R}$ such that $\gamma > -1$. Then fractional integral and derivative of a power function are given by*

$$\mathfrak{I}^{\mathfrak{l}}_{b-}\,(b-u)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+l)}\,(b-u)^{\gamma+l}.$$

and

$$_{RL}\mathfrak{D}^{l}_{b-}(b-u)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-l)}\,(b-u)^{\gamma-l}.$$

Moreovere, If $0 < l < 1$, then

$$_{RL}\mathfrak{D}^{l}_{b-}(b-u)^{l} = 0$$

Lemma 2.9. *Let $l,\ \sigma \in \mathbb{R}_+$ with $\sigma < 1$ and $q = [l]$. If $\mathfrak{S} \in C_\sigma[a,b]$ and $\mathfrak{I}^{q-l}_{b-}\mathfrak{S} \in C^{q}_{\sigma}[a,b]$. Then,*

$$\mathfrak{I}^{l}_{b-}\left[{}_{RL}\mathfrak{D}^{l}_{b-}\mathfrak{S}(u)\right] = (-1)^{q+1}\left[\mathfrak{S}(u) - \sum_{i=1}^{q}\frac{(b-u)^{l-i}}{\Gamma(l-i+1)}\left[\left(\frac{d}{du}\right)^{q-i}\mathfrak{I}^{q-l}_{b-}\mathfrak{S}(u)\right]_{u=b}\right].$$

Proof. First, by the lemma 2.6-(2). Taking $q = 1$, we have

$$\left(-\frac{d}{du}\right)\mathfrak{I}^{1}_{b-}\,\mathfrak{S}(u) = \mathfrak{S}(u). \tag{1}$$

Using the equation (1) with $\mathfrak{S}(u)$ replaced by $\mathfrak{I}^{l}_{b-}\,{}_{RL}\mathfrak{D}^{l}_{b-}\mathfrak{S}(u)$, we have

$$\mathfrak{I}^{l}_{b-}\,\left({}_{RL}\mathfrak{D}^{l}_{b-}\mathfrak{S}(u)\right) = \left(-\frac{d}{du}\right)\mathfrak{I}^{1}_{b-}\left[\mathfrak{I}^{l}_{b-}\,\left({}_{RL}\mathfrak{D}^{l}_{b-}\mathfrak{S}(u)\right)\right]. \tag{2}$$

From the relation (2) and the definition of the right Riemann-liouville fractional integrals, and derivative, we have

$$\begin{aligned}\mathfrak{I}^{l}_{b-}\,\left({}_{RL}\mathfrak{D}^{l}_{b-}\,\mathfrak{S}(u)\right) &= \left(-\frac{d}{du}\right)\left[\,\mathfrak{I}^{1+l}_{b-}\,\left({}_{RL}\mathfrak{D}^{l}_{b-}\,\mathfrak{S}(u)\right)\right]\\ &= \left(-\frac{d}{du}\right)\left[\frac{1}{\Gamma(l+1)}\int_u^b (t-u)^l\,{}_{RL}\mathfrak{D}^{l}_{b-}\,\mathfrak{S}(t)dt\right]\\ &= \left(-\frac{d}{du}\right)\left[\frac{(-1)^q}{\Gamma(l+1)}\int_u^b (t-u)^l\left(\frac{d}{dt}\right)^q\mathfrak{I}^{q-l}_{b-}\,\mathfrak{S}(t)dt\right].\end{aligned} \tag{3}$$

Integration by parts the relation

$$\begin{aligned}&\frac{1}{\Gamma(l+1)}\int_u^b (t-u)^l\left(\frac{d}{dt}\right)^q\mathfrak{I}^{q-l}_{b-}\,\mathfrak{S}(t)dt \\ &= \frac{1}{\Gamma(l+1)}\int_u^b (t-u)^l\frac{d}{dt}\left[\left(\frac{d}{dt}\right)^{q-1}\mathfrak{I}^{q-l}_{b-}\,\mathfrak{S}(t)\right]dt\\ &= \frac{1}{\Gamma(l+1)}\,(b-u)^l\left[\left(\frac{d}{dt}\right)^{q-1}\mathfrak{I}^{q-l}_{b-}\mathfrak{S}(t)\right]_{t=b}\\ &- \frac{1}{\Gamma(l)}\int_u^b (t-u)^{l-1}\frac{d}{dt}\left[\left(\frac{d}{dt}\right)^{q-2}\mathfrak{I}^{q-l}_{b-}\,\mathfrak{S}(t)\right]dt.\end{aligned} \tag{4}$$

Repeating the proces of integration by parts n^{th} step, we have

$$(4) = \sum_{j=1}^{j=q} \frac{1}{\Gamma(l-j+2)}(b-t)^{(l-j+1)} \left[\left(\frac{d}{dt}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\, \mathfrak{S}(t) \right]_{t=b} - \frac{1}{\Gamma(l-(q-1))} \int_u^b (t-u)^{l-q} \left(\frac{d}{dt}\right)^{q-q} \mathfrak{I}_{b-}^{q-l}\, \mathfrak{S}(t)dt.$$

Now, consider using the definition of right Riemann-liouville fractional integrale, its semi groupe property, we have

$$\begin{aligned}
(4) &= \sum_{j=1}^{j=q} \frac{1}{\Gamma(l-j+2)}(b-u)^{(l-j+1)} \left[\left(\frac{d}{dt}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\, \mathfrak{S}(t) \right]_{t=b} \\
&\quad - \frac{1}{\Gamma(l+1-q)} \int_u^b (t-u)^{(l+1-q-1)} \mathfrak{I}_{b-}^{q-l}\, \mathfrak{S}(t)dt. \\
&= \sum_{j=1}^{j=q} \frac{1}{\Gamma(l-j+2)}(b-u)^{(l-j+1)} \left[\left(\frac{d}{dt}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\mathfrak{S}(t) \right]_{t=b} \\
&\quad - \mathfrak{I}_{b-}^{l+1-q}\, \mathfrak{I}_{b-}^{q-l}\mathfrak{S}(u) \\
&= \sum_{j=1}^{j=q} \frac{1}{\Gamma(l-j+2)}(b-u)^{(l-j+1)} \left[\left(\frac{d}{dt}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\mathfrak{S}(t) \right]_{t=b} \\
&\quad - \int_u^b \mathfrak{S}(t)dt. \qquad (5)
\end{aligned}$$

Using the equation (3) in the equation (5), we get

$$\begin{aligned}
\mathfrak{I}_{b-}^{l}\, {}_{RL}\mathfrak{D}_{b-}^{l}\mathfrak{S}(u) &= (-1)^{q+1}\frac{d}{du}\Big\{ -\int_u^b \mathfrak{S}(t)dt \\
&\quad + \sum_{j=1}^{j=q} \frac{1}{\Gamma(l-j+2)}(b-u)^{(l-j+1)} \left[\left(\frac{d}{dt}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\mathfrak{S}(t) \right]_{t=b} \Big\} \\
&= (-1)^{q+1}\left[\mathfrak{S}(u) - \sum_{j=1}^{j=q} \frac{1}{\Gamma(l-j+1)}(b-u)^{(l-j)} \left[\left(\frac{d}{dt}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\mathfrak{S}(t) \right]_{t=b} \right] \\
&= (-1)^{q+1}\left[\mathfrak{S}(u) - \sum_{j=1}^{j=q} \frac{(b-u)^{l-j}}{\Gamma(l-j+1)} \left[\left(\frac{d}{du}\right)^{q-j} \mathfrak{I}_{b-}^{q-l}\mathfrak{S}(u) \right]_{u=b} \right].
\end{aligned}$$

□

Lemma 2.10. *Let $p \in \mathbb{R}$ with $p < 1$, $\nu \in [0,1]$ and $\gamma = p + \nu - \nu p$.*
Suppose $R : [a,b[\times \mathbb{R} \to \mathbb{R}$ is a function such that $R(.,\vartheta(.)) \in C_{1-\gamma}[a,b]$,
for any $\vartheta \in C_{1-\gamma}[a,b]$.
If $\vartheta \in C_{1-\gamma}^{\gamma}[a,b]$ then ϑ satisfies the problem $(\mathbb{P})$

$$(\mathbb{P}) \begin{cases} {}_{H}\mathfrak{D}_{b-}^{p,\nu}\vartheta(u) = R(u,\vartheta(u)), \quad u \in [a,b[, \\ \mathfrak{I}_{b-}^{1-\gamma}\vartheta(b) = \vartheta_b, \end{cases}$$

if and only if ϑ satisfiees the following fractional integral equation

$$\vartheta(u) = \frac{(b-u)^{\gamma-1}}{\Gamma(\gamma)}\,\vartheta_b + \frac{1}{\Gamma(p)}\int_u^b (t-u)^{p-1}\,R(t,\vartheta(t))dt,\ u \in [a,b[. \tag{6}$$

Theorem 2.11. *Let $l,\ \sigma \in \mathbb{R}_+$with $\sigma < 1$. Then, the right Riemann-Liouville fractional integral $\mathfrak{T}^{l}_{b-}(.)$ is bounded from $C_{1-\gamma}[a,b]$ into $C_{1-\gamma}[a,b]$.*

Theorem 2.12. *Let $l > 0,\ \sigma > 0$, with $\sigma \le \mathfrak{l} < 1$. Then, the right Riemann-Liouville fractional integral $\mathfrak{T}^{\mathfrak{l}}_{b-}$ is bounded from $C_{\sigma}[a,b]$ and to $C[a,b]$.*

Theorem 2.13. *Let $0 < l$, $0 < \gamma \le 1$ and $\mathfrak{S} \in C_{1-\gamma}[a,b]$. If $l > \gamma$, then $\mathfrak{T}^{l}_{b-}(.) \in C[a,b]$ and*

$$\mathfrak{T}^{l}_{b-}\mathfrak{S}(b) = \lim_{u\to b-}\ \mathfrak{T}^{l}_{b-}\mathfrak{S}(u) = 0.$$

Theorem 2.14. *Let $p > 0,\ 0 \le \nu \le 1$ and $\mathfrak{S} \in L^1[a,b]$. Assume that ${}_{RL}\mathfrak{D}^{\nu(1-p)}_{b-}\ \mathfrak{S}(u)$ exists it lies in $L^1[a,b]$. Then,*

$${}_{H}\mathfrak{D}^{p,\nu}_{b-}\ \mathfrak{T}^{p}_{b-}\mathfrak{S}(u) = \ \mathfrak{T}^{\nu(1-p)}_{b-}\ {}_{RL}\mathfrak{D}^{\nu(1-p)}_{b-}\ \mathfrak{S}(u), \qquad u \in [a,b).$$

Moreovere, if $\mathfrak{S} \in C_{1-\gamma}[a,b]$, $\mathfrak{T}^{\nu(1-p)}_{b-}\mathfrak{S} \in C^1_{1-\gamma}[a,b]$, then ${}_{H}\mathfrak{D}^{p,\nu}_{b-}\ \mathfrak{T}^{p}_{b-}\mathfrak{S}(.)$ exists on $[a,b)$ and

$${}_{H}\mathfrak{D}^{p,\nu}_{b-}\ \mathfrak{T}^{p}_{b-}\mathfrak{S}(u) = \mathfrak{S}(u), \qquad u \in [a,b[.$$

Theorem 2.15. *Let $0 < p < 1,\ 0 \le \nu \le 1$ and $\gamma = p + \nu - \nu p$. If $\mathfrak{S} \in C^{\gamma}_{1-\gamma}[a,b]$, then*

$$\mathfrak{T}^{\gamma}_{b-}\ {}_{RL}\mathfrak{D}^{\gamma}_{b-}\mathfrak{S}(u) = \ \mathfrak{T}^{p}_{b-}\ {}_{H}\mathfrak{D}^{p,\nu}_{b-}\mathfrak{S}(u)$$

Theorem 2.16. *(Krasnoselskii's fixed point theorem) Let Ω be a closed convex and nonempty subset of a Banach space X, let $T_1,\ T_2$ be the operators such that*
a) $T_1x + T_2\vartheta \in \Omega$ for every pair $x,\ \vartheta \in \Omega$.
b) T_1 is compact and continous.
c) T_2 is a contraction mapping.
Then there exists $z \in \Omega$ such that $z = T_1z + T_2z$.

Theorem 2.17. *(Banach fixed point theorem) Let (X ; d) be a nonempty complete metric space with $T : X \to X$ is a contraction mapping. Then map T has a fixed point.*

Lemma 2.18. *Let $\gamma = p + \nu - \nu p$ where $0 < p < 1$, $\nu \in [0,1]$. Assume that $R :\ [a,b[\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function such that $R(.,\vartheta(.),\vartheta(\tau.)) \in C_{1-\gamma}[a,b]$, for any $\vartheta \in C_{1-\gamma}[a,b]$. If $\vartheta \in C^{\gamma}_{1-\gamma}[a,b]$ then ϑ satisfies the problem $(\mathbb{P}\tau)$ if and only if ϑ satisfies the following fractional integral equation*

$$\vartheta(u) = \frac{(b-u)^{\gamma-1}}{\Gamma(\gamma) - \sum_{i=1}^{m}\lambda_i[(b-\mu_i)]^{\gamma-1}}\Bigg[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}(t-\mu_i)^{p-1}$$

$$\times R(t,\vartheta(t),\vartheta(\tau t))dt + \vartheta_b\Bigg] + \frac{1}{\Gamma(p)}\int_u^b (t-u)^{p-1}\,R(t,\vartheta(t),\vartheta(\tau t))dt,\ u \in [a,b[.$$

For the sake of convenience, we use the following notations :

$$\mathbf{\Theta}^{\gamma}_{-}\,(b,.) = (b-.)^{\gamma-1}, \qquad\qquad R(t,\vartheta(t),\vartheta(\tau t)) = \mathfrak{S}(t),$$

and

$$\mathbf{\Theta}^{-\gamma}_{-}\,(b,.) = (b-.)^{-\gamma+1}, \qquad £ = \Gamma(\gamma) - \sum_{i=1}^{m}\lambda_i\mathbf{\Theta}^{\gamma}_{-}\,(b,\mu_i) \neq 0.$$

Now, we can write

$$\vartheta(u) = \frac{\mathbf{\Theta}^{\gamma}_{-}(b,u)}{£}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt + \vartheta_b\right] + \frac{1}{\Gamma(p)}\int_{u}^{b}\mathbf{\Theta}^{p}_{-}(t,u)\,\mathfrak{S}(t)dt. \tag{7}$$

Proof. Assume that $\vartheta \in C^{\gamma}_{1-\gamma}[a,b]$ is a solution to the problem ($\mathbb{P}\tau$). We show that ϑ is also a solution of fractional integrale Equation (7).
Since $\vartheta \in C^{\gamma}_{1-\gamma}[a,b]$, we have $\vartheta \in C_{1-\gamma}[a,b]$ and

$$\left[-\frac{d}{u}\right]\mathfrak{T}^{1-\gamma}_{b-}\,\vartheta = {}_{RL}\mathfrak{D}^{\gamma}_{b-}\,\vartheta \in C_{1-\gamma}[a,b]. \tag{8}$$

Further, by applying Theorem 2.12 with $l = 1-\gamma$, we get

$$\mathfrak{T}^{1-\gamma}_{b-}\vartheta \in C[a,b]. \tag{9}$$

According to Equation (8) and (9) and using the definition of the space $C^{q}_{1-\gamma}[a,b]$, we obtain

$$\mathfrak{T}^{1-\gamma}_{b-}\vartheta \in C^{1}_{1-\gamma}[a,b].$$

Sinc $\vartheta \in C_{1-\gamma}[a,b]$ and $\mathfrak{T}^{1-\gamma}_{b-}\vartheta \in C^{1}_{1-\gamma}[a,b]$, by applying the lemma 2.9 with $\sigma = 1-\gamma$, $l=\gamma$ and $q=[\gamma]=1$, we get

$$\mathfrak{T}^{\gamma}_{b-}\,{}_{RL}\mathfrak{D}^{\gamma}_{b-}\,\vartheta(u) = \vartheta(u) - \frac{\mathbf{\Theta}^{\gamma}_{-}(b,u)}{\Gamma(\gamma)}\,[\mathfrak{T}^{1-\gamma}_{b-}\vartheta(u)]_{u=b}. \tag{10}$$

By hypothesis ${}_{RL}\mathfrak{D}^{\gamma}_{b-}\,\vartheta \in C_{1-\gamma}[a,b]$, using Lemma 2.7 and Equation (1), we have

$$\mathfrak{T}^{\gamma}_{b-}\,{}_{RL}\mathfrak{D}^{\gamma}_{b-}\,\vartheta(u) = \mathfrak{T}^{p}_{b-}\,{}_{H}\mathfrak{D}^{p,\nu}_{b-}\,\vartheta(u) = \mathfrak{T}^{p}_{b-}\,\mathfrak{S}(u). \tag{11}$$

Comparing Equation (10) and (11), we see that

$$\vartheta(u) = \frac{\mathbf{\Theta}^{\gamma}_{-}(b,u)}{\Gamma(\gamma)}\,[\mathfrak{T}^{1-\gamma}_{b-}\vartheta(u)]_{u=b} + \mathfrak{T}^{p}_{b-}\,\mathfrak{S}(u) \tag{12}$$

Now, we substitute $u=\mu_i$ in (12) and multiply by λ_i we can write

$$\lambda_i\vartheta(\mu_i) = \lambda_i\frac{\mathbf{\Theta}^{\gamma}_{-}(b,\mu_i)}{\Gamma(\gamma)}\,[\mathfrak{T}^{1-\gamma}_{b-}\vartheta(u)]_{u=b} + \lambda_i\,\mathfrak{T}^{p}_{b-}\,\mathfrak{S}(\mu_i).$$

The last equality with the nonlocal condition (1), gives us

$$\mathfrak{T}^{1-\gamma}_{a+}\vartheta(b) = \sum_{i=1}^{m}\lambda_i\vartheta(\mu_i) + \vartheta_b = \sum_{i=1}^{m}\lambda_i\frac{\mathbf{\Theta}^{\gamma}_{-}(b,\mu_i)}{\Gamma(\gamma)}\,[\mathfrak{T}^{1-\gamma}_{b-}\vartheta(u)]_{u=b} + \sum_{i=1}^{m}\lambda_i\mathfrak{T}^{p}_{b-}\,\mathfrak{S}(\mu_i).$$

We find

$$\mathfrak{T}^{1-\gamma}_{b-}\vartheta(b) = \frac{\Gamma(\gamma)\left(\sum_{i=1}^{m}\lambda_i\,\mathfrak{T}^{p}_{b-}\,\mathfrak{S}(\mu_i) + \vartheta_b\right)}{£}. \tag{13}$$

Substituting (13) into (12), we conclude that $\vartheta(u)$ satisfies (7).
Conversly, suppose that $\vartheta \in C^{\gamma}_{1-\gamma}[a,b]$ satisfying equation (7). Then,

$$\vartheta(u) = \frac{\mathbf{\Theta}^{\gamma}_{-}(b,u)}{£}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt + \vartheta_b\right] + \frac{1}{\Gamma(p)}\int_{u}^{b}\mathbf{\Theta}^{p}_{-}(t,u)\,\mathfrak{S}(t)dt.$$

Inserting ${}_{RL}\mathfrak{D}^{\gamma}_{b-}$ on both sides of above equation, we get

$$ {}_{RL}\mathfrak{D}^{\gamma}_{b-}\ \vartheta(u) = \frac{{}_{RL}\mathfrak{D}^{\gamma}_{b-}\boldsymbol{\Theta}^{\gamma}_{-}\ (b,u)}{\pounds}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\boldsymbol{\Theta}^{p}_{-}\ (t,\mu_i)\ \mathfrak{S}(t)dt + \vartheta_b\right] + {}_{RL}\mathfrak{D}^{\gamma}_{b-}\ \mathfrak{T}^{p}_{b-}\ \mathfrak{S}(u). $$

By applying the Lemma 2.6 and Lemma 2.7 we obtain

$$ {}_{RL}\mathfrak{D}^{\gamma}_{b-}\vartheta(u) = {}_{RL}\mathfrak{D}^{\nu(1-p)}_{b-}\mathfrak{S}(u). \tag{14} $$

Since $\vartheta \in C^{\gamma}_{1-\gamma}[a,b]$, and by Definition of $C^{\gamma}_{1-\gamma}[a,b]$, we have ${}_{RL}\mathfrak{D}^{\gamma}_{b-}\vartheta \in C_{1-\gamma}[a,b]$.
Therefore, from (14) it follows that

$$ {}_{RL}\mathfrak{D}^{\nu(1-p)}_{b-}R(u,\vartheta(u),\vartheta(\tau u)) \in C_{1-\gamma}[a,b]. \tag{15} $$

Since $p < 1$, $\nu \in [0,1]$ and $0 < 1-p < 1$, we obtain $\nu(1-p) < 1$. Therefore

$$ [\nu(1-p)] = 1. $$

In this case the definition of Riemann-Liouville derivative reduce to

$$ {}_{RL}\mathfrak{D}^{\nu(1-p)}_{b-}\ R(u,\vartheta(u),\vartheta(\tau u)) = \left[-\frac{d}{du}\right]\ \mathfrak{T}^{1-\nu(1-p)}_{b-}R(u,\vartheta(u),\vartheta(\tau u)). \tag{16} $$

Clearly, by (15) and (16), we obtain

$$ \left[-\frac{d}{du}\right]\ \mathfrak{T}^{1-\nu(1-p)}_{b-}R(u,\vartheta(u),\vartheta(\tau u)) \in C_{1-\gamma}[a,b]. \tag{17} $$

Since $\gamma = p + \nu - \nu p > \nu(1-p)$, we have $1-\gamma < 1-\nu+\nu p$.
Since $\mathfrak{S}(.) \in C_{1-\gamma}[a,b]$, by applying Theorem 2.12, we get

$$ \mathfrak{T}^{1-\nu(1-p)}_{b-}\mathfrak{S}(.) \in C[a,b]. \tag{18} $$

Using the definition of the space $C^{q}_{1-\gamma}[a,b]$, from equation (17) and (18), it follows that

$$ \mathfrak{T}^{1-\nu+\nu p}_{b-}\mathfrak{S}(.) \in C^{1}_{1-\gamma}[a,b]. $$

By applying $\mathfrak{T}^{\nu(1-p)}_{b-}$ on both sided of equation (14) and using Lemma 2.9 with
$l = \nu - \nu p$ and $q = 1$, we have

$$ \begin{aligned} \mathfrak{T}^{\nu(1-p)}_{b-}\ {}_{RL}\mathfrak{D}^{\gamma}_{b-}\ \vartheta(u) &= \mathfrak{T}^{\nu(1-p)}_{b-}\ {}_{RL}\mathfrak{D}^{\nu(1-p)}_{b-}\ \mathfrak{S}(t) \\ &= \mathfrak{S}(u) - \frac{\boldsymbol{\Theta}^{\nu(1-p)}_{-}\ (b,u)}{\Gamma(\nu(1-p))}\left[\mathfrak{T}^{1-\nu(1-p)}_{b-}\mathfrak{S}(b)\right]. \end{aligned} \tag{19} $$

Using the theorem 2.13 with $l = 1-\nu(1-p)$, we obtain

$$ \left[\mathfrak{T}^{1-\nu(1-p)}_{b-}\mathfrak{S}(b)\right] = 0. \tag{20} $$

Comparing the last equality with (1), we get

$$ {}_{H}\mathfrak{D}^{p,\nu}_{b-}\vartheta(u) = \mathfrak{S}(u), \quad u \in [a,b). $$

which means that (1) holds. Next, we show that if $\vartheta \in C^{\gamma}_{1-\gamma}[a,b]$ satisfies (7), it also satisfies the condition (1).
To this end, we multiply both sides of (7) by $\mathfrak{T}^{1-\gamma}_{b-}$and use Proposition 2.8 and Lemma 2.5, we have

$$\mathfrak{T}^{1-\gamma}_{b-}\vartheta(u) = \frac{\Gamma(\gamma)}{\pounds}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt + \vartheta_b\right] + \frac{1}{\Gamma(p-\gamma+1)}\int_{u}^{b}\mathbf{\Theta}^{p-\gamma+1}_{-}(u,t)\,\mathfrak{S}(t)dt.$$

Since $1-\gamma < p-\gamma+1$, Theorem 2.13 can be used when taking the limit $u \to b$,

$$\mathfrak{T}^{1-\gamma}_{a+}\vartheta(b) = \frac{\Gamma(\gamma)}{\pounds}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt + \vartheta_b\right]. \tag{21}$$

Substituting $u = \mu_i$ into (7), we have

$$\vartheta(\mu_i) = \frac{\mathbf{\Theta}^{\gamma}_{-}(b,\mu_i)}{\pounds}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt + \vartheta_b\right] + \frac{1}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt.$$

Then, we drive

$$\begin{aligned}\sum_{i=1}^{m}\lambda_i\vartheta(\mu_i) &= \sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt\left[1+\sum_{i=1}^{m}\frac{\lambda_i}{\pounds}\mathbf{\Theta}^{\gamma}_{-}(b,\mu_i)\right] + \sum_{i=1}^{m}\frac{\lambda_i}{\pounds}\mathbf{\Theta}^{\gamma}_{-}(b,\mu_i)\vartheta_b \\ &= \frac{\Gamma(\gamma)}{\pounds}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt\right] + \left[\frac{\Gamma(\gamma)}{\pounds}-1\right]\vartheta_b.\end{aligned}$$

which gives,

$$\sum_{i=1}^{m}\lambda_i\vartheta(\mu_i) + \vartheta_b = \frac{\Gamma(\gamma)}{\pounds}\left[\sum_{i=1}^{m}\frac{\lambda_i}{\Gamma(p)}\int_{\mu_i}^{b}\mathbf{\Theta}^{p}_{-}(t,\mu_i)\,\mathfrak{S}(t)dt + \vartheta_b\right]. \tag{22}$$

It follows (21) and (22) that

$$\mathfrak{T}^{1-\gamma}_{b-}\vartheta(b) = \sum_{i=1}^{m}\lambda_i\vartheta(\mu_i) + \vartheta_b.$$

This proves the initial condition 1 is verified. □

3. Existence and uniqueness results for problem ($\mathbb{P}\tau$)

In this section, we present existence and uniqueness results for the considered problem ($\mathbb{P}\tau$).

Theorem 3.1. *Assume that the hypotheses following two* (A_1) *and* (A_2) *are fulfilled*
(A_1) *Let* $R: \ [a,b[\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be a function such that* $R(.,\vartheta(.),\vartheta(\tau.)) \in C^{\nu(1-p)}_{1-\gamma}[a,b]$,
for any $\vartheta \in C_{1-\gamma}[a,b]$, *and there exists* $L > 0$ *such that*

$$|R(u,\vartheta_1,x_1) - R(u,\vartheta_2,x_2)| \le L\left[|\vartheta_1-\vartheta_2| + |x_1-x_2|\right], \tag{23}$$

for all $u \in [a,b[$ *and* $\vartheta_i, x_i \in \mathbb{R}(i=1,2)$.
(A_2) *The constant*

$$\sigma := \left[\sum_{i=1}^{m} \frac{\lambda_i}{£} \mathbf{\Theta}_-^{p+\gamma}(b,\mu_i) + \mathbf{\Theta}_-^{p+1}(a,b)\right] \frac{2LB(\gamma,p)}{\Gamma(p)} < 1. \tag{24}$$

where $B(.,.)$ *is defined as in Definition 2.1.*
The first result is based on Theorem 2.16. And so there exists at least one solution for the Hilfer problem $(\mathbb{P}\tau)$ *in the space* $C^{\gamma}_{1-\gamma}[a,b] \subset C^{p,\nu}_{1-\gamma}[a,b]$.

Proof. We use the Krasnoselskii's fixed point theorem to prove the existence of solution ϑ in the weighted space $C^{\gamma}_{1-\gamma}[a,b]$. Define the operator $T : C_{1-\gamma}[a,b] \to C_{1-\gamma}[a,b]$ by

$$(Ty)(u) = \frac{\mathbf{\Theta}_-^{\gamma}(b,u)}{£}\left[\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(p)} \int_{\mu_i}^{b} \mathbf{\Theta}_-^{p}(t,\mu_i)\, F_\vartheta(t)dt + \vartheta_b\right] + \frac{1}{\Gamma(p)} \int_u^b \mathbf{\Theta}_-^{p}(t,u)\, F_\vartheta(t)dt. \tag{25}$$

Where $F_\vartheta(t) := R(t, \vartheta(t), \vartheta(\tau t))$.
Setting $\tilde{R}(t) = F_0(t) := R(t,0,0)$, and suppose the ball $\mathcal{B}_r = \left\{\vartheta \in C_{1-\gamma}[a,b] : \|\vartheta\|_{C_{1-\gamma}} \le r\right\}$, having $r \ge \frac{w}{1-\sigma}$, $\sigma < 1$ where

$$w := \frac{B(\gamma,p)}{\Gamma(p)}\left[\sum_{i=1}^{m} \frac{\lambda_i}{£} \mathbf{\Theta}_-^{p+\gamma}(b,\mu_i) + \mathbf{\Theta}_-^{p+1}(a,b)\right] \|\tilde{R}\|_{C_{1-\gamma}} + \frac{|\vartheta_b|}{£}.$$

First, surmise the operator T into sum two operators $T_1 + T_2$ as follows

$$T_1\vartheta(u) = \frac{\mathbf{\Theta}_-^{\gamma}(b,u)}{£}\left[\sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(p)} \int_{\mu_i}^{b} \mathbf{\Theta}_-^{p}(t,\mu_i)\, F_\vartheta(t)dt + \vartheta_b\right],$$
$$T_2\vartheta(u) = \frac{1}{\Gamma(p)} \int_u^b \mathbf{\Theta}_-^{p}(t,u)\, F_\vartheta(t)dt.$$

The proof will be demonstrated by the accompanying three steps.
Step 1 : we show that $T_1\vartheta + T_2x \in \mathcal{B}_r$ for every $\vartheta, x \in \mathcal{B}_r$.
For operator T_1, by our hypotheses, we have

$$\begin{aligned}
&|\mathbf{\Theta}_-^{-\gamma}(b,u) T_1\vartheta(u)| \\
&\le \frac{1}{£} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(p)} \int_{\mu_i}^{b} \mathbf{\Theta}_-^{p}(t,\mu_i)\, [|F_\vartheta(t) - F_0(t)| + |F_0(t)|]\, dt + \frac{|\vartheta_b|}{£} \\
&\le \frac{1}{£} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(p)} \int_{\mu_i}^{b} \mathbf{\Theta}_-^{p}(t,\mu_i) \Big(L(|\vartheta(t)| + |\vartheta(\tau t)|) + |\tilde{R}(t)|\Big) dt + \frac{|\vartheta_b|}{£} \\
&\le \frac{B(\gamma,p)}{£} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(p)} \mathbf{\Theta}_-^{p+\gamma}(b,\mu_i) \Big(2L\|\vartheta\|_{C_{1-\gamma}} + \|\tilde{R}\|_{C_{1-\gamma}}\Big) + \frac{|\vartheta_b|}{£}
\end{aligned}$$

hence, for every $\vartheta \in \mathcal{B}_r$, we find that

$$\|T_1\vartheta\|_{C_{1-\gamma}} \le \frac{B(\gamma,p)}{£} \sum_{i=1}^{m} \frac{\lambda_i}{\Gamma(p)} \mathbf{\Theta}_-^{p+\gamma}(b,\mu_i) \Big(2L\|\vartheta\|_{C_{1-\gamma}} + \|\tilde{R}\|_{C_{1-\gamma}}\Big) + \frac{|\vartheta_b|}{£}. \tag{26}$$

For operator T_2, we have

$$\mid \Theta_{-}^{-\gamma}(b,u) T_2 x(u) \mid \leq \frac{\Theta_{-}^{-\gamma}(b,u)}{\Gamma(p)} \int_u^b \Theta_{-}^{p}(t,u) \; [\mid F_x(t) - F_0(t) \mid + \mid F_0(t) \mid] \, dt$$
$$\leq \frac{\Theta_{-}^{-\gamma}(b,u)}{\Gamma(p)} \Theta_{-}^{p+\gamma}(b,u) \, B(\gamma,p) \left(2L \parallel \vartheta \parallel_{C_{1-\gamma}} + \|\tilde{R}\|_{C_{1-\gamma}}\right).$$

Thus we get :

$$\|T_2 x\|_{C_{1-\gamma}} \leq \frac{B(\gamma,p)}{\Gamma(p)} \left(2L\|\vartheta\|_{C_{1-\gamma}} + \|\tilde{R}\|_{C_{1-\gamma}}\right) \left[\Theta_{-}^{p+1}(b,u)\right]. \tag{27}$$

By Definitions of σ and r with (27) and (26), we get

$$\|T_1 x + T_2 \vartheta\|_{C_{1-\gamma}} \leq \|T_1 \vartheta\|_{C_{1-\gamma}} + \|T_2 x\|_{C_{1-\gamma}} \leq \sigma r + w \leq r.$$

This proves that $T_1\vartheta + T_2 x \in \mathcal{B}_r$ for every $\vartheta, x \in \mathcal{B}_r$.
Step 2 : The operator T_1 is a contration mapping on $\mathcal{B}_r$.
For any $\vartheta, x \in \mathcal{B}_r$, and for any $u \in [a,b)$, we have

$$\mid \Theta_{-}^{-\gamma}(b,u) T_1 \vartheta(u) - \Theta_{-}^{-\gamma}(b,u) T_1 x(u) \mid$$
$$\leq \sum_{i=1}^{m} \frac{\lambda_i}{\pounds} \frac{1}{\Gamma(p)} \int_{\mu_i}^{b} \Theta_{-}^{p}(t,\mu_i) \mid F_\vartheta(t) - F_x(t) \mid dt$$
$$\leq \sum_{i=1}^{m} \frac{\lambda_i}{\pounds} \frac{1}{\Gamma(p)} \int_{\mu_i}^{b} \Theta_{-}^{p}(t,\mu_i) \, 2L \mid \vartheta(t) - x(t) \mid dt$$
$$\leq \frac{B(\gamma,p)}{\Gamma(p)} 2L \sum_{i=1}^{m} \frac{\lambda_i}{\pounds} \Theta_{-}^{p+\gamma}(b,\mu_i) \|\vartheta - x\|_{C_{1-\gamma}}.$$

This yields

$$\|T_1\vartheta - T_1 x\|_{C_{1-\gamma}} \leq \frac{B(\gamma,p)}{\Gamma(p)} 2L \sum_{i=1}^{m} \frac{\lambda_i}{\pounds} \Theta_{-}^{p+\gamma}(b,\mu_i) \|\vartheta - x\|_{C_{1-\gamma}}.$$

The operator T_1 is contraction mapping. Thus, condition (A_2) of Theorem 3.1 is satisfied.
Step 3 : The operator T_2 is completely continuous on $\mathcal{B}_r$.
Now, we will prove that the opertor T_2 is continuous.
Now, we prove that $(T_2\mathcal{B}_r)$ is uniformly bounded. Indeed, it is enough to show that for some $r > 0$, there exists a positive constant l such that $\|T_2\vartheta\|_{C_{1-\gamma}} \leq l$.
According to step 1, for $\vartheta \in \mathcal{B}_r$, we know that

$$\|T_2\vartheta\|_{C_{1-\gamma}} \leq \left[\frac{B(\gamma,p)}{\Gamma(p)} \left(2L\|\vartheta\|_{C_{1-\gamma}} + \|\tilde{R}\|_{C_{1-\gamma}}\right)\right] \left[\Theta_{-}^{p+1}(a,b)\right] := l.$$

Hence, $\|T_2\vartheta\|_{C_{1-\gamma}} \leq l$. Which shows that the operator T_2 is uniformly bounded on $\mathcal{B}_r$. Finaly, we show that $(T_2\mathcal{B}_r)$ is equicontinuous in $\mathcal{B}_r$.

Let $\vartheta \in \mathcal{B}_r$ and $u_1,\ u_2 \in [a,b]$ with $u_1 < u_2$, we have

$$\begin{aligned}
&\mid \mathbf{\Theta}_{-}^{-\gamma}\ (b,u_2)T_2\vartheta(u_2) - \mathbf{\Theta}_{-}^{-\gamma}\ (b,u_1)T_2\vartheta(u_1) \mid \\
&\leq \left| \frac{\mathbf{\Theta}_{-}^{-\gamma}\ (b,u_2)}{\Gamma(p)} \int_{u_2}^{b} \mathbf{\Theta}_{-}^{p}\ (t,u_2)\ \left[\mathbf{\Theta}_{-}^{\gamma}\ (b,t)\right] \max_{t\in[a,b]} \mid \mathbf{\Theta}_{-}^{-\gamma}\ (b,t)F_{\vartheta}(t) \mid dt \right. \\
&\left. - \frac{\mathbf{\Theta}_{-}^{-\gamma}\ (b,u_1)}{\Gamma(p)} \int_{u_1}^{b} \mathbf{\Theta}_{-}^{p}\ (t,u_1)\ \left[\mathbf{\Theta}_{-}^{\gamma}\ (b,t)\right] \max_{t\in[a,b]} \mid \mathbf{\Theta}_{-}^{-\gamma}\ (b,t)F_{\vartheta}(t) \mid dt \right| \\
&\leq \|F_{\vartheta}\|_{C_{1-\gamma}[a,b]} \frac{B(\gamma,p)}{\Gamma(p)} \left| \left[\mathbf{\Theta}_{-}^{p+\gamma}\ (b,u_2)\right] - \left[\mathbf{\Theta}_{-}^{p+\gamma}\ (b,u_1)\right] \right|.
\end{aligned}$$

As $\mid u_2 - u_1 \mid \longrightarrow 0$, the right-hand side of the above inequality tends to zero, independent of ϑ. Thus, (T_2) is equicontinuis. Therefore, it follows by Arzela-Ascoli Theorem, that T_2 is a completly continuous operator on $\mathcal{B}_r$. As a consequence of Theorem 2.16, we conclude that the problem $(\mathbb{P}\tau)$ has at least one solution in $C_{1-\gamma}[a,b]$.
Finaly, we show that such a solution is indeed in $C_{1-\gamma}^{\gamma}[a,b]$ By applying ${}_{RL}\mathfrak{D}_{b-}^{\gamma}$ on both sides of (7), we get

$$\begin{aligned}
{}_{RL}\mathfrak{D}_{b-}^{\gamma}\vartheta(u) &= {}_{RL}\mathfrak{D}_{b-}^{\gamma}\ \mathfrak{I}_{b-}^{p}\mathfrak{S}(u) = \ {}_{RL}\mathfrak{D}_{b-}^{\gamma-p}R(u,\vartheta(u),\vartheta(\tau u)) \\
&= \ {}_{RL}\mathfrak{D}_{b-}^{\nu(1-p),\psi}R(u,\vartheta(u),\vartheta(\tau u)).
\end{aligned}$$

Since $R(.,\vartheta(.),\vartheta(\tau.)) \in C_{1-\gamma}^{\nu(1-p)}[a,b]$, it follows by definition of the space $C_{1-\gamma}^{\nu(1-p)}[a,b]$ that ${}_{RL}\mathfrak{D}_{b-}^{\gamma}\vartheta(u) \in C_{1-\gamma}[a,b]$ which implies that $\vartheta(u) \in C_{1-\gamma}^{\gamma}[a,b]$. □

Theorem 3.2. *Assume that hypotheses (A_1) and (A_2) are fulfilled.*
If $\sigma < 1$. Then, the problem $(\mathbb{P}\tau)$ has a unique solution where σ is defined as in Theorem 3.1.

Proof. For the proof of Theorem 3.2, one can adopt the same technique as we did Theorem 3.1 and easily prove that the operator $T : C_{1-\gamma}[a,b] \to C_{1-\gamma}[a,b]$ stated in equation (25) is completely continuous. In view of Theorem 3.1, we know that the fixed point of T are solutions of problem $(\mathbb{P}\tau)$. Now, we prove that T has a unique fixed point, which is a solution of problem $(\mathbb{P}\tau)$. Indeed, by hypotheses $(A_1) - (A_2)$, Proposition 2.8, then for $\vartheta, x \in C_{1-\gamma}[a,b],\ u \in [a,b)$, we have

$$\begin{aligned}
&\left|\mathbf{\Theta}_{-}^{-\gamma}\ (b,u)T\vartheta(u) - \mathbf{\Theta}_{-}^{-\gamma}\ (b,u)Tx(u)\right| \\
&\leq \sum_{i=1}^{m} \frac{\lambda_i}{\pounds} \frac{1}{\Gamma(p)} \int_{\mu_i}^{b} \mathbf{\Theta}_{-}^{p}\ (t,\mu_i)\,|F_{\vartheta}(t) - F_x(t)|\, dt \\
&+ \frac{\mathbf{\Theta}_{-}^{-\gamma}\ (b,u)}{\Gamma(p)} \int_{u}^{b} \mathbf{\Theta}_{-}^{p}\ (t,u)\,|F_{\vartheta}(t) - F_x(t)|\, dt \\
&\leq \sum_{i=1}^{m} \frac{\lambda_i}{\pounds} \frac{2L}{\Gamma(p)} \int_{\mu_i}^{b} \mathbf{\Theta}_{-}^{p}\ (t,\mu_i)\left[\mathbf{\Theta}_{-}^{\gamma}(b,t)\right] \left[\|\vartheta - x\|_{C_{1-\gamma}} dt \right.\\
&\left. + \frac{2L\mathbf{\Theta}_{-}^{-\gamma}\ (b,u)}{\Gamma(p)} \int_{u}^{b} \mathbf{\Theta}_{-}^{p}\ (t,u)\mathbf{\Theta}_{-}^{\gamma}\ (b,t)\ \|\vartheta - x\|_{C_{1-\gamma}}\right] dt \\
&\leq \left[\sum_{i=1}^{m} \frac{\lambda_i}{\pounds}\mathbf{\Theta}_{-}^{p+\gamma}\ (b,\mu_i) + \left[\mathbf{\Theta}_{-}^{p+1}\ (a,b)\right]\right] \frac{2LB(\gamma,p)}{\Gamma(p)} \|\vartheta - x\|_{C_{1-\gamma}}.
\end{aligned}$$

This gives, $\| T\vartheta - Tx \|_{C_{1-\gamma}} \leq \sigma \| \vartheta - x \|_{C_{1-\gamma}}$.
Since $\sigma < 1$, the operator $T : C_{1-\gamma}[a,b] \to C_{1-\gamma}[a,b]$ is a contraction mapping. Hence by Banach fixed point theorem, it follows that T has a unique fixed point. which is a solution of problem $(\mathbb{P}\tau)$. This completed the proof. □

4. An example

This section provide illustrative example of the justness and applicability of the main results. We consider the following problem of the left Hilfer fractional differential equations of the following form:

$$(\wp)\begin{cases} {}_{H}\mathfrak{D}_{2-}^{\frac{3}{6},\frac{3}{9}}\,\vartheta(u) = \frac{30^{-8}}{e^u}\left(\frac{|\vartheta(\tau u)|}{(30^8+|\vartheta(u)|)}\right) + \sqrt{1989u}, & 0<\tau<1,\ u\in[1,2[,\\ \mathfrak{I}_{2-}^{1-\frac{6}{9}}\,\vartheta(2) = \frac{6}{15}\vartheta(\frac{6}{9}) + \vartheta_2, & \gamma = \frac{3}{6} + (\frac{3}{9}\times 1) - (\frac{3}{9}\times\frac{3}{6}) = \frac{6}{9}. \end{cases}$$

From Example, we have $a = 1$, $b = 2$, $p = 3/6$, $\nu = 3/9$, $\gamma = 6/9$, $1-\gamma = 3/9$, $\lambda_1 = \frac{6}{15}$, $\mu_1 = \frac{6}{9}$ and ϑ_2 is a constant. Thanks to (24) under the given data, this takes the value

$$£ = \Gamma(\gamma) - \lambda_1 \mathbf{\Theta}_{-}^{\gamma}\,(b,\mu_1) = \Gamma(\frac{6}{9}) - \frac{6}{15}(\frac{6}{9})^{\frac{-1}{3}} \simeq 0.9.$$

Given the continuous function:

$$R(u,\vartheta(u),\vartheta(\tau u)) = \frac{30^{-8}}{e^u}\left(\frac{|\vartheta(\tau u)|}{(30^8+|\vartheta(u)|)}\right) + \sqrt{1989u}.$$

For each $\vartheta, x \in \mathbb{R}^+$ and $u \in [1,2)$, we obtain

$$|R(u,\vartheta(u),\vartheta(\tau u) - R(u,x(u),x(\tau u))| \leq \frac{1}{30^8}\,[|\vartheta - x| + |\vartheta(\tau u) - x(\tau u)|]$$

The assumptions (A_1) and (A_2) in Theorem 3.1 are verified, we obtain

$$L = \frac{1}{30^8}.$$

Furtheremore, by simple computations we get

$$\begin{aligned} \sigma := {} & \left(\frac{\lambda_1}{£}\mathbf{\Theta}_{-}^{p+\gamma}\,(b,\mu_1) + \mathbf{\Theta}_{-}^{p+1}\,(a,b)\right)\frac{B(\gamma,p)}{\Gamma(p)}2L \\ \approx {} & \left(\frac{0.4}{0.9}\mathbf{\Theta}_{-}^{\frac{3}{6}+\frac{6}{9}}\,(2,\frac{6}{9}) + \mathbf{\Theta}_{-}^{\frac{3}{6}+1}\,(1,2)\right)\frac{B(\frac{6}{9},\frac{3}{6})}{\Gamma(\frac{3}{6})} \times 1.5\times 10^{-12} < 1, \end{aligned}$$

since all conditions of Theorem 3.2 are satisfied. Then the problem $(\wp)$ has a unique solution on $[1,2)$.

Credit authorship contribution statement

Belqassim Azzouz: Conceptualization, Investigation, Methodology, Resources, Supervision, Validation, Visualization, Writing–original draft, Writing–review and editing.

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