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Complete forcing numbers of multiple hexagonal chains

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Abstract. Let G be a graph with a perfect matching. The complete forcing number of G is the minimum cardinality of an edge subset S of G such that for each perfect matching M of G, the intersection of S and M forms a forcing set of M. Chan et al. showed that the complete forcing number of a catacondensed hexagonal system is equal to the sum of the number of hexagons and its Clar number. Closed formulas for the complete forcing numbers of certain peri-condensed hexagonal systems, such as parallelograms, have also been established. In this paper, we consider the complete forcing number of a multiple hexagonal chain (MHC), as we identified a mistake in a published result. We establish an upper bound on the complete forcing number of an MHC and a lower bound on that of a normal hexagonal system via face coloring. Using these bounds, we obtain explicit expressions for the complete forcing numbers of MHCs with $3m$ columns of hexagons (where m is a positive integer), as well as for zigzag MHCs and chevrons.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A matching of G is a set of edges with no shared end-vertices. A *perfect matching* (or 1-factor) of G is a matching in which each vertex in G is incident with exactly one edge in this matching. A perfect matching corresponds to a Kekulé structure in organic chemistry.

In 1985, Randić and Klein [14, 16] found that a Kekulé structure of a molecule can be uniquely determined by a fixed subset of double bonds, and the minimum number of such double bonds required is called the *innate degree of freedom* of the Kekulé structure. This concept was later generalized by Harary and Klein [8] as the forcing number of a perfect matching M in a graph G. A *forcing set* of M is a subset $S \subseteq M$ such that no other perfect matching of G contains S. The *forcing number* of M is the minimum cardinality among all its forcing sets. Further information on this topic and related concepts, including the anti-forcing number of a perfect matching and the global forcing number of a graph, can be found in the surveys [5, 22].

In 2015, Xu, Zhang and Cai [18] introduced the concept of "complete forcing" for all perfect matchings of a graph G. A subset $S \subseteq E(G)$ is called a *complete forcing set* if for each perfect matching M of G, the intersection $S \cap M$ is a forcing set of M. Among all such sets, one with the minimum cardinality is called a *minimum complete forcing set*, and its cardinality is referred to as the *complete forcing number* of G, denoted

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by $cf(G)$. Meanwhile, they also provided a characterization for complete forcing sets of G. Subsequent studies revealed that a complete forcing set of G not only forces but also antiforces each perfect matching of G [13]. The complete forcing number of G is bounded above by twice its cyclomatic number [11], given by $|E(G)| - |V(G)| + \omega(G)$, where $\omega(G)$ denotes the number of connected components of G. Additionally, explicit formulas for the complete forcing numbers have been obtained for grids [4], cylinders [11], complete multipartite graphs [12], the Rook's graphs [1] and several types of graphs with chemical significance [13, 15, 17, 19]. Recently, Ebrahimi et al. [7] proposed a new method for constructing complete forcing sets of G and established two upper bounds on the complete forcing number of G in terms of its degeneracy and spectral radius, respectively.

A hexagonal system (HS) is a finite connected plane graph without cut-vertices, in which all interior faces are regular hexagons [2]. If the system has a perfect matching, it models the carbon framework of a benzenoid hydrocarbon. Xu et al. [18] provided an expression for the complete forcing number of a hexagonal chain and a recurrence relation for this number in a catacondensed HS. Moreover, Chan et al. [3] proved that the complete forcing number of a catacondensed HS equals the sum of the number of hexagons and its Clar number, a crucial parameter for assessing the stability of benzenoid hydrocarbons [6]. In earlier works [9, 10], we used elementary edge-cuts to construct complete forcing sets for HSs, and derived some bounds on the complete forcing numbers of normal HSs. These results were then applied to derive expressions for the complete forcing numbers of specific types of pericondensed HSs.

In 2022, Xue et al. [20] proved that the complete forcing number of a multiple hexagonal chain (MHC) is at most the number of its vertical edges (see Proposition 3.1), and the complete forcing number of each MHC attains this upper bound. In this paper, however, we show that the above upper bound can not be reached by each MHC and further study the complete forcing numbers of MHCs. In Section 3, we establish a new upper bound on the complete forcing number of an MHC in terms of the smallest number of parallelograms that it can be split into, and a lower bound on the complete forcing number of a normal HS based on face coloring. In the final section, using the bounds provided in Section 3, we prove that the complete forcing number of an MHC with $3m$ (m is a positive integer) hexagons in each row is equal to the number of its hexagons plus one and present some closed formulas for the complete forcing numbers of zigzag MHCs and chevrons.

2. Preliminaries

Let G be a graph. For a nonempty subset $V_1 \subseteq V(G)$, the subgraph of G *induced* by V_1 is the subgraph of G with vertex set V_1 and edge set consisting of all the edges $xy \in E(G)$ with $x, y \in V_1$. We write $G - V_1$ for the subgraph induced by $V(G) \setminus V_1$. For an even cycle C in G, if $G - V(C)$ has a perfect matching, then we say that C is a *nice cycle* of G. C has two perfect matchings, each one is called a *frame* of C. Let V_2 be a nonempty proper subset of $V(G)$. The set of edges $xy \in E(G)$ with $x \in V_2$ and $y \in V(G) \setminus V_2$ is called an *edge-cut* of G.

Let H be an HS. An edge cut T of H is called an *elementary edge-cut* (or simply an e-cut) of H if it satisfies the following two conditions: (1) removing the edges in T from H results in a graph with exactly two components G_1 and G_2, and (2) all edges in T are incident with black vertices in G_1 and white vertices in G_2 (see Fig. 1).

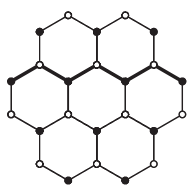


Figure 1: An elementary edge-cut of an HS with 7 hexagons.

The dual graph H^* of H is obtained by placing a vertex inside each face of H (including the exterior face), and connecting two vertices if their corresponding faces in H share an edge. We use h^* to represent the vertex in H^* that corresponds to a hexagon h in H. By definition, an e-cut T of H is a minimal edge-cut, meaning that the edges in H^* crossing T form a cycle in H^*. The following theorem provides a method to construct a complete forcing set of an HS.

Theorem 2.1. *[9] Let H be an HS that admits a perfect matching. If there is a sequence of e-cuts $T_1, T_2, \ldots, T_s$ of H such that for each nice cycle C in H, there is at least one of these e-cuts intersects $E(C)$, then $\bigcup_{i=1}^{s} T_i$ is a complete forcing set of H.*

A sequence of e-cuts $T_1, T_2, \ldots, T_s$ of H is said to *cover H* if the edge set of each facial cycle intersects at least one of these e-cuts. We also refer to such a sequence of e-cuts, or $\bigcup_{i=1}^{s} T_i$, as an *e-cut cover* of H.

An HS is called *normal* if each edge of which is contained in a perfect matching. It is known that all facial cycles of a normal HS are nice [21]. The following lemma can be used to determine whether an HS is normal.

Lemma 2.2. *[21] An HS is normal if an only if its exterior facial cycle is a nice cycle.*

An HS is called a *multiple hexagonal chain* (MHC) if it can be embedded in the plane such that certain edges are vertical and intersected by a sequence of parallel horizontal lines L_i ($i = 1, 2, \ldots, p$), thereby dividing the HS into $p + 1$ horizontal zigzag paths $P_1, P_2, \ldots, P_{p+1}$ that satisfy the following conditions: (1) Each L_i passes through q hexagons; (2) The length of both P_1 and P_{p+1} are equal to $2q$; (3) The length of P_j ($j = 2, 3, \ldots, p$) is equal to $2q + 1$ (see Fig. 2).

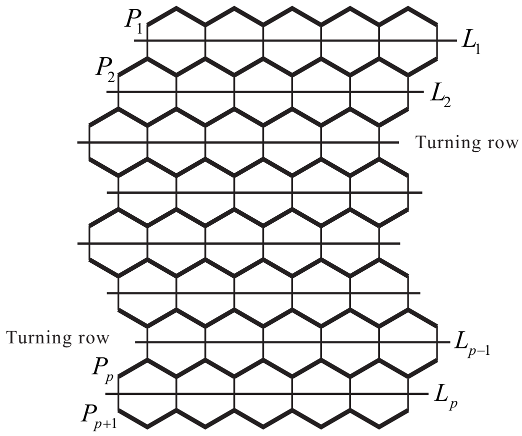


Figure 2: A multiple hexagonal chain.

Let H be an MHC. We define the linear hexagonal chain formed by all hexagons intersected by L_i ($i = 1, 2, \ldots, p$) as the *i-th row* of H, denoted by R_i. We denote by $h_{i,j}$ ($j = 1, 2, \ldots, q$) the j-th hexagon in R_i from left to right. The six edges of the boundary cycle of $h_{i,j}$ are denote by $e_{ll}(h_{i,j})$, $e_l(h_{i,j})$, $e_{ul}(h_{i,j})$, $e_{ur}(h_{i,j})$, $e_r(h_{i,j})$ and $e_{lr}(h_{i,j})$, corresponding respectively to the lower left edge, the left vertical edge, the upper left edge, the upper right edge, the right vertical edge and the lower right edge. The hexagonal chain which consists of all hexagons whose second index is equal to j is called the *j-th column* of H. Directly, H contains exactly q columns. In R_i ($i \geq 2$), if the degree of the upper end-vertex of the leftmost (resp. rightmost) vertical edge is 3, we say R_i *turns to the right (resp. left)*. If both end-vertices of either the leftmost or the rightmost vertical edge in R_i have degree 3, then R_i is defined as a *turning row*. An MHC without turning rows is called a *parallelogram*. A parallelogram is called *right monotonic* (resp. *left monotonic)* if each row of it turns to the right (resp. left) except the top row.

Let H be an MHC with p rows and q columns of hexagons, and let C be the exterior facial cycle of H. If $p = 1$ or $q = 1$, then C is a spanning cycle of H. Otherwise, $H - V(C)$ contains $p - 1$ disjoint horizontal zigzag paths, each of which is of length $2q - 3$. The union of the perfect matchings of these paths yields a perfect matching of $H - V(C)$. So C is a nice cycle, by Lemma 2.2, we have

Proposition 2.3. *An MHC is normal.*

3. Bounds

In this section, we first establish an upper bound on the complete forcing number of an MHC by graph decomposition, and then derive a new lower bound on the complete forcing number of a normal HS via face coloring.

3.1. Upper bounds

Xue et al. obtained the following upper bound on the complete forcing number of an MHC by the definition of the complete forcing set, and they claimed that the complete forcing number of each MHC attains this upper bound.

Proposition 3.1. *[20] Let H be an MHC with p rows and q columns. Then* $cf(H) \leq p(q + 1)$.

However, by Proposition 3.5 at the end of this subsection, the equality in Proposition 3.1 holds if and only if $p = 1$. In the following, we derive another upper bound on the complete forcing numbers of MHCs.

Let $P_{i_1}, P_{i_2}, \ldots, P_{i_t}$ $(1 < i_1 < i_2 < \ldots < i_t < p+1)$ be some horizontal zigzag paths of an MHC H. We denote by H_1 the subsystem of H that consists of the rows of hexagons between P_1 and P_{i_1}, by H_j the subsystem of H that consists of the rows of hexagons between $P_{i_{j-1}}$ and P_{i_j} $(j = 2, 3, \ldots, t)$ and by H_{t+1} the subsystem of H that consists of the rows of hexagons between P_{i_t} and P_{p+1}. Then we say that H can be *split* into $H_1, H_2, \ldots, H_{t+1}$ by $P_{i_1}, P_{i_2}, \ldots, P_{i_t}$ (see Fig. 3). Specially, if each of $H_1, H_2, \ldots, H_{t+1}$ is a parallelogram, then we say that H is split into parallelograms.

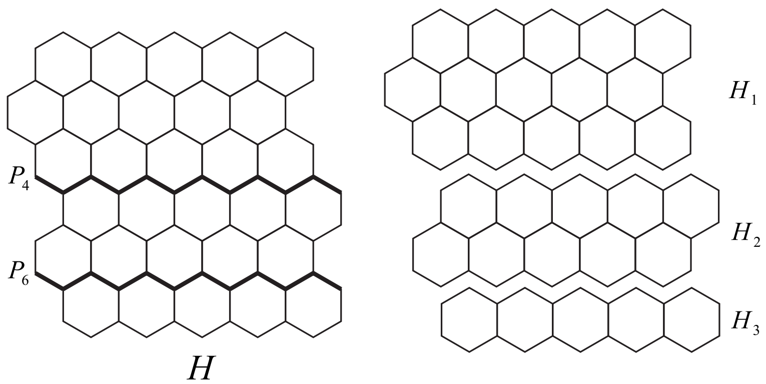


Figure 3: An MHC is split into three subsystems by two horizontal zigzag paths.

Let $B(p, q)$ be a right monotonic parallelogram of p rows and q columns of hexagons, where $p \geq 1$ and $q \geq 1$. We define three paths in $B^*(p, q)$ by $p \equiv 0, 1, 2 \pmod 3$.

If $p \equiv 0 \pmod 3$, let P_1^* be the path in $B^*(p, q)$ that traverses the corresponding vertices of the first three rows of hexagons of $B(p, q)$ in the order: $h^*_{1,q}, h^*_{1,q-1}, \ldots, h^*_{1,1}, h^*_{2,1}, h^*_{3,1}, h^*_{2,2}, h^*_{3,2}, \ldots, h^*_{2,q}, h^*_{3,q}$. This pattern is repeated every three rows until the path reaches $h^*_{p,q}$ (see the dash lines in Fig. 4 (a)).

If $p \equiv 1 \pmod 3$, for $p = 1$, let $P_2^* = (h^*_{1,q}, h^*_{1,q-1}, \ldots, h^*_{1,1})$. For $p \geq 4$, define P_2^* by repeating the same pattern as P_1^* every three rows, and by traversing the final row in the order: $h^*_{p,q}, h^*_{p,q-1}, \ldots, h^*_{p,1}$ (see Fig. 4 (b)).

If $p \equiv 2 \pmod 3$, let P_3^* first traverse the vertices of the first two rows in the order: $h_{1,1}^*, h_{2,1}^*, h_{1,2}^*, h_{2,2}^*, \dots, h_{1,p}^*, h_{2,p}^*$. If $p \geq 5$, then P_3^* continues by applying the same pattern as P_1^* to each subsequent group of three rows (see Fig. 4 (c)).

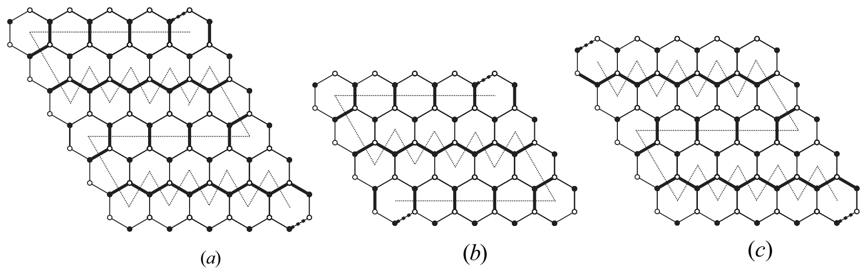


Figure 4: Illustration for P_1^*, P_2^*, P_3^* and minimum complete forcing sets of a parallelogram $B(p,q)$ for (a) $p \equiv 0 \pmod 3$; (b) $p \equiv 1 \pmod 3$; (c) $p \equiv 2 \pmod 3$.

Let S_1, S_2 and S_3 be the set of edges of $B(p,q)$ that crosses P_1^*, P_2^* and P_3^*, respectively. The following Proposition gives an expression for the complete forcing number of a parallelogram by its face number, and presents some minimum complete forcing sets of it by constructing an e-cut that satisfies Theorem 2.1.

Proposition 3.2. *[9] Let $B(p,q)$ be a right monotonic parallelogram with p rows and q columns of hexagons. Then $cf(B(p,q)) = pq + 1$ and*

(1) If $p \equiv 0 \pmod 3$, then $S_1 \cup \{e_1, e_2\}$ is a minimum complete forcing set of $B(p,q)$, where $e_1 \in \{e_r(h_{1,q}), e_{ul}(h_{1,q})\}$ and $e_2 \in \{e_{ur}(h_{p,q}), e_{lr}(h_{p,q})\}$ (see the bold lines in Fig. 4 (a)).

(2) If $p \equiv 1 \pmod 3$, then $S_2 \cup \{e_1, e_2\}$ is a minimum complete forcing set of $B(p,q)$, where $e_1 \in \{e_r(h_{1,q}), e_{ul}(h_{1,q})\}$ and $e_2 \in \{e_l(h_{p,1}), e_{lr}(h_{p,1})\}$ (see Fig. 4 (b)).

(3) If $p \equiv 2 \pmod 3$, then $S_3 \cup \{e_1, e_2\}$ is a minimum complete forcing set of $B(p,q)$, where $e_1 \in \{e_{ll}(h_{1,1}), e_{ul}(h_{1,1})\}$ and $e_2 \in \{e_{ur}(h_{p,q}), e_{lr}(h_{p,q})\}$ (see Fig. 4 (c)).

Remark 3.3. *(1) If $B(p,q)$ is left monotonic, we can use Proposition 3.2 to construct a minimum complete forcing set of $B(p,q)$ according to $p \equiv 0,1,2 \pmod 3$, since a left monotonic parallelogram $B(p,q)$ is symmetric to a right monotonic parallelogram $B(p,q)$. (2) We can also use Proposition 3.2 to construct a minimum complete forcing set of $B(p,q)$ according to $q \equiv 0,1,2 \pmod 3$, since a right (resp. left) monotonic parallelogram $B(p,q)$ can be obtained by rotating a left (resp. right) monotonic parallelogram $B(q,p)$ in the plane.*

Theorem 3.4. *Let H be an MHC with p rows and q columns of hexagons, and $r(H)$ be the smallest number of parallelograms that H can be split into. Then $cf(H) \leq pq + r(H)$.*

Proof. Let $H_1, H_2, \dots, H_{r(H)}$ be the parallelograms that H can be split into. Suppose that H_1 has p_1 rows of hexagons. If H_1 is right monotonic. And if $p_1 \equiv 0 \pmod 3$, by Proposition 3.2 (1), there exists a minimum complete forcing set S_1 of H_1 that contains $e_r(h_{1,q})$ and $e_{ur}(h_{p_1,q})$. If $p_1 \equiv 1 \pmod 3$, by Proposition 3.2 (2), there exists a minimum complete forcing set S_1 of H_1 that contains $e_r(h_{1,q})$ and $e_l(h_{p_1,1})$. If $p_1 \equiv 2$ (mod 3), by Proposition 3.2 (3), there exists a minimum complete forcing set S_1 of H_1 that contains $e_{ll}(h_{1,1})$ and $e_{ur}(h_{p_1,q})$. In each of the above cases, S_1 contains two peripheral edges of H and S_1 is an e-cut of H whose cardinality is equal to the number of hexagons of H_1 plus 1. If H_1 is left monotonic, by Remark 3.3 (1), there is also a minimum complete forcing set S_1 of H_1 with the same property as the above. Similarly, for H_i ($i = 2, 3, \dots, r(H)$), there is a minimum complete forcing set S_i of H_i with two peripheral edges of H, such that S_i is an e-cut of H and the cardinality of S_i is equal to the number of hexagons of H_i plus 1. Further, $S = \bigcup_{i=1}^{r(H)} S_i$ is an e-cut cover of H that intersects the edge set of each cycle of H and $|S| = pq + r(H)$. By Theorem 2.1, S is a complete forcing set of H, and $cf(H) \leq |S| = pq + r(H)$. □

From Proposition 3.2 and Theorem 3.4, we give a characterization for an MHC whose complete forcing number reaches the upper bound given in Proposition 3.1.

Proposition 3.5. *Let H be an MHC with p rows and q columns of hexagons. Then $cf(H) = p(q + 1)$ if and only if $p = 1$.*

Proof. If $p = 1$, H is a parallelogram with one row of hexagons, and $cf(H) = q + 1$ by Proposition 3.2. If $p = 2$, H is a also a parallelogram, by Proposition 3.2, $cf(H) = pq + 1 < p(q + 1)$. If $p > 2$, the first two rows of H forms a parallelogram, and H can be split into $p - 1$ parallelograms by the horizontal zigzag paths P_3, $P_4, \ldots, P_p$. By Theorem 3.4, $cf(H) \le pq + r(H) \le pq + (p - 1) < p(q + 1)$. □

3.2. Lower bounds

Previously, two lower bounds on the complete forcing number of a normal HS were established in [9], based respectively on its face number and the matching numbers of certain subgraphs of its dual graph.

Theorem 3.6. *[9] If a normal HS H contains n hexagons, then $cf(H) \ge n + 1$.*

Let $B(p, q)$ $(p, q \ge 3)$ be a parallelogram. Without loss of generality, suppose that $B(p, q)$ is left monotonic. In the following, we assign three colors 1, 2, 3 to the hexagons of $B(p, q)$ such that each pair of hexagons that are separated by an edge obtain different colors. Specifically, for a hexagon $h_{1,j}$ in the first row of $B(p, q)$, if $j \equiv 1 \pmod 3$, we assign 1 to $h_{1,j}$. If $j \equiv 2 \pmod 3$, we assign 2 to $h_{1,j}$. If $j \equiv 0 \pmod 3$, we assign 3 to $h_{1,j}$ (see Fig. 5). For a hexagon $h_{2,j}$ $(j \ge 2, 3, \ldots, n)$ in the second row of $B(p, q)$, the two hexagons above $h_{2,j}$ have been assigned different colors, we assign remaining one of the above three colors to $h_{2,j}$, so the color assigned to $h_{2,2}$ is 3. Then we assign color 2 to $h_{2,1}$. We can use this method to extend the assignment of the three colors to the hexagons of $B(p, q)$ row-by-row. Generally, since each HS H is a subsystem of $B(p, q)$ when p and q are sufficiently large, we can obtain a 3-coloring of hexagons of H by directly applying the 3-coloring of hexagons of $B(p, q)$, we call such 3-coloring of hexagons of H a *frame coloring* of H.

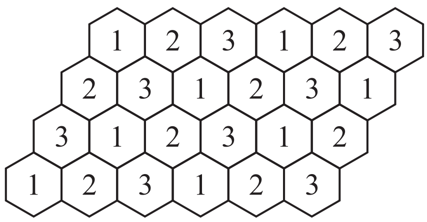


Figure 5: A frame coloring of $B(4, 6)$.

Given a frame coloring of an HS H and let V_1^*, V_2^* and V_3^* be the three sets of vertices of H^* corresponding to the hexagons of H that are colored by the three colors of a frame coloring, respectively. Then $V_1^* \cup V_2^*$, $V_1^* \cup V_3^*$ and $V_2^* \cup V_3^*$ induce three subgraphs of H^*, respectively (see Fig. 6 for example). Each of these subgraphs is a bipartite graph, and each component of one of the above subgraphs is called a *frame dual subgraph* of H. This concept coincide with the definition in [10], where it was introduced from the perspective of edge partition. For a graph G, the *matching number* of G, denoted by $\nu(G)$, is the maximum cardinality among all matchings of G.

Theorem 3.7. *[9] For a normal HS H with n hexagons, let F_1^*, $F_2^*, \ldots, F_s^*$ be all frame dual subgraphs of H. Then*

$$cf(H) \ge 2n - \sum_{i=1}^{s} \nu(F_i^*).$$

In the following, we derive a new lower bound on the complete forcing number of a normal HS by face coloring.

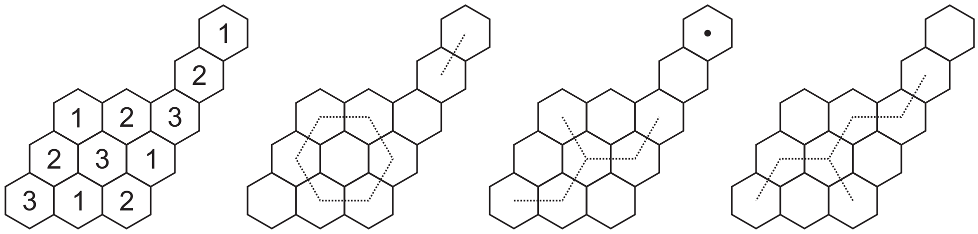


Figure 6: The three subgraphs of an HS induced by $V_1^* \cup V_2^*$, $V_1^* \cup V_3^*$ and $V_2^* \cup V_3^*$, respectively.

Theorem 3.8. *For a normal HS H with n hexagons, let $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ ($|\mathcal{F}_1| \geq |\mathcal{F}_2| \geq |\mathcal{F}_3|$) be the three set of hexagons of H that are colored by the three colors in a frame coloring, respectively. Then*

$$cf(H) \geq n + |\mathcal{F}_1| - |\mathcal{F}_3|.$$

Proof. Let V_1^*, V_2^* and V_3^* be the set of vertices of H^* that corresponds to the hexagons of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$, respectively. Let H_1^*, H_2^*, H_3^* be the three subgraphs of H^* induced by $V_1^* \cup V_2^*$, $V_1^* \cup V_3^*$ and $V_2^* \cup V_3^*$, respectively. Since V_2^* is a vertex cover of H_1^*, V_3^* is a vertex cover of each of H_2^* and H_3^*, we have

$$\nu(H_1^*) \leq |V_2^*|, \nu(H_2^*) \leq |V_3^*|, \nu(H_3^*) \leq |V_3^*|.$$

Let $F_1^*, F_2^*, \ldots, F_s^*$ be all frame dual subgraphs of H. Then

$$\sum_{i=1}^{3} \nu(H_i^*) = \sum_{i=1}^{s} \nu(F_i^*).$$

By Theorem 3.7 and $|V_1^*| + |V_2^*| + |V_3^*| = n$,

$$cf(H) \geq 2n - \sum_{i=1}^{3} \nu(H_i^*) \geq 2n - (|V_2^*| + 2|V_3^*|) = n + |V_1^*| - |V_3^*| = n + |\mathcal{F}_1| - |\mathcal{F}_3|.$$

This completes the proof. □

Clearly, it is more straightforward to determine the lower bound on the complete forcing number for a normal HS using Theorem 3.8 than using Theorem 3.7.

4. Complete forcing numbers of MHCs

In this section, we consider the complete forcing number of an MHC with p rows and q columns of hexagons, where $p \geq 3$ and $q \geq 2$. Since when $p \leq 2$ or $q = 1$, such an MHC is a catacondensed HS or a parallelogram, and its complete forcing number has been discussed in [3, 9, 18]. From the bounds provided in Section 3, we prove that the complete forcing number of an MHC with $3m$ (m is a positive integer) colums of hexagons is equal to its face number, and derive some expressions for the complete forcing numbers of zigzag MHCs and chevrons [2].

4.1. MHC with $3m$ colums

Theorem 4.1. *Let H be an MHC with p rows and q columns of hexagons, where $p \geq 3$ and $q \equiv 0 \pmod 3$. Then $cf(H) = pq + 1$.*

Proof. By Theorem 3.6, $cf(H) \geq pq + 1$. Below, we construct a complete forcing set S for H such that the cardinality of S is $pq + 1$.

Without loss of generality, we suppose that the second row of H turns to the left. Let $S_1 = \{e_{lr}(h_{1,3k+1}), e_r(h_{1,3k+1}), e_r(h_{1,3k+2}), e_{ll}(h_{1,3k+3}) \mid k = 0, 1, \ldots, \frac{q-3}{3}\}$. For $2 \leq i \leq p-1$, let $S_i = \{e_{ul}(h_{i,3k+1}), e_{lr}(h_{i,3k+1}), e_{ur}(h_{i,3k+2}), e_r(h_{i,3k+2}), e_{ll}(h_{i,3k+3}) \mid k = 0, 1, \ldots, \frac{q-3}{3}\}$ if R_i turns to the right, and $S_i = \{e_{lr}(h_{i,3k+1}), e_r(h_{i,3k+1}), e_{ul}(h_{i,3k+2}), e_{ll}(h_{i,3k+3}), e_{ur}(h_{i,3k+3}) \mid k = 0, 1, \ldots, \frac{q-3}{3}\}$ if R_i turns to the left. Let $S_p = \{e_l(h_{p,3k+1}), e_r(h_{p,3k+1}), e_{ul}(h_{p,3k+2}), e_{ur}(h_{p,3k+3}) \mid k = 0, 1, \ldots, \frac{q-3}{3}\} \cup \{e_r(h_{p,q})\}$ if R_p turns to the left, and $S_p = \{e_l(h_{p,3k+1}), e_{ul}(h_{p,3k+1}), e_{ur}(h_{p,3k+2}), e_r(h_{p,3k+2}) \mid k = 0, 1, \ldots, \frac{q-3}{3}\} \cup \{e_r(h_{p,q})\}$ if R_p turns to the right (see Fig. 7 (a)). And let $S = \bigcup_{k=1}^{p} S_k$ (see the edges crossed by the dashed line as shown in Fig. 7 (b)). Then S is an e-cut of H that intersects the edge set of each cycle in H. By Theorem 2.1, S is a complete forcing set of H. Moreover, each edge of S is shared by two facial cycles (including the exterior facial cycle) of H, and every facial cycle of H contains exactly two edges from S. As a result, $|S|$ equals the number of faces in H, which is $pq + 1$. Therefore, $cf(H) = pq + 1$. □

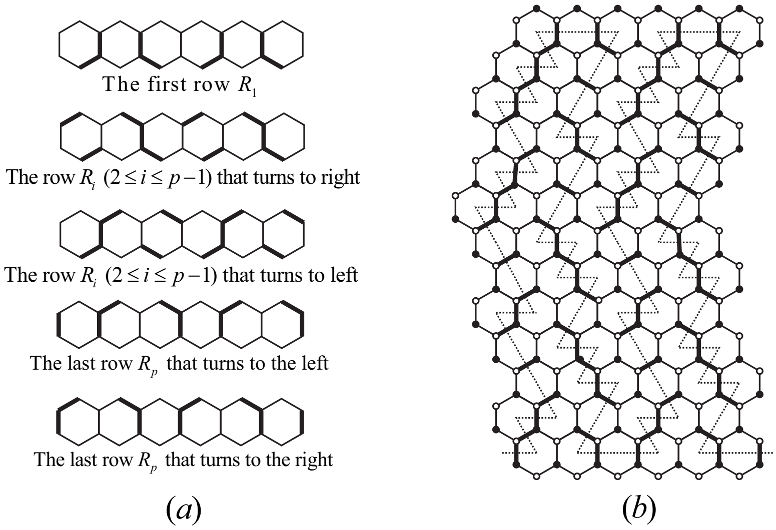


Figure 7: (a) Some specified edge sets of some particular rows; (b) A minimum complete forcing set of an MHC with p rows and q ($q \equiv 0 \pmod 3$) columns of hexagons.

4.2. *Zigzag MHC*

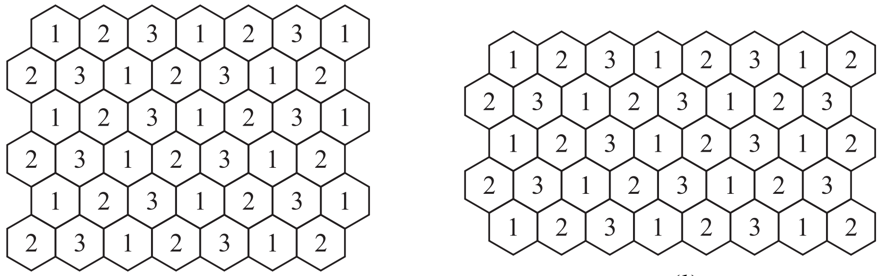
An MHC with p ($p \geq 3$) rows and q ($q \geq 2$) columns of hexagons is called *zigzag*, denoted by $Z(p,q)$, if all its rows, except the first and the last rows, are turning rows.

Theorem 4.2. *Let $Z(p,q)$ be a zigzag MHC with p rows and q columns of hexagons. Then*

$$cf(Z(p,q)) = \begin{cases} pq + 1, & \text{if } q \equiv 0 \pmod 3, \\ pq + \lceil \frac{p}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. If $q \equiv 0 \pmod 3$, by Theorem 4.1, $cf(Z(p,q)) = pq + 1$.

If $q \equiv i \pmod 3$ ($i = 1, 2$), let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ (with $|\mathcal{F}_1| \geq |\mathcal{F}_2| \geq |\mathcal{F}_3|$) be the three sets of hexagons of $Z(p,q)$ that are colored by the three colors of a frame coloring (see Fig. 8). In the first $q - i$ columns of $Z(p,q)$ from left to right, the numbers of hexagons that are colored by each of the three colors are equal.



(*a*) (*b*)

Figure 8: Two frame colorings of two zigzag MHCs.

The difference $|\mathcal{F}_1| - |\mathcal{F}_3|$ in $Z(p,q)$ is the same as the difference in the subsystem of $Z(p,q)$ consisting of the last i columns. Hence, if p is even, $|\mathcal{F}_1| = |\mathcal{F}_3| + \frac{p}{2}$. If p is odd, $|\mathcal{F}_1| = |\mathcal{F}_3| + \frac{p+1}{2}$. By Theorem 3.8, we have $cf(Z(p,q)) \geq pq + |\mathcal{F}_1| - |\mathcal{F}_3| = pq + \lceil \frac{p}{2} \rceil$.

On the other hand, if p is even, then $Z(p,q)$ can be split into $\frac{p}{2}$ parallelograms by the horizontal zigzag paths P_3, P_5, $\ldots$, P_{p-1}. If p is odd, $Z(p,q)$ can be split into $\frac{p+1}{2}$ parallelograms by the horizontal zigzag paths P_3, P_5, $\ldots$, P_p. So $r(H) \leq \lceil \frac{p}{2} \rceil$. By Theorem 3.4, $cf(Z(p,q)) \leq pq + r(H) \leq pq + \lceil \frac{p}{2} \rceil$. Consequently, $cf(Z(p,q)) = pq + \lceil \frac{p}{2} \rceil$. □

4.3. *Chevron*

An chevron is an MHC of p ($p \geq 3$) rows and q ($q \geq 2$) columns of hexagons with exactly one turning row. We denote a chevron by $Ch(p_1, p_2, q)$, where p_1 (resp. p_2) is the number of rows above (resp. under) the turning row.

Theorem 4.3. *Let $Ch(p_1, p_2, q)$ be a chevron. Then*

$$cf(Ch(p_1,p_2,q)) = \begin{cases} q(p_1+p_2+1)+2, \text{if } p_1 \equiv 1 \pmod 3,\ p_2 \equiv 1 \pmod 3 \text{ and} \\ \qquad\qquad q \equiv 1 \pmod 3 \text{ or } q \equiv 2 \pmod 3, \\ q(p_1+p_2+1)+1, \text{otherwise}. \end{cases}$$

Proof. We divide our proof into the following cases.

Case 1. $q \equiv 0 \pmod 3$.

By Theorem 4.1, $cf(Ch(p_1,p_2,q)) = q(p_1+p_2+1)+1$.

Case 2. $q \equiv 1 \pmod 3$ or $q \equiv 2 \pmod 3$ and $p_1 \equiv 0 \pmod 3$ or $p_1 \equiv 2 \pmod 3$.

Let H_1 be the parallelogram formed by the first p_1 rows of hexagons from top to bottom and let H_2 be the parallelogram formed by the remaining p_2+1 rows of hexagons. Without loss of generality, we suppose that H_1 is right monotonic. By Theorem 3.6, $cf(Ch(p_1,p_2,q)) \geq q(p_1+p_2+1)+1$. In each of the following subcases, we construct a complete forcing set S of $Ch(p_1,p_2,q)$ such that $|S| = q(p_1+p_2+1)+1$.

Subcase 2.1. $q \equiv 1 \pmod 3$ and $p_1 \equiv 0 \pmod 3$.

For $1 \leq i \leq p_1$, let $S_i = \{e_{ur}(h_{i,1})\} \cup \{e_r(h_{i,k}) \mid k = 1,2,\ldots,q-1\} \cup \{e_{ll}(h_{i,q})\}$ if $i \equiv 1 \pmod 3$ (see Fig. 9 (a)), $S_i = \{e_{lr}(h_{i,1}),\ e_r(h_{i,1}),\ e_{ll}(h_{i,2})\} \cup \{e_{ll}(h_{i,k}),\ e_{lr}(h_{i,k}) \mid k = 3,4,\ldots,q-2 \text{ for } q > 4\} \cup \{e_{ll}(h_{i,q-1}),\ e_{ur}(h_{i,q-1}),\ e_{ll}(h_{i,q}),\ e_{lr}(h_{i,q})\}$ if $i \equiv 2 \pmod 3$ (see Fig. 9 (b)), and $S_i = \{e_{ul}(h_{i,1}),\ e_{ur}(h_{i,1}),\ e_{ur}(h_{i,2})\} \cup \{e_{ul}(h_{i,k}),\ e_{ur}(h_{i,k}) \mid k = 3,4,\ldots,q-2 \text{ for } q > 4\} \cup \{e_{ur}(h_{i,q-1}),\ e_r(h_{i,q-1}),\ e_{ul}(h_{i,q})\}$ if $i \equiv 0 \pmod 3$ (see Fig. 9 (c)). By Remark 3.3 and Proposition 3.2 (2), there exist a minimum complete forcing set S' of H_2 that contains $e_{ur}(h_{p_1+1,1})$ and $e_r(h_{p_1+p_2+1,q})$. Then $S = (\bigcup_{i=1}^{p_1} S_i) \cup S'$ is an e-cut cover of $Ch(p_1,p_2,q)$ (see Fig. 10 (a)). And each cycle that shares no common edges with any one of the above e-cut is the boundary of a subsystem formed by three hexagons sharing a common vertex, and such a cycle is not a nice cycle. By Theorem 2.1, S is a complete forcing set of $Ch(p_1, p_2, q)$. Moreover, each edge of S is shared by two facial cycles (including the exterior one) of $Ch(p_1, p_2, q)$, and each facial cycle of $Ch(p_1, p_2, q)$ contains exactly two edges from S. As a result, $|S|$ is equal to the number of faces in $Ch(p_1, p_2, q)$, which is $q(p_1 + p_2 + 1) + 1$. Therefore, $cf(Ch(p_1, p_2, q)) = q(p_1 + p_2 + 1) + 1$.

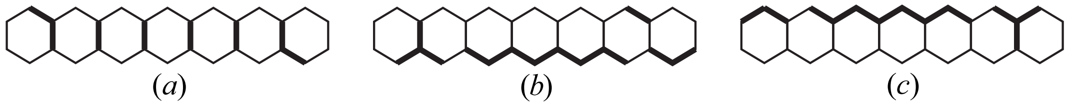


Figure 9: Illustration for the edge set S_i in R_i.

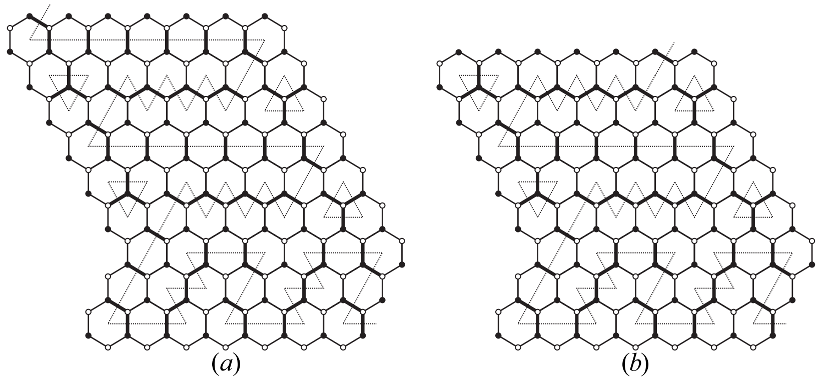


Figure 10: A complete forcing set of $Ch(p_1, p_2, q)$ for $q \equiv 1 \pmod 3$ and (a) $p_1 \equiv 0 \pmod 3$; (b) $p_1 \equiv 2 \pmod 3$.

Subcase 2.2. $q \equiv 1 \pmod 3$ and $p_1 \equiv 2 \pmod 3$.

For $1 \le i \le p_1$, let $S_i = \{e_{lr}(h_{i,1}), e_r(h_{i,1}), e_{ll}(h_{i,2})\} \cup \{e_{ll}(h_{i,k}), e_{lr}(h_{i,k}) \mid k = 3, 4, \ldots, q-2 \text{ for } q > 4\} \cup \{e_{ll}(h_{i,q-1}), e_{ur}(h_{i,q-1}), e_{ll}(h_{i,q}), e_{lr}(h_{i,q})\}$ if $i \equiv 1 \pmod 3$ (see Fig. 9 (b)), $S_i = \{e_{ul}(h_{i,1}), e_{ur}(h_{i,1}), e_{ur}(h_{i,2})\} \cup \{e_{ul}(h_{i,k}), e_{ur}(h_{i,k}) \mid k = 3, 4, \ldots, q-2 \text{ for } q > 4\} \cup \{e_{ur}(h_{i,q-1}), e_r(h_{i,q-1}), e_{ul}(h_{i,q})\}$ if $i \equiv 2 \pmod 3$ (see Fig. 9 (c)), and $S_i = \{e_{ur}(h_{i,1})\} \cup \{e_r(h_{i,k}) \mid k = 1, 2, \ldots, q-1\} \cup \{e_{ll}(h_{i,q})\}$ if $i \equiv 0 \pmod 3$ (see Fig. 9 (a)). By Remark 3.3 and Proposition 3.2 (2), there exists a minimum complete forcing set S' of H_2 that contains $e_{ur}(h_{p_1+1,1})$ and $e_r(h_{p_1+p_2+1,q})$. Then $S = (\bigcup_{i=1}^{p_1} S_i) \cup S'$ is an e-cut-cover of $Ch(p_1, p_2, q)$ (see Fig. 10 (b)). Each cycle that shares no common edges with any one of the above e-cut is the boundary of a subsystem formed by three hexagons sharing a common vertex, and such a cycle is not a nice cycle. By Theorem 2.1, S is a complete forcing set of $Ch(p_1, p_2, q)$. Moreover, each edge of S is shared by two facial cycles (including the exterior one) of $Ch(p_1, p_2, q)$, and each facial cycle of $Ch(p_1, p_2, q)$ contains exactly two edges from S. As a result, $|S| = q(p_1 + p_2 + 1) + 1$. Therefore, $cf(Ch(p_1, p_2, q)) = q(p_1 + p_2 + 1) + 1$.

Subcase 2.3. $q \equiv 2 \pmod 3$ and $p_1 \equiv 0 \pmod 3$.

By Proposition 3.2 (1), there exists a minimum complete forcing set S_1 of H_1 that contains $e_{ul}(h_{1,q})$ and $e_{lr}(h_{p_1,q})$ (see Fig. 11 (a)). By Remark 3.3 and Proposition 3.2 (3), there exists a minimum complete forcing set S_2 of H_2 that contains $e_{lr}(h_{p_1,q})$ and $e_{lr}(h_{p_1+p_2+1,1})$. It follows that $S = S_1 \cup S_2$ is an e-cut of $Ch(p_1, p_2, q)$ that intersects the edge set of each cycle in $Ch(p_1, p_2, q)$. By Theorem 2.1, S is a complete forcing set of $Ch(p_1, p_2, q)$. Moreover, each edge of S is shared by two facial cycles (including the exterior one) of $Ch(p_1, p_2, q)$, and each facial cycle of $Ch(p_1, p_2, q)$ contains exactly two edges from S. So $|S| = q(p_1 + p_2 + 1) + 1$. Therefore, $cf(Ch(p_1, p_2, q)) = q(p_1 + p_2 + 1) + 1$.

Subcase 2.4. $q \equiv 2 \pmod 3$ and $p_1 \equiv 2 \pmod 3$.

By Proposition 3.2 (3), there exists a minimum complete forcing set S_1 of H_1 that contains $e_{ul}(h_{1,1})$ and $e_{lr}(h_{p_1,q})$ (see Fig. 11 (b)). By Remark 3.3 and Proposition 3.2 (3), there exists a minimum complete forcing set S_2 of H_2 that contains $e_{lr}(h_{p_1,q})$ and $e_{lr}(h_{p_1+p_2+1,1})$. It follows that $S = S_1 \cup S_2$ is an e-cut of $Ch(p_1, p_2, q)$ that

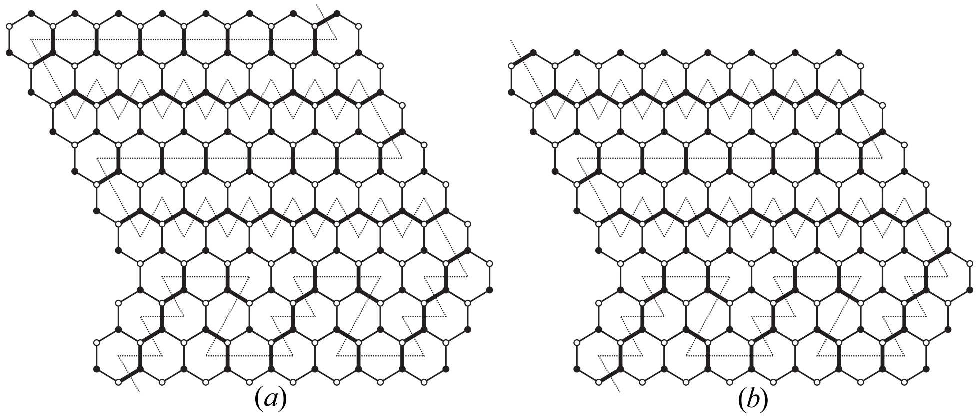


Figure 11: A complete forcing set of $Ch(p_1, p_2, q)$ for $q \equiv 2 \pmod 3$ and (a) $p_1 \equiv 0 \pmod 3$; (b) $p_1 \equiv 2 \pmod 3$.

intersects the edge set of each cycle in $Ch(p_1, p_2, q)$. By Theorem 2.1, S is a complete forcing set of $Ch(p_1, p_2, q)$. Moreover, each edge of S is shared by two facial cycles (including the exterior one) of $Ch(p_1, p_2, q)$ and each facial cycle of $Ch(p_1, p_2, q)$ contains exactly two edges from S. So $|S| = q(p_1 + p_2 + 1) + 1$. Therefore, $cf(Ch(p_1, p_2, q)) = q(p_1 + p_2 + 1) + 1$.

Case 3. $q \equiv 1 \pmod 3$ or $q \equiv 2 \pmod 3$ and $p_2 \equiv 0 \pmod 3$ or $p_2 \equiv 2 \pmod 3$.

Since a chevron $Ch(p_1, p_2, q)$ is symmetric to a chevron $Ch(p_2, p_1, q)$, $cf(Ch(p_1, p_2, q)) = q(p_1 + p_2 + 1)$ by the results in Case 2.

Case 4. $q \equiv 1 \pmod 3$ or $q \equiv 2 \pmod 3$ and $p_1 \equiv 1 \pmod 3$ and $p_2 \equiv 1 \pmod 3$.

Let $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ (with $|\mathcal{F}_1| \geq |\mathcal{F}_2| \geq |\mathcal{F}_3|$) be the three sets of hexagons of $Ch(p_1, p_2, q)$ that are colored by each of the three colors of a frame coloring. Suppose that $q \equiv i \pmod 3$ ($i = 1, 2$). Then in the first $q - i$ columns of $Ch(p_1, p_2, q)$ from left to right, the numbers of hexagons colored by each of the three colors are equal. The difference $|\mathcal{F}_1| - |\mathcal{F}_3|$ in $Z(p, q)$ is the same as the difference in the subsystem of $Z(p, q)$ consisting of the last i columns. Hence, we have $|\mathcal{F}_1| = |\mathcal{F}_3| + 2$. By Theorem 3.8, we have $cf(Ch(p_1, p_2, q)) \geq q(p_1+p_2+1)+|\mathcal{F}_1|-|\mathcal{F}_3| = q(p_1+p_2+1)+2$. On the other hand, $Ch(p_1, p_2, q)$ can be split into two parallelograms by the horizontal zigzag path P_{p_1+1}. By Theorem 3.4, $Ch(p_1, p_2, q) \leq q(p_1 + p_2 + 1) + r(H) \leq q(p_1 + p_2 + 1) + 2$. Consequently, $Ch(p_1, p_2, q) = q(p_1 + p_2 + 1) + 2$. □

Declaration of competing interest

The author has no relevant financial or non-financial interests to disclose.

Data availability

No data was used for the research described in the article.

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