Filomat 40:1 (2026), 51–73
https://doi.org/10.2298/FIL2601051K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Asymptotic analysis of increasing solutions of cyclic second-order difference systems

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Abstract.
The existence and asymptotic behavior of positive increasing solutions of the cyclic second-order non-linear difference system

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1), \quad i = \overline{1,N},$$

are studied, where $x_{N+1} = x_1$, and the sequences $p_i = p_i(n)$ and $q_i = q_i(n)$ are positive for all $n \in \mathbb{N}$, while the constants α_i and β_i, $i = \overline{1,N}$, are positive and satisfy the sublinearity condition $\alpha_1\alpha_2\cdots\alpha_N > \beta_1\beta_2\cdots\beta_N$. We consider two types of positive increasing solutions: those converging to a positive constant and those diverging to infinity, whose associated quasi-differences tend to a positive constant. For both classes of solutions, necessary and sufficient conditions for existence are established using fixed point methods. In addition, under the assumption that the coefficient sequences are regularly varying, we investigate positive increasing solutions for which both the solution components and their quasi-differences tend to infinity. In this case, the corresponding existence conditions are also derived, along with precise asymptotic formulas, based on the theory of discrete regular variation.

1. Introduction

In recent decades, difference equations have emerged as a powerful tool for modeling and solving problems across various fields, including statistics, engineering, natural and social sciences. This progress has been achieved through advancements in digital computing, which have enabled their application to diverse systems such as electrical circuits, mechanical structures, heat transfer, and wave filters.

This paper is devoted to the analysis of the following cyclic second-order nonlinear system of difference equations

$$\Delta(p_i(n)|\Delta x_i(n)|^{\alpha_i-1}\Delta x_i(n)) = q_i(n)|x_{i+1}(n+1)|^{\beta_i-1}x_{i+1}(n+1), \tag{SE}$$

where $i = \overline{1,N},\ x_{N+1} = x_1,\ n \in \mathbb{N}$, under the following assumptions:

2020 *Mathematics Subject Classification*. Primary 39A22, 39A12, 40A05; Secondary 39A20, 26A12.

Keywords. Cyclic system of difference equations, Emden-Fowler type difference equation, nonlinear difference equations, increasing solutions, asymptotic behavior, regularly varying sequence.

Received: 06 August 2025; Revised: 24 September 2025; Accepted: 23 October 2025

Communicated by Miljana Jovanović

Both authors are supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia – Grant Number 51-03-137/2025-03/ 200124.

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(a) The constants α_i and β_i, $i = \overline{1,N}$ are positive and satisfies the inequality

$$\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N > \beta_1\beta_2 \cdot \ldots \cdot \beta_N;$$

(b) The real sequences $p_i = \{p_i(n)\}$ and $q_i = \{q_i(n)\}$ are positive;

(c) All the sequences p_i, $i = \overline{1,N}$ simultaneously satisfy either

$$S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} < \infty. \tag{I}$$

or

$$S_i = \sum_{n=1}^{\infty} \frac{1}{p_i(n)^{1/\alpha_i}} = \infty, \tag{II}$$

In the case of condition (I), the following notation will be used

$$\pi_i(n) = \sum_{k=n}^{\infty} \frac{1}{p_i(k)^{1/\alpha_i}}, \quad i = \overline{1,N}, \tag{1.1}$$

while if condition (II) holds, we use the following notation

$$P_i(n) = \sum_{k=1}^{n-1} \frac{1}{p_i(k)^{1/\alpha_i}}, \quad i = \overline{1,N}. \tag{1.2}$$

System (SE) is classified as sublinear, superlinear, or half-linear depending on whether the product condition in (a) is a strict inequality (sublinear), an opposite inequality (superlinear), or an equality (half-linear).

The qualitative analysis of the second-order Emden-Fowler type differential equation

$$\left(p(t)|x'(t)|^{\alpha-1}x'(t)\right)' \pm q(t)|x(t)|^{\beta-1}x(t) = 0,$$

(see [7, 8, 32, 49] and monographs [13, 25]) and its discrete analog

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) \pm q(n)|x(n+1)|^{\beta-1}x(n+1) = 0,$$

(see [9–12] and monographs [1, 3]) serves as the foundation for the research of cyclic second-order systems of difference equations (SE). This equations have been the subject of extensive research concerning existence, uniqueness, and oscillatory behavior of their solutions.

The qualitative analysis of second-order nonlinear difference equations has also been extended to two-dimensional first-order and second-order nonlinear systems [4, 19, 26, 28, 38], as well as to symmetric and close-to-symmetric systems [42–44, 46]. As a continuation of the study some periodic difference equations, an investigation of cyclic second-order systems was proposed and conducted in [15], and later it was continued in some works dealing with both cyclic and close-to-cyclic systems (see, e.g., [27, 36, 37, 39–41, 45, 47], in addition to the already cited references). Systems of the form (SE) are beneficial for modeling numerical methods in heat and fluid transfer in layered materials equations, such as cylindrical thermal insulation or geological structures.

In the continuous case, cyclic systems of differential equations were studied by Jaroš and Kusano [16–18] and Řehák [33], while the asymptotic analysis of cyclic second-order difference systems remains less explored, with notable work by Kapešić [20] and Kapešić and Manojlović [23].

The primary objective of this study is to establish a comprehensive classification of positive increasing solutions, which is the foundation for a deeper understanding of the solution space. The second objective focuses on identifying the necessary and sufficient conditions under which various types of positive increasing solutions can exist. Finally, the most complex and demanding objective involves deriving precise asymptotic formulas for these solutions.

When N is even, the obtained results can be applied to cyclic systems of N first-order difference equations.

2. Classification of positive decreasing solutions

By a solution of (SE), we refer to a vector sequence

$$\mathbf{x} = (x_1, x_2, \ldots, x_N) \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}, \quad x_i = \{x_i(n)\}_{n\in\mathbb{N}}$$

where ${}^{\mathbb{N}}\mathbb{R} = \{f \,|\, f : \mathbb{N} \to \mathbb{R}\}$, whose components $x_i = \{x_i(n)\}_{n\in\mathbb{N}}$, $i = \overline{1,N}$ satisfy (SE). In the following analysis, we will focus on the behavior of the sequences x_i for sufficiently large values of n, i.e., $n \geq n_0$, for some $n_0 \in \mathbb{N}$. To formalize this, we introduce the notation ${}^{\mathbb{N}_{n_0}}\mathbb{R} = \{f \,|\, f : \mathbb{N}_{n_0} \to \mathbb{R}\}$, where $\mathbb{N}_{n_0} = \{n \in \mathbb{N} \;\big|\; n \geq n_0\}$.

A solution $\mathbf{x}$ is termed nonoscillatory if all its components are eventually of one sign. Because of the sign condition on the coefficients, if one component is nonoscillatory, all components are nonoscillatory and eventually monotone, and therefore they have a limit. A nonoscillatory solution is considered positive if all its components are eventually positive. Our primary objective is to investigate the existence and asymptotic behavior of positive increasing solutions of (SE), that is, solutions whose components are eventually positive and increasing, i.e., satisfying

$$x_i(n) > 0, \quad \Delta x_i(n) > 0, \quad \text{for } n \geq n_0, \quad i = \overline{1,N}. \tag{2.1}$$

Let us denote by $\mathcal{IS}$ the set of all solutions of (SE) whose components are eventually positive and increasing. For every component of any solution $\mathbf{x}$ of (SE), let us denote by $x_i^{[1]} = \left\{x_i^{[1]}(n)\right\}$ its quasi-difference, $x_i^{[1]}(n) = p_i(n)|\Delta x_i(n)|^{\alpha-1}\Delta x_i(n)$, $i = \overline{1,N}$. From (2.1), for x_i, $i = \overline{1,N}$, one of the following two cases holds:

$$(i)\ \lim_{n\to\infty} x_i(n) = k_i,\ k_i > 0 \quad \text{or} \quad (ii)\ \lim_{n\to\infty} x_i(n) = \infty,$$

while for $x_i^{[1]}$, $i = \overline{1,N}$, one of the following two cases holds:

$$(iii)\ \lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0 \quad \text{or} \quad (iv)\ \lim_{n\to\infty} x_i^{[1]}(n) = \infty.$$

If (I) holds, then in the case when (iii) holds, we have that there exists $m_0 \in \mathbb{N}$ such that $x_i^{[1]}(n) \leq c_i, n \geq m_0$ for $i = \overline{1,N}$. Therefore, it follows that

$$x_i(n) \leq x_i(m_0) + c_i^{\frac{1}{\alpha_i}} \sum_{k=m_0}^{n-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}, \quad i = \overline{1,N}.$$

From the last inequality, letting $n \to \infty$, we conclude that only (i) may hold for some positive constants $k_i, i = \overline{1,N}$. Accordingly, if (ii) holds, it is concluded that only (iv) may hold.

If (II) holds, since $x_i^{[1]}, i = \overline{1,N}$ are increasing, then there exist $m_0 \in \mathbb{N}$ such that $p_i(n)\Delta x_i(n)^{\alpha_i} \geq x_i^{[1]}(m_0), n \geq m_0$, implying that

$$x_i(n) \geq x_i(m_0) + x_i^{[1]}(m_0)^{\frac{1}{\alpha_i}} \sum_{k=m_0}^{n-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}, \quad i = \overline{1,N}.$$

Letting $n \to \infty$, we conclude that only (ii) may hold.

This leads to the following classification of a positive increasing solution: if (I) holds each component x_i of positive increasing solution $\mathbf{x}$ satisfies:

(SI) $\lim_{n\to\infty} x_i(n) = \lim_{n\to\infty} x_i^{[1]}(n) = \infty$,

(AC) $\lim_{n\to\infty} x_i(n) = k_i > 0 \;\Leftrightarrow\; x_i(n) \sim k_i,\ n \to \infty$,

while if (II) holds each component x_i of positive increasing solution $\mathbf{x}$ satisfies either (SI) or

(P) $\lim_{n\to\infty} x_i(n) = \infty$, $\lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0$

where the following asymptotic relation has been used

$$f(n) \sim g(n),\ n \to \infty \ \Leftrightarrow\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1,$$

Using Stolz-Cesaro Theorem (see Theorem 4.5), if (II) holds, solutions satisfying (SI) and (P) can be characterized as

$$(SI2)\quad x_i(n) \succ P_i(n),\ n \to \infty, \qquad (P)\quad x_i(n) \sim \omega_i P_i(n),\ n \to \infty,\ \ \omega_i = c_i^{1/\alpha_i},$$

and if (I) holds, solution satisfying (SI) can be characterized as

$$(SI1)\quad x_i(n) \succ \pi_i(n),\ n \to \infty,$$

where the following asymptotic relation has been used

$$f(n) \succ g(n),\ n \to \infty \ \Leftrightarrow\ \lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$$

Solutions of type (AC) and (P) are termed *primitive solutions*, while those of type $(SI1)$ and $(SI2)$ are known as *strongly increasing*. Section 5 establishes some necessary and sufficient conditions for the existence of primitive solutions. On the other hand, deriving necessary and sufficient conditions for the existence and precise asymptotic representations of strongly increasing solutions is generally more challenging. Thus, in Section 6, we limit our investigation to cases where the system coefficients are regularly varying sequences in order to solve this problem, and we focus on regularly varying strongly increasing solutions of (SE).

3. Regularly Varying Sequences

The theory of regularly varying sequences, often called Karamata sequences (see [24]), was developed during the seventies by Galambos, Seneta and Bojanić in [6, 14]. However, until the appearance of the paper of Matucci and Rehak [29], the connection between regularly varying sequences and difference equations was not considered. In this paper, as well as in the following ones [30, 31, 34, 35], the theory of regularly varying sequences is further developed and applied in the asymptotic analysis of linear and half-linear difference equations of the second-order, giving necessary and sufficient conditions for the existence of regularly varying solutions of these equations. After this, further development of the discrete theory of regular variation and its application to nonlinear difference equations of type Emden-Fowler type can be found in [21].

This section presents basic definitions and properties of regularly varying sequences that will be utilized to establish the main results of this paper. For a thorough discussion of regular variation, the reader is referred to Bingham et al. [5].

There are two main approaches in the basic theory of regularly varying sequences: the approach due to Karamata [24], based on a definition that can be understood as a direct discrete counterpart of elegant and straightforward continuous definition (Definition 3.1), and the approach due to Galambos and Seneta [14], based on purely sequential definition (Definition 3.2). Bojanić and Seneta have shown in [6] the equivalence of these two definitions.

Definition 3.1. (Karamata [24]) A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{k\to\infty} \frac{y([\lambda k])}{y(k)} = \lambda^{\rho} \quad \textit{for} \quad \forall \lambda > 0,$$

where $[n]$ denotes the integer part of n.

Definition 3.2. (Galambos and Seneta [14]) A positive sequence $y = \{y(k)\}, n \in \mathbb{N}$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if there exists a positive sequence $\{\alpha(k)\}$ satisfying

$$\lim_{k\to\infty} \frac{y(k)}{\alpha(k)} = C,\ 0 < C < \infty \qquad \lim_{k\to\infty} k\,\frac{\Delta\alpha(k-1)}{\alpha(k)} = \rho\,.$$

If $\rho = 0$, then y is said to be *slowly varying*. The sets of regularly varying sequences with index ρ and slowly varying sequences are denoted $\mathcal{RV}(\rho)$ and $\mathcal{SV}$, respectively.

The concept of normalized regularly varying sequences was introduced by Matucci and Rehak in [30], where they also offered a modification of Definition 3.2, i.e., they proved that the second limit in Definition 3.2 can be replaced with

$$\lim_{k\to\infty} k\,\frac{\Delta\alpha(k)}{\alpha(k)} = \rho\,.$$

Definition 3.3. A positive sequence $y = \{y(k)\}, k \in \mathbb{N}$ is said to be *normalized regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{k\to\infty} \frac{k\Delta y(k)}{y(k)} = \rho.$$

If $\rho = 0$, then y is called a *normalized slowly varying sequence.*

In what follows, $\mathcal{NRV}(\rho)$ and $\mathcal{NSV}$ will be used to denote the set of all normalized regularly varying sequences of the index ρ and the set of all normalized slowly varying sequences.

Typical examples are:

$$\{\log k\} \in \mathcal{NSV}, \quad \{k^\rho \log k\} \in \mathcal{NRV}(\rho), \quad \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}\,.$$

In order to present results for a system of difference equations, we need to define a regularly varying vector $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, where ${}^{\mathbb{N}}\mathbb{R} = \{f \,|\, f : \mathbb{N} \to \mathbb{R}\}$.

Definition 3.4. *A vector* $\mathbf{x} \in {}^{\mathbb{N}}\mathbb{R} \times \ldots \times {}^{\mathbb{N}}\mathbb{R}$, $\mathbf{x} = (\{x_1(n)\}, \ldots, \{x_N(n)\})$ *is said to be regularly varying of index* $(\rho_1, \rho_2, \ldots, \rho_N)$ *if* $x_i = \{x_i(n)\} \in \mathcal{RV}(\rho_i)$ *for* $i = \overline{1, N}$. *If all* ρ_i *are positive (or negative), then* $\mathbf{x}$ *is called a regularly varying vector sequence of positive (or negative) index* $(\rho_1, \rho_2, \ldots, \rho_N)$. *The set of all regularly varying vectors of index* $(\rho_1, \rho_2, \ldots, \rho_N)$ *is denoted by* $\mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$.

Various necessary and sufficient conditions for a sequence of positive numbers to be regularly varying have been established (see [6, 14, 29, 30]). Consequently, any of these can define a regularly varying sequence. The one that is the most important is the following Representation theorem (see [6, Theorem 3]), while some other representation formula for regularly varying sequences was established in [30, Lemma 1].

Theorem 3.1. (Representation theorem) *A positive sequence* $\{y(k)\}, k \in \mathbb{N}$ *is said to be regularly varying of index* $\rho \in \mathbb{R}$ *if and only if there exists sequences* $\{c(k)\}$ *and* $\{\delta(k)\}$ *such that*

$$\lim_{k\to\infty} c(k) = c_0 \in (0,\infty) \quad \textit{and} \quad \lim_{k\to\infty} \delta(k) = 0,$$

and

$$y(k) = c(k)\,k^\rho\,\exp\left(\sum_{i=1}^{k} \frac{\delta(i)}{i}\right).$$

In [6], a very useful embedding theorem was proved, which gives possibility of using continuous theory in developing. However, as noted in [6], such development is not generally straightforward and sometimes far from a simple imitation of arguments for regularly varying functions.

Theorem 3.2. (Embedding Theorem) *If positive sequence $y = \{y(n)\}$ is regularly varying of index $\rho \in \mathbb{R}$, then function $Y(t)$ defined on $[0, \infty)$ by $Y(t) = y([t])$ is a regularly varying function of index ρ. Conversely, if a positive function $Y(t)$, $t \in [0, \infty)$ is a regularly varying of index ρ, then a positive sequence $\{y(k)\}$, $y(k) = Y(k)$, $k \in \mathbb{N}$ is regularly varying of index ρ.*

Next, we state some important properties of $\mathcal{RV}$ sequences helpful in developing the asymptotic behavior of solutions of (*SE*) in the subsequent sections (for more properties and proofs, see [6, 29]).

Theorem 3.3. *The following properties hold:*

(i) $y \in \mathcal{RV}(\rho)$ *if and only if* $y(k) = k^{\rho}\, l(k)$, *where* $l = \{l(k)\} \in \mathcal{SV}$.

(ii) *Let* $x \in \mathcal{RV}(\rho_1)$ *and* $y \in \mathcal{RV}(\rho_2)$. *Then,* $xy \in \mathcal{RV}(\rho_1 + \rho_2)$, $x + y \in \mathcal{RV}(\rho)$, $\rho = \max\{\rho_1, \rho_2\}$ *and* $1/x \in \mathcal{RV}(-\rho_1)$.

(iii) *If* $y \in \mathcal{RV}(\rho)$, *then* $\lim_{k\to\infty} \dfrac{y(k+1)}{y(k)} = 1$.

(iv) *If* $l \in \mathcal{SV}$ *and* $l(k) \sim L(k)$, $k \to \infty$, *then,* $L \in \mathcal{SV}$.

(v) *If* $l \in \mathcal{SV}$, *then for any* $\varepsilon > 0$,

$$\lim_{k\to\infty} k^{\varepsilon} l(k) = \infty, \qquad \lim_{k\to\infty} k^{-\varepsilon} l(k) = 0,$$

(vi) *If* $y \in \mathcal{RV}(\rho)$, *then* $\{k^{-\sigma} y(k)\}$ *is eventually increasing for each* $\sigma < \rho$ *and* $\{k^{-\mu} y(k)\}$ *is eventually decreasing for each* $\mu > \rho$.

The following theorem can be seen as *the discrete analog of Karamata's integration theorem* and plays a central role in proving the main results of this paper. Proof of this Theorem can be found in [21]. Also, some parts of this theorem's proof can be found in [6] and [34].

Theorem 3.4. *Let* $l = \{l(n)\} \in \mathcal{SV}$.

$$(i) \quad \text{If } \alpha > -1, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{n^{\alpha+1} l(n)} \sum_{k=1}^{n} k^{\alpha} l(k) = \frac{1}{1+\alpha};$$

$$(ii) \quad \text{If } \alpha < -1, \quad \text{then} \quad \lim_{n\to\infty} \frac{1}{n^{\alpha+1} l(n)} \sum_{k=n}^{\infty} k^{\alpha} l(k) = -\frac{1}{1+\alpha};$$

$$(iii) \quad \text{If } \sum_{k=1}^{\infty} \frac{l(k)}{k} < \infty, \quad \text{then} \quad S_{\star}(n) = \sum_{k=n}^{\infty} \frac{l(k)}{k}, \quad S_{\star} \in \mathcal{SV} \quad \text{and} \quad \lim_{n\to\infty} \frac{S_{\star}(n)}{l(n)} = \infty;$$

$$(iv) \quad \text{If } \sum_{k=1}^{\infty} \frac{l(k)}{k} = \infty, \quad \text{then} \quad S^{\star}(n) = \sum_{k=1}^{n} \frac{l(k)}{k}, \quad S^{\star} \in \mathcal{SV} \quad \text{and} \quad \lim_{n\to\infty} \frac{S^{\star}(n)}{l(n)} = \infty\,.$$

Remark 3.1. It is easy to see, because of Theorem 3.3-(*iii*) and Theorem 3.4-(*i*), that for $l \in \mathcal{SV}$, if $\alpha > -1$, we have

$$\sum_{k=1}^{n-1} k^{\alpha} l(k) \sim \frac{(n-1)^{\alpha+1} l(n-1)}{\alpha+1} \sim \frac{n^{\alpha+1} l(n)}{\alpha+1} \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty,$$

and since $\lim_{n\to\infty} \sum_{k=1}^{n-1} k^{\alpha} l(k) = \infty$, we also get

$$\sum_{k=n_0}^{n} k^{\alpha} l(k) \sim \sum_{k=1}^{n} k^{\alpha} l(k), \qquad n \to \infty.$$

If $\lim_{n\to\infty}\sum_{k=1}^{n} k^{-1}l(k) = \infty$, we have

$$\sum_{k=n_0}^{n} k^{-1}l(k) \sim \sum_{k=1}^{n} k^{-1}l(k), \qquad n\to\infty.$$

4. Basic concepts

This section introduces the fundamental notation and statements essential for proving the main results presented in subsequent sections.

Fixed-point methods will be used to establish the existence of solutions. The following two fixed-point theorems will serve as the primary tools throughout the paper.

Theorem 4.1. (Knaster-Tarski fixed point theorem [2]) *Let X be a partially ordered Banach space with ordering $\leq$. Let M be a subset of X with the following properties: the infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $\mathcal{F} : M \to M$ be an increasing mapping, i.e. $x \geq y$ implies $\mathcal{F}x \geq \mathcal{F}y$. Then $\mathcal{F}$ has a fixed point in M.*

Theorem 4.2. (Schauder-Tychonoff fixed point theorem [2]) *Let S be a closed, convex, nonempty subset of a locally convex topological vector space X. Let T be a continuous mapping from S to itself, such that TS is relatively compact. Then T has a fixed point.*

We will apply the following theorem to prove that the appropriately constructed operator T is continuous.

Theorem 4.3. (Discrete Lebesgue's dominated convergence theorem [1]) *Let $\{a^{(m)}(k)\}$ be a double real sequence, $a^{(m)}(k) \geq 0$ for $m,k \in \mathbb{N}$ such that $\lim_{m\to\infty} a^{(m)}(k) = A(k)$, for every $k \in \mathbb{N}$. Assume that the series $\sum_{k=1}^{\infty} a^{(m)}(k)$ is totally convergent, that is, there exists a sequence $\{\alpha(k)\}$ such that $a^{(m)}(k) \leq \alpha(k)$ for all $m,k \in \mathbb{N}$ with $\sum_{k=1}^{\infty}\alpha(k) < \infty$. Then, the series $\sum_{k=1}^{\infty} A(k)$ converges and*

$$\lim_{m\to\infty}\sum_{k=1}^{\infty} a^{(m)}(k) = \sum_{k=1}^{\infty} A(k).$$

To apply the Schauder-Tychonoff fixed point theorem, the relatively compactness of the set TS must be verified, and for that purpose, the following statement will be used. This theorem represents a discrete version of the Arzela-Ascoli theorem, known as the Cheng-Patula theorem (see [12]).

Theorem 4.4. *A bounded, uniformly Cauchy subset Ω of l^∞ is relatively compact.*

The Stolz-Cesaro Theorem will be used to prove the regularity of solutions. For completeness, we recall the following variant (see [48]).

Theorem 4.5. *If $f = \{f(n)\}$ is a strictly increasing sequence of positive real numbers, such that $\lim_{n\to\infty} f(n) = \infty$, then for any sequence $g = \{g(n)\}$ of positive real numbers one has the inequalities:*

$$\liminf_{n\to\infty}\frac{\Delta f(n)}{\Delta g(n)} \leq \liminf_{n\to\infty}\frac{f(n)}{g(n)} \leq \limsup_{n\to\infty}\frac{f(n)}{g(n)} \leq \limsup_{n\to\infty}\frac{\Delta f(n)}{\Delta g(n)}.$$

In particular, if the sequence $\{\Delta f(n)/\Delta g(n)\}$ has a limit, then

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\Delta f(n)}{\Delta g(n)}. \tag{4.1}$$

In Section 6, analyzing the regularly varying solutions of the system with regularly varying coefficients, we assume $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$ and express them as follows:

$$p_i(n) = n^{\lambda_i} l_i(n), \quad q_i(n) = n^{\mu_i} m_i(n), \quad l_i, m_i \in \mathcal{SV}, \quad i = \overline{1,N}, \tag{4.2}$$

while components of the regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ of the observed system are expressed in the form

$$x_i(n) = n^{\rho_i} \xi_i(n), \quad \xi_i \in \mathcal{SV}, \quad i = \overline{1,N}. \tag{4.3}$$

We also assume that all sequences p_i, $i = \overline{1,N}$ satisfy either (I) or (II). Condition (I) is satisfied if and only if index of regularity λ_i satisfies either

$$\lambda_i > \alpha_i, \tag{4.4}$$

or

$$\lambda_i = \alpha_i \text{ and } S_i = \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} < \infty. \tag{4.5}$$

If (4.4) holds, using Theorem 3.4, the following asymptotic relation is obtained for the sequence $\pi_i = \{\pi_i(n)\}$ defined by (1.1):

$$\pi_i(n) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \ \ n \to \infty, \tag{4.6}$$

implying that $\pi_i \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right)$. Condition (II) is satisfied if and only if either

$$\lambda_i < \alpha_i, \tag{4.7}$$

or

$$\lambda_i = \alpha_i \text{ and } S_i = \sum_{n=1}^{\infty} n^{-1} l_i(n)^{-\frac{1}{\alpha_i}} = \infty, \tag{4.8}$$

Using Theorem 3.4, if (4.7) holds, the following asymptotic relation is obtained for the sequence $P_i = \{P_i(n)\}$ defined by (1.2):

$$P_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}}, \ \ n \to \infty, \tag{4.9}$$

implying that $P_i \in \mathcal{RV}\left(\frac{\alpha_i - \lambda_i}{\alpha_i}\right)$.

Also, to simplify notation we denote $A_N = \alpha_1 \alpha_2 \cdot \ldots \cdot \alpha_N$, $B_N = \beta_1 \beta_2 \cdot \ldots \cdot \beta_N$ and use matrix

$$M = \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \dots & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-2}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_1\beta_2\cdot\ldots\cdot\beta_{N-1}}{\alpha_1\alpha_2\cdot\ldots\cdot\alpha_{N-1}} \\ \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_N}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_N} & 1 & \frac{\beta_2}{\alpha_2} & \dots & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-2}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_2\beta_3\cdot\ldots\cdot\beta_{N-1}}{\alpha_2\alpha_3\cdot\ldots\cdot\alpha_{N-1}} \\ \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_N}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_N} & \frac{\beta_3\cdot\ldots\cdot\beta_N\beta_1}{\alpha_3\cdot\ldots\cdot\alpha_N\alpha_1} & 1 & \dots & \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_{N-2}}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_{N-2}} & \frac{\beta_3\beta_4\cdot\ldots\cdot\beta_{N-1}}{\alpha_3\alpha_4\cdot\ldots\cdot\alpha_{N-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\beta_{N-1}\beta_N}{\alpha_{N-1}\alpha_N} & \frac{\beta_{N-1}\beta_N\beta_1}{\alpha_{N-1}\alpha_N\alpha_1} & \frac{\beta_{N-1}\beta_N\beta_1\beta_2}{\alpha_{N-1}\alpha_N\alpha_1\alpha_2} & \dots & 1 & \frac{\beta_{N-1}}{\alpha_{N-1}} \\ \frac{\beta_N}{\alpha_N} & \frac{\beta_N\beta_1}{\alpha_N\alpha_1} & \frac{\beta_N\beta_1\beta_2}{\alpha_N\alpha_1\alpha_2} & \dots & \frac{\beta_N\beta_1\cdot\ldots\cdot\beta_{N-2}}{\alpha_N\alpha_1\cdot\ldots\cdot\alpha_{N-2}} & 1 \end{pmatrix}, \tag{4.10}$$

whose elements will be denoted by $M = (M_{ij})$. In fact, the i−th row of (M_{ij}) is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \dots, \frac{\beta_i\beta_{i+1}\cdot\ldots\cdot\beta_{i+(N-2)}}{\alpha_i\alpha_{i+1}\cdot\ldots\cdot\alpha_{i+(N-2)}}\right), \quad \alpha_{N+j} = \alpha_j,\ \beta_{N+j} = \beta_j,\ \ j = \overline{1, N-2}$$

$(i-1)-$times to the right cyclically, so that the lower triangular elements $M_{ij}, i > j$, satisfy the relation

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N}, \quad i > j, \quad i = \overline{2,N}.$$

It is easy to see that elements of matrix M satisfy for $i = \overline{1,N}$, $j = \overline{1,N}$

$$M_{i+1,i}\frac{\beta_i}{\alpha_i} = \frac{B_N}{A_N}, \quad M_{i+1,j}\frac{\beta_i}{\alpha_i} = M_{ij} \text{ for } \quad j \neq i, \quad M_{N+1,j} = M_{1,j}. \tag{4.11}$$

The next matrix also plays an important role in the proof of the main results:

$$A = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \ldots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -\frac{\beta_{N-1}}{\alpha_{N-1}} \\ -\frac{\beta_N}{\alpha_N} & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}. \tag{4.12}$$

Since,

$$\det(A) = 1 - \frac{\beta_1\beta_2 \cdot \ldots \cdot \beta_N}{\alpha_1\alpha_2 \cdot \ldots \cdot \alpha_N} > 0\,,$$

the matrix A is invertible, and its inverse matrix is given by

$$A^{-1} = \frac{A_N}{A_N - B_N}M\,. \tag{4.13}$$

Throughout the text, $n \geq n_0$ means that n is sufficiently large so that n_0 need not be the same at each occurrence.

5. Existence of primitive increasing solutions

The classification itself directly reveals the asymptotic behavior of primitive solutions. In the following theorems, using fixed-point theory, we establish the necessary and sufficient conditions for the existence of solutions of type (AC) and $(P1)$ of the system (SE) with arbitrary coefficients satisfying (b).

Theorem 5.1. *Let* (I) *holds. The system* (SE) *has a solution* $\mathbf{x} \in \mathcal{IS}$ *in which every component satisfies* (AC) *if and only if*

$$J_i^1 = \sum_{n=2}^{\infty}\left(\frac{1}{p_i(n)}\sum_{k=1}^{n-1} q_i(k)\right)^{\frac{1}{\alpha_i}} < \infty, \quad i = \overline{1,N}. \tag{5.1}$$

Proof. *The "only if" part:* Let $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ be an eventually positive and increasing solution of (SE) in which every component satisfies (AC). Then, there exist $n_0 \in \mathbb{N}$ such that $l_i = x_i(n_0) \leq x_i(n) \leq k_i$, $n \geq n_0$, $i = \overline{1,N}$. Summing the equations of (SE) first from n_0 to $n-1$, and then from n_0 to m, we get for $i = \overline{1,N}$

$$\begin{aligned} x_i(m) - x_i(n_0) &= \sum_{n=n_0}^{m}\left(\frac{1}{p_i(n)}\left(x_i^{[1]}(n_0) + \sum_{k=n_0}^{n-1} q_i(k)x_{i+1}(k+1)^{\beta_i}\right)\right)^{\frac{1}{\alpha_i}} \\ &\geq l_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{n=n_0}^{m}\left(\frac{1}{p_i(n)}\sum_{k=n_0}^{n-1} q_i(k)\right)^{\frac{1}{\alpha_i}}. \end{aligned}$$

As $x_i(m)$, is bounded for $m \geq n_0$, the last inequality implies that the partial sums of the series J_i^1, $i = \overline{1,N}$, are bounded, from which we conclude that condition (5.1) holds.

The "if" part: Suppose that (5.1) holds. Then, there exists $n_0 > 1$ such that

$$\sum_{k=n_0}^{\infty} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) \right)^{\frac{1}{\alpha_i}} < 1, \quad i = \overline{1,N}. \tag{5.2}$$

Denote with $\mathcal{L}_{n_0}$ the space of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, such that $x_i = \{x_i(n)\} \in {}^{\mathbb{N}_{n_0}}\mathbb{R}$, $i = \overline{1,N}$ are bounded. Then, $\mathcal{L}_{n_0}$ is a Banach space endowed with the norm

$$\|\mathbf{x}\| = \max_{1 \leq i \leq N} \left\{ \sup_{n \geq n_0} |x_i(n)| \right\}. \tag{5.3}$$

Set

$$\Lambda_1 = \left\{ \mathbf{x} \in \mathcal{L}_{n_0} \;\middle|\; \frac{c_i}{2} \leq x_i(n) \leq c_i, \quad n \geq n_0, \quad i = \overline{1,N} \right\}, \tag{5.4}$$

where c_i, $i = \overline{1,N}$ are positive constants such that

$$c_i \geq 2 c_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad c_{N+1} = c_1. \tag{5.5}$$

Define operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_0}}\mathbb{R} \to {}^{\mathbb{N}_{n_0}}\mathbb{R}$ by

$$\mathcal{F}_i x(n) = \frac{c_i}{2} + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) x(s+1)^{\beta_i} \right)^{1/\alpha_i}, \quad n > n_0, \quad i = \overline{1,N}, \tag{5.6}$$

and define the mapping $\Theta : \Lambda_1 \to \mathcal{L}_{n_0}$ by

$$\Theta(x_1, x_2, \ldots, x_N) = \left(\mathcal{F}_1 x_2, \mathcal{F}_2 x_3, \ldots, \mathcal{F}_N x_{N+1} \right), \quad x_{N+1} = x_1. \tag{5.7}$$

We will show that Θ has a fixed point by using the Schauder-Tychonoff fixed point theorem. Namely, the operator Θ has the following properties:

(i) Θ *maps* Λ_1 *into itself*: Let $\mathbf{x} \in \Lambda_1$. Then, using (5.2), (5.4), (5.5) and (5.6), we see that

$$\frac{c_i}{2} \leq \mathcal{F}_i x_{i+1}(n) \leq \frac{c_i}{2} + c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) \right)^{\frac{1}{\alpha_i}} \leq \frac{c_i}{2} + \frac{c_i}{2} = c_i,$$

for $i = \overline{1,N}$ and $n > n_0$.

(ii) Θ *is continuous*: Let $\varepsilon_i > 0$, $i = \overline{1,N}$ and $\{\mathbf{x}^{(m)}\}_{m \in \mathbb{N}} = \left\{ (x_1^{(m)}, x_2^{(m)}, \ldots, x_N^{(m)}) \right\}_{m \in \mathbb{N}}$, be a sequence in Λ_1 which converges to $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ as $m \to \infty$. Since, Λ_1 is closed, $\mathbf{x} \in \Lambda_1$. The rest of the proof does not depend on i, so let $i \in \{1, 2, \ldots, N\}$ be arbitrary fixed. For every $n > n_0$, we have

$$\begin{aligned} &\left| \mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n) \right| \\ &\quad \leq \sum_{k=n_0}^{n-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}} \left| \left(\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}^{(m)}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} - \left(\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \right| \\ &\quad = \sum_{k=n_0}^{n-1} a_i^{(m)}(k). \end{aligned}$$

Since

$$\begin{aligned} a_i^{(m)}(k) &\le \left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}^{(m)}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} + \left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \\ &\le 2c_{i+1}^{\frac{\beta_i}{\alpha_i}}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)\right)^{\frac{1}{\alpha_i}}, \end{aligned} \tag{5.8}$$

the assumption (5.1) implies that series $\sum_{k=n_0}^{\infty} a_i^{(m)}(k)$ is totally convergent. Also, $\lim_{m\to\infty} a_i^{(m)}(k) = 0$, so that a discrete analogue of the Lebesgue dominated convergence theorem (Theorem 4.3) yields

$$\lim_{m\to\infty} \sup_{n\ge n_0} \left|\mathcal{F}_i x_{i+1}^{(m)}(n) - \mathcal{F}_i x_{i+1}(n)\right| = 0.$$

Therefore, $\|\Theta \mathbf{x}^{(m)} - \Theta \mathbf{x}\| \to 0$ as $m \to \infty$, i.e. Θ is continuous.

(iii) $\Theta(\Lambda_1)$ *is relatively compact*: To show this, by Theorem 4.4, it is sufficient to show that $\Theta(\Lambda_1)$ is uniformly Cauchy in the topology of $\mathcal{L}_{n_0}$. For $\mathbf{x} \in \Lambda_1$ and $m > n > n_0$ we have

$$\begin{aligned} |\mathcal{F}_i x_{i+1}(m) - \mathcal{F}_i x_{i+1}(n)| &= \left|\sum_{k=n}^{m-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}\right| \\ &\le \sum_{k=n}^{m-1} \frac{1}{p_i(k)^{\frac{1}{\alpha_i}}}\left(\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \\ &\le c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n}^{m-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)\right)^{\frac{1}{\alpha_i}}. \end{aligned}$$

According to the assumption (5.1), it follows that $\Theta(\Lambda_1)$ is uniformly Cauchy.

Therefore, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, implying the existence of a fixed point $\mathbf{x} \in \Lambda_1$ of the mapping Θ, which satisfies

$$x_i(n) = \frac{c_i}{2} + \sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad n \ge n_0, \quad i = \overline{1,N}.$$

It is clear that $\mathbf{x}$ is a positive increasing solution of (SE) whose all components tend to constants. □

In order to prove the existence of a solution of type (P) for the system (SE), we consider the system

$$\Delta\left(\frac{1}{q_i(n)^{1/\beta_i}} |\Delta y_i(n)|^{\frac{1}{\beta_i}-1} \Delta y_i(n)\right) = \frac{1}{p_{i+1}(n+1)^{\frac{1}{\alpha_{i+1}}}} |y_{i+1}(n+1)|^{\frac{1}{\alpha_{i+1}}-1} y_{i+1}(n+1), \tag{RSE}$$

where $i = \overline{1,N}$, $y_{N+1} = y_1$, $n \in \mathbb{N}$. This system is of the same type as the original one, but its coefficients are obtained by interchanging $p_i(n)$ with $q_i(n)^{-1/\beta_i}$, and $q_i(n)$ with $p_{i+1}(n+1)^{-1/\alpha_{i+1}}$, $i = \overline{1,N}$. So, the system (RSE) is referred as the reciprocal system of the system (SE) (see [9]). If $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is a solution of the system (SE), then the vector $\mathbf{y} = \mathbf{x}^{[1]} = (x_1^{[1]}, x_2^{[1]}, \ldots, x_N^{[1]})$, whose components are the quasi-differences of the components of $\mathbf{x}$, is a solution of the system (RSE). Moreover,

$$\mathbf{x} \in \mathcal{IS} \iff \mathbf{y} \in \mathcal{IS}.$$

Theorem 5.2. *Let* (*II*) *holds. The system* (*SE*) *has a solution* $\mathbf{x} \in \mathcal{IS}$ *in which every component satisfies* (*P*) *if and only if*

$$J_i^2 = \sum_{n=1}^{\infty} q_i(n) \left(\sum_{k=1}^{n} \frac{1}{p_{i+1}(k)^{1/\alpha_{i+1}}} \right)^{\beta_i} < \infty, \quad i = \overline{1,N}. \tag{5.9}$$

Since the series J_i^2, $i = \overline{1,N}$ for the system (*SE*) plays the same role as the series J_i^1, $i = \overline{1,N}$ for the system (*RSE*) (and vice versa), the proof of the theorem follows by applying Theorem 5.1 to the reciprocal system (*RSE*) .

6. Asymptotic behavior of strongly increasing regularly varying solutions

Strongly increasing solution of (*SE*) of type (*SI*1) as well as of type (*SI*2) is the solution of the system

$$x_i(n) = a_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(n)} \left(b_i + \sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right) \right)^{\frac{1}{\alpha_i}}, \quad i = \overline{1,N} \tag{6.1}$$

for some constant $n_0 \in \mathbb{N}, n_0 > 1$ and $a_i = x_i(n_0) > 0, b_i = x_i^{[1]}(n_0 - 1) \geq 0$. In view of (*SI*), the strongly increasing solution is required to satisfy

$$\sum_{n=n_0}^{\infty} q_i(n) x_{i+1}(n+1) = \infty, \quad i = \overline{1,N}. \tag{6.2}$$

To solve the system of equations (6.1) with (6.2) in the class of regularly varying sequences we will analyse the system of asymptotic relations

$$x_i(n) \sim \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n \to \infty \quad i = \overline{1,N}. \tag{6.3}$$

which can be regarded as an approximation of the system (6.1). Our objective in this section is to provide the necessary and sufficient conditions for the existence of regularly varying solutions of this system with positive indices of regularity. We exclude slowly varying solutions from our analysis in this section, due to computational complexity.

To accomplish the goal, coefficients p_i and q_i are assumed to be regularly varying sequences. Accordingly, we use the expressions (4.2) for p_i and q_i, while for the components x_i of the solution vector $\mathbf{x}$ of (*SE*) , we utilize the expression (4.3).

6.1. Regularly varying solutions of the system (6.3)

The solution of the problem of determining the necessary and sufficient conditions for the system of asymptotic relations (6.3) to have a regularly varying solution $\mathbf{x}$ of the positive regularity index $(\rho_1, \rho_2, \ldots, \rho_N)$ in the case (*I*) is given by the following Theorem.

Theorem 6.1. *Let* $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$ *and suppose that* (*I*) *holds. The system of asymptotic relations* (6.3) *has a regularly varying solution* $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ *with* $\rho_i > 0$, $i = \overline{1,N}$ *if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} > 0 \, , \tag{6.4}$$

in which case ρ_i are given by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = \overline{1,N} \tag{6.5}$$

and the asymptotic behavior of any such solution is governed by the unique formula

$$x_i(n) \sim \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j} \right)^{M_{ij}} \right]^{\frac{A_N}{A_N-B_N}}, \quad n \to \infty, \quad i = \overline{1,N} \tag{6.6}$$

with D_j, $j = \overline{1,N}$ given by

$$D_j = \left(\lambda_j - \alpha_j + \alpha_j \rho_j\right)^{\frac{1}{\alpha_j}} \rho_j, \quad j = \overline{1,N}. \tag{6.7}$$

Proof. *The 'only if' part:* Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with all $\rho_i > 0$ be a solution of (6.3). Then, by Theorem 3.3 - (v) and (vi) we have that all components of the solution $\mathbf{x}$ satisfies (2.1) and that $\lim_{n\to\infty} x_i(n) = \infty$, $i = \overline{1,N}$. Since $x_i^{[1]}$, $i = \overline{1,N}$ are positive and increasing it follows that

$$\text{(a)} \quad \lim_{n\to\infty} x_i^{[1]}(n) = c_i > 0 \quad \text{or} \quad \text{(b)} \quad \lim_{n\to\infty} x_i^{[1]}(n) = \infty.$$

From classification, we see that case (a) implies that $\lim_{n\to\infty} x_i(n) = const.$, that is $x_i \in \mathcal{SV}$, which is impossible. Thus, for $x_i^{[1]}$, $i = \overline{1,N}$ we have that (b) holds.

Using (4.2) and (4.3), we obtain for all $i = \overline{1,N}$

$$\sum_{k=n_0-1}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \sim \sum_{k=n_0-1}^{n-1} k^{\mu_i + \beta_i \rho_{i+1}} m_i(k) \xi_{i+1}(k)^{\beta_i}, \quad n > n_0. \tag{6.8}$$

Then, (b) implies that $\mu_i + \beta_i \rho_{i+1} \geq -1$, $i = \overline{1,N}$. If the equality holds for some i, then from (6.8) we obtain

$$\left(\frac{1}{p_i(n)} \sum_{k=n_0-1}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim n^{-\frac{\lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} K_i(n)^{\frac{1}{\alpha_i}}, \quad n \to \infty, \tag{6.9}$$

where

$$K_i(n) = \sum_{k=n_0-1}^{n-1} k^{-1} m_i(k) \xi_{i+1}(k)^{\beta_i}, \quad K_i \in \mathcal{SV}.$$

Summing (6.9) from n_0 to $n-1$ we get

$$x_i(n) \sim \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \sum_{k=n_0}^{n-1} k^{-\frac{\lambda_i}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} K_i(k)^{\frac{1}{\alpha_i}}, \quad n \to \infty. \tag{6.10}$$

Since $x_i(n) \to \infty, n \to \infty$, it must be $-\frac{\lambda_i}{\alpha_i} \geq -1$, i.e. $\lambda_i \leq \alpha_i$. On the other hand, (I) is satisfed so that either (4.4) or (4.5) holds. Therefore, it must be $\lambda_i = \alpha_i$, i.e. $x_i \in \mathcal{SV}$, contradicting with the assumption $\rho_i > 0$. Accordingly, in (6.8) $\mu_i + \beta_i \rho_{i+1} > -1$ for all i and application of Theorem 3.4 gives

$$\left(\frac{1}{p_i(n)} \sum_{k=n_0-1}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i + \beta_i \rho_{i+1} + 1)^{\frac{1}{\alpha_i}}}, \tag{6.11}$$

as $n \to \infty$. Summing (6.11) from n_0 to $n-1$ and using (6.3) we have that

$$x_i(n) \sim \sum_{k=n_0}^{n-1} \frac{k^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}} l_i(k)^{-\frac{1}{\alpha_i}} m_i(k)^{\frac{1}{\alpha_i}} \xi_{i+1}(k)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i+\beta_i\rho_{i+1}+1)^{\frac{1}{\alpha_i}}}, \quad i=\overline{1,N}. \tag{6.12}$$

Since $x_i(n) \to \infty$, $n \to \infty$, from (6.12) we conclude that $(-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1)/\alpha_i \geq -1$, $i=\overline{1,N}$. All inequalities should be strict, because if the equality holds for some i, it leads to the contradiction that $x_i \in \mathcal{SV}$.

Applying Theorem 3.4, from (6.12) we get

$$x_i(n) \sim \frac{n^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \xi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\mu_i+\beta_i\rho_{i+1}+1)^{\frac{1}{\alpha_i}} \left(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1\right)}, \quad n \to \infty, \quad i=\overline{1,N}. \tag{6.13}$$

From the previous relation, since $x_i \in \mathcal{RV}(\rho_i)$, $i=\overline{1,N}$, we see that

$$\rho_i = \frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1, \quad i=\overline{1,N}, \quad \rho_{N+1}=\rho_1 \tag{6.14}$$

which is equivalent to a linear cyclic system of equations

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\alpha_i-\lambda_i+\mu_i+1}{\alpha_i}, \quad i=\overline{1,N}, \quad \rho_{N+1}=\rho_1. \tag{6.15}$$

The matrix of the system (6.15) is given by (4.12). As shown in Section 4, the matrix A is invertible, implying that the system (6.15) has the unique solution $(\rho_1,\dots,\rho_N)$. Using (4.13), we derive that these ρ_i are given explicitly by (6.5). It is obvious that $\rho_i > 0$, $i=\overline{1,N}$ if and only if (6.4) holds. Using (4.2) and (4.3) we can transform (6.13) in the form

$$x_i(n) \sim \frac{n^{\frac{\alpha_i+1}{\alpha_i}} p_i(n)^{-\frac{1}{\alpha_i}} q_i(n)^{\frac{1}{\alpha_i}} x_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty, \tag{6.16}$$

where $D_i, i=\overline{1,N}$ are given by (6.7). It is easy to obtain from (6.16) that each component x_i of $\mathcal{RV}$ solution **x** satisfies the explicit asymptotic formula (6.6).

The 'if' part: Suppose now that (6.4) holds. Define ρ_i with (6.5) and sequences X_i, $i=\overline{1,N}$, by

$$X_i(n) = \left[\prod_{j=1}^{N} \left(\frac{n^{\frac{\alpha_j+1}{\alpha_j}} p_j(n)^{-\frac{1}{\alpha_j}} q_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}, \quad i=\overline{1,N}. \tag{6.17}$$

where D_j are given by (6.7). We will show that sequences X_i, $i=\overline{1,N}$, satisfy the system of asymptotic relations (6.3), for arbitrary $n_0 \in \mathbb{N} \setminus \{1\}$, where $X_{N+1}=X_1$. Clearly, $X_i \in \mathcal{RV}(\rho_i)$, $i=\overline{1,N}$, so it can be represent as

$$X_i(n) = n^{\rho_i}\chi_i(n), \quad i=\overline{1,N}, \tag{6.18}$$

where

$$\chi_i(n) = \left[\prod_{j=1}^{N} \left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}}, \quad i=\overline{1,N}.$$

Using (6.4) and (6.5) we have that $\rho_i > 0$, $i = \overline{1,N}$. From "the only if" part we conclude that $(\rho_1, \dots, \rho_N)$, with ρ_i given by (6.5), is the unique solution of the linear cyclic system of equations (6.15) (or equivalent (6.14)). In order to apply Theorem 3.4, it must hold that $\mu_i + \rho_{i+1}\beta_i > -1$, $i = \overline{1,N}$. From the fact $\rho_i > 0$ and (6.14) we have

$$\mu_i + \rho_{i+1}\beta_i = (\rho_i - 1)\alpha_i + \lambda_i - 1 > -\alpha_i + \lambda_i - 1 \geq -1.$$

Therefore, using (6.14) and applying Theorem 3.4, we obtain

$$\left(\frac{1}{p_i(n)}\sum_{s=n_0-1}^{n-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i-1} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{(\alpha_i\rho + \lambda_i - \alpha_i)^{\frac{1}{\alpha_i}}}, \quad n \to \infty,$$

and

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \sim \frac{n^{\rho_i} l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}} \chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}}}{D_i}, \quad n \to \infty. \tag{6.19}$$

Using relation (4.11) for matrix elements M_{ij}, the right hand side of the relation (6.19) can be transformed as follows

$$\frac{l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}}}{D_i}\chi_{i+1}(n)^{\frac{\beta_i}{\alpha_i}} = \frac{l_i(n)^{-\frac{1}{\alpha_i}} m_i(n)^{\frac{1}{\alpha_i}}}{D_i}\left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}}\right]^{\frac{A_N}{A_N-B_N}}$$

$$= \left[\prod_{j=1}^{N}\left(\frac{l_j(n)^{-\frac{1}{\alpha_j}} m_j(n)^{\frac{1}{\alpha_j}}}{D_j}\right)^{M_{ij}}\right]^{\frac{A_N}{A_N-B_N}} = \chi_i(n), \quad i = \overline{1,N}, \quad \chi_{N+1} = \chi_N,$$

so from (6.19), we obtain that X_i, $i = \overline{1,N}$ satisfy (6.3). □

Next we consider the case (*II*). Since (*II*) holds if and only if (4.7) or (4.8) is satisfied, we will distinguish two cases. Namely, in the following two theorems we establish necessary and sufficient conditions for the existence of a solution $\mathbf{x}$ of the system of asymptotic relations (6.3), with an index $(\rho_1, \rho_2, \dots, \rho_N)$ such that $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if (4.7) holds, and such that $\rho_i > 0$ if (4.8) holds. In both theorems, the precise asymptotic behavior of the solutions will be determined.

Theorem 6.2. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose* (4.7) *holds. The system of asymptotic relations* (6.3) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij}\left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j\alpha_{j+1}}\right) > 0, \quad i = \overline{1,N} \tag{6.20}$$

holds, where $\alpha_{N+1} = \alpha_1$, $\lambda_{N+1} = \lambda_1$, in which case ρ_i are uniquely determined by (6.5) *and the asymptotic behavior of any such solution is governed by the unique formulas* (6.6) *where D_j, $j = \overline{1,N}$ are given by* (6.7).

Proof. *The "only if" part:* Let $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \dots, \rho_N)$ with all $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$ be a solution of (6.3). Then, all x_i satisfy (2.1) by Theorem 3.3 - (*vi*). Since indices of regularity of x_i/P_i, $i = \overline{1,N}$ are greater then zero, from Theorem 3.3 - (*v*), we have that $\lim_{n\to\infty} x_i(n)/P_i(n) = \infty$ and $\lim_{n\to\infty} x_i(n) = \infty$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = \infty$. As in the previous Theorem, we obtain (6.8), which due to (6.2) implies that $\mu_i + \beta_i\rho_{i+1} \geq -1$, $i = \overline{1,N}$. If for some i

equality holds, then we have (6.9) and (6.10), which using the Theorem 3.4 and assumption $\lambda_i < \alpha_i$, $i = \overline{1,N}$ gives

$$x_i(n) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} n^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(n)^{-\frac{1}{\alpha_i}} K_i(n)^{\frac{1}{\alpha_i}}, n \to \infty.$$

This implies that $\rho_i = \frac{\alpha_i - \lambda_i}{\alpha_i}$, which is a contradiction. Therefore, $\mu_i + \beta_i \rho_{i+1} > -1$ for $i = \overline{1,N}$. Proceeding exactly as in the proof of Theorem 6.1, we obtain that for each component x_i of the solution $\mathbf{x}$, (6.12) holds. Since $x_i(n) \to \infty$, $n \to \infty$, from (6.12) we conclude that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \geq -1$, $i = \overline{1,N}$. All inequalities should be strict because the equality for some i would imply that $0 < \mu_i + \beta_i \rho_{i+1} + 1 = \lambda_i - \alpha_i$, which is impossible. Therefore, $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i > -1$, $i = \overline{1,N}$. Applying Theorem 3.4, from (6.12) we get (6.13). The relation (6.13) implies that ρ_i, $i = \overline{1,N}$, satisfy (6.14) i.e. ρ_i, $i = \overline{1,N}$, will be determined as a unique solution of the linear cyclic system (6.15). Thus, ρ_i, $i = \overline{1,N}$ are given explicitly by (6.5). Let us denote $d_i = \rho_i - \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$. Then, the system (6.15) becomes

$$d_i - \frac{\beta_i}{\alpha_i} d_{i+1} = \frac{\mu_i + 1}{\alpha_i} + \frac{\beta_i(\alpha_{i+1} - \lambda_{i+1})}{\alpha_i \alpha_{i+1}}, \quad i = \overline{1,N}, \quad d_{N+1} = d_1. \tag{6.21}$$

Matrix of the system (6.21) is given by (4.12). Since A is nonsingular matrix, the system (6.21) has a unique solution d_i, $i = \overline{1,N}$, where

$$d_i = \sum_{j=1}^{N} M_{ij} \left(\frac{\mu_j + 1}{\alpha_j} + \frac{\beta_j(\alpha_{j+1} - \lambda_{j+1})}{\alpha_j \alpha_{j+1}} \right), \quad i = \overline{1,N}. \tag{6.22}$$

Using that $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if $d_i > 0$, we conclude that the condition (6.20) is satisfied. Like in the proof of the previous Theorem, transformation of the asymptotic relation (6.13) gives that each x_i satisfies the asymptotic relation (6.6).

The "if" part: Suppose now that (6.20) holds, define ρ_i and D_i with (6.5) and (6.7), respectively, and let $X_i \in \mathcal{RV}(\rho_i)$, $i = \overline{1,N}$ be sequences defined with (6.17). That X_i, $i = \overline{1,N}$, satisfy the system of asymptotic relations (6.3) can be verified as in the proof of previous Theorem. The regularity indices ρ_i, $i = \overline{1,N}$ of X_i are the unique solution of the linear cyclic system (6.15), which is equivalent to the system (6.21). Thus, the assumption (6.20) implies that solutions d_i, $i = \overline{1,N}$, of the system (6.21) are positive, implying that $\rho_i > (\alpha_i - \lambda_i)/\alpha_i$, $i = \overline{1,N}$. □

Theorem 6.3. *Let $p_i \in \mathcal{RV}(\lambda_i)$, $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose* (4.8) *holds. The system of asymptotic relations* (6.3) *has a regularly varying solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i > 0$, $i = \overline{1,N}$ if and only if*

$$\sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j} > 0, \quad i = \overline{1,N} \tag{6.23}$$

in which case ρ_i are uniquely determined by

$$\rho_i = \frac{A_N}{A_N - B_N} \sum_{j=1}^{N} M_{ij} \frac{\mu_j + 1}{\alpha_j}, \quad i = \overline{1,N} \tag{6.24}$$

and the asymptotic behavior of any such solution is governed by the unique formulas (6.6) *with $D_j = \left(\alpha_j \rho_j^{\alpha_j + 1}\right)^{1/\alpha_j}$, $j = \overline{1,N}$.*

Proof. *The "only if" part:* Suppose that the system (6.3) has a solution $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{RV}(\rho_1, \ldots, \rho_N)$, with $\rho_i > 0, i = \overline{1,N}$. Due to fact that $P_i \in \mathcal{SV}$, for all $i = \overline{1,N}$, index of regularity of x_i/P_i is $\rho_i > 0$. Thus, from Theorem 3.3 - (v), we have that $\lim_{n\to\infty} x_i(n)/P_i(n) = \infty$ and $\lim_{n\to\infty} x_i(n) = \infty$, implying that $\lim_{n\to\infty} x_i^{[1]}(n) = \infty$. Using (4.2) and (4.3), we obtain (6.8), which due to (6.2) implies that $\mu_i + \beta_i\rho_{i+1} \geq -1$, $i = \overline{1,N}$. As previously all inequalities are strict, because if equality holds for some i, then (6.10) implies that $x_i \in \mathcal{SV}$, which is a contradiction. Therefore, with $\mu_i + \beta_i\rho_{i+1} > -1$, $i = \overline{1,N}$, application of Theorem 3.4 to (6.8) gives (6.11) and since $x_i(n) \to \infty$, $n \to \infty$, it must be $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i \geq -1, i = \overline{1,N}$. If equality holds for any i, then $\mu_i + \beta_i\rho_{i+1} = -1$, which is impossible. Thus, $(-\lambda_i + \mu_i + \beta_i\rho_{i+1} + 1)/\alpha_i > -1, i = \overline{1,N}$. Summing (6.11) from n_0 to $n-1$ and using Theorem 3.4, we get (6.13). Using assumption $\lambda_i = \alpha_i$, $i = \overline{1,N}$, from (6.13) we obtain the following cyclic system

$$\rho_i - \frac{\beta_i}{\alpha_i}\rho_{i+1} = \frac{\mu_i + 1}{\alpha_i}, \quad i = \overline{1,N}, \quad \rho_{N+1} = \rho_1. \tag{6.25}$$

Matrix of the system (6.25) is given by (4.12), and therefore, the system has a unique solution $\rho_i, i = \overline{1,N}$ given by (6.24). All ρ_i are positive if and only if (6.23) holds. Proceeding exactly as in the proof of the Theorem 6.1, we conclude that the asymptotic behavior of regularly varying solution $\mathbf{x}$ is given by (6.6), with $D_j = \left(\alpha_j\rho_j^{\alpha_j+1}\right)^{1/\alpha_j}, j = \overline{1,N}$.

The "if" part: The proof of the "if" part of the Theorem is the same as the proof of Theorem 6.1. □

6.2. Regularly varying solutions of the system (*SE*)

Now, we proceed to the main results of this section. The following three theorems provide the necessary and sufficient conditions for the system (*SE*) with regularly varying coefficients p_i and q_i to have a strongly increasing regularly varying solution of of positive indices.

Theorem 6.4. *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose that* (*I*) *holds. The system* (*SE*) *possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i > 0$, $i = \overline{1,N}$, if and only if* (6.4) *holds, in which case ρ_i are given by* (6.5) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (6.6)*, with D_j, $j = \overline{1,N}$ given by* (6.7)*.*

Theorem 6.5. *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose* (4.7) *holds. The system* (*SE*) *possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i}$, $i = \overline{1,N}$, if and only if* (6.20) *holds, in which case ρ_i are given by* (6.5) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (6.6)*, with D_j, $j = \overline{1,N}$ given by* (6.7)*.*

Theorem 6.6. *Let $p_i \in \mathcal{RV}(\lambda_i)$ and $q_i \in \mathcal{RV}(\mu_i)$, $i = \overline{1,N}$. Suppose* (4.8) *holds. The system* (*SE*) *possesses a solution $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$ with $\rho_i > 0$, $i = \overline{1,N}$, if and only if* (6.23) *holds, in which case ρ_i are given by* (6.24) *and the asymptotic behavior of any such solution $\mathbf{x}$ is governed by the unique formula* (6.6)*, with $D_j = \left(\alpha_j\rho_j^{\alpha_j+1}\right)^{1/\alpha_j}$, $j = \overline{1,N}$.*

We remark that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorem 6.1, Theorem 6.2 and Theorem 6.3.

Proof of the "if" part of Theorem 6.4: Suppose (6.4) is satisfied. Let we define the sequences $X_i = \{X_i(n)\} \in \mathcal{RV}(\rho_i)$ by (6.17), where D_j for $j = \overline{1,N}$ are given by (6.7). As we have shown in the proof of the Theorem 6.1, X_i, $i = \overline{1,N}$ satisfy the system of asymptotic relations (6.3), implying that there exists $n_1 > n_0$ such that

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \leq 2X_i(n), \quad n > n_1, \quad i = \overline{1,N}. \tag{6.26}$$

As regularly varying function of positive index is asymptotic to an increasing sequence, we may assume, without loss of generality, that X_i is eventually increasing. It is possible to choose $n_2 > n_1 + 1$ so large that

$$\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \geq \frac{1}{2}X_i(n), \quad n \geq n_2, \quad i = \overline{1,N}. \tag{6.27}$$

Let we choose positive constants c_i and C_i so that

$$c_i \leq \frac{1}{2}c_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad C_i \geq 4C_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad c_{N+1} = c_1, \quad C_{N+1} = C_1. \tag{6.28}$$

An example of such choices is

$$c_i = \left(\frac{1}{2}\right)^{\frac{A_N}{A_N-B_N}\sum_{j=1}^N M_{ij}}, \quad C_i = 4^{\frac{A_N}{A_N-B_N}\sum_{j=1}^N M_{ij}} \tag{6.29}$$

for $i = \overline{1,N}$. Clearly $c_i \leq 1 \leq C_i$. Constants c_i and C_i can be chosen so that

$$2c_iX_i(n_2) \leq C_iX_i(n_1), \quad i = \overline{1,N}, \tag{6.30}$$

because these constants are independent of X_i as well as of the choice of n_1 and n_2.

Consider the space Υ_{n_1} of all vectors $\mathbf{x} = (x_1, x_2, \ldots, x_N)$, $x_i \in {}^{\mathbb{N}_{n_1}}\mathbb{R}$, $i = \overline{1,N}$, such that $\{x_i(n)/X_i(n)\}$, $i = \overline{1,N}$ are bounded. Then, Υ_{n_1} is a Banach space with the norm

$$\|\mathbf{x}\| = \max_{1\leq i\leq N}\left\{\sup_{n\geq n_1}\left|\frac{x_i(n)}{X_i(n)}\right|\right\}.$$

Further, Υ_{n_1} is partially ordered, with the usual pointwise ordering $\leq$: For $\mathbf{x}, \mathbf{y} \in \Upsilon_{n_1}$, $\mathbf{x} \leq \mathbf{y}$ means $x_i(n) \leq y_i(n)$ for all $n \geq n_1 + 1$ and $i = \overline{1,N}$. Define the subset $\mathcal{X} \subset \Upsilon_{n_1}$ with

$$\mathcal{X} = \left\{ \mathbf{x} \in \Upsilon_{n_1} \,\middle|\, c_iX_i(n) \leq x_i(n) \leq C_iX_i(n), \quad n > n_1, \quad i = \overline{1,N}\right\}. \tag{6.31}$$

It is easy to see that for any $\mathbf{x} \in \mathcal{X}$, the norm of $\mathbf{x}$ is finite and that for any subset $\mathcal{B} \subset \mathcal{X}$, $\inf \mathcal{B} \in \mathcal{X}$ and $\sup \mathcal{B} \in \mathcal{X}$. Define the operators $\mathcal{F}_i : {}^{\mathbb{N}_{n_1}}\mathbb{R} \to {}^{\mathbb{N}_{n_1}}\mathbb{R}$ by

$$\mathcal{F}_ix(n) = b_i + \sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)x(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}, \quad n \geq n_1, \quad i = \overline{1,N}, \tag{6.32}$$

where b_i are positive constants such that

$$c_iX_i(n_2) \leq b_i \leq \frac{1}{2}C_iX_i(n_1), \quad i = \overline{1,N}, \tag{6.33}$$

and define the mapping $\Phi : \mathcal{X} \to \Upsilon_{n_1}$ by

$$\Phi(x_1, x_2, \ldots, x_N) = \left(\mathcal{F}_1x_2, \mathcal{F}_2x_3, \ldots, \mathcal{F}_Nx_{N+1}\right), \ x_{N+1} = x_1. \tag{6.34}$$

We will show that Φ has a fixed point by using Theorem 4.1. Namely, the operator Φ has the following properties:

(i) Φ *maps* $\mathcal{X}$ *into itself*: Let $\mathbf{x} \in \mathcal{X}$. Then, using (6.26)-(6.34), we see that

$$\begin{aligned}
\mathcal{F}_ix_{i+1}(n) &\leq \frac{1}{2}C_iX_i(n_1) + C_{i+1}^{\frac{\beta_i}{\alpha_i}}\sum_{k=n_0}^{n-1}\left(\frac{1}{p_i(k)}\sum_{s=n_0-1}^{k-1} q_i(s)X_{i+1}(s+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}} \\
&\leq \frac{1}{2}C_iX_i(n_1) + 2C_{i+1}^{\frac{\beta_i}{\alpha_i}}X_i(n) \leq \frac{1}{2}C_iX_i(n) + \frac{1}{2}C_iX_i(n) = C_iX_i(n)
\end{aligned}$$

for $n > n_1$ and

$$\mathcal{F}_i x_{i+1}(n) \geq b_i \geq c_i X_i(n_2) \geq c_i X_i(n), \quad \text{for} \quad n_1 < n < n_2,$$

$$\mathcal{F}_i x_{i+1}(n) \geq c_{i+1}^{\frac{\beta_i}{\alpha_i}} \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}} \geq \frac{1}{2} c_{i+1}^{\frac{\beta_i}{\alpha_i}} X_i(n) \geq c_i X_i(n), \quad n \geq n_2.$$

This shows that $\Phi \mathbf{x} \in \mathcal{X}$, that is, Φ is a self-map on $\mathcal{X}$.

(ii) Φ *is increasing*, i.e. for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$, $\mathbf{x} \leq \mathbf{y}$ implies $\Phi \mathbf{x} \leq \Phi \mathbf{y}$.

Thus, all the hypotheses of Theorem 4.1 are fulfilled implying the existence of a fixed point $\mathbf{x} \in \mathcal{X}$ of Φ, which satisfies

$$x_i(n) = \mathcal{F}_i x_{i+1}(n) = b_i + \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) x_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \quad n > n_1,$$

for $i = \overline{1,N}$. This shows that $\mathbf{x} \in \mathcal{X}$ is a positive and increasing solution of system (SE).

It remains to verify that $\mathbf{x} \in \mathcal{RV}(\rho_1, \rho_2, \ldots, \rho_N)$. We define

$$u_i(n) = \sum_{k=n_0}^{n-1} \left(\frac{1}{p_i(k)} \sum_{s=n_0-1}^{k-1} q_i(s) X_{i+1}(s+1)^{\beta_i} \right)^{\frac{1}{\alpha_i}}, \; i = \overline{1,N},$$

and put

$$r_i = \liminf_{n\to\infty} \frac{u_i(n)}{x_i(n)}, \quad R_i = \limsup_{n\to\infty} \frac{u_i(n)}{x_i(n)}.$$

Using (6.31) and

$$u_i(n) \sim X_i(n), \quad n \to \infty, \quad i = \overline{1,N}, \tag{6.35}$$

it follows that $0 < r_i \leq R_i < \infty$, $i = \overline{1,N}$. Using Theorem 4.5 we obtain

$$\begin{aligned}
r_i &\geq \liminf_{n\to\infty} \frac{\Delta u_i(n)}{\Delta x_i(n)} = \liminf_{n\to\infty} \frac{\left(\frac{1}{p_i(n)} \sum_{k=n_0-1}^{n-1} q_i(k) X_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}}{\left(\frac{1}{p_i(n)} \sum_{k=n_0-1}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i}\right)^{\frac{1}{\alpha_i}}} \\
&= \liminf_{n\to\infty} \left(\frac{\sum_{k=n_0-1}^{n-1} q_i(k) X_{i+1}(k+1)^{\beta_i}}{\sum_{k=n_0-1}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \left(\liminf_{n\to\infty} \frac{\sum_{k=n_0-1}^{n-1} q_i(k) X_{i+1}(k+1)^{\beta_i}}{\sum_{k=n_0-1}^{n-1} q_i(k) x_{i+1}(k+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} \\
&\geq \left(\liminf_{n\to\infty} \frac{q_i(n) X_{i+1}(n+1)^{\beta_i}}{q_i(n) x_{i+1}(n+1)^{\beta_i}} \right)^{\frac{1}{\alpha_i}} = \liminf_{n\to\infty} \left(\frac{X_{i+1}(n+1)}{x_{i+1}(n+1)} \right)^{\frac{\beta_i}{\alpha_i}} = r_{i+1}^{\frac{\beta_i}{\alpha_i}}
\end{aligned}$$

where (6.35) has been used in the last step. Thus, r_i satisfies the cyclic system of inequalities

$$r_i \geq r_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad r_{N+1} = r_1. \tag{6.36}$$

If we take the upper limits instead of the lower limits, we are led to the cyclic system of inequalities

$$R_i \leq R_{i+1}^{\frac{\beta_i}{\alpha_i}}, \quad i = \overline{1,N}, \quad R_{N+1} = R_1. \tag{6.37}$$

From (6.36) and (6.37) we easily see

$$r_i \geq r_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}}, \quad R_i \leq R_i^{\frac{\beta_1\beta_2\cdots\beta_N}{\alpha_1\alpha_2\cdots\alpha_N}},$$

whence, because of the hypothesis $\beta_1\beta_2 \dots \beta_N/\alpha_1\alpha_2 \dots \alpha_N < 1$, we find that $r_i \geq 1$ and $R_i \leq 1$, $i = \overline{1,N}$. It follows therefore that $r_i = R_i = 1$ i.e. $\lim_{n\to\infty} u_i(n)/x_i(n) = 1$ for $i = \overline{1,N}$. Combined this with (6.35) implies that $x_i(n) \sim u_i(n) \sim X_i(n)$ as $n \to \infty$, which shows that each x_i is a regularly varying sequence of index $\rho_i > 0$. Thus, the proof of the "if" part of Theorem 6.5 is completed. □

The "if" part of the Theorem 6.5 and the Theorem 6.6, can be proved in essentially the same way as the "if" part of the Theorem 6.4.

Application. Obtained results can be applied to the well-known second-order difference equation of Thomas-Fermy type

$$\Delta(p(n)|\Delta x(n)|^{\alpha-1}\Delta x(n)) = q(n)|x(n+1)|^{\beta-1}x(n+1), \tag{6.38}$$

with $p \in \mathcal{RV}(\lambda)$ and $q \in \mathcal{RV}(\mu)$, which has been studied in [21, 22]. As a direct consequence of Theorem 6.4 and Theorem 6.5, we have Theorem 3.1 from [21]. However, in the existing literature the case $p \in \mathcal{RV}(\alpha)$ has not been considered, due to the calculation difficulty. For that reason, as a consequence of Theorem 6.4 and Theorem 6.6, we obtain a new result for the equation (6.38) with $p \in \mathcal{RV}(\alpha)$.

Theorem 6.7. *Let $p \in \mathcal{RV}(\alpha)$ and $q \in \mathcal{RV}(\mu)$. The equation* (6.38) *possesses a regularly varying solution of index $\rho > 0$ if and only if $\mu > -1$, in which case ρ is given by*

$$\rho = \frac{\mu+1}{\alpha-\beta}, \tag{6.39}$$

and the asymptotic behavior of any such solution x is governed by the unique formula

$$x(n) \sim \left[\frac{n^{\alpha+1}p(n)^{-1}q(n)}{\alpha\rho^{\alpha+1}}\right]^{\frac{1}{\alpha-\beta}}, \quad n \to \infty.$$

The following examples illustrates the results obtained in this section.

Example 6.1. Consider the following cyclic system of difference equations

$$\Delta\left(n^3(\log n)^2(\Delta x_1(n))^2\right) = \frac{\gamma_1(n)}{n^2(\log n)^2}(x_2(n+1))^2$$
$$\Delta\left(n^2\log n(\Delta x_2(n))^{\frac{3}{2}}\right) = \gamma_2(n)n^3(\log n)^5(x_1(n+1))^{\frac{1}{2}}, \; n \geq 2,$$

where $\gamma_1(n)$ and $\gamma_2(n)$ are positive real-valued sequences such that $\lim_{n\to\infty}\gamma_1(n) = \delta_1$ and $\lim_{n\to\infty}\gamma_2(n) = \delta_2$. From this system, we see that $\alpha_1 = 2$, $\alpha_2 = 3/2$, $\beta_1 = 2$, $\beta_2 = 1/2$, $\{p_1(n)\} \in \mathcal{RV}(3)$, $\{p_2(n)\} \in \mathcal{RV}(2)$, $\{q_1(n)\} \in \mathcal{RV}(-2)$, and $\{q_2(n)\} \in \mathcal{RV}(3)$. Also, $A_2 = 3 > 1 = B_2$ and the matrix M is

$$M = \begin{pmatrix} 1 & 1 \\ \frac{1}{3} & 1 \end{pmatrix}.$$

Since, $\lambda_1 > \alpha_1$ and $\lambda_2 > \alpha_2$ we conclude that condition (I) holds. Also,

$$\text{For } i=1: \sum_{j=1}^{N} M_{1j}\frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} = \frac{4}{3} > 0$$

$$\text{For } i=2: \sum_{j=1}^{N} M_{2j}\frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} = 2 > 0,$$

so condition (6.4) holds. Therefore, according to Theorem 6.4 we conclude that the considered system has a (strongly increasing) $\mathcal{RV}-$solution of index $(\rho_1, \rho_2) = (2, 3)$. The asymptotic behavior of components x_1 and x_2 of such a solution are

$$x_1(n) \sim \frac{\delta_1^{3/4}\delta_2}{6^{3/2}\cdot 5^{7/4}} n^2 \log n, \quad n \to \infty, \quad x_2(n) \sim \frac{\delta_1^{1/4}\delta_2}{2^{1/2}\cdot 3^{3/2}\cdot 5^{5/4}} n^3 (\log n)^3, \quad n \to \infty.$$

If

$$\gamma_1(n) = \psi_1(n)\log n(\log(n+1))^2 \left(\frac{(n+2)^4(\log(n+2))^2}{n+1} (1-\psi_1(n+1))^2 - n^3(\log n)^2 (1-\psi_1(n))^2 \right),$$
$$\gamma_2(n) = \frac{n^2(n+1)^{1/2}}{(\psi_2(n))^{5/3} (\psi_2(n+1))^{3/2}} (1-\psi_2(n+1))^{\frac{3}{2}} - \frac{(n+1)^{7/2}(\log(n+1))^4}{n(\log n)^4} (1-\psi_2(n))^{\frac{3}{2}},$$

where

$$\psi_1(n) = \left(\frac{n}{n+1}\right)^2 \frac{\log n}{\log(n+1)}, \quad \psi_2(n) = \left(\frac{n\log n}{(n+1)\log(n+1)}\right)^3,$$

then $\delta_1 = 20$, $\delta_2 = 15\sqrt{3}$ and the system has the exact solution

$$(x_1(n), x_2(n)) = \left(n^2 \log n, n^3(\log n)^3\right) \in \mathcal{RV}(2,3).$$

Example 6.2. Consider the difference equation

$$\Delta\left(n^3 \log n(\Delta x(n))^3\right) = \gamma(n) n(\log n)^3 (x(n+1))^2, \ n \geq 2, \tag{6.40}$$

where $\gamma(n)$ is positive real-valued sequence such that $\lim_{n\to\infty}\gamma(n) = \delta$. In this equation, $\alpha = 3$, $\beta = 2$, $\{p(n)\} \in \mathcal{RV}(3)$, and $\{q(n)\} \in \mathcal{RV}(1)$.

Since $\mu = 1 > -1$, by Theorem 6.7 we conclude that equation (6.40) has a (strongly increasing) $\mathcal{RV}-$solution of index $\rho = 2$. The asymptotic behavior of such a solution is

$$x(n) \sim \frac{\delta}{48} n^2 (\log n)^2, \quad n \to \infty.$$

If

$$\gamma(n) = \frac{(n+1)^5(\log(n+1))^3}{n(\log n)^3} \left(\frac{1}{\psi(n+1)} - 1\right)^3 - \frac{n^2(n+1)^2(\log(n+1))^2}{(\log n)^2} (1-\psi(n))^3,$$

where

$$\psi(n) = \left(\frac{n\log n}{(n+1)\log(n+1)}\right)^2,$$

then $\delta = 48$ and equation (6.40) has the exact solution $x(n) = n^2(\log n)^2 \in \mathcal{RV}(2)$.

Acknowledgements

The authors are indebted to the anonymous referee for valuable comments and suggestions.

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