Filomat 40:1 (2026), 355–370
https://doi.org/10.2298/FIL2601355D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ideal convergence of quantum difference sequences of bi-complex numbers

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Abstract. We investigate different properties of ideal convergent second-order quantum difference sequence spaces over bi-complex numbers. We define $Z[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ for $Z \in \{\mathcal{I}_c^*, \mathcal{I}_\theta^*, \mathcal{I}_1^*, \mathcal{I}_p^*, \mathcal{I}_\infty^*\}$, using the second order q- difference operator ∇_q^2 under the Euclidean norm. We examine their BK-space structure, symmetric property, inclusion relations, and isomorphisms with classical $\mathcal{I}-$ convergent bi-complex sequence spaces. A matrix representation of the operator is given, along with counterexamples, to demonstrate contradictions in specific inclusion cases.

1. Introduction and Priliminaries

In 1892, Segre [15] introduced *bi-complex numbers*, inspired by the works of Hamilton [7] and Clifford [4]. He studied a family of algebras, naming them *bi-complex*, *tri-complex*, and so on. A bi-complex number is of the form

$$(x_1 + i_1x_2) + i_2(x_3 + i_1x_4),$$

or equivalently

$$z_1 + i_2z_2,$$

where $i_1^2 = i_2^2 = -1$, and $z_1 = x_1 + i_1x_2$, $z_2 = x_3 + i_1x_4$ are complex numbers. Here, i_1 and i_2 are commuting imaginary units, and their product is $j = i_1i_2$.

The set of all bi-complex numbers is denoted $\mathbb{C}_2$. In 1991, Price [13] provided foundational work, followed by studies from Srivastava and Srivastava [16], Wagh [17], Rochan and Shapiro [14], Kumar and Tripathy [10–12], and Bera and Tripathy [1–3] on sequences of bi-complex numbers.

Each bi-complex number $\xi \in \mathbb{C}_2$ can be written as

$$\xi = \mu_1e_1 + \mu_2e_2,$$

2020 *Mathematics Subject Classification*. Primary 40A35, 39A13; Secondary 30G35, 40A05, 40A30, 40G15, 46A45.
Keywords. q-difference operator; $I-$convergence; bi-complex numbers; BK- space.
Received: 15 April 2025; Revised: 07 June 2025; Accepted: 27 June 2025
Communicated by Miodrag Spalević
Research supported by DST/INSPIRE Fellowship [IF220239]
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where the idempotents $e_1 = \frac{1+j}{2}$ and $e_2 = \frac{1-j}{2}$ satisfy $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1e_2 = 0$. The set $\mathbb{C}_2$ can be decomposed as

$$\mathbb{C}_2 = e_1A_1(i_1) + e_2A_2(i_1) = e_1\{z_1 - i_1z_2 : z_1, z_2 \in \mathbb{C}_1\} + e_2\{z_1 + i_1z_2 : z_1, z_2 \in \mathbb{C}_1\}.$$

The Euclidean norm on $\mathbb{C}_2$ is

$$\|\xi\|_{\mathbb{C}_2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

where $\xi = (x_1 + i_1x_2) + i_2(x_3 + i_1x_4)$.

A bi-complex number $\xi = z_1 + i_2z_2$ is *hyperbolic* if the imaginary part of z_1 and the real part of z_2 are zero. It is *singular* (or *non-invertible*) if $\|z_1^2 + z_2^2\|_{\mathbb{C}_2} = 0$; otherwise, it is *non-singular* (or *invertible*). The set of singular bi-complex numbers is O_2.

Bi-complex numbers have three types of conjugations:

- i_1*-conjugation* ($i_1 \mapsto -i_1$),
- i_2*-conjugation* ($i_2 \mapsto -i_2$),
- i_1i_2*-conjugation* ($i_1 \mapsto -i_1$ and $i_2 \mapsto -i_2$).

1.1. Sequence space

The collection of all sequences with real or complex values is represented by ω, while the set of all sequences with bi-complex values is denoted by ω^*. The spaces of absolutely summable, p-summable, bounded sequences, convergent sequences, and null sequences are denoted by $\ell_1, \ell_p, \ell_\infty, c,$ and c_0, respectively. For bi-complex valued sequence spaces, these are represented as $\ell_1^*, \ell_p^*, \ell_\infty^*, c^*,$ and c_0^*, respectively.

Definition 1.1. *A sequence space E is said to be $BK-$ space (Banach coordinate maps continuous) if $\|x^n - x\| \to 0$, as $n \to \infty$ implies $|x_k^n - x_k| \to 0$, as $n \to \infty$, where $x^n = (x_k^n)$ for each $n \in \mathbb{N}$ and $x = (x_k)$.*

Definition 1.2. *Let $A = (a_{n,k})$ be an infinite matrix with real or complex entries, and let $A_n = (a_{n,k})_{k\in\mathbb{N}_0}$. The A-transform of a sequence $z = (z_k)$ is given by the sequence $Az = \{(Az)_n\}$, where*

$$(Az)_n = \sum_{k=0}^{\infty} a_{n,k}z_k$$

provided that the series $\sum_{k=0}^{\infty} a_{n,k}z_k$ converges for each $n \in \mathbb{N}_0$.

Furthermore, if Z and U are sequence spaces and $Az \in U$ for every sequence $z \in Z$, then the matrix A is said to define a matrix mapping from Z to U. The notation (Z, U) denotes the family of all matrices that map sequences from Z to U. A triangular matrix $A = (a_{n,k})$ satisfies the conditions $a_{n,n} \neq 0$ and $a_{n,k} = 0$ for $n < k$. The domain of the matrix A in the sequence space Z, denoted by Z_A, is defined as:

$$Z_A = \{z \in \omega : Az \in Z\}.$$

Additionally, if Z is a BK-space and A is a triangular matrix, then the matrix domain $Z_A = \{z \in \omega : Az \in Z\}$ is also a BK-space when equipped with the norm

$$\|z\|_{Z_A} = \|Az\|_Z.$$

This ensures that the transformed space inherits the normed structure from Z.

Definition 1.3. *A sequence space $E \subsetneq w$ is said to be solid if for every sequence $x = (x_k) \in E$ and every sequence $y = (y_k) \in w$ satisfying $|y_k| \leq |x_k|$ for all $k \in \mathbb{N}$, it follows that $y \in E$.*

Definition 1.4. *A sequence space E is said to be symmetric if* $(x_k) \in E$ *implies* $(x_{\pi(k)}) \in E$*, where* π *is a permutation of* $\mathbb{N}$.

Definition 1.5. *Let* $K = \{k_1 < k_2 < \cdots\} \subset \mathbb{N}$ *and let E be a sequence space. A K-*step space *of E is the sequence space*

$$\lambda_K^E = \{(x_{k_n}) \in w : (x_n) \in E\}.$$

Definition 1.6. *A* canonical preimage *of a sequence* $(x_{k_n}) \in \lambda_K^E$ *is a sequence* $(y_n) \in w$ *defined by*

$$y_n = \begin{cases} x_n, & \text{if } n \in K, \\ 0, & \text{otherwise}. \end{cases}$$

A canonical preimage *of a step space* λ_K^E *is the set of all canonical preimages of elements in* λ_K^E*. That is,* $y \in w$ *belongs to the canonical preimage of* λ_K^E *if and only if y is the canonical preimage of some* $x \in \lambda_K^E$.

Definition 1.7. *A sequence space* $E \subseteq w$ *is said to be monotone if it contains the canonical preimages of its step spaces.*

Definition 1.8. *A sequence space E is said to be a sequence algebra if* $(x_k) * (y_k) = (x_k y_k) \in E$ *whenever* $(x_k), (y_k) \in E$.

Definition 1.9. *A sequence space E is said to be convergence free if* $(y_k) \in E$ *whenever* $(x_k) \in E$ *and* $x_k = 0$ *implies* $y_k = 0$.

1.2. q-Sequence Space

A q-integer is defined as

$$[a]_q = \begin{cases} \sum_{k=0}^{a-1} q^k, & \text{for } a = 1, 2, 3, \ldots \\ 0, & \text{for } a = 0. \end{cases}$$

As $q \to 1^-$, we have $[a]_q = a$.

The q-analog of the binomial coefficient is given by

$$\binom{a}{b}_q = \begin{cases} \frac{[a]_q!}{[a-b]_q![b]_q!}, & \text{for } a \geq b, \\ 0, & \text{for } b > a. \end{cases}$$

where $[a]_q!$ is defined as

$$[a]_q! = \begin{cases} \prod_{k=0}^{a} [k]_q, & \text{for } a = 1, 2, 3, \ldots, \\ 1, & \text{for } a = 0. \end{cases}$$

Additionally, the following identities hold:

$$\binom{0}{0}_q = \binom{a}{0}_q = \binom{a}{a}_q = 1, \quad \binom{a}{a-b}_q = \binom{a}{b}_q$$

For further details on q-calculus, see [6, 8, 9]. Yaying et al. [18, 19] defined the difference operator $\nabla_q^2 : \omega \to \omega$ by

$$(\nabla_q^2 x)_k = x_k - (1+q)x_{k-1} + q x_{k-2},$$

where $k \in \mathbb{N}$ and $x_k = 0$ for $k < 0$.

1.3. Ideal convergence of bi-complex numbers

Definition 1.10. *Let X be a non empty set. A collection $\mathcal{I} \subseteq 2^X$ is said to be an ideal in X if and only if for all $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$ if $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$.*

Definition 1.11. *$\mathcal{I}$ is said to be an admissible ideal if $\forall x \in X, \{x\} \in \mathcal{I}$.*

Definition 1.12. *$\mathcal{I}$ is said to be a non trivial ideal if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$.*

Definition 1.13. *A sequence of bi-complex numbers (ξ_k) is said to be $\mathcal{I}-$convergent to η if and only if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : \|\xi_k - \eta\|_{\mathbb{C}_2} \geq \varepsilon\} \in \mathcal{I}$.*

The bi-complex number ζ is $\mathcal{I}-$limit of (ξ_k) and denoted by $\mathcal{I} - \lim(\xi_k) = \zeta$. $\mathcal{I}^*_1, \mathcal{I}^*_p, \mathcal{I}^*_\infty, \mathcal{I}^*_c$, and $\mathcal{I}^*_\theta$ are denoted by $\mathcal{I}-$ absolutely summable, $\mathcal{I} - p$ summable, $\mathcal{I}-$ bounded, $\mathcal{I}-$convergent, and $\mathcal{I}-$null sequences of bi-complex numbers, respectively. For a detailed discussion one may refer to [5].

2. Main Results

Metric represantation : For the subsequent results, all ideals considered are nontrivial and admissible. Define the difference operator $\nabla^2_q : \omega^* \to \omega^*$ by

$$(\nabla^2_q \xi)_k = \xi_k - (1+q)\xi_{k-1} + q\xi_{k-2},$$

where $k \in \mathbb{N}$ and any term of the sequence with negative indices are assumed to be zero. The operator $\nabla^2_q = (\delta^2_{q_{nk}})$ can also be expressed in the form of a triangle matrix as follows:

$$\delta^2_{q_{nk}} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -(1+q) & 1 & 0 & 0 & \cdots \\ q & -(1+q) & 1 & 0 & \cdots \\ 0 & q & -(1+q) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We observe that the operator ∇^2_q reduces to ∇^2 when $q \to 1^-$. Using some elementary calculation, we derive the inverse of the operator ∇^2_q as

$$\nabla^{-2}_q = \begin{cases} \binom{n-k+1}{n-k}_q & , \text{ if } 0 < k \leq n, \\ 0 & , k > n. \end{cases}$$

Now, We define the following ideal convergent quantum difference sequences of spaces of bi-complex numbers as $\mathcal{I}^*_1[, \nabla^m_q, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}^*_p[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}]$,$\mathcal{I}^*_\infty[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}^*_c[\nabla^\alpha_q, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}^*_0[\nabla^\alpha_q, \|\cdot\|_{\mathbb{C}_2}]$ as $\mathcal{I}-$ absolutely summable, $\mathcal{I}-$p-summable, $\mathcal{I}-$ bounded, $\mathcal{I}-$ convergent, and $\mathcal{I}-$ null q-difference sequences of bi-complex numbers using q-difference operator ∇^m_q.

These sequence spaces are defined as:

$$\mathcal{I}^*_1[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}] = \{\xi = (\xi_k) \in \omega^* : \sum_{k \in K \subseteq \mathcal{F}(\mathcal{I})} \|\nabla^m_q \xi_k\|_{\mathbb{C}_2} < \infty\}$$

$$\mathcal{I}^*_p[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}] = \{\xi = (\xi_k) \in \omega^* : \sum_{k \in K \subseteq \mathcal{F}(\mathcal{I})} \|\nabla^m_q \xi_k\|^p_{\mathbb{C}_2} < \infty\}$$

$$\mathcal{I}^*_\infty[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}] = \{\xi = (\xi_k) \in \omega^* : \sup_{k \in K \subseteq \mathcal{F}(\mathcal{I})} \|\nabla^m_q \xi_k\|_{\mathbb{C}_2} < \infty\},$$

$$\mathcal{I}^*_c[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}] = \{\xi = (\xi_k) \in \omega^* : \{k \in \mathbb{N} : \|\nabla^m_q \xi_k - \zeta\|_{\mathbb{C}_2} \geq \varepsilon\} \in \mathcal{I}\}, \text{ and}$$

$$\mathcal{I}^*_\theta[\nabla^m_q, \|\cdot\|_{\mathbb{C}_2}] = \{\xi = (\xi_k) \in \omega^* : \{k \in \mathbb{N} : \|\nabla^m_q \xi_k\|_{\mathbb{C}_2} \geq \varepsilon\} \in \mathcal{I}\}.$$

Theorem 2.1. *The spaces $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ are sequence spaces.*

Proof. Let (ξ_k) and (η_k) be two sequences in $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$. Then:

$$\nabla_q^2 \xi_k = \xi_k - (1+q)\xi_{k-1} + q\xi_{k-2} \text{ is is } \mathcal{I}-\text{convergent to } \zeta_1,$$

$$\nabla_q^2 \eta_k = \eta_k - (1+q)\eta_{k-1} + q\eta_{k-2} \text{ is is } \mathcal{I}-\text{convergent to } \zeta_2,$$

which means, for every $\varepsilon > 0$,

$$\left\{k \in \mathbb{N} : \left\|\nabla_q^2 \xi_k - \zeta_1\right\|_{\mathbb{C}_2} \geq \varepsilon\right\} \in \mathcal{I}, \quad \left\{k \in \mathbb{N} : \left\|\nabla_q^2 \eta_k - \zeta_2\right\|_{\mathbb{C}_2} \geq \varepsilon\right\} \in \mathcal{I}.$$

Let $\alpha, \beta \in \mathbb{C}_2$, and define a new sequence (μ_k) by:

$$\mu_k = \alpha\xi_k + \beta\eta_k.$$

Then the generalized second-order q-backward difference of (η_k) is given by:

$$\begin{aligned}
\nabla_q^2 \mu_k &= \mu_k - (1+q)\mu_{k-1} + q\mu_{k-2}, \\
&= \alpha\xi_k + \beta\eta_k - (1+q)(\alpha\xi_{k-1} + \beta\eta_{k-1}) + q(\alpha\xi_{k-2} + \beta\eta_{k-2}), \\
&= \alpha(\xi_k - (1+q)\xi_{k-1} + q\xi_{k-2}) + \beta(\eta_k - (1+q)\eta_{k-1} + q\eta_{k-2}), \\
&= \alpha\nabla_q^2 \xi_k + \beta\nabla_q^2 \eta_k.
\end{aligned}$$

Since $(\nabla_q^2 \xi_k)$ is $\mathcal{I}-$convergent to ζ_1 and $(\nabla_q^2 \eta_k)$ is $\mathcal{I}-$convergent to ζ_2, it follows that:

$$(\nabla_q^2 \mu_k) \text{ is } \mathcal{I}-\text{convergent to } \alpha\zeta_1 + \beta\zeta_2,$$

that is, for every $\varepsilon > 0$,

$$\left\{k \in \mathbb{N} : \left\|\nabla_q^2 \eta_k - (\alpha\zeta_1 + \beta\zeta_2)\right\|_{\mathbb{C}_2} \geq \varepsilon\right\} \in \mathcal{I}.$$

Hence, $(\mu_k) \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

Therefore, the class $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is closed under addition and scalar multiplication. As it also contains the zero sequence, it forms a sequence space over complex numbers.
A comparable approach can be used to demonstrate the result for the spaces $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$. □

Theorem 2.2. *The space $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, endowed with the $\mathcal{I}-$sup norm, is a Banach space, where the $\mathcal{I}-$sup norm is*

$$\|\xi\|_{\mathcal{I}_\infty^*} = \sup\left\{\|\xi_k\|_{\mathbb{C}_2} : k \notin A, \, A \in \mathcal{I}\right\}.$$

Proof. We aim to prove that $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is a Banach space.

It is easy to verify that $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is an ideal normed linear space, because the space is closed under addition and scalar multiplication. The $\mathcal{I}-$ sup norm is well-defined and satisfies the properties of a norm. Thus, $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is an ideal normed linear space.
Let $(\xi^{(n)})$ be an $\mathcal{I}$-Cauchy sequence in $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$. Then for every $\varepsilon > 0$, there exists $m(\varepsilon)$ such that

$$\left\{n \in \mathbb{N} : \|\xi_k^{(n)} - \xi_k^{(m)}\|_{\mathbb{C}_2} \geq \varepsilon\right\} \in \mathcal{I}.$$

Thus, for each fixed $k \in \mathbb{N}$, $(\xi_k^{(n)})_n$ is a Cauchy sequence in $\mathbb{C}_2$, and hence converges to some $\xi_k \in \mathbb{C}_2$.
Define $\xi = (\xi_k)$ as the pointwise $\mathcal{I}$-limit. We now show that $\xi \in \mathcal{I}^*_\infty[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, i.e., $(\nabla_q^2 \xi_k)$ is $\mathcal{I}$-bounded.
From linearity and continuity:

$$\nabla_q^2 \xi_k = \lim_{n\to\infty, n\in K\in F(K)} \nabla_q^2 \xi_k^{(n)}.$$

Since each $(\nabla_q^2 \xi_k^{(n)})$ is $\mathcal{I}$-bounded and the limit of $\mathcal{I}$-bounded sequences is also $\mathcal{I}$-bounded, it follows that $(\nabla_q^2 \xi_k)$ is $\mathcal{I}$-bounded.
Hence, $\xi \in \mathcal{I}^*_\infty[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and the limit lies in the space.
Hence, the space $\mathcal{I}^*_\infty[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, equipped with the $\mathcal{I}-$ sup norm $\|\cdot\|_{\mathcal{I}^*_\infty}$, is a Banach space. □

Theorem 2.3. *The spaces $\mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}^*_\theta[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, endowed with the $\mathcal{I}-$ sup norm, are Banach spaces.*

Proof. We aim to show that each of the spaces $\mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ and $\mathcal{I}^*_\theta[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, endowed with the $\mathcal{I}$-supremum norm, are Banach space.
First, note that both spaces are normed linear spaces. The second-order q-difference operator ∇_q^2 is linear, and the $\mathcal{I}$-supremum norm

$$\|\xi\|_{\mathcal{I}^*_\infty} = \sup\{\|\xi_k\|_{\mathbb{C}_2} : k \notin A,\ A \in \mathcal{I}\}.$$

Let $(\xi^{(n)})$ be an $\mathcal{I}$-Cauchy sequence in either space. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$\left\{k \in \mathbb{N} : \|\xi_k^{(n)} - \xi_k^{(m)}\|_{\mathbb{C}_2} \geq \varepsilon\right\} \in \mathcal{I}.$$

This implies that for each fixed $k \in \mathbb{N}$, the sequence $(\xi_k^{(n)})$ is Cauchy in $\mathbb{C}_2$, which is complete. Hence, there exists $\xi_k \in \mathbb{C}_2$ such that $(\xi_k^{(n)}) \to \xi_k$. Define $\xi = (\xi_k)$.
Since ∇_q^2 is linear and continuous, we may write

$$\nabla_q^2 \xi_k = \lim_{n\to\infty} \nabla_q^2 \xi_k^{(n)}.$$

For each n, the sequence $(\nabla_q^2 \xi_k^{(n)})$ is $\mathcal{I}$-convergent to some $\zeta_n \in \mathbb{C}_2$. Furthermore, since the sequence $(\xi^{(n)})$ is $\mathcal{I}$-Cauchy, the sequence $(\nabla_q^2 \xi_k^{(n)})$ is also $\mathcal{I}$-Cauchy in norm, and hence converges $\mathcal{I}$-uniformly to $\nabla_q^2 \xi_k$. Therefore, by the preservation of $\mathcal{I}$-limits, we conclude that $(\nabla_q^2 \xi_k)$ is $\mathcal{I}$-convergent to some $\zeta \in \mathbb{C}_2$.
Thus, $\xi \in \mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$. An analogous argument applies for $\mathcal{I}^*_\theta$, where the limit is $\zeta = 0$.
Finally, for every $\varepsilon > 0$, and for all $n \geq N$, we have

$$\left\{k \in \mathbb{N} : \|\xi_k^{(n)} - \xi_k\|_{\mathbb{C}_2} \geq \varepsilon\right\} \in \mathcal{I},$$

so $\|\xi^{(n)} - \xi\|_{\mathcal{I}^*_\infty} < \varepsilon$, showing that $\xi^{(n)} \to \xi$ in the norm.
Hence, the spaces are complete under the $\mathcal{I}$-sup norm, and thus are Banach spaces. □

Remark 2.4. *The spaces $\mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}^*_\theta[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}^*_\infty[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ are not sequence algebras.*

The following examples illustrate the above remark.

Example 2.5. *Consider two bi-complex sequences (ξ_k) and (η_k) in $\mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ such that their product sequence does not belong to $\mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, proving that $\mathcal{I}^*_c[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is not closed under multiplication.*
We define the sequences as follows:

$$\xi_k = i_1 k + j,$$

$$\eta_k = 1 + \frac{(-1)^k}{k} i_2.$$

where i_1, i_2 are imaginary units satisfying $i_1^2 = i_2^2 = -1$ and $j = i_1 i_2$ with $j^2 = 1$.
For ξ_k,

$$\begin{aligned}\nabla^2 \xi_k &= \xi_k - 2\xi_{k-1} + \xi_{k-2}\\ &= (i_1 k + j) - 2(i_1(k-1) + j) + (i_1(k-2) + j)\\ &= i_1 k + j - 2i_1 k + 2i_1 + 2j + i_1 k - 2i_1 + j\\ &= 0.\end{aligned}$$

Therefore, $\{k \in \mathbb{N} : \|\nabla^2 \xi_k - 0\|_{\mathbb{C}_2} \geq \varepsilon\} \in \mathcal{I}_c^*$. Thus, $(\xi_k) \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.
For η_k,

$$\begin{aligned}\nabla^2 \eta_k &= \eta_k - 2\eta_{k-1} + \eta_{k-2}\\ &= \left(1 + \frac{(-1)^k}{k} i_2\right) - 2\left(1 + \frac{(-1)^{k-1}}{k-1} i_2\right) + \left(1 + \frac{(-1)^{n-2}}{k-2} i_2\right)\\ &= 0 + \frac{(-1)^k}{k} i_2 - \frac{2(-1)^{k-1}}{n-1} i_2 + \frac{(-1)^{n-2}}{n-2} i_2.\end{aligned}$$

Since the terms $\frac{(-1)^k}{k} \to 0$ as, $k \to \infty$, $\{k \in \mathbb{N} : \|\nabla^2 \xi_k - 0\|_{\mathbb{C}_2} \geq \varepsilon\} \in \mathcal{I}_c^*$. We conclude $(\eta_k) \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.
On computing $(\mu_k) = (\xi_k \eta_k)$, we get:

$$\begin{aligned}\mu_k &= (i_1 k + j)\left(1 + \frac{(-1)^k}{k} i_2\right)\\ &= i_1 k + j + i_1 k \cdot \frac{(-1)^k}{k} i_2 + j \cdot \frac{(-1)^k}{k} i_2.\end{aligned}$$

By using $i_1 i_2 = j$, we get:

$$\mu_k = i_1 k + j + j\frac{(-1)^k}{k} + j\frac{(-1)^k}{k} i_2.$$

On simplifying we get:

$$\mu_k = i_1 k + j\left(1 + \frac{(-1)^k}{k} + \frac{(-1)^k}{k} i_2\right).$$

Now, computing its second-order backward difference for q=1:

$$\begin{aligned}\nabla^2 \mu_k &= \mu_k - 2\mu_{k-1} + \mu_{k-2}\\ &= \left(i_1 k + j(1 + \frac{(-1)^k}{k} + \frac{(-1)^k}{k} i_2)\right)\\ &\quad - 2\left(i_1(k-1) + j(1 + \frac{(-1)^{k-1}}{k-1} + \frac{(-1)^{k-1}}{k-1} i_2)\right)\\ &\quad + \left(i_1(k-2) + j(1 + \frac{(-1)^{k-2}}{k-2} + \frac{(-1)^{k-2}}{k-2} i_2)\right).\end{aligned}$$

On expanding we get:

$$\nabla^2 \mu_k = 0 + j\left(\frac{(-1)^k}{k} - \frac{2(-1)^{k-1}}{k-1} + \frac{(-1)^{k-2}}{k-2}\right) \quad + ji_2\left(\frac{(-1)^k}{k} - \frac{2(-1)^{k-1}}{k-1} + \frac{(-1)^{k-2}}{k-2}\right).$$

Since the expression inside the brackets oscillates, $(\nabla^2 \mu_k)$ does not converge.
Thus, $(\mu_k) \notin \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, proving that $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is not a sequence algebra.

Example 2.6. *We define the bi-complex sequences* (ξ_k) *and* (η_k) *as followes:*

$$\xi_k = \frac{i_1 k}{k+1} + \frac{j}{k+1}, \ \eta_k = \frac{1}{k+1} + \frac{(-1)^k}{k+1} i_2.$$

It can be easily verified that these are null sequences.

The ***second-order q-difference for*** q***=1,*** *is defined as:*

$$\nabla^2 \xi_k = \xi_k - 2\xi_{k-1} + \xi_{k-2}.$$

On expanding we get:

$$\nabla^2 \xi_k = \left(\frac{i_1 k}{k+1} + \frac{j}{k+1}\right) - 2\left(\frac{i_1(k-1)}{k} + \frac{j}{k}\right) + \left(\frac{i_1(k-2)}{k-1} + \frac{j}{k-1}\right).$$

For large k, it can be shown that:

$$\nabla^2 \xi_k \to 0.$$

Then, $(\nabla^2 \xi_k) \in \mathcal{I}^*_\theta$*, as every convergent sequence is* $\mathcal{I}-$*convergent. Thus,* $(\xi_k) \in \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$.

Similarly, for η_k*, we compute:*

$$\nabla^2 \eta_k = \eta_k - 2\eta_{k-1} + \eta_{k-2}.$$

we have

$$\nabla^2 \eta_k \to 0 \quad \text{as } k \to \infty.$$

Thus, $(\eta_k) \in \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$.

Now,

$$\zeta_k = \xi_k \eta_k = \left(\frac{i_1 k}{k+1} + \frac{j}{k+1}\right)\left(\frac{1}{k+1} + \frac{(-1)^k}{k+1} i_2\right).$$
$$= \frac{i_1 k}{(k+1)^2} + \frac{(-1)^k jk}{(k+1)^2} + \frac{j}{(k+1)^2} + \frac{(-1)^k j i_2}{(k+1)^2}.$$

Since, $i_1 i_2 = j$*, we rewrite the above expression as follows* :

$$\zeta_k = \frac{i_1 k}{(k+1)^2} + j\left(\frac{(-1)^k k}{(k+1)^2} + \frac{1}{(k+1)^2} + \frac{(-1)^k i_2}{(k+1)^2}\right).$$

On computing $\nabla^2 \zeta_k$*, we get:*

$$\nabla^2 \zeta_k = \zeta_k - 2\zeta_{k-1} + \zeta_{k-2}.$$

For large k, the dominant terms oscillate rather than tending to zero, To observe the oscillation, we present the values for different k.

Observations

- *The sign of* $(-1)^k$ *in the terms involving j alternates as k increases.*
- *This sign change affects the values of* (ζ_k)*, and more importantly, the second-order* $q-$*difference* $(\nabla^2 \zeta_k)$.
- *Since the dominant terms retain alternating signs,* $(\nabla^2 \zeta_k)$ *does not tend to zero.*
- *The sequence* (ζ_k) *oscillates in the imaginary unit components and fails to belong to* $\mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$.

k	ζ_k	ζ_{k-1}	$\nabla^2\zeta_k$	$\nabla^2\zeta_{k-1}$
1	$\frac{i_1}{4}+\frac{j}{4}+(-1)^1\frac{j}{4}+(-1)^1\frac{i_2 j}{4}$	$\frac{i_1}{9}+\frac{j}{9}+(-1)^0\frac{j}{9}+(-1)^0\frac{i_2 j}{9}$	$\nabla^2\zeta_1$	$\nabla^2\zeta_0$
2	$\frac{2i_1}{9}+\frac{j}{9}+(-1)^2\frac{j}{9}+(-1)^2\frac{i_2 j}{9}$	$\frac{i_1}{16}+\frac{j}{16}+(-1)^1\frac{j}{16}+(-1)^1\frac{i_2 j}{16}$	$\nabla^2\zeta_2$	$\nabla^2\zeta_1$
3	$\frac{3i_1}{16}+\frac{j}{16}+(-1)^3\frac{j}{16}+(-1)^3\frac{i_2 j}{16}$	$\frac{2i_1}{25}+\frac{j}{25}+(-1)^2\frac{j}{25}+(-1)^2\frac{i_2 j}{25}$	$\nabla^2\zeta_3$	$\nabla^2\zeta_2$
4	$\frac{4i_1}{25}+\frac{j}{25}+(-1)^4\frac{j}{25}+(-1)^4\frac{i_2 j}{25}$	$\frac{3i_1}{36}+\frac{j}{36}+(-1)^3\frac{j}{36}+(-1)^3\frac{i_2 j}{36}$	$\nabla^2\zeta_4$	$\nabla^2\zeta_3$
5	$\frac{5i_1}{36}+\frac{j}{36}+(-1)^5\frac{j}{36}+(-1)^5\frac{i_2 j}{36}$	$\frac{4i_1}{49}+\frac{j}{49}+(-1)^4\frac{j}{49}+(-1)^4\frac{i_2 j}{49}$	$\nabla^2\zeta_5$	$\nabla^2\zeta_4$
6	$\frac{6i_1}{49}+\frac{j}{49}+(-1)^6\frac{j}{49}+(-1)^6\frac{i_2 j}{49}$	$\frac{5i_1}{64}+\frac{j}{64}+(-1)^5\frac{j}{64}+(-1)^5\frac{i_2 j}{64}$	$\nabla^2\zeta_6$	$\nabla^2\zeta_5$

Table 1: To observe the oscillation, we present the values for different k.

The oscillatory nature of (ζ_k) confirms that $(\zeta_k) \notin \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ due to the persistent sign alternations in $(\nabla^2\zeta_k)$.
Which implies :

$$(\nabla^2\zeta_k) \nrightarrow 0.$$

Thus, $(\zeta_k) \notin \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$. Since $(\xi_k), (\eta_k) \in \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ but $(\xi_k\eta_k) \notin \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$, we conclude that the $\mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ is not closed under multiplication. Hence, it is not a sequence algebra. The same example works for $\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$

Remark 2.7. *The spaces $\mathcal{I}^*_c[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ and $\mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ are not symmetric.*

The following example illustrates the above remark.

Example 2.8. *Consider the bicomplex sequence $\xi = (\xi_k)$ and I' be an arbitrary element of $\mathcal{I}$ given by*

$$\xi_k = \begin{cases} 1 + i_1 + i_2 + j, & \text{if } k \notin I', \\ \frac{1}{k} + i_1 - i_2 - j, & \text{if } k \in I'. \end{cases}$$

The first-order backward difference sequence is

$$\nabla_q \xi_k|_{q=1} = \nabla\xi_k = \begin{cases} 0, & \text{if both } k, k-1 \notin I', \\ \frac{1}{k} - 1 - 2i_2 - 2j, & \text{if } k \in I', k-1 \notin I', \\ 1 - \frac{1}{k-1} + 2i_2 + 2j, & \text{if } k \notin I', k-1 \in I', \\ \frac{1}{k} - \frac{1}{k-1}, & \text{if } k, k-1 \in I'. \end{cases}$$

The second-order backward difference sequence is

$$\nabla^2_q \xi_k|_{q=1} = \nabla^2\xi_k = \begin{cases} 0, & \text{if } k, k-1, k-2 \notin I', \\ \frac{1}{k} - \frac{1}{k-1} - 1 + 2i_2 + 2j, & \text{if } k \notin I', k-1 \notin I', k-2 \in I', \\ \frac{1}{k} - \frac{1}{k-1} - \frac{1}{k-2} + 2i_2 + 2j, & \text{if } k \in I', k-1, k-2 \in I', \\ \frac{1}{k} - \frac{1}{k-1} - 1 + 2i_2 + 2j, & \text{if } k \notin I', k-1 \in I', k-2 \in I', \\ \frac{1}{k} - \frac{1}{k-2}, & \text{if } k \in I', k-1 \notin I', k-2 \notin I', \\ \frac{1}{k} - \frac{1}{k-1}, & \text{if } k \notin I', k-1 \in I', k-2 \notin I', \\ \frac{1}{k} - \frac{1}{k-2} - 1 + 2i_2 + 2j, & \text{if } k \in I', k-1 \notin I', k-2 \in I', \\ \frac{1}{k} - \frac{1}{k-1} - \frac{1}{k-2}, & \text{if } k, k-1, k-2 \in I'. \end{cases}$$

Since the deviations occur only on the ideal, the sequence (ξ_k) belongs to $\mathcal{I}^*_c[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$.

Now consider a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ that rearranges the indices such that the locations of nonzero second-order $q-$ differences are spread across an infinite subset of $\mathbb{N}$ not belonging to $\mathcal{I}$. In this case,

$$\nabla^2 \xi_{\sigma(k)} = \xi_{\sigma(k)} - 2\xi_{\sigma(k)-1} + \xi_{\sigma(k)-2}$$

may no longer be ideally convergent. Thus, the permuted sequence fails to satisfy the membership criteria for $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, proving that the space is not symmetric.

Similarly, we can easily show that the space $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is not symmetric by modifying the above example and taking $\xi_k = 0$ for all $k \notin I'$, while assigning nonzero values to ξ_k for all $k \in I'$.

Remark 2.9. *The sequence space $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, are not monotone.*

The following example illustrates the above remark.

Example 2.10. *Let $\mathcal{I}$ be a nontrivial admissible ideal of $\mathbb{N}$. For instance, let $I' \in \mathcal{I}$ be the set of even natural numbers and a bicomplex sequence $\xi = (\xi_k)$ is given by*

$$\xi_k = \begin{cases} \frac{1}{k} + i_1 - i_2 - j, & \text{if } k \in I', \\ 0, & \text{otherwise.} \end{cases}$$

Then $\xi \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ since the second-order q-differences of the nonzero terms decay suitably and are ideal-convergent with respect to $\mathcal{I}$.

Now, choose a finite subset $K \subset \mathbb{N}$, for example, $K = \{2, 4\} \subset I'$, and define the canonical preimage $\eta = (\eta_k)$ of ξ by

$$\eta_k = \begin{cases} \xi_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

$$\eta = \left(0, \frac{1}{2} + i_1 - i_2 - j, 0, \frac{1}{4} + i_1 - i_2 - j, 0, 0, \dots\right).$$

This sequence has only two nonzero terms and zeros elsewhere. When computing the second-order q-difference $\nabla_q^2 \eta_k$, the large gaps between nonzero and zero terms create sharp fluctuations. As a result, the norm $\|\nabla_q^2 \eta_k\|_{\mathbb{C}_2}$ does not converge to zero in the ideal sense. Therefore, $\eta \notin \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

This shows that although $\xi \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, its canonical preimage $\eta \notin \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

Hence, the space $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is not monotone.

A similar construction can be used to show that the spaces $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ and $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ are also not monotone.

Remark 2.11. *The sequence spaces , $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, are not convergence-free.*

The following example illustrates the validity of the above remark.

Example 2.12. *Consider $I' \in \mathcal{I}$ and the sequence $\xi = (\xi_k)$ given by*

$$\xi_k = \begin{cases} 1 + i_1 + i_2 + j, & \text{if } k \in I', \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\xi \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

Now define the sequence $\eta = (\eta_k)$ by

$$\eta_k = \begin{cases} \frac{1}{k} + i_1 + i_2 + j, & \text{if } k \in I', \\ 1 + i_1 + i_2 + j, & \text{otherwise.} \end{cases}$$

Clearly, $\xi_k = 0 \Rightarrow \eta_k = 1 + i_1 + i_2 + j \neq 0$. *So the implication* $\xi_k = 0 \Rightarrow \eta_k = 0$ *fails, which does not yet violate the convergence-free property.*

However, for the convergence-free condition, we require the reverse: if $\xi_k = 0 \Rightarrow \eta_k = 0$*, and still* $\eta \notin \mathcal{I}_c^*$*, then the space is not convergence-free.*

So instead, define:

$$\eta_k = \begin{cases} \frac{1}{k} + i_1 + i_2 + j, & \text{if } k \in I', \\ 0, & \text{otherwise.} \end{cases}$$

Now observe that:

$$\xi_k = 0 \Rightarrow \eta_k = 0,$$

so the condition for convergence-free space is satisfied.

Next, compute the second-order difference:

$$\nabla^2 \eta_k = \eta_k - 2\eta_{k-1} + \eta_{k-2}.$$

Let us assume I' *is the set of all even numbers, i.e.,* $I' = \{2, 4, 6, \dots\}$. *Then for* $k = 6$*, we have:*

$$\eta_6 = \frac{1}{6} + i_1 + i_2 + j, \quad \eta_5 = 0, \quad \eta_4 = \frac{1}{4} + i_1 + i_2 + j.$$

Therefore,

$$\nabla^2 \eta_6 = \eta_6 - 2\eta_5 + \eta_4 = \left(\frac{1}{6} + \frac{1}{4}\right) + 2(i_1 + i_2 + j) = \frac{5}{12} + 2(i_1 + i_2 + j).$$

Then,

$$\|\nabla^2 \eta_6\|_{\mathbb{C}_2} = \left\| \frac{5}{12} + 2(i_1 + i_2 + j) \right\|_{\mathbb{C}_2} > 0.$$

This kind of non-zero second-order difference occurs for infinitely many even-indexed k (since I' *is infinite), and hence the sequence* $(\nabla^2 \eta_k)$ *does not converge ideally to* 0.

So,

$$\eta \notin \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}].$$

Thus, although $\xi_k = 0$ *implies* $\eta_k = 0$ *and* $\xi \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$*, the sequence* η *does not belong to the space. Therefore, the space* $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ *is not convergence-free.*
A similar construction can be used to show that the spaces $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ *and* $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ *are also not convergence-free.*

Remark 2.13. *The sequence spaces* $\mathcal{I}_p^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}_\infty^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$*, and* $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ *are, in general, not solid. This is due to the fact that the second-order q-difference operator* ∇_q^2 *does not preserve coordinatewise domination. That is, even if* $\|\eta_k\|_{\mathbb{C}_2} \leq \|\xi_k\|_{\mathbb{C}_2}$ *for all* $k \in \mathbb{N}$*, it does not necessarily follow that* $\|\nabla_q^2 \eta_k\|_{\mathbb{C}_2} \leq \|\nabla_q^2 \xi_k\|_{\mathbb{C}_2}$. *Therefore, the condition* $\xi \in \mathcal{I}^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ *does not imply* $\eta \in \mathcal{I}^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

The following example illustrates the above remark.

Example 2.14. *Consider* $q = 1$ *, the bicomplex sequence* $\xi = (\xi_k)$ *given by*

$$\xi_k = 1 + i_1 + i_2 + j \quad, \text{for all } k \in \mathbb{N}.$$

This is a constant sequence. Therefore, the second-order difference satisfies

$$\nabla^2 \xi_k = \xi_k - 2\xi_{k-1} + \xi_{k-2} = 0 \quad \text{for all } k \geq 2.$$

Thus, $\nabla^2 \xi_k = 0$ *for* $k \geq 2$*, implying that*

$$\|\nabla^2 \xi_k\|_{\mathbb{C}_2} = 0,$$

so $(\nabla^2 \xi_k)$ *converges ideally to* 0*, i.e.,* $\xi \in \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

Now define another sequence $\eta = (\eta_k)$ *by*

$$\eta_k = \frac{1}{k} + i_1 + i_2 + j \quad \text{for all } k \in \mathbb{N}.$$

Then clearly, for each k,

$$\|\eta_k\|_{\mathbb{C}_2} \leq \|\xi_k\|_{\mathbb{C}_2},$$

because $\frac{1}{k} \leq 1$ *and the imaginary parts are the same.*

However, we now compute the second-order difference of η*:*

$$\nabla^2 \eta_k = \eta_k - 2\eta_{k-1} + \eta_{k-2} = \left(\frac{1}{k} - \frac{2}{k-1} + \frac{1}{k-2}\right).$$

The imaginary parts cancel because they are constant, so we get:

$$\nabla^2 \eta_k = \frac{1}{k} - \frac{2}{k-1} + \frac{1}{k-2}.$$

Let us compute this explicitly at $k = 5$*:*

$$\eta_5 = \frac{1}{5} + i_1 + i_2 + j, \quad \eta_4 = \frac{1}{4} + i_1 + i_2 + j, \quad \eta_3 = \frac{1}{3} + i_1 + i_2 + j,$$

$$\nabla^2 \eta_5 = \left(\frac{1}{5} - 2 \cdot \frac{1}{4} + \frac{1}{3}\right) = \frac{1}{5} - \frac{1}{2} + \frac{1}{3} = \frac{1}{30}.$$

Therefore,

$$\|\nabla^2 \eta_5\|_{\mathbb{C}_2} = \frac{1}{30} > 0.$$

Since $(\nabla^2 \eta_k)$ *does not converge to zero ideally, it follows that*

$$\eta \notin \mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}].$$

Thus, although $\|\eta_k\|_{\mathbb{C}_2} \leq \|\xi_k\|_{\mathbb{C}_2}$ *for all k and* $\xi \in \mathcal{I}_c^*$*, the sequence* η *does not belong to the space. Hence,* $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ *is not solid.*

Theorem 2.15. *The relations* $\mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}] \cong \mathcal{I}_\theta^*$ *and* $\mathcal{I}_c^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}] \cong \mathcal{I}_c^*$ *holds good.*

Proof. Define the mapping $\Omega : \mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}] \to \mathcal{I}_\theta^*$ by $\Omega\xi = \eta = \nabla_q^2 \xi$,
for all $\xi \in \mathcal{I}_\theta^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.
Clearly, Ω is linear and injective operator.
Let the sequence (ξ_k) be given by

$$\xi_k = \sum_{i=0}^{k} \binom{k-i+1}{k-i}_q \eta_k$$

where (η_k) is an arbitrary sequence in $\mathcal{I}^*_\theta$. i.e, for every $\varepsilon > 0$, $\exists\, I' = \{k \in \mathbb{N} : \|\eta_k\|_{\mathbb{C}_2} \geq \varepsilon\} \in \mathcal{I}$ or $\mathcal{I} - \lim_{k\to\infty} \eta_k = 0$.
Then, we obtain $\nabla^2_q \xi_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{2}{k}_q \xi_{n-k} = \eta_n$.
Therefore, for any $\varepsilon > 0$,

$$\{n \in \mathbb{N} : \|\nabla^2_q \xi_n\|_{\mathbb{C}_2} \geq \varepsilon\}$$
$$\implies \mathcal{I} - \lim_{n\to\infty} (\nabla^2_q \xi_n) = \theta$$
$$\implies (\nabla^2_q \xi_n) \in \mathcal{I}^*_\theta$$
$$\implies (\xi_n) \in \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}].$$

The mapping Ω is both onto and norm-preserving. Hence, we establish the isomorphism:

$$\mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \cong \mathcal{I}^*_\theta.$$

similarly we can show that $\mathcal{I}^*_c[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \cong \mathcal{I}^*_c$. □

Theorem 2.16. *The inclusion* $\mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \subseteq \mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$, *for* $1 \leq p < \infty$, *holds.*

Proof. Since $\sum_{k,\, k\in K\in\mathcal{F}(\mathcal{I})}^{\infty} \|\xi_k\|^p_{\mathbb{C}_2} < \infty \implies \sup_{k,\, k\in K\in\mathcal{F}(\mathcal{I})} \|\xi_k\|_{\mathbb{C}_2} < \infty, \mathcal{I}^*_p \subset \mathcal{I}^*_\infty$ holds, this suffices for the inclusion relation. Taking example that $(\xi_k) = (i_1)$, strictness of the relation is clear
To see the inclusion is strict, consider the sequence (i_1), showing that $\mathcal{I}^*_p \subset \mathcal{I}^*_\infty$ is strict. So we can take a sequence $(\xi_k) \in \mathcal{I}^*_\infty \setminus \mathcal{I}^*_p$.
We define a sequence (ξ'_k) such that $\xi'_k = \sum_{i=0}^{k} \binom{k-i+1}{k-i}_q \xi_i$ for each $k \in \mathbb{N}$. Then the sequence $(\nabla^2_q \xi'_k) = (\xi'_k \in \mathcal{I}^*_\infty \setminus \mathcal{I}^*_p)$.
Consequently, $(\xi'_k) \in \mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \setminus \mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$. □

Theorem 2.17. *The relations*

$$\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \cong \mathcal{I}^*_\infty \quad \textit{and} \quad \mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \cong \mathcal{I}^*_p,$$

holds good where $1 \leq p < \infty$.

Proof. We prove the result for the case $\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$.
Let, $\sigma : \mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}] \to \mathcal{I}^*_\infty$ defined by
$\sigma(\xi_k) = (\nabla^2_q \xi_k) = (\eta_k)$.
σ is linear and as q-difference operator ∇^2_q can be written as triangular invertible matrix and so σ is linear.
Now, if $\sigma(\xi_k) = \theta$, Then $(\eta_k) = \sigma(\xi_k) = \theta \implies (\xi_k - (1+q)\xi_{k-1} + q\xi_{k-2}) = 0\ \forall k \in \mathbb{N}. \implies T$ is injective.

Let the sequence ξ_k be given by

$$\xi_k = \sum_{i=0}^{k} \binom{k-i+1}{k-i}_q \eta_k$$

where (η_k) is an arbitrary sequence in $\mathcal{I}^*_\infty$. i.e for every $G > 0$, there exists $K \in \mathcal{F}(\mathcal{I})$ such that $\eta_k > G,\ \forall\, k \in K \in \mathcal{F}(\mathcal{I})$. Then, we obtain $\nabla^2_q \xi_n = \sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \binom{2}{k}_q \xi_{n-k} = \eta_n$. Which implies for every $G > 0$, there exists $K \in \mathcal{F}(\mathcal{I})$ such that $\nabla^2_q \xi_k > G,\ \forall\, k \in K \in \mathcal{F}(\mathcal{I})$.
Thus,
$(\nabla^2_q \xi_k) \in \mathcal{I}^*_\infty$ and so $(\xi_k) \in \mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ and $\mathcal{I}^*_\infty$ are bijective.
Similarly, we can prove for $\mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$, for, $1 \leq 0 < \infty$. □

We state the following result without proofs.

Lemma 2.18. *The following inclusions hold and are strict.*

(1) $\mathcal{I}^*_p \subset \mathcal{I}^*_p[\nabla^2, \|\cdot\|_{\mathbb{C}_2}] \subset \mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$,

(2) $\mathcal{I}^*_\infty \subset \mathcal{I}^*_\infty[\nabla^2, \|\cdot\|_{\mathbb{C}_2}] \subset \mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$.

Theorem 2.19. *The following spaces are BK-spaces equipped with their respective norms:*

(1) *The space $\mathcal{I}^*_p$, $(1 \le p < \infty)$ is BK-space with the norm*

$$\|\xi\|_{\mathcal{I}^*_p} = \left(\sum_{k \in K \in \mathcal{F}(\mathcal{I})} \|\xi_k\|^p_{\mathbb{C}_2} \right)^{1/p}.$$

(2) *The spaces $\mathcal{I}^*_\infty, \mathcal{I}^*_c$, and $\mathcal{I}^*_\theta$ are BK-spaces with the norm*

$$\|\xi\|_{\mathcal{I}^*_\infty} = \sup_{k \in K \in \mathcal{F}(\mathcal{I})} \|\xi_k\|_{\mathbb{C}_2}.$$

Proof. *Let, $\|\xi^n - \xi\|_{\mathcal{I}^*_\infty}$ is $\mathcal{I}-$convergent to 0, as $n \to \infty$, $n \in K \in \mathcal{F}(\mathcal{I})$.*
Then, for given $\varepsilon > 0$,

$$\begin{aligned} & \|\xi^n - \xi\|_{\mathcal{I}^*_\infty} < \varepsilon, \qquad \forall n \in K \in \mathcal{F}(\mathcal{I}), \\ \implies & \sup_{k \in K \in \mathcal{F}(\mathcal{I})} \|\xi^n_k - \xi_k\|_{\mathbb{C}_2} < \varepsilon, \qquad \forall n \in K \in \mathcal{F}(\mathcal{I}), \\ \implies & \|\xi^n_k - \xi_k\|_{\mathbb{C}_2} < \varepsilon, \qquad \forall n \in K \in \mathcal{F}(\mathcal{I}) \end{aligned}$$

Therefore, $\|\xi^n_k - \xi_k\|_{\mathbb{C}_2}$ is $\mathcal{I}-$ convergent to 0, as n tends to 0, $\forall n \in K \in \mathcal{F}(\mathcal{I})$ □

Since the newly defined sequence spaces $\mathcal{I}^*_c[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}], \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ are isomorphic to the BK-spaces $\mathcal{I}^*_c, \mathcal{I}^*_\theta$, and $\mathcal{I}^*_\infty$, it follows that $\mathcal{I}^*_c[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}], \mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$, and $\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ are also BK-spaces.

We present the following theorems without proof.

Theorem 2.20. *The sequence space $\mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}], (1 \le p < \infty)$ are BK-space with the norm*

$$\|\xi\|_{\mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]} = \left(\sum_{k \in K \in \mathcal{F}(\mathcal{I})} \|\xi_k - (1+q)\xi_{k-1} - q\xi_{k-2}\|^p_{\mathbb{C}_2} \right)^{1/p}.$$

Theorem 2.21. *Let $\nabla^2_q \xi_k = \xi_k - (1+q)\xi_{k-1} + q\xi_{k-2}$ be the second-order q-difference operator. Then the following sequence spaces are BK-spaces with their respective norms:*

(1) *For $1 \le p < \infty$, the sequence space $\mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ is a BK-space under the norm*

$$\|\xi\|_{\mathcal{I}^*_p[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]} = \left(\sum_{k \in K \in \mathcal{F}(\mathcal{I})} \|\xi_k - (1+q)\xi_{k-1} + q\xi_{k-2}\|^p_{\mathbb{C}_2} \right)^{1/p}.$$

(2) *The sequence space $\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$, $\mathcal{I}^*_c[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ and $\mathcal{I}^*_\theta[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]$ are BK-spaces under the norm*

$$\|\xi\|_{\mathcal{I}^*_\infty[\nabla^2_q, \|\cdot\|_{\mathbb{C}_2}]} = \sup_{k \in K \in \mathcal{F}(\mathcal{I})} \|\xi_k - (1+q)\xi_{k-1} + q\xi_{k-2}\|_{\mathbb{C}_2}.$$

Remark 2.22. *The sequence spaces $\mathcal{I}_p^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}], (0 \le p < 1)$ are not BK-space, but quasi norm space with the norm*

$$\|\xi\|_{\mathcal{I}_p^*[\nabla_q^2,\|\cdot\|_{\mathbb{C}_2}],0\le p<1} = \left(\sum_{k\in K\in\mathcal{F}(\mathcal{I})} \|\xi_k - (1+q)\xi_{k-1} - q\xi_{k-2}\|_{\mathbb{C}_2}^p \right)^{1/p}$$

Example 2.23. *Let the bicomplex sequence $\xi = (\xi_k)$ be defined by*

$$\xi_k = \frac{1}{k^\alpha}(1 + i_1 - i_2 - j), \quad \textit{for all } k \in \mathbb{N},$$

where $0 < \alpha < \frac{1}{p}$ and $0 < p < 1$. Then the quasi-norm

$$\|\xi\|_{\mathcal{I}_p^*[\nabla_q^2,\|\cdot\|_{\mathbb{C}_2}],0\le p<1} = \left(\sum_{k\in K\in F(I)} \left\|\xi_k - (1+q)\xi_{k-1} - q\xi_{k-2}\right\|_{\mathbb{C}_2}^p \right)^{1/p}$$

is finite, since each term in the summation is of the form $\frac{C}{k^\alpha}$, where C is a constant in $\mathbb{C}_2$, and the series $\sum \frac{1}{k^{\alpha p}}$ converges due to $\alpha p < 1$. Thus, $\xi \in I_p^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$.

However, since the functional defined for $0 < p < 1$ does not satisfy the triangle inequality, it is not a norm. Therefore, the space $I_p^*[\nabla_q^2, \|\cdot\|_{\mathbb{C}_2}]$ is a quasi-normed space but not a BK-space.

3. Conclusion

This paper defines second-order quantum difference sequence spaces over bi-complex numbers and examines their *BK*-space structure, symmetry, inclusion relations, and isomorphisms with classical $\mathcal{I}$-convergent spaces. A matrix representation of the operator ∇_q^2 is provided, and counterexamples highlight contradictions in specific inclusion cases. The study advances understanding of quantum difference sequence spaces and sets the stage for future research.

Acknowledgement

The first author gratefully acknowledges the financial support received from the Department of Science and Technology, Government of India, under the DST/INSPIRE Fellowship [IF220239], Ministry of Science and Technology, Technology Bhawan, New Mehrauli Road, New Delhi–110016.

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