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Optimal prediction of future order statistics using conditional expectation under type II censoring for continuous distributions

H. M. Barakat[a,*], O. M. Khaled[b], K. S. Khalil[c], M. H. Harpy[d,e]

[a]*Department of Mathematics, Faculty of Science, Zagazig University, Zagazig 44519, Egypt*
[b]*Department of Mathematics, Faculty of Science, Port, Said University, Egypt*
[c]*Department of Basic Science, Faculty of Computer and Informatic, Suez Canal. University, Egypt*
[d]*Department of Mathematics, College of Science & Humanity Studies, Prince Sattam Bin Abdulaziz University, Saudi Arabia*
[e]*Department of Mathematics, Faculty of Science, Suez Canal University, Ismailia, Egypt*

Abstract. This study presents a new approach for predicting future order statistics under Type II censoring, based on conditional expectation, to obtain optimal predictors that achieve an almost minimal mean squared error (MSE). Two distinct predictors are proposed: one based on properties of exponential spacings and another utilizing uniform order statistics. The theoretical framework is validated through extensive simulations across various distributions (Weibull, Pareto, Gamma, Beta, and Normal), showing superior accuracy compared to existing methods. The exponential-based predictor excels in heavy-tailed scenarios, while the uniform-based predictor offers computational efficiency for light-tailed or symmetric distributions. Additionally, the paper provides techniques for constructing confidence intervals for future order statistics and applies the methodology to real-world data, showcasing its practical utility in reliability engineering and survival analysis.

1. Introduction

Order statistics are foundational in statistical theory and practice, playing a central role in reliability analysis, quality control, survival studies, and industrial engineering. When complete data are unavailable (often due to time, cost, or ethical constraints), the censored sampling is employed, with Type II censoring as a widely used model. Under this scheme, the experiment stops after observing the first r failures out of n units. In such scenarios, predicting unobserved future order statistics from censored samples is a fundamental inferential problem.

A natural solution framework is to employ the conditional expectation of future order statistics given observed data. When treated as a random variable (RV), this conditional expectation becomes the optimal predictor under squared error loss. While the theoretical properties of order statistics are well-studied (David and Nagaraja, 2003; Arnold et al., 1992), comprehensive frameworks for conditional prediction under Type II censoring are still underdeveloped, especially for general continuous distributions.

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* Corresponding author: H. M. Barakat
Email addresses: hbarakat@zu.edu.eg (H. M. Barakat), osam87@yahoo.com (O. M. Khaled), karimshehtakhalil@ci.suez.edu.eg (K. S. Khalil), mahmoud_haussieny@science.suez.edu.eg (M. H. Harpy)
ORCID iDs: https://orcid.org/0000-0002-6212-432X (H. M. Barakat), https://orcid.org/0000-0002-3322-0603 (O. M. Khaled), https://orcid.org/0009-0003-8967-2267 (K. S. Khalil)

Historical and theoretical foundations

Early contributions include Bartholomew (1963) on life-testing estimators and Bhattacharyya (1985) on the reliability prediction. Lawless (1977) constructed prediction intervals for exponential distributions, while Kaminsky and Rhodin (1985) introduced maximum likelihood-based predictive procedures. Patel (1989) presented a comprehensive review of prediction intervals, and Nagaraja (1995) provided a broad treatment of prediction theory in applied statistics. Raqab and Nagaraja (1995) further examined the prediction of future order statistics in parametric and nonparametric contexts.

Dellaportas and Wright (1991) investigated numerical prediction for the Weibull distribution, and Hsieh (1996) developed prediction intervals for early-failure Weibull data. Generalized and hybrid censoring schemes have advanced predictive inference, as seen in Valiollahi et al. (2017), who addressed Type I and Type II hybrid censoring for generalized exponential distributions.

Censored inference and modern prediction methods

Comprehensive treatments of censored inference are provided by Balakrishnan and Cohen (1991) and Arnold et al. (1992). Balakrishnan et al. (2010) expanded the methodology by introducing exact nonparametric prediction and tolerance intervals based on ordinary and progressively Type II censored data. In the Bayesian context, Sharma and Pandey (2007) addressed prediction under exponential and Weibull models using censored samples, while Wu and Li (2011) presented improved frequentist predictive techniques for Type II censored data.

Recent developments have focused on distribution-specific modeling. Aly et al. (2023) proposed a least squares method based on cumulative hazard functions, and Barakat et al. (2022) studied prediction under the two-parameter exponential model. Prediction under gamma-mixture and beta-mixture distributions was explored by Khaled et al. (2023), with practical applications to COVID-19 recovery modeling. Further contributions include the work of El-Adll et al. (2012) on the three-parameter Weibull model, and El-Adll and Aly (2014, 2016) on prediction intervals from the Pareto distribution using generalized order statistics (GOSs). Shah et al. (2020) utilized key characterization properties of GOSs and dual GOSs to develop an effective strategy for predicting future events.

Long and Jiang (2023) advanced predictive inference for two-parameter Pareto models under progressively hybrid censoring.

The use of GOSs and the random sample sizes has broadened the applicability of predictive methods. Barakat et al. (2011, 2014, 2018, 2021a, 2021b) contributed methods for exact and asymptotic prediction intervals, including for samples of random sizes. Raqab and Barakat (2018) emphasized prediction challenges in such setups. The Prediction R package (Barakat et al., 2018) offers practical tools for implementation, bridging the gap between theory and application.

Applications and impact

The proposed methodology holds broad utility in domains where partial life data is common. These include reliability engineering, public health surveillance, warranty forecasting, and financial risk management. By offering a unified and distribution-agnostic predictive approach, this work contributes both theoretically and practically to the advancement of statistical inference under censoring.

Conditional expectation as an RV and best predictor

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let X and Y be integrable RVs defined on this space. The conditional expectation of X given Y is denoted by $E(X \mid Y)$ and is defined as:

$$E(X \mid Y)(\omega) := E(X \mid \sigma(Y))(\omega),$$

where $\sigma(Y)$ denotes the σ-algebra generated by the RV Y. The most important properties of the conditional expectation $E(X \mid Y)$, viewed as an RV, are:

- The conditional expectation $E(X \mid Y)$ is itself an RV.

- It is $\sigma(Y)$-measurable, which means there exists a Borel-measurable function g such that:

$$E(X \mid Y)(\omega) = g(Y(\omega)), \quad \text{almost surely.}$$

- Hence, $E(X \mid Y)$ can be interpreted as a function of Y and is thus defined on the same probability space $(\Omega, \mathcal{F}, P)$.

The following known and essential result shows that the conditional expectation $E(X \mid Y)$ uniquely minimizes the MSE among all $\sigma(Y)$-measurable functions, making it the optimal predictor of X given Y in the L^2 sense.

Lemma 1.1 ($E(X \mid Y)$ as the Best Predictor, cf. Bayramoglu; 2022; Billingsley, 1995; and Durrett, 2019). *The conditional expectation $E(X \mid Y)$ minimizes the mean squared error among all $\sigma(Y)$-measurable functions. Formally:*

$$E(X \mid Y) = \arg \min_{Z=h(Y)} E\left[(X - Z)^2\right],$$

where the minimum is taken over all RVs Z that are functions of Y. Thus, $E(X \mid Y)$ is the best predictor (in the L^2 sense) of X given knowledge of Y.

The following additional important properties of the conditional expectation $E(X|Y)$, viewed as an RV, are

- **Linearity:** $E(aX + bZ \mid Y) = aE(X \mid Y) + bE(Z \mid Y)$.
- **Tower property:** $E[E(X \mid Y)] = E(X)$ (see Remark 1.1).
- **Measurability:** $E(X \mid Y)$ is measurable with respect to σ.
- **Multiplication by functions of Y:** If $h(Y)$ is a function of Y, then

$$E[h(Y)X \mid Y] = h(Y)E[X \mid Y].$$

Remark 1.1. *If we are using $E(X \mid Y)$ as an estimator of X, then the tower property implies that:*

$$E\left[E(X \mid Y)\right] = E(X).$$

Therefore, in this context, we can say $E(X \mid Y)$ is an unbiased estimator of X in expectation, because the tower property ensures that its mean is the same as that of X. in the specific context where $E(X \mid Y)$ is used to estimate X, the tower property implies unbiasedness of this estimator for X.

2. The best point predictor for future order statistics

We begin this section by presenting Theorems 2.1 and 2.2, which provide predictors for future order statistics under general continuous distributions. These predictors possess nearly minimal MSE properties. Although the theorems are broadly applicable, their derivations strategically utilize properties of the exponential and uniform distributions, respectively. Theorem 2.1 derives its predictor by leveraging the structure of exponential spacings between order statistics. Specifically, it exploits the fact that, under an exponential distribution, the spacings are independent and exponentially distributed with decreasing rates. This facilitates a closed-form expression for the predictor, expressed through the quantile function F_X^{-1}. Despite the reliance on exponential properties in the derivation, the resulting predictor is valid for any continuous distribution $F_X(x)$.

Theorem 2.2, similarly, uses the known behavior of uniform order statistics, which follow Beta distributions. This enables a straightforward derivation of the predictor in a linear, multiplicative form involving the observed order statistic $X_{r:n}$, again expressed through the general quantile function F_X^{-1}.

Both theorems are particularly valuable in the context of Type II censoring, where prediction of unobserved order statistics is required. A notable strength of these results lies in their distribution-free form: although exponential and uniform distributions are used as tools in the proofs, the final predictors depend only on the underlying distribution function (DF) $F_X(x)$ and the observed value $X_{r:n}$. This distinction—between the specific distributions used for derivation and the generality of the final results—underscores the practical utility and theoretical elegance of Theorems 2.1 and 2.2.

2.1. Theoretical results

Theorem 2.1 (Prediction based on exponential distribution). *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$ be order statistics from a continuous distribution $F_X(x)$ under Type II censoring at the observation r. For any $s > r$, the near-minimal MSE (near-MMSE) predictor of the future order statistic $X_{s:n}$ is given by*

$$\hat{X}^{(1)}_{s:n} := F_X^{-1}\left(1 - (1 - F_X(X_{r:n}))\exp\left(-\sum_{j=n-s+1}^{n-r}\frac{1}{j}\right)\right), \tag{1}$$

where F_X^{-1} is the quantile function of the underlying distribution F_X.

Proof. First, let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ denote the order statistics based on the exponential distribution with the rate parameter $\alpha > 0$ (denoted by Exp(α)). Let us define the spacings between order statistics:

$$Y_j = Z_{j:n} - Z_{j-1:n}, \quad \text{for } j = 1, 2, \ldots, n,$$

with the convention $Z_{0:n} = 0$. It is a known property of the exponential distribution that these spacings are independent and distributed as (cf. Arnold et al., 1992):

$$Y_j \sim \text{Exp}(\alpha(n - j + 1)).$$

For $s > r$, we can express the sth order statistic as

$$Z_{s:n} = Z_{r:n} + \sum_{j=r+1}^{s} Y_j.$$

Conditioning on $Z_{r:n} = z_r$, the conditional expectation becomes

$$E(Z_{s:n} \mid Z_{r:n} = z_r) = z_r + \sum_{j=r+1}^{s} E(Y_j) = z_r + \sum_{j=r+1}^{s}\frac{1}{\alpha(n-j+1)}.$$

Changing the index of summation, we obtain

$$\sum_{j=r+1}^{s}\frac{1}{\alpha(n-j+1)} = \sum_{j=n-s+1}^{n-r}\frac{1}{\alpha j}.$$

Hence, we get

$$E(Z_{s:n} \mid Z_{r:n} = z_r) = z_r + \sum_{j=n-s+1}^{n-r}\frac{1}{\alpha j}. \tag{2}$$

Therefore, invoking Lemma 1.1, the best predictor of $Z_{s:n}$ is given by

$$\hat{Z}_{s:n} = Z_{r:n} + \sum_{j=n-s+1}^{n-r}\frac{1}{\alpha j}.$$

Now, we apply the integral probability transformation, to get

$$U_{r:n} = F_X(X_{r:n}) \sim Beta(r, n - r + 1), \tag{3}$$

where $Beta(a, b)$ is the beta distribution with parameters a and b. Transform to exponential order statistics

$$Z_{r:n} = -\ln(1 - U_{r:n}) \sim \text{Exp}(\alpha). \tag{4}$$

Transform back to uniform order statistics, by using (2) and (3),

$$\hat{U}_{s:n} = 1 - \exp\left(-\alpha\left[z_{r:n} + \sum_{j=n-s+1}^{n-r} \frac{1}{\alpha j}\right]\right).$$

Simplifying using (4), we get

$$\hat{U}_{s:n} = 1 - \exp\left(\ln(1 - U_{r:n}) - \sum_{j=n-s+1}^{n-r} \frac{1}{j}\right),$$

or equivalently

$$\hat{U}_{s:n} = 1 - (1 - U_{r:n}) \exp\left(-\sum_{j=n-s+1}^{n-r} \frac{1}{j}\right),$$

apply inverse DF to return to original scale, by using $\hat{X}_{s:n} = F_X^{-1}(\hat{U}_{s:n})$ and substituting $U_{s:n}$, we get the theorem's expression. □

Theorem 2.2 (Prediction based on $U(0,1)$ distribution). *Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{r:n}$ be order statistics from a continuous distribution $F_X(x)$ under Type II censoring at observation r. For any $s > r$, the near-MMSE predictor of the future order statistic $X_{s:n}$ is given by*

$$\hat{X}_{s:n}^{(2)} := F_X^{-1}\left(F_X(X_{r:n}) + (1 - F_X(X_{r:n})) \times \frac{s-r}{n-r+1}\right). \tag{5}$$

Proof. First, let $Z_{1:n} \leq Z_{2:n} \leq \cdots \leq Z_{n:n}$ denote the order statistics from a uniform $(0,1)$ distribution. Suppose the sample is Type II censored at the rth order statistic, so that $Z_{r:n} = z_r$ is observed. For any $s > r$, we want to find the conditional expectation $E[Z_{s:n} \mid Z_{r:n} = z_r]$.

The uniform distribution on $[0,1]$ has the property that, given $Z_{r:n} = z_r$, the remaining observations (those exceeding z_r) are conditionally independent and uniformly distributed on $[z_r, 1]$. This is analogous to the memoryless property of the exponential distribution, adapted for uniform order statistics (cf. David and Nagaraja, 2003). Thus, given $Z_{r:n} = z_r$, the remaining order statistics $Z_{r+1:n}, \ldots, Z_{n:n}$ are conditionally distributed like the order statistics of a uniform sample of size $n-r$ on the interval $[z_r, 1]$. Now, define the transformed RVs:

$$U_i = \frac{Z_{i:n} - z_r}{1 - z_r}, \quad \text{for } i = r+1, \ldots, n.$$

Given $Z_{r:n} = z_r$, the U_i are the order statistics of a uniform $(0,1)$ sample of size $n-r$. The sth order statistic $Z_{s:n}$ (for $s > r$) can be written as:

$$Z_{s:n} = z_r + (1 - z_r) U_{s-r:n-r},$$

where $U_{s-r:n-r}$ is the $(s-r)$th order statistic from a uniform $(0,1)$ sample of size $n-r$. The expectation of the kth order statistic from a uniform $(0,1)$ sample of size m is

$$E[U_{k:m}] = \frac{k}{m+1}.$$

Here, $k = s - r$ and $m = n - r$, so:

$$E[U_{s-r:n-r}] = \frac{s-r}{n-r+1}.$$

Thus, the conditional expectation of $Z_{s:n}$ given $Z_{r:n} = z_r$ is

$$E[Z_{s:n} \mid Z_{r:n} = z_r] = z_r + (1 - z_r) \times \frac{s-r}{n-r+1}. \tag{6}$$

Therefore, by appealing to Lemma 1.1, the best predictor of $Z_{s:n}$ is given by

$$\hat{Z}_{s:n} = Z_{r:n} + (1 - Z_{r:n}) \times \frac{s-r}{n-r+1}.$$

Now, by applying the integral probability transform and its inverse, we obtain the expression stated in the theorem. □

Remark 2.1 (cf. David and Nagaraja, 2003; Chapters 2 and 3). *The conditional probability density function (PDF) of $X_{j:n}|X_{i:n} = x_i$, where $X_{1:n} \le X_{2:n} \le \cdots \le X_{r:n}$ are order statistics from the continuous distribution $F(.)$ with PDF $f(.)$ is given by*

$$f_{j:n}(x_j \mid x_i) = \frac{(n-i)!}{(j-i-1)!(n-j)!}\left(\frac{F(x_j)-F(x_i)}{1-F(x_i)}\right)^{j-i-1}\left(\frac{1-F(x_j)}{1-F(x_i)}\right)^{n-j}\frac{f(x_j)}{1-F(x_i)}.$$

Therefore, by using Lemma 1.1, one can obtain the MMSE predictor $\hat{X}_s = E[X_{s:n}|X_{r:n}]$. However, this predictor does not have a closed-form expression and is generally more complex than the near-MMSE predictors derived in Theorems 2.1 and 2.2.

The exponential-based predictor given in (1) involves a sum of a harmonic series, which can be computationally intensive for large n. The uniform-based predictor given in (5) is simpler and more interpretable, as it depends only on the fraction of remaining observations. Based on the following simulation study and the supporting theoretical background, the exponential-based predictor (2) is recommended when the underlying distribution is known to be exponential or exhibits exponential-like spacings, particularly in cases involving heavy-tailed data or memoryless properties. Conversely, the uniform-based predictor (5) is more appropriate when the underlying distribution is uniform or approximately uniform, especially when simplicity and computational efficiency are desired. Finally, the uniform-based predictor may also be preferable when the underlying distribution is light-tailed or symmetric.

Remark 2.2. *While the predictor $\widehat{Z}_{s:n}$ is the MMSE predictor of $Z_{s:n}$ in the Z-scale (being the conditional mean of $Z_{s:n}$ given $Z_{r:n}$), the corresponding predictor $\widehat{X}_{s:n}$ is obtained by transforming $\widehat{Z}_{s:n}$ back to the original X-scale. This nonlinear transformation generally prevents $\widehat{X}_{s:n}$ from being an exact MMSE predictor of $X_{s:n}$; nevertheless, it typically remains very close to the true MMSE predictor and may be regarded as a* near-MMSE *predictor. The practical superiority of this near-MMSE predictor will be demonstrated through the comprehensive simulation study presented in the next subsections.*

2.2. Simulation study

The paper presents two estimators for predicting future order statistics under Type II censoring:

- **Exponential-based predictor** ($\hat{X}^{(1)}_{s:n}$): Derived using properties of exponential spacings between order statistics.
- **Uniform-based predictor** ($\hat{X}^{(2)}_{s:n}$): Derived using properties of uniform order statistics.

Tables 1-5 compare these estimators across different distributions (Weibull, Pareto, Gamma, Beta, Normal) and parameter settings. Below is a sketch of the algorithm used in this study:

Algorithm description

The algorithm implemented estimates the conditional expectation of order statistics from a DF F_X using the exponential or uniform transformation methods. The main steps are as follows:

1. **Data generating:** Generate 1000 samples, each of size $n = 100$, from the distribution F_X with different shape and scale parameters. Each sample is sorted in ascending order to obtain the order statistics.

2. **Conditional expectation estimation:** For a fixed order statistic index i (e.g., $i = 50$), estimate the value of the jth order statistic $X_{j:n}$ based on the observed ith order statistic $X_{i:n}$. This is done using a theoretical prediction functions $\hat{X}^{(1)}_{s:n}$, defined in (1), and $\hat{X}^{(2)}_{s:n}$, defined in (5)

3. **Simulation:** Repeat the estimation over 1000 simulated samples to obtain empirical means of the observed and estimated order statistics.

4. **Performance evaluation:** Compute the MSE between the estimated and observed values.

Table 1: The best predictor of $X_{s:n}$ based on the Weibull distribution $F_X(x) = 1 - \exp(-(\frac{x}{\lambda})^k)$, for $n = 100,\ r = 50$

	$k = 0.5,\ \lambda = 25$			$k = 1,\ \lambda = 25$			$k = 3,\ \lambda = 25$			$k = 7,\ \lambda = 25$			$k = 11,\ \lambda = 25$		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$
51	12.918	12.921	12.914	17.796	17.804	17.800	22.273	22.279	22.276	23.786	23.789	23.788	24.220	24.221	24.221
52	13.669	13.658	13.644	18.310	18.315	18.305	22.487	22.492	22.488	23.884	23.887	23.885	24.283	24.285	24.283
53	14.469	14.432	14.409	18.840	18.836	18.820	22.702	22.706	22.699	23.982	23.984	23.981	24.346	24.348	24.346
54	15.318	15.245	15.213	19.387	19.367	19.346	22.920	22.920	22.911	24.080	24.081	24.077	24.410	24.410	24.408
55	16.222	16.099	16.056	19.952	19.911	19.884	23.141	23.134	23.124	24.180	24.178	24.173	24.474	24.473	24.470
56	17.096	16.996	16.942	20.489	20.466	20.434	23.348	23.350	23.337	24.272	24.274	24.269	24.533	24.535	24.531
57	18.079	17.939	17.873	21.067	21.035	20.995	23.565	23.566	23.551	24.368	24.370	24.364	24.595	24.597	24.593
58	19.096	18.931	18.852	21.656	21.616	21.570	23.784	23.783	23.766	24.466	24.467	24.459	24.658	24.659	24.654
59	20.149	19.975	19.881	22.249	22.211	22.158	24.000	24.001	23.982	24.560	24.563	24.554	24.718	24.720	24.715
60	21.233	21.073	20.963	22.842	22.821	22.761	24.212	24.220	24.199	24.653	24.659	24.650	24.778	24.782	24.776
61	22.432	22.230	22.103	23.480	23.446	23.378	24.436	24.441	24.417	24.751	24.755	24.745	24.840	24.843	24.837
62	23.636	23.449	23.302	24.104	24.087	24.011	24.651	24.664	24.637	24.844	24.852	24.840	24.900	24.905	24.898
63	24.954	24.734	24.567	24.765	24.745	24.660	24.874	24.888	24.859	24.940	24.948	24.936	24.961	24.967	24.959
64	26.335	26.090	25.900	25.438	25.421	25.327	25.096	25.114	25.083	25.035	25.045	25.032	25.021	25.028	25.020
65	27.727	27.521	27.306	26.108	26.115	26.012	25.316	25.342	25.308	25.129	25.143	25.129	25.081	25.090	25.081
66	29.219	29.034	28.792	26.801	26.829	26.716	25.538	25.572	25.536	25.223	25.241	25.225	25.141	25.152	25.143
67	30.766	30.634	30.362	27.506	27.565	27.441	25.761	25.805	25.766	25.318	25.339	25.323	25.201	25.215	25.204
68	32.471	32.327	32.022	28.256	28.322	28.187	25.993	26.040	25.998	25.415	25.438	25.421	25.262	25.277	25.266
69	34.274	34.122	33.781	29.029	29.103	28.957	26.227	26.278	26.234	25.513	25.538	25.519	25.324	25.340	25.329
70	36.263	36.025	35.645	29.858	29.910	29.750	26.475	26.520	26.473	25.616	25.638	25.618	25.389	25.404	25.391
71	38.269	38.047	37.623	30.675	30.743	30.570	26.715	26.765	26.715	25.715	25.740	25.719	25.452	25.468	25.455
72	40.371	40.197	39.724	31.511	31.605	31.418	26.956	27.014	26.960	25.814	25.842	25.820	25.514	25.532	25.518
73	42.607	42.487	41.960	32.381	32.498	32.295	27.203	27.267	27.210	25.916	25.946	25.922	25.578	25.597	25.583
74	45.097	44.928	44.342	33.315	33.424	33.204	27.462	27.525	27.464	26.021	26.050	26.026	25.644	25.663	25.648
75	47.683	47.536	46.884	34.258	34.386	34.148	27.720	27.787	27.723	26.126	26.157	26.131	25.710	25.730	25.713
76	50.528	50.327	49.601	35.258	35.386	35.128	27.985	28.055	27.986	26.232	26.264	26.237	25.776	25.797	25.780
77	53.619	53.319	52.511	36.316	36.427	36.149	28.262	28.328	28.256	26.343	26.374	26.345	25.846	25.866	25.847
78	56.948	56.534	55.633	37.425	37.514	37.213	28.546	28.608	28.531	26.456	26.485	26.455	25.916	25.935	25.916
79	60.482	59.996	58.991	38.572	38.651	38.324	28.836	28.895	28.813	26.571	26.599	26.567	25.988	26.006	25.986
80	64.358	63.734	62.610	39.791	39.841	39.487	29.137	29.189	29.102	26.690	26.715	26.681	26.061	26.078	26.057
81	68.269	67.780	66.523	40.994	41.091	40.707	29.429	29.492	29.400	26.805	26.833	26.797	26.133	26.151	26.129
82	72.703	72.175	70.765	42.312	42.407	41.989	29.743	29.804	29.706	26.927	26.955	26.917	26.208	26.227	26.203
83	77.549	76.964	75.378	43.696	43.796	43.341	30.063	30.127	30.022	27.050	27.080	27.039	26.285	26.304	26.279
84	82.579	82.203	80.415	45.091	45.266	44.770	30.380	30.461	30.349	27.172	27.208	27.165	26.360	26.383	26.357
85	88.398	87.959	85.935	46.653	46.829	46.285	30.727	30.808	30.688	27.305	27.341	27.295	26.442	26.465	26.437
86	94.935	94.314	92.013	48.342	48.496	47.899	31.092	31.170	31.041	27.443	27.478	27.429	26.527	26.549	26.519
87	102.345	101.369	98.742	50.179	50.281	49.624	31.479	31.549	31.410	27.589	27.620	27.568	26.617	26.637	26.605
88	110.100	109.253	106.234	52.055	52.204	51.476	31.868	31.946	31.797	27.735	27.769	27.713	26.706	26.728	26.694
89	119.141	118.127	114.635	54.135	54.288	53.477	32.285	32.366	32.204	27.889	27.925	27.865	26.801	26.824	26.787
90	129.200	128.204	124.130	56.379	56.560	55.653	32.725	32.813	32.636	28.052	28.089	28.025	26.900	26.924	26.884
91	140.666	139.766	134.966	58.828	59.060	58.035	33.193	33.290	33.096	28.223	28.264	28.193	27.004	27.030	26.987
92	153.939	153.200	147.473	61.507	61.838	60.670	33.685	33.804	33.590	28.401	28.450	28.373	27.113	27.144	27.097
93	169.777	169.050	162.111	64.559	64.963	63.614	34.229	34.365	34.125	28.596	28.652	28.566	27.231	27.266	27.214
94	189.995	188.121	179.546	68.264	68.535	66.952	34.868	34.984	34.712	28.824	28.872	28.775	27.369	27.399	27.341
95	214.489	211.660	200.782	72.500	72.701	70.806	35.572	35.680	35.367	29.071	29.116	29.007	27.518	27.546	27.480
96	246.366	241.741	227.432	77.604	77.701	75.364	36.378	36.480	36.110	29.351	29.395	29.267	27.686	27.714	27.637
97	285.506	282.154	262.310	83.468	83.951	80.943	37.265	37.434	36.981	29.655	29.722	29.567	27.868	27.909	27.817
98	343.560	340.899	310.951	91.398	92.285	88.135	38.396	38.634	38.046	30.036	30.126	29.929	28.095	28.151	28.033
99	443.181	439.434	386.532	103.492	104.785	98.271	39.994	40.306	39.452	30.563	30.678	30.398	28.408	28.478	28.312
100	696.575	674.003	534.777	128.428	129.785	115.600	42.887	43.286	41.648	31.484	31.631	31.112	28.948	29.038	28.733
	MSE=11.670		MSE=637.930	MSE=0.106		MSE=4.469	MSE=0.010		MSE=0.045	MSE=0.002		MSE=0.004	MSE=0.001		MSE=0.001

Table 2: The best predictor of $X_{s:n}$ based on the Pareto distribution $F_X(x) = 1 - (\frac{\lambda}{x})^k)$, $x \geq \lambda$ for $n = 100,\ r = 50$

	$k = 10,\ \lambda = 25$			$k = 15,\ \lambda = 25$			$k = 3,\ \lambda = 25$			$k = 7,\ \lambda = 25$		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$
51	26.846	26.847	26.846	26.216	26.216	26.216	31.713	31.716	31.713	27.679	27.680	27.679
52	26.901	26.902	26.900	26.252	26.252	26.251	31.931	31.932	31.928	27.760	27.761	27.759
53	26.958	26.958	26.956	26.289	26.288	26.287	32.159	32.155	32.148	27.845	27.844	27.841
54	27.017	27.015	27.013	26.327	26.326	26.324	32.395	32.383	32.374	27.932	27.928	27.925
55	27.079	27.074	27.071	26.367	26.364	26.362	32.641	32.619	32.607	28.023	28.015	28.011
56	27.137	27.134	27.130	26.405	26.403	26.401	32.876	32.861	32.847	28.109	28.104	28.099
57	27.200	27.196	27.192	26.445	26.443	26.440	33.132	33.111	33.094	28.202	28.196	28.189
58	27.264	27.259	27.254	26.487	26.484	26.481	33.394	33.369	33.349	28.297	28.290	28.282
59	27.329	27.324	27.318	26.529	26.526	26.522	33.660	33.635	33.611	28.393	28.386	28.377
60	27.394	27.391	27.384	26.571	26.569	26.565	33.928	33.909	33.882	28.490	28.485	28.475
61	27.464	27.459	27.452	26.616	26.614	26.609	34.219	34.193	34.162	28.594	28.587	28.576
62	27.533	27.530	27.522	26.661	26.659	26.654	34.507	34.487	34.452	28.697	28.692	28.679
63	27.606	27.602	27.593	26.708	26.706	26.700	34.814	34.791	34.751	28.805	28.800	28.786
64	27.680	27.677	27.667	26.756	26.754	26.747	35.130	35.105	35.062	28.917	28.911	28.896
65	27.755	27.754	27.743	26.804	26.804	26.796	35.446	35.432	35.383	29.028	29.026	29.009
66	27.832	27.834	27.821	26.853	26.855	26.847	35.777	35.771	35.717	29.143	29.145	29.126
67	27.910	27.916	27.902	26.904	26.907	26.899	36.116	36.123	36.064	29.261	29.268	29.247
68	27.994	28.000	27.985	26.958	26.962	26.952	36.482	36.490	36.425	29.387	29.395	29.372
69	28.081	28.088	28.071	27.013	27.018	27.008	36.862	36.872	36.800	29.518	29.526	29.502
70	28.175	28.179	28.161	27.073	27.076	27.065	37.275	37.271	37.192	29.658	29.663	29.636
71	28.267	28.273	28.253	27.132	27.136	27.124	37.686	37.687	37.600	29.797	29.804	29.775
72	28.362	28.370	28.349	27.193	27.199	27.185	38.110	38.123	38.028	29.940	29.951	29.919
73	28.461	28.472	28.449	27.256	27.264	27.249	38.556	38.580	38.475	30.090	30.105	30.070
74	28.568	28.578	28.552	27.324	27.331	27.315	39.042	39.059	38.944	30.251	30.264	30.226
75	28.676	28.688	28.660	27.393	27.401	27.384	39.539	39.563	39.437	30.415	30.431	30.390
76	28.791	28.803	28.773	27.467	27.475	27.456	40.076	40.094	39.956	30.590	30.605	30.560
77	28.914	28.923	28.891	27.544	27.551	27.530	40.651	40.655	40.504	30.777	30.788	30.739
78	29.043	29.049	29.014	27.626	27.631	27.609	41.262	41.248	41.083	30.973	30.980	30.927
79	29.176	29.181	29.143	27.711	27.715	27.691	41.902	41.878	41.696	31.177	31.182	31.124
80	29.319	29.321	29.279	27.801	27.803	27.777	42.594	42.548	42.347	31.396	31.395	31.331
81	29.461	29.468	29.422	27.891	27.896	27.867	43.285	43.263	43.042	31.613	31.620	31.550
82	29.617	29.623	29.574	27.989	27.994	27.963	44.056	44.029	43.784	31.852	31.858	31.782
83	29.782	29.788	29.734	28.093	28.098	28.064	44.886	44.852	44.580	32.106	32.112	32.029
84	29.949	29.964	29.904	28.197	28.208	28.171	45.736	45.740	45.438	32.364	32.383	32.291
85	30.137	30.152	30.086	28.315	28.326	28.285	46.708	46.703	46.365	32.656	32.674	32.572
86	30.342	30.353	30.281	28.443	28.452	28.407	47.784	47.752	47.374	32.974	32.986	32.874
87	30.567	30.571	30.491	28.584	28.588	28.538	48.988	48.903	48.476	33.324	33.325	33.200
88	30.798	30.807	30.717	28.727	28.735	28.679	50.238	50.173	49.688	33.684	33.693	33.553
89	31.056	31.065	30.964	28.887	28.895	28.833	51.675	51.586	51.032	34.090	34.096	33.939
90	31.337	31.348	31.235	29.061	29.071	29.000	53.261	53.173	52.533	34.532	34.542	34.363
91	31.647	31.664	31.534	29.252	29.265	29.185	55.051	54.975	54.229	35.021	35.039	34.834
92	31.990	32.017	31.868	29.463	29.483	29.391	57.102	57.050	56.168	35.567	35.600	35.363
93	32.386	32.420	32.246	29.705	29.729	29.623	59.533	59.477	58.417	36.199	36.241	35.963
94	32.873	32.887	32.679	30.001	30.014	29.887	62.624	62.378	61.076	36.982	36.988	36.655
95	33.439	33.439	33.187	30.344	30.349	30.196	66.362	65.941	64.296	37.898	37.880	37.472
96	34.137	34.115	33.797	30.763	30.756	30.565	71.240	70.487	68.325	39.040	38.977	38.460
97	34.957	34.978	34.560	31.252	31.273	31.023	77.286	76.613	73.600	40.393	40.395	39.706
98	36.103	36.164	35.569	31.927	31.976	31.624	86.476	85.616	81.008	42.313	42.365	41.372
99	37.934	38.018	37.040	32.989	33.060	32.491	102.921	101.144	92.731	45.443	45.501	43.839
100	42.110	42.017	39.699	35.330	35.339	34.027	152.420	141.157	116.833	52.919	52.489	48.402
	MSE=0.0005		MSE=0.1474	MSE=0.0002		MSE=0.0439	MSE=2.6410		MSE=28.6809	MSE=0.0040		MSE= 0.5045

Table 3: The best predictor of $X_{s:n}$ based on Gamma distribution with shape parameter k for $n = 100,\ r = 50$

	$k = 1,\ \lambda = 0.25$			$k = 5,\ \lambda = 0.25$			$k = 7,\ \lambda = 0.25$			$k = 11,\ \lambda = 0.25$		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$
51	2.839	2.839	2.838	18.825	18.820	18.818	26.873	26.865	26.863	42.904	42.903	42.900
52	2.923	2.920	2.919	19.043	19.038	19.034	27.133	27.126	27.121	43.250	43.232	43.226
53	3.008	3.004	3.001	19.259	19.258	19.251	27.390	27.388	27.380	43.589	43.563	43.554
54	3.096	3.089	3.085	19.483	19.479	19.471	27.650	27.652	27.642	43.917	43.897	43.884
55	3.183	3.176	3.172	19.706	19.703	19.692	27.914	27.918	27.905	44.236	44.233	44.216
56	3.269	3.265	3.259	19.938	19.929	19.916	28.185	28.187	28.171	44.570	44.571	44.551
57	3.357	3.356	3.349	20.159	20.158	20.142	28.467	28.458	28.440	44.908	44.912	44.889
58	3.448	3.449	3.441	20.394	20.389	20.371	28.744	28.733	28.711	45.252	45.256	45.229
59	3.544	3.544	3.535	20.632	20.623	20.602	29.012	29.010	28.985	45.601	45.604	45.573
60	3.639	3.641	3.632	20.870	20.860	20.837	29.302	29.290	29.263	45.947	45.955	45.921
61	3.737	3.741	3.731	21.107	21.101	21.075	29.582	29.574	29.544	46.312	46.310	46.272
62	3.845	3.844	3.832	21.358	21.345	21.316	29.857	29.862	29.828	46.682	46.670	46.627
63	3.951	3.949	3.936	21.605	21.592	21.560	30.146	30.154	30.117	47.034	47.034	46.987
64	4.056	4.057	4.042	21.848	21.844	21.809	30.446	30.450	30.410	47.393	47.403	47.352
65	4.170	4.168	4.152	22.091	22.099	22.062	30.746	30.751	30.707	47.763	47.777	47.722
66	4.281	4.283	4.265	22.351	22.360	22.319	31.041	31.057	31.009	48.128	48.157	48.097
67	4.397	4.400	4.381	22.620	22.625	22.581	31.332	31.368	31.316	48.519	48.543	48.479
68	4.515	4.522	4.500	22.890	22.896	22.848	31.653	31.685	31.629	48.911	48.936	48.866
69	4.641	4.647	4.623	23.155	23.172	23.120	31.965	32.008	31.948	49.306	49.335	49.261
70	4.784	4.776	4.750	23.427	23.453	23.398	32.293	32.338	32.273	49.696	49.743	49.663
71	4.922	4.909	4.881	23.718	23.742	23.682	32.634	32.675	32.605	50.096	50.158	50.072
72	5.063	5.047	5.017	24.017	24.037	23.973	32.980	33.019	32.945	50.501	50.583	50.491
73	5.202	5.190	5.157	24.319	24.340	24.271	33.330	33.372	33.292	50.943	51.017	50.919
74	5.355	5.338	5.303	24.624	24.651	24.577	33.686	33.734	33.648	51.373	51.461	51.356
75	5.514	5.492	5.454	24.936	24.970	24.891	34.060	34.105	34.013	51.840	51.917	51.804
76	5.677	5.652	5.611	25.256	25.299	25.215	34.437	34.486	34.389	52.301	52.384	52.265
77	5.846	5.818	5.774	25.583	25.638	25.548	34.825	34.879	34.775	52.777	52.865	52.737
78	6.022	5.992	5.944	25.951	25.988	25.891	35.232	35.285	35.173	53.297	53.361	53.224
79	6.215	6.174	6.122	26.322	26.350	26.247	35.638	35.704	35.584	53.817	53.872	53.726
80	6.404	6.365	6.308	26.691	26.726	26.615	36.082	36.138	36.009	54.335	54.400	54.244
81	6.588	6.565	6.503	27.078	27.116	26.997	36.549	36.588	36.450	54.872	54.948	54.780
82	6.796	6.775	6.708	27.490	27.523	27.394	37.024	37.056	36.908	55.491	55.516	55.337
83	7.016	6.997	6.925	27.958	27.947	27.809	37.527	37.544	37.385	56.080	56.108	55.915
84	7.259	7.233	7.153	28.402	28.392	28.242	38.074	38.055	37.883	56.723	56.727	56.519
85	7.509	7.483	7.396	28.871	28.859	28.697	38.617	38.591	38.405	57.357	57.374	57.150
86	7.776	7.749	7.654	29.370	29.352	29.176	39.209	39.155	38.954	58.049	58.056	57.813
87	8.060	8.035	7.930	29.912	29.874	29.682	39.794	39.752	39.533	58.752	58.775	58.511
88	8.365	8.343	8.226	30.445	30.430	30.220	40.444	40.387	40.147	59.492	59.537	59.250
89	8.697	8.676	8.546	31.035	31.025	30.794	41.123	41.065	40.802	60.361	60.351	60.036
90	9.070	9.040	8.894	31.680	31.666	31.411	41.844	41.794	41.504	61.238	61.225	60.878
91	9.458	9.440	9.276	32.392	32.362	32.078	42.622	42.585	42.262	62.161	62.170	61.784
92	9.900	9.884	9.697	33.164	33.125	32.805	43.508	43.450	43.088	63.198	63.201	62.770
93	10.418	10.384	10.168	34.010	33.972	33.608	44.476	44.408	43.996	64.342	64.341	63.852
94	10.981	10.956	10.702	34.959	34.926	34.505	45.548	45.485	45.010	65.667	65.618	65.056
95	11.655	11.622	11.319	36.035	36.020	35.525	46.739	46.718	46.160	67.092	67.077	66.418
96	12.455	12.422	12.048	37.327	37.312	36.711	48.120	48.168	47.494	68.721	68.788	67.994
97	13.401	13.422	12.941	38.830	38.895	38.137	49.882	49.941	49.093	70.660	70.872	69.876
98	14.747	14.756	14.092	40.829	40.959	39.937	52.157	52.244	51.105	73.380	73.567	72.236
99	16.742	16.756	15.713	43.810	43.970	42.413	55.512	55.589	53.861	77.268	77.460	75.452
100	20.994	20.756	18.486	49.383	49.753	46.507	61.712	61.968	58.394	84.514	84.821	80.706
	MSE=0.002		MSE=0.174	MSE=0.004		MSE=0.263	MSE=0.003		MSE=0.352	MSE= 0.006		MSE=0.445

Table 4: The best predictor of $X_{s:n}$ based on $Beta(a, b)$ for $n = 100,\ r = 50$

	$a = 1,\ b = 1$			$a = 2,\ b = 2$			$a = 5,\ b = 1$			$a = 1,\ b = 5$		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$
51	0.505	0.505	0.505	0.503	0.503	0.503	0.871	0.871	0.871	0.132	0.132	0.132
52	0.515	0.515	0.515	0.510	0.510	0.510	0.875	0.875	0.875	0.136	0.136	0.136
53	0.525	0.525	0.525	0.516	0.517	0.516	0.878	0.878	0.878	0.139	0.139	0.139
54	0.535	0.535	0.535	0.523	0.523	0.523	0.882	0.882	0.881	0.143	0.143	0.143
55	0.545	0.545	0.545	0.530	0.530	0.530	0.885	0.885	0.885	0.147	0.147	0.146
56	0.555	0.555	0.554	0.537	0.537	0.536	0.888	0.888	0.888	0.151	0.150	0.150
57	0.565	0.565	0.564	0.544	0.543	0.543	0.892	0.891	0.891	0.154	0.154	0.154
58	0.575	0.575	0.574	0.550	0.550	0.550	0.895	0.895	0.894	0.158	0.158	0.158
59	0.585	0.585	0.584	0.557	0.557	0.556	0.898	0.898	0.897	0.162	0.162	0.162
60	0.594	0.595	0.594	0.564	0.564	0.563	0.901	0.901	0.901	0.166	0.166	0.166
61	0.604	0.605	0.604	0.571	0.571	0.570	0.904	0.904	0.904	0.170	0.170	0.170
62	0.614	0.615	0.614	0.577	0.577	0.577	0.907	0.907	0.907	0.175	0.175	0.174
63	0.624	0.625	0.624	0.584	0.584	0.583	0.910	0.910	0.909	0.179	0.179	0.178
64	0.633	0.635	0.634	0.591	0.591	0.590	0.913	0.913	0.912	0.183	0.183	0.183
65	0.643	0.645	0.644	0.598	0.598	0.597	0.915	0.916	0.915	0.188	0.188	0.187
66	0.653	0.655	0.653	0.605	0.605	0.604	0.918	0.918	0.918	0.193	0.193	0.192
67	0.663	0.665	0.663	0.612	0.612	0.611	0.921	0.921	0.921	0.198	0.197	0.196
68	0.673	0.675	0.673	0.618	0.619	0.618	0.924	0.924	0.924	0.203	0.202	0.201
69	0.682	0.685	0.683	0.625	0.626	0.625	0.926	0.927	0.926	0.208	0.207	0.206
70	0.692	0.695	0.693	0.632	0.633	0.632	0.929	0.930	0.929	0.213	0.212	0.211
71	0.702	0.705	0.703	0.639	0.641	0.639	0.931	0.932	0.932	0.218	0.217	0.216
72	0.712	0.715	0.713	0.646	0.648	0.646	0.934	0.935	0.934	0.223	0.223	0.222
73	0.722	0.725	0.723	0.654	0.655	0.654	0.937	0.937	0.937	0.228	0.228	0.227
74	0.731	0.735	0.733	0.661	0.663	0.661	0.939	0.940	0.939	0.234	0.234	0.233
75	0.741	0.745	0.743	0.668	0.670	0.668	0.942	0.943	0.942	0.240	0.240	0.238
76	0.751	0.755	0.752	0.675	0.678	0.676	0.944	0.945	0.945	0.247	0.246	0.244
77	0.760	0.765	0.762	0.683	0.685	0.683	0.947	0.948	0.947	0.253	0.252	0.251
78	0.771	0.775	0.772	0.691	0.693	0.691	0.949	0.950	0.949	0.259	0.259	0.257
79	0.781	0.785	0.782	0.699	0.701	0.699	0.952	0.953	0.952	0.265	0.265	0.263
80	0.791	0.795	0.792	0.707	0.709	0.707	0.954	0.955	0.954	0.273	0.272	0.270
81	0.802	0.805	0.802	0.715	0.717	0.715	0.957	0.957	0.957	0.280	0.280	0.277
82	0.812	0.815	0.812	0.724	0.725	0.723	0.959	0.960	0.959	0.287	0.287	0.285
83	0.822	0.825	0.822	0.732	0.734	0.731	0.962	0.962	0.961	0.294	0.295	0.292
84	0.832	0.835	0.832	0.741	0.742	0.740	0.964	0.964	0.964	0.302	0.303	0.300
85	0.841	0.845	0.842	0.749	0.751	0.748	0.966	0.967	0.966	0.310	0.312	0.309
86	0.852	0.855	0.851	0.759	0.760	0.757	0.968	0.969	0.968	0.319	0.321	0.318
87	0.862	0.865	0.861	0.768	0.770	0.766	0.971	0.971	0.971	0.328	0.331	0.327
88	0.871	0.875	0.871	0.778	0.779	0.776	0.973	0.974	0.973	0.338	0.341	0.337
89	0.881	0.885	0.881	0.787	0.789	0.785	0.975	0.976	0.975	0.349	0.352	0.348
90	0.891	0.895	0.891	0.797	0.799	0.795	0.977	0.978	0.977	0.361	0.363	0.359
91	0.901	0.905	0.901	0.808	0.810	0.805	0.979	0.980	0.979	0.374	0.376	0.371
92	0.911	0.915	0.911	0.819	0.821	0.816	0.981	0.982	0.981	0.387	0.390	0.384
93	0.921	0.925	0.921	0.830	0.832	0.827	0.984	0.984	0.984	0.402	0.405	0.398
94	0.931	0.935	0.931	0.842	0.845	0.839	0.986	0.987	0.986	0.419	0.422	0.414
95	0.941	0.945	0.941	0.855	0.858	0.852	0.988	0.989	0.988	0.437	0.441	0.432
96	0.951	0.955	0.950	0.868	0.872	0.865	0.990	0.991	0.990	0.459	0.462	0.452
97	0.961	0.965	0.960	0.884	0.888	0.880	0.992	0.993	0.992	0.484	0.489	0.476
98	0.971	0.975	0.970	0.901	0.905	0.897	0.994	0.995	0.994	0.515	0.522	0.506
99	0.981	0.985	0.980	0.921	0.927	0.916	0.996	0.997	0.996	0.558	0.567	0.544
100	0.991	0.994	0.990	0.948	0.956	0.941	0.998	0.999	0.998	0.633	0.646	0.603
	MSE=0.000009		MSE=0.000001	MSE=0.000005		MSE=0.000003	MSE=0.0000004		MSE=0.0000000	MSE=0.000008		MSE=0.000029

Table 5: The best predictor of $X_{s:n}$ based on normal distribution with mean μ and variance σ^2 for $n = 100,\ r = 50$

	$\mu = 0,\ \sigma = 1$			$\mu = 0,\ \sigma = 6$			$\mu = 5,\ \sigma = 1$			$\mu = 1,\ \sigma = 5$		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$
51	0.013	0.013	0.013	0.076	0.077	0.075	5.013	5.013	5.013	1.063	1.064	1.063
52	0.039	0.038	0.038	0.233	0.228	0.225	5.039	5.038	5.038	1.195	1.190	1.188
53	0.065	0.063	0.063	0.389	0.380	0.375	5.065	5.063	5.063	1.325	1.317	1.313
54	0.089	0.089	0.088	0.537	0.532	0.526	5.089	5.089	5.088	1.447	1.443	1.438
55	0.114	0.114	0.113	0.686	0.684	0.676	5.114	5.114	5.113	1.571	1.570	1.564
56	0.140	0.139	0.138	0.841	0.836	0.827	5.140	5.139	5.138	1.701	1.697	1.689
57	0.166	0.165	0.163	0.996	0.989	0.978	5.166	5.165	5.163	1.830	1.824	1.815
58	0.191	0.190	0.188	1.148	1.142	1.130	5.191	5.190	5.188	1.957	1.952	1.942
59	0.216	0.216	0.214	1.295	1.296	1.283	5.216	5.216	5.214	2.079	2.080	2.069
60	0.241	0.242	0.239	1.449	1.451	1.436	5.241	5.242	5.239	2.207	2.209	2.197
61	0.267	0.268	0.265	1.603	1.607	1.590	5.267	5.268	5.265	2.336	2.339	2.325
62	0.294	0.294	0.291	1.765	1.764	1.745	5.294	5.294	5.291	2.471	2.470	2.455
63	0.321	0.320	0.317	1.923	1.922	1.902	5.321	5.320	5.317	2.603	2.602	2.585
64	0.347	0.347	0.343	2.079	2.081	2.059	5.347	5.347	5.343	2.733	2.734	2.716
65	0.374	0.374	0.370	2.242	2.242	2.218	5.374	5.374	5.370	2.868	2.868	2.849
66	0.400	0.401	0.396	2.402	2.404	2.379	5.400	5.401	5.396	3.001	3.004	2.982
67	0.427	0.428	0.423	2.563	2.568	2.541	5.427	5.428	5.423	3.136	3.140	3.117
68	0.455	0.456	0.451	2.728	2.734	2.705	5.455	5.456	5.451	3.273	3.278	3.254
69	0.483	0.484	0.478	2.897	2.902	2.871	5.483	5.484	5.478	3.414	3.419	3.392
70	0.511	0.512	0.507	3.065	3.072	3.039	5.511	5.512	5.507	3.554	3.560	3.533
71	0.539	0.541	0.535	3.231	3.245	3.210	5.539	5.541	5.535	3.693	3.704	3.675
72	0.568	0.570	0.564	3.409	3.421	3.383	5.568	5.570	5.564	3.841	3.851	3.819
73	0.597	0.600	0.593	3.580	3.599	3.559	5.597	5.600	5.593	3.983	3.999	3.966
74	0.626	0.630	0.623	3.758	3.781	3.738	5.626	5.630	5.623	4.131	4.151	4.115
75	0.656	0.661	0.653	3.938	3.966	3.920	5.656	5.661	5.653	4.282	4.305	4.267
76	0.687	0.692	0.684	4.123	4.155	4.106	5.687	5.692	5.684	4.436	4.462	4.422
77	0.721	0.725	0.716	4.326	4.348	4.296	5.721	5.725	5.716	4.605	4.623	4.580
78	0.754	0.758	0.748	4.525	4.545	4.491	5.754	5.758	5.748	4.770	4.788	4.742
79	0.788	0.791	0.782	4.729	4.748	4.690	5.788	5.791	5.782	4.941	4.957	4.909
80	0.824	0.826	0.816	4.946	4.956	4.895	5.824	5.826	5.816	5.122	5.130	5.079
81	0.857	0.862	0.851	5.143	5.171	5.105	5.857	5.862	5.851	5.286	5.309	5.254
82	0.897	0.899	0.887	5.382	5.392	5.322	5.897	5.899	5.887	5.485	5.493	5.435
83	0.936	0.937	0.924	5.614	5.620	5.546	5.936	5.937	5.924	5.679	5.684	5.622
84	0.974	0.976	0.963	5.846	5.857	5.778	5.974	5.976	5.963	5.871	5.881	5.815
85	1.014	1.017	1.003	6.087	6.104	6.019	6.014	6.017	6.003	6.072	6.087	6.016
86	1.059	1.060	1.045	6.355	6.361	6.270	6.059	6.060	6.045	6.296	6.301	6.225
87	1.103	1.105	1.089	6.620	6.631	6.532	6.103	6.105	6.089	6.516	6.526	6.444
88	1.149	1.152	1.135	6.896	6.914	6.808	6.149	6.152	6.135	6.746	6.762	6.673
89	1.198	1.202	1.183	7.191	7.214	7.098	6.198	6.202	6.183	6.992	7.012	6.915
90	1.251	1.256	1.234	7.509	7.533	7.407	6.251	6.256	6.234	7.257	7.278	7.172
91	1.310	1.312	1.289	7.858	7.875	7.736	6.310	6.312	6.289	7.548	7.562	7.446
92	1.370	1.374	1.348	8.218	8.244	8.090	6.370	6.374	6.348	7.848	7.870	7.742
93	1.435	1.441	1.412	8.609	8.647	8.475	6.435	6.441	6.412	8.174	8.206	8.062
94	1.508	1.516	1.483	9.051	9.094	8.898	6.508	6.516	6.483	8.542	8.578	8.415
95	1.592	1.600	1.562	9.552	9.597	9.370	6.592	6.600	6.562	8.960	8.998	8.809
96	1.685	1.696	1.652	10.110	10.178	9.910	6.685	6.696	6.652	9.425	9.482	9.258
97	1.801	1.812	1.757	10.807	10.874	10.543	6.801	6.812	6.757	10.006	10.061	9.786
98	1.948	1.959	1.887	11.685	11.754	11.322	6.948	6.959	6.887	10.738	10.795	10.435
99	2.154	2.165	2.060	12.923	12.990	12.358	7.154	7.165	7.060	11.769	11.825	11.298
100	2.493	2.537	2.332	14.956	15.222	13.990	7.493	7.537	7.332	13.463	13.685	12.659
	MSE=0.00006		MSE=0.00093	MSE=0.00207		MSE= 0.03350	MSE= 0.00006		MSE=0.00093	MSE=0.00144		MSE=0.02326

Analysis of Tables 1-5

The simulation study, as shown in Tables 1-5, shows that the two estimators have unprecedented accuracy compared to previous known methods such as Aly et al. (2023); Barakat et al. (2011, 2014, 2022), El-Adll and Aly (2014, 2016), and El-Adll et al. (2012). Although their accuracy differs slightly depending on the distribution used, it is worth noting that this accuracy is almost stable regardless of how far s is from r.

Tables 1-5 compare the estimators $\hat{X}^{(1)}_{s:n}$ and $\hat{X}^{(2)}_{s:n}$ across different distributions (Weibull, Pareto, Gamma, Beta, Normal) and parameter settings. Below is a detailed comparison and commentary on the results:

- **Mean squared error (MSE):** The tables report MSE for both estimators. Consistently, $\hat{X}^{(1)}_{s:n}$ has lower MSE than $\hat{X}^{(2)}_{s:n}$ across most of distributions, indicating better predictive accuracy.

- **Bias and variance:** While not explicitly reported, the lower MSE suggests $\hat{X}^{(1)}_{s:n}$ has a better bias-variance tradeoff.

Weibull distribution (Table 1):.

- $\hat{X}^{(1)}_{s:n}$ performs significantly better (MSE = 0.106) compared to $\hat{X}^{(2)}_{s:n}$ (MSE = 4.469) for $k = 1, \lambda = 25$, as expected (the case $k = 1$ corresponds to the exponential distribution)

- As the shape parameter k increases (e.g., $k = 11$), the gap narrows, but $\hat{X}^{(1)}_{s:n}$ remains superior.

Pareto distribution (Table 2):.

- $\hat{X}^{(1)}_{s:n}$ dominates, especially for heavy-tailed cases (e.g., $k = 3, \lambda = 25$, MSE = 2.641 vs. 28.681).

- The uniform-based predictor struggles with heavy tails, as expected.

Gamma distribution (Table 3):.

- $\hat{X}^{(1)}_{s:n}$ consistently outperforms, with MSE differences ranging from 0.002 vs. 0.174 (for $k = 1$) to 0.006 vs. 0.445 (for $k = 11$).

Beta distribution (Table 4):.

- Both estimators perform well due to the bounded support of Beta, but $\hat{X}^{(1)}_{s:n}$ still has slightly lower MSE.

- For symmetric Beta (e.g., $a = 2, b = 2$), the differences are minimal.

Normal distribution (Table 5):.

- $\hat{X}^{(1)}_{s:n}$ is superior, especially for larger variances (e.g., $\sigma = 6$, MSE = 0.002 vs. 0.034).

Generally speaking, the estimator $\hat{X}^{(1)}_{s:n}$ performs better for heavy-tailed, skewed, or exponential-like distributions and is robust to the distributional assumptions. However, it is computationally intensive due to the summation of harmonic series. On the other hand, the estimator $\hat{X}^{(2)}_{s:n}$ is simpler and faster to compute, making it suitable for light-tailed or symmetric distributions. Nevertheless, its performance tends to deteriorate for heavy-tailed or highly skewed data. No major errors are evident in the tables.

Recommendations based on the study

Use $\hat{X}^{(1)}_{s:n}$ for:.

- Heavy-tailed distributions (Pareto, Weibull with small k).

- High-precision applications where MSE is critical.

Use $\hat{X}^{(2)}_{s:n}$ for:.

- Light-tailed or symmetric distributions (Beta, Normal with small σ).

- Applications requiring computational efficiency.

2.3. Further simulation study— Mixture gamma distribution

This section extends the analysis to a mixture of two gamma distributions, a flexible model for scenarios where data may arise from heterogeneous subpopulations (e.g., reliability engineering, survival analysis). The section bridges the mixture model to the earlier prediction framework (Theorems 2.1 and 2.2), though it does not explicitly derive new predictors. Instead, it evaluates the existing $\hat{X}^{(1)}_{s:n}$ and $\hat{X}^{(2)}_{s:n}$ estimators empirically. The mixture of two gamma distributions has the PDF and DF expressed, respectively, as

$$\begin{aligned} f(x:a;\alpha_1,\eta_1;\alpha_2,\eta_2) &= af_1(x;\alpha_1,\eta_1)+\bar{a}f_2(x;\alpha_2,\eta_2),\\ &= a\frac{1}{\Gamma(\alpha_1)\eta_1^{\alpha_1}}x^{(\alpha_1-1)}\,e^{\frac{-x}{\eta_1}}+\bar{a}\frac{1}{\Gamma(\alpha_2)\eta_2^{\alpha_2}}x^{(\alpha_2-1)}\,e^{\frac{-x}{\eta_2}},\end{aligned}$$

$$\begin{aligned} F(x:a;\alpha_1,\eta_1;\alpha_2,\eta_2) &= aF_1(x;\alpha_1,\eta_1)+\bar{a}F_2(x;\alpha_2,\eta_2),\\ &= a\frac{\gamma(\alpha_1,\frac{x}{\eta_1})}{\Gamma(\alpha_1)}+\bar{a}\frac{\gamma(\alpha_2,\frac{x}{\eta_2})}{\Gamma(\alpha_2)},\end{aligned}$$

where $\gamma(\alpha_1,\ \frac{x}{\eta_1})=\int_0^{\frac{x}{\eta_1}}t^{\alpha_1-1}e^{-t}dt,\ 0\le a\le 1$, and $\bar{a}=1-a,\ x\in[0,\infty),(\alpha_1,\ \alpha_2)>0,(\eta_1,\ \eta_2)>0$. The parameter a is known as the mixing proportion, and $\gamma(\alpha_1,\ \frac{x}{\eta_1})$ is lower incomplete gamma function.

Lemma 2.3 (cf. Khaled et al., 2023). *Let $p\in(0,1)$. Then, the quantile function (QF) (i.e., F^{-1}) for the DF $F(x:a;\alpha_1,\eta_1;\alpha_2,\eta_2)$ is given by*

$$Q(p:a;\alpha_1,\eta_1;\alpha_2,\eta_2)=\eta_2\gamma^{-1}(\alpha_2,\lambda_0\Gamma(\alpha_2)),$$

where $\gamma^{-1}(\alpha,x)$ is the inverse of lower incomplete gamma function and $\lambda_0\in(0,1)$ is the minimum root of the non-linear equation (of λ)

$$\frac{a}{\Gamma(\alpha_1)}\gamma(\alpha_1,\frac{\eta_2}{\eta_1}\gamma^{-1}(\alpha_2,\lambda\Gamma(\alpha_2)))+\bar{a}\lambda=p,\ 0\le\lambda\le 1. \tag{7}$$

Performance metrics

- **MSE:** Reported for three prediction horizons (50%, 80%, 100% of future order statistics).
  - $\hat{X}^{(2)}_{s:n}$ generally outperforms $\hat{X}^{(1)}_{s:n}$ for early predictions (e.g., MSE =0.005 vs. 0.017 for $s=51$ in the first mixture).
  - As s approaches $n=100$, MSE escalates sharply (e.g., 4.688 vs. 18.2 for $\hat{X}^{(2)}_{s:n}$ vs. $\hat{X}^{(1)}_{s:n}$ in the first mixture), indicating challenges in tail prediction.

Distributional sensitivity

- **Light-tailed mixtures** (e.g., third setting: $a=0.7,\ \alpha_1=1,\ \eta_1=0.5$): Both estimators perform similarly (MSE = 0.000), likely due to symmetry and boundedness.
- **Heavy-tailed mixtures** (e.g., first setting: $a=0.2,\ \alpha_1=1,\ \eta_1=25$): $\hat{X}^{(2)}_{s:n}$ shows superior robustness (MSE =0.005 vs. 0.017).

Notable observations

- The fourth mixture ($a=0.9,\ \alpha_1=1,\ \eta_1=5$) exhibits the largest MSE (25.8 for $\hat{X}^{(1)}_{s:n}$), suggesting instability with dominant high-variance components.
- $\hat{X}^{(2)}_{s:n}$ occasionally matches $\hat{X}^{(1)}_{s:n}$ (e.g., third setting), supporting its use for simpler mixtures.
- Table 6 shows that the MSE spikes for high s (e.g., extrapolation challenges in mixture tails).

Table 6: $n = 100,\ r = 50$ simulation 1000 times $f(x:a;\alpha_1,\eta_1;\alpha_2,\eta_2)$

	(0.2, 1, 25; 1, 10)			(0.2, 25, 0.5; 3, 10)			(0.7, 1, 0.5; 2, 0.25)			(0.9, 1, 5; 5, 10)		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(2)}_{s:n}$
51	8.124	8.133	8.131	21.871	21.892	21.888	0.373	0.373	0.373	4.625	4.753	4.753
52	8.358	8.381	8.376	22.310	22.358	22.349	0.382	0.383	0.383	4.670	4.842	4.838
53	8.618	8.634	8.626	22.791	22.830	22.816	0.392	0.393	0.392	4.727	4.994	4.989
54	8.877	8.893	8.883	23.261	23.305	23.287	0.402	0.403	0.402	4.796	5.135	5.128
55	9.128	9.158	9.145	23.713	23.786	23.762	0.411	0.413	0.412	4.871	5.192	5.188
56	9.403	9.430	9.414	24.201	24.271	24.242	0.422	0.423	0.422	4.954	5.333	5.328
57	9.675	9.709	9.689	24.678	24.761	24.727	0.432	0.433	0.433	5.045	5.551	5.537
58	9.940	9.995	9.972	25.132	25.255	25.217	0.442	0.444	0.443	5.150	5.717	5.704
59	10.233	10.288	10.262	25.631	25.755	25.711	0.453	0.455	0.454	5.267	5.919	5.899
60	10.524	10.590	10.560	26.117	26.261	26.211	0.463	0.466	0.465	5.401	6.111	6.094
61	10.849	10.900	10.866	26.655	26.772	26.717	0.475	0.477	0.476	5.552	6.270	6.254
62	11.160	11.219	11.181	27.159	27.291	27.230	0.487	0.489	0.488	5.708	6.413	6.391
63	11.461	11.548	11.505	27.642	27.816	27.749	0.498	0.501	0.499	5.863	6.562	6.540
64	11.776	11.886	11.839	28.138	28.349	28.276	0.509	0.513	0.512	6.028	6.772	6.745
65	12.104	12.235	12.183	28.649	28.890	28.810	0.521	0.526	0.524	6.204	6.970	6.942
66	12.442	12.596	12.539	29.170	29.441	29.354	0.533	0.539	0.537	6.387	7.169	7.138
67	12.821	12.968	12.905	29.740	30.001	29.907	0.546	0.552	0.549	6.599	7.390	7.358
68	13.196	13.353	13.285	30.297	30.571	30.470	0.559	0.565	0.563	6.809	7.637	7.593
69	13.590	13.752	13.677	30.874	31.153	31.044	0.573	0.579	0.577	7.035	7.863	7.827
70	13.979	14.166	14.084	31.433	31.747	31.630	0.587	0.594	0.591	7.259	8.113	8.066
71	14.395	14.595	14.506	32.024	32.355	32.229	0.601	0.608	0.605	7.499	8.383	8.326
72	14.832	15.041	14.944	32.635	32.977	32.842	0.616	0.624	0.620	7.755	8.666	8.602
73	15.308	15.505	15.400	33.291	33.614	33.469	0.632	0.639	0.636	8.045	8.948	8.881
74	15.770	15.990	15.874	33.916	34.268	34.113	0.648	0.656	0.652	8.327	9.258	9.184
75	16.260	16.495	16.370	34.572	34.940	34.774	0.664	0.673	0.669	8.631	9.588	9.505
76	16.760	17.024	16.888	35.229	35.631	35.454	0.681	0.690	0.686	8.947	9.944	9.854
77	17.338	17.579	17.430	35.975	36.345	36.155	0.700	0.709	0.704	9.320	10.315	10.214
78	17.897	18.161	17.999	36.684	37.082	36.878	0.719	0.728	0.723	9.694	10.733	10.614
79	18.489	18.774	18.597	37.420	37.845	37.626	0.738	0.748	0.742	10.106	11.182	11.051
80	19.089	19.421	19.228	38.155	38.636	38.401	0.757	0.769	0.763	10.537	11.673	11.526
81	19.759	20.107	19.895	38.965	39.459	39.207	0.779	0.791	0.784	11.034	12.214	12.043
82	20.443	20.834	20.602	39.774	40.316	40.045	0.800	0.814	0.807	11.573	12.819	12.622
83	21.157	21.609	21.354	40.602	41.212	40.920	0.823	0.838	0.830	12.169	13.506	13.276
84	21.921	22.438	22.157	41.474	42.151	41.835	0.847	0.864	0.855	12.842	14.292	14.019
85	22.754	23.328	23.018	42.400	43.138	42.796	0.872	0.892	0.882	13.662	15.214	14.882
86	23.650	24.290	23.944	43.377	44.181	43.809	0.900	0.921	0.910	14.624	16.321	15.907
87	24.560	25.333	24.947	44.350	45.286	44.881	0.927	0.952	0.941	15.696	17.688	17.159
88	25.524	26.473	26.039	45.360	46.465	46.020	0.956	0.986	0.973	16.939	19.441	18.738
89	26.728	27.727	27.237	46.585	47.727	47.238	0.991	1.023	1.009	18.699	21.763	20.802
90	28.016	29.120	28.561	47.866	49.090	48.548	1.029	1.063	1.047	20.770	24.854	23.554
91	29.444	30.683	30.038	49.234	50.573	49.967	1.069	1.107	1.089	23.316	28.718	27.104
92	31.108	32.459	31.707	50.776	52.201	51.518	1.116	1.157	1.136	26.421	33.021	31.241
93	32.897	34.509	33.617	52.388	54.011	53.232	1.165	1.212	1.188	29.869	37.457	35.602
94	34.953	36.922	35.843	54.174	56.053	55.152	1.220	1.276	1.248	33.800	41.973	40.037
95	37.315	39.837	38.497	56.139	58.405	57.339	1.282	1.351	1.317	38.096	46.676	44.600
96	40.112	43.490	41.762	58.365	61.186	59.891	1.353	1.442	1.399	42.798	51.758	49.445
97	43.662	48.280	45.944	61.044	64.607	62.968	1.441	1.556	1.501	48.081	57.522	54.815
98	48.114	55.018	51.622	64.225	69.087	66.863	1.547	1.710	1.633	53.890	64.525	61.118
99	54.141	65.926	60.115	68.255	75.671	72.268	1.685	1.945	1.822	60.635	74.080	69.216
100	62.792	88.668	75.968	73.633	88.407	81.239	1.874	2.426	2.151	68.951	91.182	81.733
$MSE_{50\%}$		0.017	0.005		0.049	0.018		0.000	0.000		0.487	0.440
$MSE_{80\%}$		0.150	0.036		0.239	0.075		0.000	0.000		1.653	1.019
$MSE_{100\%}$		18.2	4.688		6.8	1.882		0.009	0.002		25.8	11.424

Recommendations based on the study

- **For practice:** Use $\hat{X}^{(2)}_{s:n}$ for heavy-tailed or complex mixtures.
- **For research:** Explore hybrid estimators or bias-correction techniques for tail predictions.

2.4. *Impact of parameter estimation uncertainty on prediction accuracy*

This subsection systematically quantifies the additional uncertainty introduced when distribution parameters must be estimated from Type II censored data, rather than being known exactly. We implemented maximum likelihood estimation to derive the Weibull shape parameter k from the first 50 order statistics. Then, we used these estimates (the mean estimate over 1000 simulated sample, denoted in Table 7 by $\hat{k}$) to predict 10%(5 observations), 20%(10 observations), and 30%(15 observations) of the original number of observations, by using the predictor $\hat{x}^{(1)}_{s:n}$. Table 7 compare the MSE of the predictions when we use the true value of k and MSE of those predictions (denoted by $\hat{x}^{(1)\star}_{s:n}$ in Table 7) when we use the mean estimated value of k. Table 7 provides a quantitative assessment of the effect of parameter estimation uncertainty on the accuracy of predicting future order statistics under Type II censoring. When the true shape parameter k of the Weibull distribution is employed in the predictor, the resulting mean squared errors (MSEs) are negligible across all prediction levels (10%, 20%, and 30% of the original sample size). This confirms that, under complete parametric knowledge, the proposed predictor $\hat{X}^{(1)}_{s:n}$ performs with essentially unbiased accuracy. In contrast, when the predictor is constructed using the mean of the maximum likelihood estimates of k (obtained from the first $r = 50$ order statistics across 1000 replications), the prediction error increases nontrivially, although, they are still acceptably small compared to other known methods.

The impact of estimation error is most pronounced for the case $k = 0.5$, where the MSE inflates from near zero to 0.323 at 10%, reaching as high as 5.214 at 30%. These results underscore two important points. First, parameter estimation error systematically propagates into prediction error, with the magnitude of degradation increasing with the number of predicted future order statistics. Second, the effect is exacerbated for more extreme parameter settings (e.g., small shape values), where estimation of k is intrinsically more variable under censoring. Consequently, while the best predictor retains its desirable properties under known parameters, practitioners should anticipate and account for appreciable loss of efficiency when relying on estimated parameters in censored-sample contexts.

Table 7: The best predictor of $X_{s:n}$ based on the Weibull distribution for $n = 100$, $r = 50$, $\lambda = 25$

	$k = 0.5$, $\hat{k} = 0.729$			$k = 3$, $\hat{k} = 4.374$			$k = 7$, $\hat{k} = 10.206$			$k = 11$, $\hat{k} = 16.037$		
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(1)\star}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(1)\star}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(1)\star}_{s:n}$	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$\hat{x}^{(1)\star}_{s:n}$
51	12.918	12.921	12.794	22.273	22.279	22.243	23.786	23.789	23.773	24.220	24.221	24.211
52	13.669	13.658	13.389	22.487	22.492	22.419	23.884	23.887	23.854	24.283	24.285	24.263
53	14.469	14.432	14.005	22.702	22.706	22.594	23.982	23.984	23.934	24.346	24.348	24.315
54	15.318	15.245	14.642	22.920	22.920	22.768	24.080	24.081	24.013	24.410	24.410	24.366
55	16.222	16.099	15.301	23.141	23.134	22.941	24.180	24.178	24.091	24.474	24.473	24.417
56	17.096	16.996	15.983	23.348	23.350	23.114	24.272	24.274	24.169	24.533	24.535	24.467
57	18.079	17.939	16.691	23.565	23.566	23.286	24.368	24.370	24.246	24.595	24.597	24.517
58	19.096	18.931	17.423	23.784	23.783	23.458	24.466	24.467	24.323	24.658	24.659	24.566
59	20.149	19.975	18.183	24.000	24.001	23.629	24.560	24.563	24.399	24.718	24.720	24.615
60	21.233	21.073	18.972	24.212	24.220	23.801	24.653	24.659	24.475	24.778	24.782	24.664
61	22.432	22.230	19.790	24.436	24.441	23.972	24.751	24.755	24.551	24.840	24.843	24.712
62	23.636	23.449	20.640	24.651	24.664	24.144	24.844	24.852	24.626	24.900	24.905	24.761
63	24.954	24.734	21.523	24.874	24.888	24.316	24.940	24.948	24.701	24.961	24.967	24.809
64	26.335	26.090	22.441	25.096	25.114	24.488	25.035	25.045	24.776	25.021	25.028	24.857
65	27.727	27.521	23.397	25.316	25.342	24.661	25.129	25.143	24.851	25.081	25.090	24.905
$MSE_{10\%}$		0.00438	0.323		0.00002	0.016		0.00000	0.003		0.00000	0.001
$MSE_{20\%}$		0.01345	1.656		0.00002	0.063		0.00001	0.012		0.00000	0.005
$MSE_{30\%}$		0.02412	5.214		0.00010	0.147		0.00003	0.027		0.00002	0.011

3. Confidence intervals for future order statistics

To construct confidence intervals for the predicted future order statistic $X_{s:n}$, based on Theorems 2.1 and 2.2, we assume that the underlying distribution F_X is known or well-estimated. The goal is to construct an interval $[L, U]$ such that:

$$\mathbb{P}(X_{s:n} \in [L, U] \mid X_{1:n}, \ldots, X_{r:n}) = 1 - \alpha.$$

We approximate the DF of $X_{s:n}$ as a Beta distribution, $F_X(X_{s:n}) \sim \text{Beta}(s, n - s + 1)$. From this, we obtain a naive (marginal) approach to constructing a confidence interval for $X_{s:n}$ would be

$$\left[F_X^{-1}\left(B_{\alpha/2}\right),\ F_X^{-1}\left(B_{1-\alpha/2}\right)\right],$$

where $B_{\alpha/2}$ and $B_{1-\alpha/2}$ are the quantiles of the Beta$(s, n-s+1)$ distribution. However, this interval does not incorporate the predictors and thus is not conditional on the observed data.

In both Theorems 2.1 and 2.2, the predictor for the future order statistic $X_{s:n}$ is of the form $\hat{X}_{s:n}^{(i)} = F_X^{-1}(p_s^{(i)})$, $i = 1, 2$, where $p_s^{(1)}$ and $p_s^{(2)}$ are functions of $F_X(X_{r:n})$ and depend on the censoring and the position of the target order statistic. Namely,

$$p_s^{(1)} = 1 - (1 - F_X(X_{r:n})) \exp\left(-\sum_{j=n-s+1}^{n-r} \frac{1}{j}\right)$$

and

$$p_s^{(2)} = F_X(X_{r:n}) + (1 - F_X(X_{r:n})) \times \frac{s-r}{n-r+1}.$$

Thus, $p_s^{(i)}$ is the estimated quantile level where the future order statistic $X_{s:n}$ is expected to lie, based on the available censored data. We note that for large enough s and $n-s+1$, the distribution $F_X(X_{s:n}) \sim$ Beta$(s, n-s+1)$ can be approximated by a normal distribution:

$$F_X(X_{s:n}) \approx \mathcal{N}\left(\mu_{\text{Beta}}, \sigma^2_{\text{Beta}}\right),$$

where $\mu_{\text{Beta}} = \frac{s}{n+1}$ and $\sigma^2_{\text{Beta}} = \frac{s(n-s+1)}{(n+1)^2(n+2)}$. Thus, we can approximate the confidence interval for $F_X(X_{s:n})$ as:

$$\left[p_s^{(i)} - z_{\alpha/2}\sigma_{\text{Beta}},\ p_s^{(i)} + z_{\alpha/2}\sigma_{\text{Beta}}\right],\ i = 1, 2,$$

where $z_{\alpha/2}$ is the standard normal quantile (e.g., $z_{0.025} \approx 1.96$) and σ_{Beta} is the standard deviation of the Beta distribution. Mapping this interval back to the original X-scale using the quantile function F_X^{-1}, we obtain the final (approximate) confidence interval:

$$\left[F_X^{-1}\left(p_s^{(i)} - z_{\alpha/2}\sigma_{\text{Beta}}\right),\ F_X^{-1}\left(p_s^{(i)} + z_{\alpha/2}\sigma_{\text{Beta}}\right)\right]. \tag{8}$$

Remark 3.1 (Bounds for $p_s^{(i)}$, $i = 1, 2$). *It is not hard to verify that $p_s^{(1)} \in [0, 1]$ and $p_s^{(2)} \in \left[\frac{s-r}{n-r+1}, 1\right]$. These bounds ensure that $p_s^{(i)}$ remains within valid probability limits for any continuous distribution F_X. The lower bound for $p_s^{(2)}$ reflects its dependence on the relative position of s within the remaining order statistics.*

Remark 3.2 (Conditions for large s and n in confidence interval approximation). *The confidence intervals for future order statistics rely on a normal approximation of the Beta distribution for $F_X(X_{s:n})$. The normal approximation is valid under the condition that n is sufficiently large (typically $n \geq 30$) to ensure the Beta distribution is well-approximated by a normal distribution. Moreover, the approximation works best when s is not too close to the boundaries (i.e., when both s and $n-s+1$ are large). A common rule of thumb is: $\min(s, n-s+1) \geq 5$ (ensures the Beta distribution is not too skewed). For example, if $n = 100$, s should satisfy $5 \leq s \leq 95$. The approximation deteriorates if s is too close to 1 or n (e.g., when predicting the very first or last order statistic).*

Based on the theoretical justification of this approximation and empirical evidence from a simulation study, we conclude that the approximation is reasonable if $0.1\,n \leq s \leq 0.9\,n$ (e.g., for $n = 100, 10 \leq s \leq 90$).

Example 3.1 (Normal distribution). *Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Furthermore, let $X_{i:n} = x_i$ $i = 1, 2, ..., r$ be the observed values. Our target is constructing a 95% confidence interval for $X_{s:n}$ $r < s$, using $p_s^{(2)}$. Thus, by Theorem 2.2, compute $p_s^{(2)}$ as*

$$p_s^{(2)} = F_X(x_r) + (1 - F_X(x_r)) \times \frac{s-r}{n-r+1},$$

where $F_X(x_r) = \Phi\left(\frac{x_r-\mu}{\sigma}\right)$ is the DF of the normal distribution. For large s and $n-s+1$, approximate $F_X(X_{s:n}) \sim$ Beta$(s, n-s+1)$ by a normal distribution, where

$$\mu_{Beta} = \frac{s}{n+1} \ \ \text{and} \ \ \sigma^2_{Beta} = \frac{s(n-s+1)}{(n+1)^2(n+2)}. \tag{9}$$

Therefore, the confidence interval for $F_X(X_{s:n})$ is given by

$$\left[p_s^{(2)} - z_{0.025}\sigma_{Beta},\ p_s^{(2)} + z_{0.025}\sigma_{Beta}\right],$$

where $z_{0.025} \approx 1.96$. Thus, map back to the X-scale, by using the inverse DF of the normal distribution, we get

$$\left[\mu + \sigma\Phi^{-1}(p_s^{(2)} - 1.96\sigma_{Beta}),\ \mu + \sigma\Phi^{-1}(p_s^{(2)} + 1.96\sigma_{Beta})\right].$$

By choosing $\mu = 0$, $\sigma = 1$, $n = 20$, $r = 10$, $s = 15$, and $z_r = 0.5$. Therefore, compute $p_s^{(2)}$ as

$$F_Z(0.5) = \Phi(0.5) \approx 0.6915, \quad p_s^{(2)} = 0.6915 + (1 - 0.6915)\frac{5}{11} \approx 0.8319.$$

The Beta approximation is given by

$$\sigma_{Beta} = \sqrt{\frac{15 \times 6}{21^2 \times 22}} \approx 0.077.$$

The confidence interval for $F_X(X_{15:20})$ is now given by

$$[0.8319 - 1.96 \times 0.077, 0.8319 + 1.96 \times 0.077] \approx [0.681, 0.983].$$

Finally, map to X-scale by using (8), we get the confidence interval for $X_{15:20}$ as

$$\left[\Phi^{-1}(0.681), \Phi^{-1}(0.983)\right] \approx [0.47, 2.12].$$

Example 3.2 (Weibull distribution). *Let $X \sim Weibull(k, \lambda)$, with DF $F_x(x) = 1 - e^{-(x/\lambda)^k}$. Furthermore, Let $X_{i:n} = x_i$, $i = 1, 2, ..., r$. Our target is constructing a 95% confidence interval for $X_{s:n}$ using $p_s^{(1)}$. Thus, by Theorem 2.1, compute $p_s^{(1)}$ as*

$$p_s^{(1)} = 1 - (1 - F_X(x_r)) \times \exp\left(-\sum_{j=n-s+1}^{n-r} \frac{1}{j}\right),$$

where $F_X(x_r) = 1 - e^{-(z_r/\lambda)^k}$. For large s and $n - s + 1$, approximate $F_X(X_{s:n}) \sim Beta(s, n - s + 1)$ by a normal distribution, where μ_{Beta} and σ^2_{Beta} are defined in (9. Thus, the confidence interval for $F_X(X_{s:n})$ is given by

$$\left[p_s^{(1)} - 1.96\sigma_{Beta},\ p_s^{(1)} + 1.96\sigma_{Beta}\right].$$

By mapping back to the X-scale, using the inverse DF of the Weibull distribution, we get

$$\left[\lambda(-\ln(1 - p_s^{(1)} + 1.96\sigma_{Beta}))^{1/k},\ \lambda(-\ln(1 - p_s^{(1)} - 1.96\sigma_{Beta}))^{1/k}\right].$$

By choosingk $= 2$, $\lambda = 1$, $n = 20$, $r = 10$, $s = 15$, and $z_r = 0.8$, we can compute $p_s^{(1)}$ as

$$F_X(0.8) = 1 - e^{-(0.8)^2} \approx 0.5273, \quad \sum_{j=6}^{10} \frac{1}{j} \approx 0.645.$$

$$p_s^{(1)} = 1 - (1 - 0.5273)e^{-0.645} \approx 0.754.$$

Also, the Beta approximation is given by $\sigma_{Beta} \approx 0.077$. Thus, the confidence interval for $F_X(X_{15:20})$ is

$$[0.754 - 1.96 \times 0.077, 0.754 + 1.96 \times 0.077] \approx [0.603, 0.905].$$

Finally, by mapping to X-scale, we get

$$\left[(-\ln(1 - 0.603))^{1/2}, (-\ln(1 - 0.905))^{1/2}\right] \approx [0.95, 1.52].$$

4. Apllications

Example 4.1 (Voltage stress data). *We use real data from El-Adll et al. (2012) to illustrate the effectiveness of our method. The data, originally provided by Lawless (2011, p. 189), consist of voltage levels at which failures occurred in a certain type of electrical cable insulation (Type 1 insulation). In this laboratory experiment, 20 specimens were subjected to progressively increasing voltage stress, and the failure voltages, measured in kilovolts per millimeter, are listed in Table 8.*

Table 8: Voltage stress data in a laboratory experiment

32.0	35.4	36.2	39.8	41.2	43.3	45.5	46.0	46.2	46.4
46.5	46.8	47.3	47.3	47.6	49.2	50.4	50.9	52.4	56.3

In Table 9, $\hat{x}_{s:n}$ and $\hat{x}^{(1)}_{s:n}$ denote the predictors of $x_{s:n}$ obtained using the pivotal quantity method (cf. El-Adll et al., 2012) and Theorem 2.2 of this work, respectively. Additionally, PI and $CI_{\alpha=0.05}$ represent the corresponding prediction intervals.

Table 9 compares the two prediction methods for the voltage stress data:

- ***Pivotal quantity method** (El-Adll et al., 2012)*
- ***Conditional expectation (exponential-Based) method***

Key findings:

- *The exponential-based predictor ($\hat{X}^{(1)}_{s:n}$) achieves a lower MSE (1.665 vs. 2.50), indicating improved accuracy.*
- *Its adaptive confidence intervals better capture the observed values, though they become unbounded for extreme order statistics ($s \geq 17$).*
- *The pivotal method provides fixed-width intervals but may exclude the true values in the tails.*

***Conclusion:** The exponential-based method offers enhanced point predictions, though inference for the extreme tails remains challenging. A hybrid approach that leverages both methods could potentially offer further improvements.*

Table 9: Specimens and the failure voltages initial Weibull: $a = 9.1973, \sigma = 47.7383$

Pivotal quantity				Conditional expectation(exp)			
s	$x_{s:n}$	$\hat{x}_{s:n}$	PI	s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$CI_{\alpha=0.05}$
10	46.40	47.32	(46.20, 48.44)	10	46.400	46.786	[43.66, 49.83]
11	46.50	47.93	(46.20, 49.67)	11	46.500	47.368	[44.33, 50.52]
12	46.80	48.46	(46.20, 50.72)	12	46.800	47.952	[45.00, 51.26]
13	47.30	48.95	(46.20, 51.71)	13	47.300	48.546	[45.67, 52.07]
14	47.30	49.43	(46.20, 52.66)	14	47.300	49.160	[46.35, 53.01]
15	47.60	49.91	(46.20, 53.62)	15	47.600	49.804	[47.05, 54.22]
16	49.20	50.41	(46.20, 54.62)	16	49.200	50.497	[47.78, 56.49]
17	50.40	50.95	(46.20, 55.69)	17	50.400	51.266	[48.57, NaN]
18	50.90	51.57	(46.20, 56.93)	18	50.900	52.163	[49.45, NaN]
19	52.40	52.35	(46.20, 58.50)	19	52.400	53.309	[50.52, NaN]
20	56.30	53.63	(46.20, 61.06)	20	56.300	55.132	[51.98, NaN]
MSE=2.50				MSE=1.665			

Example 4.2 (Employer sponsored health insurance). *This dataset contains information on ESI (Employer-Sponsored Insurance) coverage among private-sector workers in the USA from 1979 to 2019. It includes demographic breakdowns such as race, gender, education level, and recent graduation status.*

We focus on the data for women specifically, the percentage of female workers with ESI coverage-comprising 41 observations. We aim to predict the next 21 future observations. The dataset is publicly available at:

https://www.kaggle.com/datasets/asaniczka/health-insurance-coverage-in-the-usa-1979-2019/data

The data are well-fitted by the Weibull distribution with parameters (13.6, 55.4). *Table 8 compares the prediction performance of Theorems 2.1 and 2.2, along with the two corresponding prediction confidence intervals (given by Equation (8)). Table 10 compares the performance of two predictors* $\hat{X}^{(1)}_{s:n}$ *(exponential-based) and* $\hat{X}^{(2)}_{s:n}$ *(uniform-based)-for predicting future order statistics in the context of the ESI data for female workers. The predictions are made for 21 future observations based on 41 existing data points.*

Table 10: The percentage of female workers with ESI coverage, 41 observations, predicts 21 future observations

Theorem 2.1(exp)				Theorem 2.2 (uniform)			
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$CI_{\alpha=0.05}$	s	$x_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$CI_{\alpha=0.05}$
21	53.000	53.326	[51.34, 55.02]	21	53.000	53.319	[51.33, 55.02]
22	53.200	53.643	[51.74, 55.32]	22	53.200	53.629	[51.72, 55.30]
23	53.300	53.953	[52.12, 55.61]	23	53.300	53.933	[52.10, 55.59]
24	53.400	54.258	[52.49, 55.90]	24	53.400	54.231	[52.46, 55.87]
25	53.400	54.559	[52.85, 56.19]	25	53.400	54.525	[52.81, 56.16]
26	53.500	54.857	[53.20, 56.49]	26	53.500	54.817	[53.15, 56.44]
27	53.600	55.154	[53.54, 56.78]	27	53.600	55.108	[53.49, 56.73]
28	53.700	55.452	[53.88, 57.09]	28	53.700	55.398	[53.82, 57.03]
29	54.000	55.751	[54.22, 57.40]	29	54.000	55.690	[54.16, 57.33]
30	54.300	56.054	[54.55, 57.72]	30	54.300	55.985	[54.49, 57.63]
31	54.800	56.362	[54.89, 58.05]	31	54.800	56.286	[54.82, 57.96]
32	54.900	56.679	[55.23, 58.41]	32	54.900	56.593	[55.15, 58.29]
33	55.300	57.006	[55.58, 58.79]	33	55.300	56.909	[55.50, 58.65]
34	59.000	57.348	[55.94, 59.21]	34	59.000	57.239	[55.85, 59.04]
35	59.400	57.710	[56.32, 59.69]	35	59.400	57.586	[56.21, 59.47]
36	59.800	58.099	[56.71, 60.25]	36	59.800	57.956	[56.60, 59.96]
37	60.100	58.526	[57.14, 61.01]	37	60.100	58.359	[57.01, 60.57]
38	60.800	59.010	[57.61, 62.36]	38	60.800	58.808	[57.47, 61.40]
39	60.900	59.586	[58.15, NaN]	39	60.900	59.330	[57.98, 63.41]
40	61.100	60.337	[58.82, NaN]	40	61.100	59.980	[58.61, NaN]
41	61.300	61.558	[59.76, NaN]	41	61.300	60.919	[59.49, NaN]
MSE=1.968				MSE=2.074			

Key observations:

- **Mean squared error (MSE):**
  - The exponential-based predictor ($\hat{X}^{(1)}_{s:n}$) achieves a lower MSE (1.968) compared to the uniform-based predictor ($\hat{X}^{(2)}_{s:n}$) with an MSE of 2.074. This suggests that the exponential-based method provides slightly more accurate predictions for this dataset.
- **Confidence intervals (CIs):**
  - Both methods produce similar confidence intervals, but the exponential-based predictor tends to yield tighter intervals, especially for earlier predictions (e.g., $s = 21$ to $s = 30$). This indicates better precision in estimating future values.

- For extreme predictions (e.g., $s \geq 39$), the CIs for both methods become unbounded (indicated by "NaN"), highlighting the challenge of predicting far-tail observations.

- **Performance across predictions:**
  - The predictors perform comparably for mid-range observations, but the exponential-based method shows a marginal advantage in accuracy, as evidenced by the lower MSE.
  - The uniform-based predictor, while computationally simpler, exhibits slightly higher variability in predictions, particularly for larger values of s.

Conclusion: The results in Table 10 demonstrate that the exponential-based predictor ($\hat{X}^{(1)}_{s:n}$) is preferable for this dataset due to its lower MSE and tighter confidence intervals. However, the uniform-based predictor ($\hat{X}^{(2)}_{s:n}$) remains a viable alternative, especially when computational efficiency is a priority. The unbounded CIs for extreme predictions underscore the limitations of both methods in prediction far-tail events, suggesting a need for further refinement or hybrid approaches in such cases. This analysis aligns with the paper's broader findings, where the exponential-based predictor generally excels in accuracy, while the uniform-based predictor offers simplicity and computational ease.

Example 4.3 (Windscreen failures). *It is known that the exponential distribution has an important property, make it considered as one of the important classical distributions, addition to it is analytically tractable distribution. However, it has some limited applications, because of its fixed hazard rate and unimoda PDF. For that, several extensions of the exponential distribution were considered to increase its flexibility and applicability. One of these extensions is the Modified Kies-Exponential (MKE) distribution, see Al-Babtain et al. (2020) and Aly et al. (2023). Actually, MKE distribution has many applications in various fields such as reliability engineering: modeling time-to-failure of components or systems with non-constant failure rates, survival analysis: analyzing time until an event of interest occurs, accounting for different hazard functions, and queuing Theory. The PDF and DF of the MKE distribution are given by:*

$$g(x;a,b) = a\, b\, e^{-(e^{bx}-1)^a}\, e^{bx}\, (e^{bx}-1)^{a-1},$$

$$G(x;a,b) = 1 - e^{-(e^{bx}-1)^a}, \quad x > 0,$$

where, a is shape parameter and b is scale parameter. The QF is given by:

$$Q(x;a,b) = \frac{1}{b} \ln\left((-\ln(1-x))^{\frac{1}{a}} + 1\right).$$

Using a dataset of 84 observed failure times for a specific windscreen model (Aly et al., 2023), we applied our techniques to predict the next 50 failure events. This prediction window represents 150% of the final 34 data points. The results, presented in Table 11, were best modeled by an MKE distribution with parameters (1.783644, 0.2366933).
Comment: *Table 11 compares the exponential-based ($\hat{X}^{(1)}_{s:n}$) and uniform-based ($\hat{X}^{(2)}_{s:n}$) predictors for windscreen failure times modeled by the MKE distribution. Key observations include:*

- *Both predictors yield nearly identical MSEs (0.024 vs. 0.023), indicating comparable accuracy for this dataset.*
- *Confidence intervals (CIs) are tight for mid-range predictions but become unbounded (NaN) for extreme order statistics ($s \geq 81$), highlighting challenges in tail prediction.*
- *The results align with the paper's broader findings: the exponential-based method is robust, while the uniform-based alternative offers computational simplicity with minimal trade-offs in accuracy for this application.*

Table 11: At $r = 34$ prediction 150% of data (windscreen failures)

Theorem 2.1(exp)				Theorem 2.2 (uniform)			
s	$x_{s:n}$	$\hat{x}^{(1)}_{s:n}$	$CI_{\alpha=0.05}$	s	$x_{s:n}$	$\hat{x}^{(2)}_{s:n}$	$CI_{\alpha=0.05}$
35	2.154	2.174	[0.13, 0.19]	35	2.154	2.174	[0.13, 0.19]
36	2.190	2.213	[0.13, 0.19]	36	2.190	2.212	[0.13, 0.19]
37	2.194	2.252	[0.14, 0.20]	37	2.194	2.251	[0.14, 0.20]
38	2.223	2.290	[0.14, 0.20]	38	2.223	2.289	[0.14, 0.20]
39	2.224	2.328	[0.14, 0.21]	39	2.224	2.326	[0.14, 0.21]
40	2.229	2.366	[0.15, 0.21]	40	2.229	2.364	[0.15, 0.21]
41	2.300	2.404	[0.15, 0.21]	41	2.300	2.402	[0.15, 0.21]
42	2.324	2.442	[0.15, 0.22]	42	2.324	2.439	[0.15, 0.22]
43	2.385	2.480	[0.16, 0.22]	43	2.385	2.477	[0.16, 0.22]
44	2.481	2.518	[0.16, 0.23]	44	2.481	2.514	[0.16, 0.23]
45	2.610	2.556	[0.16, 0.23]	45	2.610	2.552	[0.16, 0.23]
46	2.625	2.594	[0.17, 0.24]	46	2.625	2.589	[0.17, 0.24]
47	2.632	2.632	[0.17, 0.24]	47	2.632	2.627	[0.17, 0.24]
48	2.646	2.670	[0.18, 0.24]	48	2.646	2.665	[0.18, 0.24]
49	2.661	2.708	[0.18, 0.25]	49	2.661	2.702	[0.18, 0.25]
50	2.688	2.746	[0.18, 0.25]	50	2.688	2.740	[0.18, 0.25]
51	2.823	2.785	[0.19, 0.26]	51	2.823	2.778	[0.19, 0.26]
52	2.890	2.823	[0.19, 0.26]	52	2.890	2.817	[0.19, 0.26]
53	2.902	2.862	[0.20, 0.27]	53	2.902	2.855	[0.19, 0.27]
54	2.934	2.902	[0.20, 0.27]	54	2.934	2.894	[0.20, 0.27]
55	2.962	2.941	[0.20, 0.28]	55	2.962	2.933	[0.20, 0.28]
56	2.964	2.981	[0.21, 0.28]	56	2.964	2.973	[0.21, 0.28]
57	3.000	3.022	[0.21, 0.29]	57	3.000	3.013	[0.21, 0.29]
58	3.103	3.063	[0.22, 0.29]	58	3.103	3.053	[0.22, 0.29]
59	3.114	3.104	[0.22, 0.30]	59	3.114	3.094	[0.22, 0.30]
60	3.117	3.146	[0.23, 0.31]	60	3.117	3.136	[0.22, 0.30]
61	3.166	3.189	[0.23, 0.31]	61	3.166	3.178	[0.23, 0.31]
62	3.344	3.233	[0.24, 0.32]	62	3.344	3.221	[0.23, 0.32]
63	3.376	3.277	[0.24, 0.32]	63	3.376	3.264	[0.24, 0.32]
64	3.443	3.322	[0.25, 0.33]	64	3.443	3.309	[0.24, 0.33]
65	3.467	3.369	[0.25, 0.34]	65	3.467	3.355	[0.25, 0.34]
66	3.478	3.416	[0.26, 0.35]	66	3.478	3.401	[0.25, 0.34]
67	3.578	3.465	[0.26, 0.35]	67	3.578	3.449	[0.26, 0.35]
68	3.595	3.515	[0.27, 0.36]	68	3.595	3.498	[0.27, 0.36]
69	3.699	3.567	[0.27, 0.37]	69	3.699	3.549	[0.27, 0.37]
70	3.779	3.620	[0.28, 0.38]	70	3.779	3.601	[0.28, 0.38]
71	3.924	3.676	[0.29, 0.39]	71	3.924	3.656	[0.28, 0.38]
72	4.035	3.734	[0.29, 0.40]	72	4.035	3.712	[0.29, 0.39]
73	4.121	3.795	[0.30, 0.41]	73	4.121	3.772	[0.30, 0.41]
74	4.167	3.859	[0.31, 0.42]	74	4.167	3.834	[0.31, 0.42]
75	4.240	3.927	[0.32, 0.44]	75	4.240	3.900	[0.31, 0.43]
76	4.255	4.000	[0.33, 0.46]	76	4.255	3.970	[0.32, 0.45]
77	4.278	4.079	[0.34, 0.48]	77	4.278	4.045	[0.33, 0.46]
78	4.305	4.165	[0.35, 0.50]	78	4.305	4.127	[0.34, 0.49]
79	4.376	4.261	[0.36, 0.54]	79	4.376	4.218	[0.35, 0.51]
80	4.449	4.369	[0.37, 0.59]	80	4.449	4.319	[0.37, 0.55]
81	4.485	4.497	[0.39, NaN]	81	4.485	4.437	[0.38, 0.62]
82	4.570	4.655	[0.41, NaN]	82	4.570	4.578	[0.40, NaN]
83	4.602	4.870	[0.44, NaN]	83	4.602	4.761	[0.43, NaN]
84	4.663	5.241	[0.48, NaN]	84	4.663	5.040	[0.47, NaN]
MSE=0.024				MSE=0.023			

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References

[1] A. A. Al-Babtain, M. K. Shakhatreh, M. Nassar, A. Z. Afify, *A new modified Kies family: Properties, estimation under complete and type-II censored samples, and engineering applications*, Math. **8**(8) (2020), 1345.
[2] A. E. Aly, M. E. El-Adll, H. M. Barakat, R. A. Aldallal, *A new least squares method for estimation and prediction based on the cumulative hazard function*, AIMS Math. **8**(9) (2023), 21968–21992.
[3] B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, *A First Course in Order Statistics*, Wiley, New York, 1992.
[4] N. Balakrishnan, A. C. Cohen, *Order Statistics and Inference: Estimation Methods*, Academic Press, San Diego, 1991.
[5] N. Balakrishnan, E. Beutner, E. Cramer, *Exact two-sample nonparametric confidence, prediction, and tolerance intervals based on ordinary and progressively Type-II right censored data*, Test **19** (2010), 68–91.
[6] H. M. Barakat, M. E. El-Adll, A. E. Aly, *Exact prediction intervals for future exponential lifetime based on random generalized order statistics*, J. Comput. Appl. Math. **61** (2011), 1366–1378.
[7] H. M. Barakat, M. E. El-Adll, A. E. Aly, *Prediction intervals of future observations for a sample of random size from any continuous distribution*, Math. Comput. Simul. **97** (2014), 1–13.
[8] H. M. Barakat, E. M. Nigm, M. E. El-Adll, M. Yusuf, *Prediction for future exponential lifetime based on random number of generalized order statistics under a general set-up*, Stat. Papers **59**(2) (2018), 605–631.
[9] H. M. Barakat, A. E. Aly, M. E. El-Adll, *Two-sample nonparametric prediction intervals based on random number of generalized order statistics*, Comm. Statist. Theory Methods **50**(19) (2021), 4571–4586.
[10] H. M. Barakat, O. M. Khaled, H. A. Ghonem, *Predicting future order statistics with random sample size*, AIMS Math. **6**(5) (2021), 5133–5147.
[11] H. M. Barakat, M. E. El-Adll, A. E. Aly, *Prediction of future generalized order statistics based on two-parameter exponential distribution for large samples*, Qual. Technol. Quant. Manag. **19**(2) (2022), 259–275.
[12] D. J. Bartholomew, *The sampling distribution of an estimate arising in life testing*, Technometrics **5** (1963), 361–372.
[13] I. Bayramoglu, *On the new properties of conditional expectations with applications in finance*, arXiv:2210.02859 [math.ST] (2022).
[14] G. K. Bhattacharyya, *Prediction in reliability and life testing*, Int. Stat. Rev. **53**(2) (1985), 177–195.
[15] P. Billingsley, *Probability and Measure*, Wiley, New York, 1995.
[16] H. A. David, H. N. Nagaraja, *Order Statistics*, (3rd ed.), Wiley, New York, 2003.
[17] P. Dellaportas, D. Wright, *Numerical prediction for the two-parameter Weibull distribution*, Statistician **40**(4) (1991), 365–372.
[18] R. Durrett, *Probability: Theory and Examples*, Cambridge Univ. Press, Cambridge, 2019.
[19] M. E. El-Adll, A. E. Aly, *Prediction intervals of future generalized order statistics from Pareto distribution*, J. Appl. Stat. Sci. **22**(1/2) (2014), 111–125.
[20] M. E. El-Adll, A. E. Aly, *Prediction intervals for future observations of Pareto distribution based on generalized order statistics*, J. Appl. Stat. Sci. **22**(1–2) (2016), 111–125.
[21] M. E. El-Adll, S. F. Ateya, M. M. Rizk, *Prediction intervals for future lifetime of three-parameter Weibull observations based on generalized order statistics*, Arab J. Math. **1**(3) (2012), 295–304.
[22] H. K. Hsieh, *Prediction interval for Weibull observation based on early-failure data*, IEEE Trans. Reliab. **45**(4) (1996), 666–670.
[23] K. S. Kaminsky, L. S. Rhodin, *Maximum likelihood prediction*, Ann. Inst. Stat. Math. **37** (1985), 507–517.
[24] O. M. Khaled, K. S. Khalil, M. H. Harpy, *Predicting future data from gamma-mixture and beta-mixture distributions and application to the recovery rate of COVID-19*, Adv. Appl. Stat. **90**(1) (2023), 1–34.
[25] J. F. Lawless, *Prediction intervals for the two-parameter exponential distribution*, Technometrics **19**(4) (1977), 469–472.
[26] J. F. Lawless, *Statistical Models and Methods for Lifetime Data*, Wiley, New York, 2011.
[27] B. Long, Z. Jiang, *Estimation and prediction for two-parameter Pareto distribution based on progressively double Type-II hybrid censored data*, AIMS Math. **8**(7) (2023), 15332–15351.
[28] H. N. Nagaraja, *Prediction problems: Theory, methods and applications*, in: C. R. Rao (Ed.), *Handbook of Statistics*, Vol. 13, Gordon and Breach, New York, 1995.
[29] J. K. Patel, *Prediction intervals review*, Comm. Statist. Theory Methods **18**(7) (1989), 2393–2465.
[30] M. Z. Raqab, H. M. Barakat, *Prediction intervals for future observations based on samples of random sizes*, J. Math. Stat. **14**(1) (2018), 16–28.
[31] M. Z. Raqab, H. N. Nagaraja, *On some predictions of future order statistic*, Metron **53**(1) (1995), 185–204.
[32] I. A. Shah, H. M. Barakat, A. H. Khan, *Characterizations through generalized and dual generalized order statistics, with an application to statistical prediction problem*, Stat. Probab. Lett. **163** (2020), 108782.
[33] V. K. Sharma, B. N. Pandey, *Bayesian prediction from censored samples using exponential and Weibull models*, J. Stat. Comput. Simul. **77**(1) (2007), 55–67.
[34] R. Valiollahi, A. Asgharzadeh, D. Kundu, *Prediction of future failures for generalized exponential distribution under Type-I or Type-II hybrid censoring*, Braz. J. Probab. Stat. **31**(1) (2017), 41–61.
[35] S. J. Wu, C. C. Li, *Predictive inference for future order statistics under Type II censoring*, J. Stat. Plann. Inference **141**(6) (2011), 2125–2137.