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Generalizations of the Bohr inequality for certain classes of harmonic mappings

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Abstract. In this article, we study improved versions of the Bohr inequality for a subclass of close-to-convex harmonic mappings of the form $f = h + \overline{g}$, where h and g are analytic functions, define on the unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and obtain results in terms distance formulation with the quantity S_r/π. All the results are proved to be sharp.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Let $\mathcal{B}(\mathbb{D})$ denote the class of analytic functions f in the unit disk $\mathbb{D}$ such that $|f(z)| \leq 1$ in $\mathbb{D}$. One of the most celebrated classical results for the class $\mathcal{B}(\mathbb{D})$ was given by H. Bohr.

Theorem 1.1. *[14] If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{B}(\mathbb{D})$, then*

$$M_f(r) := \sum_{n=0}^{\infty} |a_n| r^n \leq 1 \quad for \quad |z| = r \leq \frac{1}{3}. \tag{1}$$

The constant $1/3$ *is best possible.*

Initially, H. Bohr showed the inequality (1) holds for $|z| \leq 1/6$, and M. Riesz, I. Schur and F. Wiener subsequently improved the inequality (1) for $|z| \leq 1/3$ and showed that the constant $1/3$ cannot be improved. It is quite natural that the constant $1/3$ and the inequality (1) are called respectively, the Bohr radius and the classical Bohr inequality for the class $\mathcal{B}(\mathbb{D})$. Moreover, for: $f_a(z) := (a - z)/(1 - az)$, $\ a \in [0, 1), z \in \mathbb{D}$, it follows easily that $M_{f_a}(r) > 1$ if, and only if, $r > 1/(1 + 2a)$, which, for $a \to 1$, shows that $1/3$ is best possible.

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Interest in the Bohr inequality was revived when Dixon (see [16]) used it to settle in the negative the conjecture that a non-unital Banach algebra that satisfies the von Neumann inequality must be isometrically isomorphic to a closed subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Subsequently, Paulson and Singh [35] have extended the Bohr inequality in the context of Banach algebra. An open problem on the Bohr inequality has been raised in [17] has been settled in [25]. Bohr radius has also been investigated by many researchers in various multidimensional spaces (see e.g., [4, 9, 13, 20, 30, 34]) and references therein.

Similar to Bohr inequalities, there is another concept known as the Bohr–Rogosinski inequalities, which were established by Kayumov *et al.* [28] for the class $\mathcal{B}(\mathbb{D})$. For analytic functions f defined on the unit disk $\mathbb{D}$, $S_r := S_r(f)$ denotes the planar integral

$$S_r = \int_{\mathbb{D}_r} |f'(z)|^2 dA(z), \text{ where } \mathbb{D}_r := \mathbb{D}(0; r) \text{ and } 0 < r < 1.$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, than the quantity S_r has the series representation $S_r = \pi \sum_{n=1}^{\infty} n|a_n|^2 r^{2n}$, and the quantity S_r plays a significant role in the study of improved Bohr inequalities. For example, Kayumov and Ponnusamy (see [26]) obtained the result improving the classical Bohr inequality for the class $\mathcal{B}(\mathbb{D})$. Later, Ismagilov *et al.* (see [23]) established another improved version of the classical Bohr inequality, and also derived a sharp inequality as an improvement of the Bohr-Rogosinski inequality for the class $\mathcal{B}(\mathbb{D})$.

1.1. Harmonic mappings and the Bohr inequality on $\mathbb{D}$

A complex-valued function of the class $\mathbb{C}^2$, that is, f which has continuous second order partial derivatives, is said to be harmonic if $\Delta f = 4f_{z\bar{z}} = 0$. We denote by $\mathrm{Har}(\Omega)$ the class of all complex-valued sense-preserving harmonic mappings in a simply connected domain Ω, and by $\mathcal{H}$ its subclass of all complex-valued harmonic mappings. It is well-known that each function $f \in \mathrm{Har}(\Omega)$ has the decomposition $f = h + \bar{g}$ defined on the unit disk $\Omega = \mathbb{D}$, where h and g are analytic in $\mathbb{D}$ with the normalization $h(0) = h'(0) - 1 = 0$ and $g(0) = 0$. Let $\mathcal{H}_0 := \{f = h + \bar{g} \in \mathcal{H} : g'(0) = 0\}$. Then each function $f = h + \bar{g} \in \mathcal{H}_0$, where h and g are given by

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \tag{2}$$

We note that the inequality (1), in turn, equivalent to

$$\sum_{n=1}^{\infty} |a_n| r^n \leq 1 - |f(0)| = d(f(0), \partial\mathbb{D}) \text{ for } |z| = r \leq \frac{1}{3}, \tag{3}$$

where $d(f(0), \partial\mathbb{D})$ is the Euclidean distance between $f(0)$ and the boundary of $\mathbb{D}$.

The Bohr inequality for a class of harmonic mappings was initiated first in [1] in terms of the Euclidean distance d and was investigated in [27] and subsequently by a number of authors, (see e.g. [2, 3, 5, 7, 8, 10, 11, 19, 22, 24, 33]).

In recent years, the study of various geometric properties of harmonic mappings has become a subject of great interest. For example, in [37], a class of k-symmetric harmonic functions involving a certain q-derivative operator is introduced, and several of its properties are established. In [36], the authors define, explore, and analyze two new families of meromorphically harmonic functions by applying a certain generalized convolution q-operator along with the concept of convolution. They investigate convolution properties and sufficiency criteria for these families of meromorphically harmonic functions. In [12], some Janowski-type harmonic q-starlike functions associated with symmetrical points are studied, and a new criterion for sense-preserving and, hence, univalence is developed in terms of the q-differential operator. Necessary and sufficient conditions for univalence of this newly defined class are also established, along with discussions on other interesting properties such as distortion bounds, convolution preservation, and convexity conditions. In [29], certain new subclasses of meromorphic harmonic functions are introduced using the principles of the q-derivative operator. New criteria for sense-preserving and univalence are obtained, and other significant aspects—such as distortion bounds, convolution preservation, and convexity conditions—are also addressed.

In this paper, our primary objective is to study improved versions of the classical Bohr inequality for certain class of harmonic mappings, in a general form, which was introduced by Chichra in [15]. In addition, Chichra [15] has shown that functions in the class $\mathcal{W}(\alpha)$, which consisting of normalized analytic functions h, satisfying the condition $\operatorname{Re}(h'(z) + \alpha z h''(z)) > 0$ for $z \in \mathbb{D}$ and $\alpha \geq 0$, constitute a subclass of close-to-convex functions in $\mathbb{D}$. In 2014, Nagpal and Ravichandran [31] obtained coefficient bounds for the functions in the class $\mathcal{W}^0_{\mathcal{H}}$ which is defined by

$$\mathcal{W}^0_{\mathcal{H}} := \{f = h + \bar{g} \in \mathcal{H} : \operatorname{Re}(h'(z) + zh''(z)) > |g'(z) + zg''(z)| \ \text{ for } z \in \mathbb{D}\}.$$

In [21], Ghosh and Vasudevarao studied the generalized class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$, where

$$\mathcal{W}^0_{\mathcal{H}}(\alpha) := \{f = h + \bar{g} \in \mathcal{H} : \operatorname{Re}(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)| \ \text{ for } z \in \mathbb{D}\}$$

and proved some growth, convolution and convex combination theorems.

1.2. Preliminary results

It is shown in [21] that $\mathcal{W}^0_{\mathcal{H}}(\alpha)$ is a subclass of the close-to-convex harmonic mappings. As our main interest is to find the sharp improved Bohr inequality for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$, henceforth, we recall here the sharp coefficient bounds and the sharp growth estimates for functions in the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$, which will play key roles in proving the Bohr inequalities in terms of distance formations.

Lemma 1.2. *[21, Theorem 4.3] Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha)$ for $\alpha \geq 0$ and be of the form* (2)*. Then for any $n \geq 2$,*

$$|a_n| + |b_n| \leq \frac{2}{\alpha n^2 + (1-\alpha)n}, \ ||a_n| - |b_n|| \leq \frac{2}{\alpha n^2 + (1-\alpha)n}, \ |a_n| \leq \frac{2}{\alpha n^2 + (1-\alpha)n}.$$

All these inequalities are sharp for the function $f = f_\alpha$ given by

$$f_\alpha(z) := z + \sum_{n=2}^{\infty} \frac{2z^n}{\alpha n^2 + (1-\alpha)n}. \tag{4}$$

Lemma 1.3. *[21, Theorem 4.4] Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha)$ and be of the form* (2) *with $0 < \alpha \leq 1$. Then*

$$|z| + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}|z|^n}{\alpha n^2 + (1-\alpha)n} \leq |f(z)| \leq |z| + \sum_{n=2}^{\infty} \frac{2|z|^n}{\alpha n^2 + (1-\alpha)n}. \tag{5}$$

Both the inequalities are sharp for the function $f = f_\alpha$ given by (4)*.*

In view of the Lemmas 1.2 and 1.3, in [11], the Bohr inequality for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$ is obtained in terms of distance formulation (3) and shown that the Bohr radius for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$ is best possible.

Theorem 1.4. *[11, Theorem 2.4] For $\alpha \geq 0$, let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha)$ be of the form* (2)*. Then*

$$|z| + \sum_{n=2}^{\infty} (|a_n| + |b_n|)|z|^n \leq d(f(0), \partial f(\mathbb{D})) \tag{6}$$

for $|z| = r \leq r_f$, where r_f is the unique root of the equation

$$r + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}$$

in the interval $(0, 1)$*. The radius r_f is best possible.*

2. Main Results

In this section, we establish Bohr-type inequalities for certain expression of the quantity S_r/π for certain class of harmonic mappings. Inspired by the results in [11] and continuing the study, the discussions above motivate us to establish improved Bohr inequalities for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$.

We show that the function dilogarithm has some pivotal role to determine the improved Bohr inequalities for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$. Hence, before stating the main results of this section, we first recall the definition of dilogarithm $\mathrm{Li}_2(z)$ which is defined by the power series

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{for} \quad |z| < 1.$$

The analytic continuation of the dilogarithm is given by

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-u)\frac{du}{u} \quad \text{for} \quad z \in \mathbb{C} \setminus [1, \infty).$$

Using the Lemmas 1.2 and 1.3, we prove the following sharp improved Bohr inequality for the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$.

Theorem 2.1. *Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha)$, where $0 < \alpha \leq 1$, be given by (2) and $\mu, \lambda : [0,\infty) \to [0,\infty)$ be monotone increasing functions. Then, for $p \geq 1$, we have*

$$\mathcal{I}_{f,p,\mu,\lambda}(r) := |z|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \mu\left(\frac{S_{|z|}}{\pi}\right) + \lambda\left(\left(\frac{S_{|z|}}{\pi}\right)^2\right) \leq d(f(0), \partial f(\mathbb{D})) \tag{7}$$

holds for $|z| = r \leq R_f(\alpha)$, where $R_f(\alpha)$ is the unique root in $(0,1)$ of the equation

$$J_1(r) := r^p + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(r)\right) + \lambda\left((M_\alpha(r))^2\right) - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n} = 0.$$

The radius $R_f(\alpha)$ is best possible.

Remark 2.2. *If $\alpha = 0$, then the class $\mathcal{W}^0_{\mathcal{H}}(\alpha)$ reduces to $\mathcal{P}^0_{\mathcal{H}}$ (see [32]), whereas when $\alpha = 1$, the class reduces to $\mathcal{W}^0_{\mathcal{H}}$ and in either case, the Bohr radius can be obtained precisely. Theorem 2.1 is an improved version of Theorem B, and in particular, if $\mu(s) = 0 = \lambda(s)$ for $s \in [0,\infty)$ and $p = 1$, then the inequality (7) coincides exactly with (6). If in particular, $\mu(s) = (16/9)s$ and $\lambda(s) = 0$ for $s \in [0,\infty)$ with $p = 1$, then we see that (7) in Theorem 2.1 is harmonic analogue of the inequality [26, Eq. (2), Theorem 1]. If $\mu(s) = (16/9)(s)$ and $\lambda(s) = (18.6095...)s$ for $s \in [0,\infty)$ with $p = 1$, then (7) in Theorem 2.1 is a harmonic analogue of the inequality [23, Eq. (3), Theorem 1]. Further, if $\mu(s) = P_k(s)$ and $\lambda(s) = 0$ for $s \in [0,\infty)$ with $p = 1$, where $P_k(s)$ is a polynomial in s of degree k given by $P_k(s) = \lambda_k s^k + \cdots + \lambda_1 s$, with coefficients $\lambda_k \neq 0$ and $\lambda_j \geq 0$ for $j = 1, \ldots, k$, then Theorem 2.1 specializes to a known result (specifically, Theorem 2.2 from [6]). This shows that our Theorem 2.1 provides a more general formulation for the study of Bohr inequality.*

In view of the sharp bounds of $|f(z)|$ in Lemma 1.3, considering integral powers of $|f(z)|$, we obtain the following result.

Theorem 2.3. *Let $f \in \mathcal{W}^0_{\mathcal{H}}(\alpha)$, where $0 < \alpha \leq 1$, be given by (2) and $\mu : [0,\infty) \to [0,\infty)$ be a monotone increasing function. Then, for $p > 0$, we have*

$$\mathcal{J}_{f,p,\mu}(r) := |f(z)|^p + \sum_{n=2}^{\infty} (|a_n| + |b_n|)\,|z|^n + \mu\left(\frac{S_{|z|}}{\pi}\right) \leq d(f(0), \partial f(\mathbb{D})) \tag{8}$$

holds for $|z| = r \leq R_f^*(\alpha)$*, where* $R_f^*(\alpha)$ *is the unique root in* $(0,1)$ *of the equation*

$$J_2(r) := \left(r + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} \right)^p + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(r)\right) - 1 - \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n} = 0.$$

The radius $R_f^*(\alpha)$ *is best possible.*

Remark 2.4. *For particular values,* $p = 1$ *and* $p = 2$ *with* $\mu(s) = 0$*, the inequality* (8) *of Theorem 2.3 is harmonic analogue of the inequalities in [28, Corollary 1]. Moreover, if* $p = 1$ *and* $\mu(s) = 2(\sqrt{5} - 1)s$*, then it is easy to see that* (8) *is a harmonic analogue of the inequality [23, Eq. (5), Theorem 3].*

2.1. *Proof of the main results*

Before, starting the proof, we need some preparation. For $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha)$, the Jacobian of f is denoted by J_f which is defined by

$$J_f(z) := |f_z(z)|^2 - |f_{\bar{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2 \quad \text{for} \quad z \in \mathbb{D}.$$

It is well-known that (see [18, p.113]) the area of the image of disk $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ under the harmonic map $f = h + \bar{g}$ is

$$S_r = \iint_{\mathbb{D}_r} J_f(z)dxdy = \iint_{\mathbb{D}_r} \left(|h'(z)|^2 - |g'(z)|^2\right) dxdy. \tag{9}$$

In view of Lemma 1.2 and (9), we obtain

$$\frac{S_r}{\pi} = \frac{1}{\pi} \iint_{\mathbb{D}_r} \left(|h'(z)|^2 - |g'(z)|^2\right) dxdy \leq r^2 + \sum_{n=2}^{\infty} \frac{4nr^{2n}}{\left(\alpha n^2 + (1-\alpha)n\right)^2} =: M_\alpha(r). \tag{10}$$

Let $f \in \mathcal{W}_{\mathcal{H}}^0(\alpha)$ be given by (2). In view of Lemma 1.3 for $|z| = r$, we have

$$r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}r^n}{\alpha n^2 + (1-\alpha)n} \leq |f(z)| \leq r + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n}. \tag{11}$$

It is easy to see that $f(0) = 0$ and hence, $|f(z) - f(0)| = |f(z)|$, and, we have

$$\begin{aligned} \liminf_{r\to 1^-} \left(r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}r^n}{\alpha n^2 + (1-\alpha)n} \right) &\leq \liminf_{r\to 1^-} |f(z) - f(0)| \\ &\leq \liminf_{r\to 1^-} \left(r + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} \right). \end{aligned} \tag{12}$$

For $0 < \alpha \leq 1$ and $n \geq 2$, since $\alpha n^2 + (1-\alpha)n \geq \alpha n^2 > 0$, hence, we obtain $\sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} \leq \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2}$. We define the functions g_n and K_n by

$$g_n(r) := \frac{2r^n}{\alpha n^2 + (1-\alpha)n} \quad \text{and} \quad K_n(r) := \frac{2r^n}{\alpha n^2}.$$

Evidently, $K_n(r) > 0$ and $|g_n(r)| \leq K_n(r)$ for each $n \geq 2$. Since $|z| = r < 1$, a simple computation shows that

$$\sum_{n=2}^{\infty} K_n(r) < \frac{2}{\alpha} \sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{2}{\alpha} \left(\frac{\pi^2}{6} - 1 \right).$$

By the comparison test, the series $\sum_{n=2}^{\infty} K_n(r)$ converges, hence, by the Weierstrass M-test for series of functions, *i.e.*, the series $\sum_{n=2}^{\infty} g_n(r) = \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2+(1-\alpha)n}$ is absolutely and uniformly convergent in $|z| = r < 1$ and hence, we see that

$$\lim_{r\to 1^-} \sum_{n=2}^{\infty} g_n(r) = \sum_{n=2}^{\infty} \lim_{r\to 1^-} g_n(r).$$

Thus, we have

$$\liminf_{r\to 1^-} \left(r + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} \right) = 1 + \sum_{n=2}^{\infty} \frac{2}{\alpha n^2 + (1-\alpha)n}.$$

On the other hand, it is easy to see that

$$r + \sum_{n=2}^{\infty} \left| \frac{2(-1)^{n-1} r^n}{\alpha n^2 + (1-\alpha)n} \right| \leq r + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2} = r + \sum_{n=2}^{\infty} K_n(r) < 1 + \frac{2}{\alpha} \left(\frac{\pi^2}{6} - 1 \right),$$

and hence, by Weierstrass M-test, the series $r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1} r^n}{\alpha n^2+(1-\alpha)n}$ is also absolutely and uniformly convergent in $|z| = r < 1$. By using the same argument used in the above, it is easy to see that

$$\liminf_{r\to 1^-} \left(r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1} r^n}{\alpha n^2 + (1-\alpha)n} \right) = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}. \tag{13}$$

Thus, it follows from (12) that

$$1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n} \leq \liminf_{r\to 1^-} |f(z) - f(0)| \leq 1 + \sum_{n=2}^{\infty} \frac{2}{\alpha n^2 + (1-\alpha)n}. \tag{14}$$

The Euclidean distance $d(f(0), \partial f(\mathbb{D}))$ between $f(0)$ and the boundary of $f(\mathbb{D})$ satisfies

$$d(f(0), \partial f(\mathbb{D})) := \liminf_{|z|=r\to 1^-} |f(z) - f(0)| \geq 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}. \tag{15}$$

2.2. **Proof of Theorem 2.1**

In view of Lemmas 1.2 and 1.3, for $|z| = r$, from (10) we obtain

$$\mathcal{I}_{f,p,\mu,\lambda}(r) \leq r^p + \sum_{n=2}^{\infty} \frac{2r^n}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(r)\right) + \lambda\left((M_\alpha(r))^2\right) := \mathcal{I}^*_{f,p,\mu,\lambda}(r). \tag{16}$$

In fact, using (15), it is easy to see that the desired inequality

$$\mathcal{I}^*_{f,p,\mu,\lambda}(r) \leq 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n} \leq d(f(0), \partial f(\mathbb{D}))$$

holds for $r \leq R_f(\alpha)$, where $R_f(\alpha)$ is the smallest root of the equation $J_1(r) = 0$, where $J_1 : [0,1] \to \mathbb{R}$ is defined in the statement of the theorem.

Next, we show that $R_f(\alpha)$ in $(0,1)$ is unique. It is clear that, $J_1(r)$ is a real valued differentiable function in $r \in (0,1)$ satisfying $J_1'(r) > 0$ for all $r \in (0,1)$. A simple computation shows that $M_\alpha(r) > 0$ and

$$\frac{d}{dr}(M_\alpha(r)) = 2r + \sum_{n=2}^{\infty} \frac{4n^2 r^{n-1}}{(\alpha n^2 + (1-\alpha)n)^2} > 0 \quad \text{for} \quad r \in (0,1).$$

Hence, for $r \in (0,1)$, given that μ and λ are monotonically increasing functions, with $\mu'(s) > 0$ and $\lambda'(s) > 0$ for all $s \in [0,\infty)$, we have

$$\frac{d}{dr}\left(J_1(r)\right) = pr^{p-1} + \sum_{n=2}^{\infty} \frac{2nr^{n-1}}{\alpha n^2 + (1-\alpha)n} + \mu'\left(M_\alpha(r)\right) \frac{d}{dr}\left(M_\alpha(r)\right) + 2\lambda'\left((M_\alpha(r))^2\right) M_\alpha(r) \frac{d}{dr}(M_\alpha(r)) > 0.$$

Then by the Intermediate value theorem, $R_f(\alpha)$ is the unique root in $(0,1)$ of equation $J_1(r) = 0$. Consequently, we have

$$\begin{aligned} &R_f^p(\alpha) + \sum_{n=2}^{\infty} \frac{2R_f^n(\alpha)}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(R_f(\alpha))\right) + \lambda\left(\left(M_\alpha(R_f(\alpha))\right)^2\right) \\ &\quad = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}. \end{aligned} \tag{17}$$

In order to show that $R_f(\alpha)$ is best possible, we consider the function $f = f_\alpha$ which is defined by (4). It is easy to see that $f_\alpha \in \mathcal{W}_{\mathcal{H}}^0(\alpha)$ and $f_\alpha(0) = 0$. At $z = -r$, a simple computation shows that

$$|f_\alpha(-r) - f_\alpha(0)| = \left| -r + \sum_{n=2}^{\infty} \frac{2(-r)^n}{\alpha n^2 + (1-\alpha)n} \right| = r + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1} r^n}{\alpha n^2 + (1-\alpha)n}. \tag{18}$$

In view of (13) and (18), we see that

$$d(f_\alpha(0), \partial f_\alpha(\mathbb{D})) = \liminf_{r \to 1^-} |f_\alpha(-r) - f_\alpha(0)| = 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}. \tag{19}$$

Using (16), (17) and (19) for $f = f_\alpha$ and $|z| = r > R_f(\alpha)$, we see that

$$\begin{aligned} \mathcal{I}_{f_\alpha,\mu,\lambda}(R_f(\alpha)) &> R_f^p(\alpha) + \sum_{n=2}^{\infty} (|a_n| + |b_n|) R_f^n(\alpha) + \mu\left(\frac{S_{R_f(\alpha)}}{\pi}\right) + \lambda\left(\left(\frac{S_{R_f(\alpha)}}{\pi}\right)^2\right) \\ &= R_f^p(\alpha) + \sum_{n=2}^{\infty} \frac{2R_f^n(\alpha)}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(R_f(\alpha))\right) + \lambda\left(\left(M_\alpha(R_f(\alpha))\right)^2\right) \\ &= 1 + \sum_{n=2}^{\infty} \frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n} \\ &= d(f_\alpha(0), \partial f_\alpha(\mathbb{D})), \end{aligned}$$

which shows that $R_f(\alpha)$ is best possible. This completes the proof.

2.3. **Proof of Theorem 2.3**

In view of Lemmas 1.2 and 1.3, for $|z| = r$, from (10) we obtain

$$\begin{aligned}\mathcal{J}_{f,p,\mu}(r) &\leq \left(r + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n\right)^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)r^n + \mu\left(M_\alpha(r)\right) \\ &\leq \left(r + \sum_{n=2}^{\infty}\frac{2r^n}{\alpha n^2 + (1-\alpha)n}\right)^p + \sum_{n=2}^{\infty}\frac{2r^n}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(r)\right) \\ &\leq 1 + \sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}\end{aligned} \tag{20}$$

for $r \leq R_f^*(\alpha)$, where $R_f^*(\alpha)$ is the smallest root of the equation $J_2(r) = 0$ in $(0, 1)$, where $J_2 : [0, 1] \to \mathbb{R}$ is defined in the statement of the theorem. Using the similar argument as used in the proof of Theorem 2.1, it is easy to show that $R_f^*(\alpha)$ is the unique root of equation $J_2(r) = 0$ in $(0, 1)$. Therefore, we must have

$$\begin{aligned}&\left(R_f^*(\alpha) + \sum_{n=2}^{\infty}\frac{2\left(R_f^*(\alpha)\right)^n}{\alpha n^2 + (1-\alpha)n}\right)^p + \sum_{n=2}^{\infty}\frac{2\left(R_f^*(\alpha)\right)^n}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(R_f^*(\alpha))\right) \\ &= 1 + \sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n}.\end{aligned} \tag{21}$$

To show $R_f^*(\alpha)$ is best possible, we consider the function $f = f_\alpha$ given in (4). In view of (20) and (21), for $f = f_\alpha$ and $z = r > R_f^*(\alpha)$, we obtain

$$\begin{aligned}\mathcal{J}_{f_\alpha,p,\mu}(R_f^*(\alpha)) &> |f_\alpha(R_f^*(\alpha))|^p + \sum_{n=2}^{\infty}(|a_n| + |b_n|)\left(R_f^*(\alpha)\right)^n + \mu\left(\frac{S_{R_f^*(\alpha)}}{\pi}\right) \\ &= \left(R_f^*(\alpha) + \sum_{n=2}^{\infty}\frac{2\left(R_f^*(\alpha)\right)^n}{\alpha n^2 + (1-\alpha)n}\right)^p + \sum_{n=2}^{\infty}\frac{2\left(R_f^*(\alpha)\right)^n}{\alpha n^2 + (1-\alpha)n} + \mu\left(M_\alpha(R_f^*(\alpha))\right) \\ &= 1 + \sum_{n=2}^{\infty}\frac{2(-1)^{n-1}}{\alpha n^2 + (1-\alpha)n} \\ &= d(f_\alpha(0), \partial f_\alpha(\mathbb{D})).\end{aligned}$$

This shows that $R_f^*(\alpha)$ is best possible. This completes the proof.

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