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# Frame multiresolution analysis of band-limited functions over locally compact Abelian groups

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**Abstract.** In this article, we introduce the notion of frame multiresolution analysis (FMRA) of band-limited functions on locally compact Abelian (LCA) groups and derive certain conditions under which the subspaces $V_j = \overline{\text{span}}\left\{D^j T_\lambda \phi : \lambda \in \Lambda\right\}$, $j \in \mathbb{Z}$, where $\hat{\phi}(\omega) = \chi_K(\omega), \omega \in \widehat{G}, K \subset \widehat{G}$, constitutes an FMRA for $L^2(G)$. Subsequently, we construct the corresponding wavelet frames of both the dyadic and arbitrary dilations on LCA groups. Nevertheless, all the results are braced with illustrative examples.

## 1. Introduction

Doubtlessly, wavelets have grabbed the attention of scientific, engineering, and research communities with their wide range of applications and lucid mathematical framework in such a way that they are now considered now as a nucleus of shared aspirations and ideas [1, 2]. The most valuable and widely-used algorithm for constructing orthonormal wavelet basis for $L^2(\mathbb{R}^n)$ is the multiresolution analysis (MRA) developed by Mallat [3] in the framework of time-frequency analysis. Over the last two decades, many lucubrations and extensions have been witnessed in the literature to harness the advantages of an MRA. For instance, biorthogonal MRA, vector-valued MRA, Riesz MRA, generalized MRA, non-uniform MRA and so on [4–7]. In particular, Benedetto and Li [8] considered the dyadic frame multiresolution analysis of $L^2(\mathbb{R})$ with a single scaling function and successfully applied the theory in the analysis of narrow band signals. Later on, Yu [9] extended the results of Benedetto and Li's theory of FMRA to higher dimensions with arbitrary integral expansive matrix dilations and has established the necessary and sufficient conditions to characterize semi-orthogonal multiresolution analysis frames for $L^2(\mathbb{R}^n)$.

Parallel developments in the construction of orthonormal wavelets have also been witnessed in the realm of abstract settings at an exponential rate. For instance, For example, Dahlke [10] constructed orthonormal wavelet basis on LCA groups by employing the generalized $B$-splines and self-similar tiles.

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Kamyabi-Gol and Tousi [11, 12] investigated the conditions under which a function generates an MRA on a locally compact Abelian group using the theory of spectral functions and shift-invariant spaces. Subsequently, Yang and Taylor [13] established the concept of an MRA on non-abelian locally compact groups $G$ with no regularity or decay requirements on the scaling and built the Haar-like wavelet bases for $L^2(G)$. Bownik and Jahan [14] built an MRA on compact Abelian groups using epimorphism as a dilation operator and characterize the scaling sequences of such an MRA for $L^p(G), 1 \leq p < \infty$. Recently, Kumar and Satyapriya [15] developed the theory of frame multiresolution analysis (FMRA) on LCA groups and studied certain aspects of multiresolution subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ which offer quantitative conditions for the construction of an FMRA for $L^2(G)$. The key characteristic of this innovative approach is that the collection $\left\{\phi(x-\lambda) : \lambda \in \Lambda\right\}$ is no longer an orthonormal basis for the core subspace $V_0$ but rather a frame for $V_0$, making it useful for a variety of signal processing applications.

As is well known that most signals in practice are band-limited in nature and a complete representation of these signals requires frequency analysis that is restricted to prescribed bands, resulting in the band-limited wavelets. The band-limited wavelets are constructed via the conventional MRA approach by choosing the scaling functions to be band-limited [16]. Some popular band-limited refinable functions and wavelets include the orthonormal Shannon's and Meyer's scaling functions and wavelets. Apart from the construction of orthonormal and band-limited wavelets together with their allies, much attention has been paid to the construction of wavelet frames or framelets which are not only easy to construct but also provide a suitable platform to obtain perfect reconstruction of a given signal in situations where redundancy, robustness, over-sampling, and irregular sampling play a role [17]. One of the commonly used method to construct wavelet frames is through an FMRA [18]. For instance, Zhang [19] showed that the number of generators in wavelet frames associated with FMRA is determined completely by the frequency domain of FMRAs. Similarly, Atreasa et al.[20] constructed many examples on compactly supported framelets when FMRAs extend into general MRAs. Zhang [21] established an explicit characterization of band-limited FMRAs in frequency domain. Recently, Kumar et al.[22, 23] have developed a novel method for the construction of wavelet frames on LCA groups via an FMRA and even formulated certain conditions for an FMRA in $L^2(G)$ to admit that a single function $\psi \in W_0$ can generate a wavelet frame for $W_0$.

In this article, we continue our investigation on the formulation of FMRA of band-limited signals over LCA groups. More precisely, we establish several conditions under which the subspaces $V_j = \overline{\text{span}}\left\{D^j T_\lambda \phi : \lambda \in \Lambda\right\}$, $j \in \mathbb{Z}$, where $\hat{\phi}(\omega) = \chi_K(\omega), \omega \in \widehat{G}, K \subset \widehat{G}$ forms an FMRA for $L^2(G)$. We next build the relevant wavelet frames for both the dyadic and arbitrary dilations on LCA groups, and all of the results are supported with examples.

The remainder of the article is organized as follows: Section 2, is devoted to the exposition of the preliminaries such as the notion of uniform lattices, annihilator, automorphism and the Fourier transforms on LCA groups. Section 3 explicitly deals with the construction of an FMRA of band-limited functions for $L^2(G)$. In Section 4, we present an explicit procedure for the construction of wavelet frames in $L^2(G)$ with several illustrative examples.

## 2. Preliminaries and Fourier Analysis on LCA Groups

This section starts with a brief overview of locally compact Abelian groups, followed by some preliminary results concerning Fourier transforms on LCA groups, which serve as the foundation for the development of semi-orthogonal wavelet frames for $L^2(G)$. The definition and characterizations of frames in Hilbert spaces are presented towards the culmination of the section.

### *2.1. Basics of LCA Groups*

A group $G$ equipped with a Hausdorff topology is called an LCA group, if it is metrizable, locally compact and can be written as a countable union of compact sets. The set of real number $\mathbb{R}$, integers $\mathbb{Z}$, unit disk

$\mathbb{T}$ and $\mathbb{Z}_N$ (the integers modulo $N$) are some prominent examples of LCA groups. These groups along with their higher dimensional variants, are called elementary LCA groups. Moreover, the family of all continuous homomorphisms from the LCA group $G$ to the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is denoted by $\widehat{G}$ and also constitutes an LCA group under a suitable topology and the composition

$$(\omega + \omega')(x) = \omega(x)\,\omega'(x), \quad x \in G, \omega, \omega' \in \widehat{G}. \tag{1}$$

This group is often referred as the dual group of $G$ and its elements are called the characters of $\widehat{G}$. It is well-known that the double-dual group $\widehat{\widehat{G}} = G$ and as such $\omega(x)$ can be interpreted as either the action of $\omega \in \widehat{G}$ on $x \in G$ or the action of $x \in G$ on $\omega \in \widehat{G}$. For the sake of brevity, we shall use the following notation:

$$(\omega, x) = \omega(x), \quad x \in G, \omega \in \widehat{G}. \tag{2}$$

*2.2. Fourier Analysis on LCA Groups*

Let $\mu_G$ and $\mu_{\widehat{G}}$ be the Haar measures on LCA groups $G$ and $\widehat{G}$, respectively. Based on the Haar measure, we define the spaces $L^p(G)$ and $L^p(\widehat{G}), 1 \leq p \leq \infty$ in the usual way. The Fourier transform of any arbitrary function $f \in L^1(G)$ is defined by

$$\mathscr{F} : L^1(G) \to C_0(\widehat{G}), \quad \mathscr{F}(f)(\omega) = \int_G f(x)\,\overline{(\omega, x)}\,d\mu_G(x), \tag{3}$$

where $C_0(\widehat{G})$ denotes the space of all continuous functions on $\widehat{G}$ vanishing at infinity. For the sake of our convenience, we will also use the notation $\hat{f}$ to denote the Fourier transform of the function $f$.

It is worth noticing that for a fixed Haar measure $d\mu_G(x)$, there exits a Haar measure $d\mu_{\widehat{G}}(x)$ on $\widehat{G}$ called the normalized Plancherel measure, such that the Fourier transform (3) is an isometric transform on $L^1(G) \cap L^2(G)$, and hence, it can be extended uniquely to a unitary isomorphism from $L^2(G)$ onto $L^2(\widehat{G})$ [24]. Therefore, each $f \in L^1(G)$ with $\mathscr{F}(f)(\omega) \in L^1(\widehat{G})$ can be reconstructed via the following formula:

$$f(x) = \int_{\widehat{G}} \hat{f}(\omega)(\omega, x)\,d\mu_{\widehat{G}}(\omega), \quad x \in G. \tag{4}$$

Moreover, the Parseval's formula corresponding to (3) reads

$$\left\langle f, g\right\rangle = \int_G f(x)\,\overline{g(x)}\,d\mu_G(x) = \int_{\widehat{G}} \hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,d\mu_{\widehat{G}}(\omega) = \left\langle \hat{f}, \hat{g}\right\rangle. \tag{5}$$

For typographical convenience, we shall denote the Haar measures $d\mu_G$ and $d\mu_{\widehat{G}}$ by $dx$ and $d\omega$, respectively.

For $y \in G$, the generalized translation operator is defined by

$$T_y : L^2(G) \to L^2(G), \;\; T_y f(x) = f(x - y), \quad x \in G. \tag{6}$$

Likewise, the generalized dilation operator $D$ in $L^2(G)$ can be defined via the dilative automorphism introduced by Dahlke [10]. An automorphism $\alpha : G \to G$ is said to be dilative if there exists $N \in \mathbb{N}$ such that $K \subseteq \alpha^n(U)$, $\forall\, n \geq N$, where $K$ is any compact set in $G$ and $U$ is an open neighbourhood at the origin. Therefore, for a dilative automorphism $\alpha$, the dilation operator $D : L^2(G) \to L^2(G)$ is defined by

$$Df(x) = \delta(\alpha)^{1/2} f\big(\alpha(x)\big), \quad x \in G, \tag{7}$$

where $\delta(\alpha)$ is a positive constant such that

$$\int_G f(x)\,dx = \delta(\alpha) \int_G f\big(\alpha(x)\big)\,dx. \tag{8}$$

Moreover, the induced automorphism $\hat{\alpha}$ of $\widehat{G}$ is given by

$$\left(\hat{\alpha}(\omega), x\right) = \left(\omega, \alpha(x)\right), \quad x \in G, \omega \in \hat{G}. \tag{9}$$

Subsequently, the dilation operator $\mathcal{D} : L^2(\widehat{G}) \to L^2(\widehat{G})$ can be defined by

$$\mathcal{D}F(\omega) = \delta(\alpha)^{1/2} F(\hat{\alpha}(\omega)). \tag{10}$$

*2.3. Lattices and Fundamental Domains in LCA Groups*

A uniform lattice in an LCA group $G$ is a discrete subgroup $\Lambda$ for which the quotient group $G/\Lambda$ is compact. In addition to this, we shall also assume that $\alpha(\Lambda) \subseteq \Lambda$. Corresponding to the lattice $\Lambda$, an annihilator $\Lambda^\perp$ is defined by

$$\Lambda^\perp = \left\{\omega \in \widehat{G} : (x, \omega) = 1, x \in \Lambda\right\}. \tag{11}$$

It is easy to verify that the annihilator $\Lambda^\perp$ is also a lattice in $\widehat{G}$ and $\hat{\alpha}\left(\Lambda^\perp\right) \subset \Lambda^\perp$, whenever $\alpha(\Lambda) \subset \Lambda$. For the classical case $G = \mathbb{R}$, we have $\Lambda = \Lambda^\perp = \mathbb{Z}$. Therefore, the inclusion $\alpha(\Lambda) \subset \Lambda$ always holds for the automorphism $x \mapsto 2x$ as $\alpha(\Lambda) = 2\mathbb{Z}$. Nevertheless, it is pertinent to mention that a lattice $\Lambda$ in $G$ can be used to obtain a splitting of the group $G$ and $\widehat{G}$ into disjoint cosets [25].

**Lemma 2.1.** *[25] Let $\Lambda$ be a lattice in an LCA group $G$. Then the following hold:*

(i). *There exists a Borel measurable relatively compact set $\mathcal{Q} \subseteq G$ such that*

$$G = \bigcup_{\lambda \in \Lambda} (\lambda + \mathcal{Q}), \;\; (\lambda + \mathcal{Q}) \cap (\lambda' + \mathcal{Q}) = \emptyset, \text{ for } \lambda \neq \lambda'; \;\; \lambda, \lambda' \in \Lambda. \tag{12}$$

(ii). *There exists a Borel measurable relatively compact set $\mathcal{S} \subseteq \widehat{G}$ such that*

$$\widehat{G} = \bigcup_{\omega \in \Lambda^\perp} (\omega + \mathcal{S}), \;\; (\omega + \mathcal{S}) \cap (\omega' + \mathcal{S}) = \emptyset, \text{ for } \omega \neq \omega'; \;\; \omega, \omega' \in \Lambda^\perp. \tag{13}$$

The sets $\mathcal{Q}$ and $\mathcal{S}$ appearing in (12) and (13) are called a fundamental domains or the tiles associated with the lattices $\Lambda$ and $\Lambda^\perp$, respectively.

We now discuss the periodic functions on $G$. For a given set $H \subset G$, a function $f : G \to \mathbb{C}$ is said to be $H$-periodic if

$$f(x + h) = f(x), \quad \forall\, x \in G, h \in H. \tag{14}$$

In particular, if we take $H = \Lambda$, then by virtue of $\Lambda$-periodicity of the functions defined on $G$, we can determine the space $L^2(G/\Lambda)$. Similarly, we can define $\Lambda^\perp$-periodic functions on $\widehat{G}$ and hence, the space $L^2(\widehat{G}/\Lambda^\perp)$ can be determined accordingly [15, 26]. If we assume that $G = \mathbb{R}$ and $\Lambda = \mathbb{Z}$, then both the quotient spaces $L^2(G/\Lambda)$ and $L^2(\widehat{G}/\Lambda^\perp)$ can be identified with the space $L^2(\mathbb{T})$. Note that a function $F \in L^2(\widehat{G}/\Lambda^\perp)$ if and only if there exists a sequence $\{c_\lambda\}_{\lambda \in \Lambda} \in l^2(\Lambda)$ such that [15]

$$F(\omega) = \sum_{\lambda \in \Lambda} c_\lambda (\omega, \lambda), \quad \forall\, \omega \in \widehat{G}. \tag{15}$$

The $\Lambda^\perp$-periodic extension of any set $H \subset \widehat{G}$ is defined by

$$P(H) = \bigcup_{\gamma \in \Lambda^\perp} (\gamma + H). \tag{16}$$

Then, we observe that

$$\hat{\alpha}\Big(P(H)\Big) = \bigcup_{\gamma\in\Lambda^\perp}\Big(\hat{\alpha}(\gamma) + \hat{\alpha}(H)\Big) \quad \text{and} \quad \hat{\alpha}\Big(P(H)\Big) \neq P\Big(\hat{\alpha}(H)\Big).$$

Before proceeding further, we make an assumption that $(\Lambda, \alpha)$ is a scaling system on $G$, i.e., $\Lambda$ and $\alpha$ satisfy $\alpha(\Lambda) \subset \Lambda$ [13, 15]. Consequently, the pair $(\Lambda^\perp, \hat{\alpha})$ will also be a scaling system on $\widehat{G}$. Therefore, the quotient groups $\Lambda/\alpha(\Lambda)$ and $\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)$ are both finite and have cardinality equal to $\delta_\alpha$. Note that the fundamental domain $\mathcal{Q}$ associated with the scaling system $(\Lambda, \alpha)$ is self-similar if

$$\mathcal{Q} = \bigcup_{\gamma\in\Lambda_f}\Big(\alpha^{-1}(\gamma) + \alpha^{-1}(\mathcal{Q})\Big), \tag{17}$$

where $\Lambda_f$ is the complete set of quotient representatives of $\alpha(\Lambda)$ in $\Lambda$ [10, 13, 27]. From here onwards, we shall assume that $\mathcal{Q}$ and $\mathcal{S}$ are both self-similar fundamental domains in $G$ and $\widehat{G}$, respectively. Nevertheless, we assume that the quotient groups $\Lambda/\alpha(\Lambda)$ and $\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)$ have the representations [27]:

$$\Lambda/\alpha(\Lambda) = \Big\{\alpha(\Lambda)\Big\} \cup \Big\{\lambda_j + \alpha(\Lambda) : 1 \le j \le \delta_\alpha - 1\Big\},$$
$$\Lambda^\perp/\hat{\alpha}(\Lambda^\perp) = \Big\{\hat{\alpha}(\Lambda^\perp)\Big\} \cup \Big\{\gamma_j + \hat{\alpha}(\Lambda^\perp) : 1 \le j \le \delta_\alpha - 1\Big\}.$$

The terms $\lambda_j$ and $\gamma_j$ satisfy the relation [9]:

$$\sum_{j=0}^{\delta_\alpha-1}\Big(\omega_j, \lambda_k - \lambda_l\Big) = \begin{cases}\delta_\alpha, & k = l\\ 0, & k \neq l\end{cases}; \quad \omega_j = \hat{\alpha}^{-1}(\gamma_j),\ 0 \le j \le \delta_\alpha - 1. \tag{18}$$

### 2.4. Frames on LCA Groups

By considering the lattice $\Lambda$ as a countable index set, we introduce the notion of frame for the space $L^2(G)$. For a detailed study on frames and related topics, we refer to [25].

**Definition 2.2.** *A family $\Big\{f_\lambda : \lambda \in \Lambda\Big\}$ is called a frame for $L^2(G)$ if there exist positive constants $0 < A \le B < \infty$ such that*

$$A\Big\|f\Big\|^2 \le \sum_{\lambda\in\Lambda}\Big|\Big\langle f, f_\lambda\Big\rangle\Big|^2 \le B\Big\|f\Big\|^2, \quad \forall\ f \in L^2(G). \tag{19}$$

The numbers $A$ and $B$ are called lower and upper frame bounds, respectively. A tight frame refers to the case when $A = B$, and a Parseval frame refers to the case when $A = B = 1$ [25]. The frame is exact if it ceases to be a frame whenever any single element is deleted. In most cases, it is extremely strenuous to determine the existence of $A$ and $B$ or to verify whether the family $\Big\{f_\lambda : \lambda \in \Lambda\Big\}$ constitutes a frame or not.

In wavelet analysis, the family family $\{T_\lambda\phi : \lambda \in \Lambda\}$ consisting of translates of a single function $\phi \in L^2(G)$ is of utmost importance and will constitute a frame for its closed linear span if [25, 28]

$$A \le \sum_{\gamma\in\Lambda^\perp}\Big|\hat{\phi}(\omega - \gamma)\Big|^2 \le B, \ \ \forall\ \omega \in \Big\{\omega \in \widehat{G} : \sum_{\gamma\in\Lambda^\perp}|\hat{\phi}(\omega - \gamma)|^2 \neq 0\Big\}. \tag{20}$$

We culminate this subsection by introducing the Paley-Wiener spaces in $L^2(G)$. The Paley-wiener space $PW_G(K)$ associated with the set $K \subset \widehat{G}$ is given by

$$PW_G(K) = \Big\{f \in L^2(G) : \hat{f}(\omega) = 0,\ \forall\ \omega \in K^c\Big\}. \tag{21}$$

## 3. Frame Multiresolution Analysis of Band-limited Functions in $L^2(G)$

In this section, firstly we shall recall the definition of frame multiresolution analysis (FMRA) on LCA groups. Then, we present an explicit construction scheme for how to construct an FMRA in $L^2(G)$ by first choosing an appropriate scaling function $\phi(x)$ and obtaining core subspace $V_0$ by taking the linear span of integer translates of $\phi(x)$. The other spaces $V_j$, $j \in \mathbb{Z}$ can be generated as the scaled versions of $V_0$.

Following is the formal definition of a frame multiresolution analysis on LCA groups.

**Definition 3.1.** *[15] A frame multiresolution analysis of $L^2(G)$ is a sequence of closed subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ of $L^2(G)$ satisfying the following properties:*

(a). $V_j \subseteq V_{j+1}$, *for all* $j \in \mathbb{Z}$;

(b). $\bigcup_{j\in\mathbb{Z}} V_j$ *is dense in* $L^2(G)$ *and* $\bigcap_{j\in\mathbb{Z}} V_j = \left\{0\right\}$;

(c). $f(\cdot) \in V_j$ *if and only if* $f(\alpha(\cdot)) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$;

(d). *the function* $\phi \in V_0$ *such that the collection* $T_\lambda f(\cdot) = f(\cdot - \lambda) \in V_0$, *for all* $\lambda \in \Lambda$;

(e). *the sequence* $\left\{T_\lambda \phi(\cdot) = \phi(\cdot - \lambda) : \lambda \in \Lambda\right\}$ *is a frame for the subspace* $V_0$.

The function $\phi$ appearing in (e) is called as the scaling function of an FMRA, where as the subspaces $V_j$'s are known as approximation spaces. An MRA is an FMRA in which the term "frame" is replaced by an "orthonormal basis" in condition (e).

We start the process of construction of an FMRA on $G$ by choosing an appropriate function $\phi \in L^2(G)$ such that the family $\{T_\lambda \phi : \lambda \in \Lambda\}$ of its $\Lambda$-translates forms a frame sequence. Define a function $\phi$ on $G$ via its Fourier transform by

$$\hat{\phi}(\omega) = \chi_K(\omega), \quad \omega \in \widehat{G}, \tag{22}$$

where $\chi_K$ is the indicator function or the characteristic function of $K \subset \widehat{G}$ given by

$$\chi_K(\omega) = \begin{cases} 1, & \omega \in K \\ 0, & \omega \notin K \end{cases}.$$

We now define the subspaces $V_j$ by

$$V_j = \overline{\operatorname{span}}\left\{D^j T_\lambda \phi : \lambda \in \Lambda\right\}, \ j \in \mathbb{Z}. \tag{23}$$

We observe that the subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ satisfies the conditions (c)-(e) of the Definition 3.1 and also satisfies the intersection property trivially. The first and foremost task is to choose the set $K$ such that the family $\{T_\lambda \phi : \lambda \in \Lambda\}$ is a frame sequence. The following lemma solves our purpose.

**Lemma 3.2.** *Let $K \subset \widehat{G}$ be non-empty and let $\phi$ be defined by* (22)*. Then, the family $\left\{T_\lambda \phi : \lambda \in \Lambda\right\}$ of $\Lambda$-translates of $\phi$ constitutes a frame for $L^2(G)$ if*

$$K \subseteq \bigcup_{\ell=1}^{L} \left(\xi_\ell + \mathcal{S}\right), \ \text{for some } \xi_\ell \in \Lambda^\perp, \ 1 \leq \ell \leq L. \tag{24}$$

*Proof.* At the outset, we show that $\phi \in L^2(G)$ and, then we prove that $\Lambda$-translates of $\phi$ forms a frame sequence. We have

$$\left\|\phi\right\|^2 = \left\|\hat{\phi}\right\|^2 = \int_{\widehat{G}} \left|\hat{\phi}(\omega)\right|^2 d\omega = \int_K d\omega.$$

Since $\left(\xi_\ell + \mathcal{S}\right) \cap \left(\xi_{\ell'} + \mathcal{S}\right) = \emptyset$, $1 \le \ell \ne \ell' \le L$ and $\mu_{\widehat{G}}(\mathcal{S}) < \infty$, so we can write

$$\int_K d\omega \le \sum_{\ell=1}^{L} \int_{\mathcal{S}} d\omega = L\, d\mu_{\widehat{G}}(\mathcal{S}) < \infty,$$

which implies that $\|\phi\|^2 < \infty$ and, hence, $\phi \in L^2(G)$. Moreover, for any $\omega \in \widehat{G}$, we note that

$$\sum_{\gamma\in\Lambda^\perp} \left|\hat{\phi}(\omega+\gamma)\right|^2 = \sum_{\gamma\in\Lambda^\perp} \chi_{(\gamma+K)}(\omega).$$

Then, we see that

$$1 \le \sum_{\gamma\in\Lambda^\perp} \chi_{(\gamma+K)}(\omega) \le L < \infty,$$

for all $\omega \in \left\{\omega \in \widehat{G} : \sum_{\gamma\in\Lambda^\perp} |\hat{\phi}(\omega+\gamma)|^2 \ne 0\right\}$. Therefore, by virtue of (20), we conclude that the family $\{T_\lambda\phi : \lambda \in \Lambda\}$ constitutes a frame for $L^2(G)$. □

For the sake of brevity, we denote

$$\Phi(\omega) = \sum_{\gamma\in\Lambda^\perp} \left|\hat{\phi}(\omega+\gamma)\right|^2, \quad \text{and } \mathcal{N} = \left\{\omega \in \widehat{G} : \Phi(\omega) = 0\right\}. \tag{25}$$

**Remark 3.3.** *It is pertinent to mention that condition* (24) *implies that* $\mu_{\widehat{G}}(K) < \infty$, *however, one can not relax the assumption* (24) *to* $\mu_{\widehat{G}}(K) < \infty$ *because if we take* $K = \bigcup_{n=0}^{\infty} \left(n, n + \frac{1}{(n+1)^2}\right)$ *as a subset of the Euclidean space* $\mathbb{R}$ *and* $\mu_{\mathbb{R}}$ *as the Lebesgue measure on* $\mathbb{R}$, *then* $\mu_{\mathbb{R}}(K) < \infty$, *but the family of translates* $\left\{T_\lambda\phi : \lambda \in \Lambda\right\}$ *given* (22) *does not form a frame for* $L^2(G)$ *as the function* $\Phi$ *corresponding to* $\phi$ *is not bounded above.*

We now investigate the remaining two conditions for being an FMRA for $L^2(G)$, that is; the density condition and the nested property of the subspaces $V_j$, $j \in \mathbb{Z}$. In the following lemma, we derive conditions under which it holds.

**Lemma 3.4.** *Let* $\left\{V_j : j \in \mathbb{Z}\right\}$ *be the subspaces of* $L^2(G)$ *as defined in* (23) *and let* $\phi$ *belongs to* $L^2(G)$ *such that the system of translates* $\{T_\lambda\phi : \lambda \in \Lambda\}$ *is a frame for* $V_0$. *Assume that* $K \subset \widehat{G}$ *is such that* (24) *holds. Then,*

(i). *If the subspaces* $V_j$ *are nested, then* $K \subseteq \hat{\alpha}(K)$.

(ii). *If* $K \subseteq \hat{\alpha}(K)$ *is such that* $P\left(\hat{\alpha}^{-1}(K)\right) \cap P\left(K \backslash \hat{\alpha}^{-1}(K)\right) = \emptyset$, *then* $V_j$ *are nested.*

*Proof.* Assume that the sequence of subspaces $\left\{V_j : j \in \mathbb{Z}\right\}$ is nested. Then, it is quite evident that $\phi \in V_0 \subset V_1$, so there exists sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that

$$\phi(x) = \sum_{\lambda\in\Lambda^\perp} c_\lambda D T_\lambda \phi(x), \ \ \forall\, x \in G.$$

The above relation can be recast in the Fourier domain as

$$\hat{\phi}\Big(\hat{\alpha}(\omega)\Big) = F(\omega)\,\hat{\phi}(\omega), \;\; \forall\, \omega \in \widehat{G},$$

where $F \in L^2(\widehat{G}/\Lambda^\perp)$. This means that

$$\chi_{\hat{\alpha}^{-1}(K)}(\omega) = F(\omega)\chi_K(\omega), \;\; \forall\, \omega \in \widehat{G}.$$

We observe that if $\omega \notin K$, then $w \notin \hat{\alpha}^{-1}(K)$. This means that $\hat{\alpha}^{-1}(K) \subseteq K$ and thus $K \subseteq \hat{\alpha}(K)$.

Let $K$ be a subset of $\hat{\alpha}(K)$ such that $P(\hat{\alpha}^{-1}(K)) \cap P(K\backslash\hat{\alpha}^{-1}(K)) = \emptyset$. Define a function $m_0$ by

$$m_0(\omega) = \begin{cases} 1, & \omega \in P(\hat{\alpha}^{-1}(K)) \\ 0, & \omega \in P(K\backslash\hat{\alpha}^{-1}(K)) \end{cases}. \tag{26}$$

Clearly, $m_0(\omega)$ is $\Lambda^\perp$-periodic, bounded, and satisfies

$$\hat{\phi}(\hat{\alpha}(\omega)) = m_0(\omega)\hat{\phi}(\omega), \;\; \forall\, \omega \in \widehat{G}. \tag{27}$$

Then, for any $f \in V_0$, there exist some $F_2 \in L^2(\widehat{G}/\Lambda^\perp)$ such that $\hat{f}(\omega) = F_2(\omega)\hat{\phi}(\omega)$, $\forall \omega \in \widehat{G}$.Equivalently, we can write

$$\hat{f}(\hat{\alpha}(\omega)) = H(\omega)\,\hat{\phi}(\omega), \;\; \text{where } H(\omega) = F_2(\hat{\alpha}(\omega))m_0(\omega), \; \forall\, \omega \in \widehat{G}.$$

Clearly, $H(\omega) \in L^2(\widehat{G}/\Lambda^\perp)$ which implies that $f \in V_1$. Thus, $V_0 \subset V_1$, which in turn implies that $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$ as the dilation operator $D$ is a unitary operator on $L^2(G)$. □

The function $m_0(\omega)$ appearing in (26) is called the two scale symbol or the refinement mask. Note that the functions $m_0(\omega)$ and $\Phi(\omega)$ can be expressed as

$$\Phi(\omega) = \sum_{\gamma \in \Lambda^\perp} \chi_{(\gamma+K)}(\omega), \quad m_0(\omega) = \chi_{P(\hat{\alpha}^{-1}(K))}(\omega), \; \forall\, \omega \in \widehat{G},$$

and have the following relationship:

$$\Phi\Big(\hat{\alpha}(\omega)\Big) = \sum_{j=0}^{\delta_\alpha - 1} \Big|m_0(\omega - \omega_j)\Big|^2 \Phi(\omega - \omega_j). \tag{28}$$

It only remains to prove the density condition of an FMRA for $L^2(G)$. We shall use the already established results of [15] for this purpose.

**Lemma 3.5.** *Let $\Big\{V_j : j \in \mathbb{Z}\Big\}$ be the subspaces of $L^2(G)$ as defined in* (23) *and let $\phi \in L^2(G)$ defined via* (22) *be such that the system of translates $\{T_\lambda \phi : \lambda \in \Lambda\}$ forms a frame for $V_0$. If there exists a neighbourhood of $0 \in \widehat{G}$ contained in $K$, then $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(G)$.*

*Proof.* Let $U$ be any neighbourhood of $0 \in \widehat{G}$ such that $U \subseteq K$. Then, $\hat{\phi}(\omega) \neq 0$ on $U$ as the function $\phi$ is defined via (22). By applying same strategy as employed in [15], it is easy to show that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(G)$. □

We now sum up all the conditions required to be imposed on the set $K$ so that the function $\phi$ defined via (22) generates an FMRA for $L^2(G)$.

**Theorem 3.6.** *Let* $\left\{V_j : j \in \mathbb{Z}\right\}$ *be the subspaces of* $L^2(G)$ *as defined in* (23) *and let* $K \subset \widehat{G}$. *Assume that the following hold:*

(i). *There exist some integer* $L > 0$ *such that*

$$K \subseteq \bigcup_{\ell=1}^{L} (\xi_\ell + \mathcal{S}), \ \textit{for some } \xi_\ell \in \Lambda^\perp, \ 1 \le \ell \le L. \tag{24}$$

(ii). $K \subseteq \hat{\alpha}(K)$ *and* $P(\hat{\alpha}^{-1}(K)) \cap P(K \backslash \hat{\alpha}^{-1}(K)) = \emptyset$.

(iii). *There exist an open neighbourhood* $U$ *of* $0 \in \widehat{G}$ *such that* $U \subseteq K$.

*Then, the function* $\phi$ *as defined by* (22) *generates an FMRA* $\{V_j : j \in \mathbb{Z}\}$ *for* $L^2(G)$

Now from onwards, we shall assume that the set $K$ satisfies all the properties of Theorem 3.6.

**Theorem 3.7.** *Let* $\phi \in L^2(G)$ *be defined via* (22) *and let* $\left\{V_j : j \in \mathbb{Z}\right\}$ *be an FMRA for* $L^2(G)$. *Then,* $V_j \subseteq PW_G(\hat{\alpha}^j(K))$, *for each* $j \in \mathbb{Z}$. *Furthermore,* $V_0 = PW_G(K)$, *whenever* $K \subseteq \mathcal{S}$.

*Proof.* At the outset, we claim that $V_0 \subseteq PW_G(K)$. To do so, we assume that $f \in V_0$, then there exist some $F \in L^2(\widehat{G}/\Lambda^\perp)$ such that $\hat{f}(\omega) = F(\omega)\hat{\phi}(\omega)$, $\forall\, \omega \in \widehat{G}$. Hence, $\hat{f}(\omega) = 0$ for every $\omega \notin K$, which in turn implies that $f \in PW_G(K)$. Therefore, it follows that $V_0 \subseteq PW_G(K)$. Next, we claim that $D(PW_G(K)) = PW_G(\hat{\alpha}(K))$. Let $f \in D(PW_G(K))$. This means that $\hat{f}(\hat{\alpha}(\omega)) = 0$, for all $\omega \notin K$. Hence, it follows that $f \in PW_G(\hat{\alpha}(K))$. The converse can also be traced on the similar lines. Thus, we conclude that $V_j \subseteq PW_G(\hat{\alpha}^j(K))$, for all $j \in \mathbb{Z}$.

For any $g \in PW_G(K)$, we observe that $\widehat{g}(\omega) = 0$, $\forall\, \omega \notin \mathcal{S}$. Therefore, there exist a sequence $\{c_\lambda\}_{\lambda\in\Lambda} \in l^2(\Lambda)$ such that [15, 26]

$$\widehat{g}(\omega) = \sum_{\lambda\in\Lambda} c_\lambda(\omega, \lambda)\chi_{\mathcal{S}}(\omega), \ \forall\, \omega \in \widehat{G}.$$

Using the fact that $\chi_K \cdot \chi_{\mathcal{S}} = \chi_K$ and $\widehat{g} = \widehat{g} \cdot \chi_K$, we have

$$\widehat{g}(\omega) = \sum_{\lambda\in\Lambda} c_\lambda(\omega, \lambda)\hat{\phi}(\omega),$$

which further implies that $g \in V_0$ and hence, we conclude that $V_0 = PW_G(K)$, whenever $K \subseteq \mathcal{S}$. This completes the proof of Theorem 3.7. □

## 4. FMRA Wavelet Frames over LCA Groups: Construction and Examples

In this section, firstly we shall present a detailed procedure for the construction of FMRA based wavelet frames in $L^2(G)$ and then present several illustrative examples.

### *4.1. FMRA Wavelet Frames with Dyadic Dilations*

In this subsection, we construct wavelet frames with dyadic dilations from a given FMRA $\left\{V_j : j \in \mathbb{Z}\right\}$. For the case of dyadic dilations, the constant $\delta_\alpha$ appearing in (7) is equal to 2 and the quotient groups $\Lambda/\alpha(\Lambda)$ and $\Lambda^\perp/\hat{\alpha}(\Lambda^\perp)$ are supposed to have the following representation [22]:

$$\Lambda/\alpha(\Lambda) = \left\{\alpha(\Lambda), \lambda_0 + \alpha(\Lambda)\right\}, \quad \Lambda^\perp/\hat{\alpha}(\Lambda^\perp) = \left\{\hat{\alpha}(\Lambda^\perp), \gamma_0 + \hat{\alpha}(\Lambda^\perp)\right\}.$$

Besides, we can write $\omega_0 = \hat{\alpha}^{-1}(\gamma_0)$, $(\omega_0, \lambda_0) = -1$ [**?** ], so that the relation (28) boils down to

$$\Phi\big(\hat{\alpha}(\omega)\big) = \left|m_0(\omega)\right|^2 \Phi(\omega) + \left|m_0(\omega + \omega_0)\right|^2 \Phi(\omega + \omega_0)$$

where $m_0(\omega)$ is the two-scale symbol associated with the scaling function of an FMRA.

Let $W_j$ denote the orthogonal complement of the subspace $V_j$ in $V_{j+1}$. Then, property (c) of the Definition 3.1 implies that the sequence of subspaces $\left\{W_j : j \in \mathbb{Z}\right\}$ is pairwise orthogonal and satisfies

$$L^2(G) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{29}$$

Moreover, it is easy to verify that these subspaces also satisfies the scaling property; that is,

$$W_j = \left\{f \in L^2(G) : f\left(\alpha^{-j}(\cdot)\right) \in W_0\right\}. \tag{30}$$

Therefore, in order to construct wavelet frames for $L^2(G)$ via FMRA, all we need is to find functions $\psi_1, \psi_2$ in $L^2(G)$ such that their $\Lambda$-translates form a frame for $W_0$ or equivalently, we can say that the family $\left\{D^j T_\lambda \psi_\ell : j \in \mathbb{Z}, \lambda \in \Lambda, \ell = 1, 2\right\}$ constitutes a frame for $L^2(G)$.

We now define the functions $\psi_1, \psi_2$ in $L^2(G)$ by

$$\psi_\ell(\hat{\alpha}(\omega)) = F_\ell(\omega)\,\hat{\phi}(\omega),\ \ell = 1, 2,\ \forall\ \omega \in \widehat{G} \tag{31}$$

where $F_1, F_2 \in L^\infty(\widehat{G}/\Lambda^\perp)$. Then, our task reduces to find suitable functions $F_1$ and $F_2$ in $L^\infty(\widehat{G}/\Lambda^\perp)$. We decompose the entire space $\widehat{G}$ into several disjoint subspaces as:

$$\mathcal{P}^{(0)} = P(K)^c \cap P(\omega_0 + K)^c, \quad \mathcal{P}^{(1)} = P(K) \cap P(\omega_0 + K)^c,$$
$$\mathcal{P}^{(2)} = P(K)^c \cap P(\omega_0 + K), \quad \mathcal{P}^{(12)} = P(K) \cap P(\omega_0 + K).$$

The set $\mathcal{P}^{(12)}$ can further be splited into

$$\mathcal{P}_1^{(12)} = \mathcal{P}^{(12)} \cap \left(P(\hat{\alpha}^{-1}(K)) \cup P(\omega_0 + \hat{\alpha}^{-1}(K))\right)^c,$$
$$\mathcal{P}_2^{(12)} = \mathcal{P}^{(12)} \cap \left(P(\hat{\alpha}^{-1}(K)) \cup P(\omega_0 + \hat{\alpha}^{-1}(K))\right)$$

Therefore, a possible choice for the functions $F_1(\omega)$ and $F_2(\omega)$ [? ] could be

$$F_1(\omega) = \begin{cases} \mathcal{T}_{\omega_0}\left(\overline{m_0}\Phi\right)(\omega)(\omega, \lambda_0), & \omega \in \mathcal{P}_2^{(12)} \\ 1, & \omega \in \mathcal{P}_1^{(12)} \\ 1, & \omega \in \mathcal{P}^{(1)}, m_0(\omega) = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{32}$$

$$F_2(\omega) = \begin{cases} (\omega, \lambda_0), & \omega \in \mathcal{P}_1^{(12)} \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

After clubbing, we obtain

$$F_1(\omega) = \chi_{A_1}(\omega) + \Big(\sum_{\gamma\in\Lambda^\perp} \chi_{(\gamma+\omega_0+K)}(\omega)\Big)(\omega, \lambda_0)\chi_{B_1}(\omega),$$
$$F_2(\omega) = (\omega, \lambda_0)\chi_{C_1}(\omega);$$

where

$$A_1 = P(K) \cap P(\hat{\alpha}^{-1}(K))^c \cap \left(P(\omega_0 + K)^c \cup P(\omega_0 + \hat{\alpha}^{-1}(K)^c)\right),$$
$$B_1 = P(K) \cap P(\omega_0 + K) \cap P(\omega_0 + \hat{\alpha}^{-1}(K)),$$
$$C_1 = P(K) \cap P(\omega_0 + K) \cap P(\hat{\alpha}^{-1}(K))^c \cap P(\omega_0 + \hat{\alpha}^{-1}(K))^c.$$

Hence, relation (31) becomes

$$\widehat{\psi_1}(\omega) = \chi_{A_2}(\omega) + \Big(\sum_{\gamma\in\Lambda^\perp} \chi_{(\hat{\alpha}(\gamma)+\gamma_0+\hat{\alpha}(K))}(\omega)\Big)(\hat{\alpha}^{-1}(\omega),\lambda_0)\chi_{B_2}(\omega),$$
$$\widehat{\psi_2}(\omega) = (\hat{\alpha}^{-1}(\omega),\lambda_0)\chi_{C_2}(\omega),$$

where

$$A_2 = \hat{\alpha}(K)\cap\hat{\alpha}(A_1), \quad B_2 = \hat{\alpha}(K)\cap\hat{\alpha}(B_1) \quad \text{and} \quad C_2 = \hat{\alpha}(K)\cap\hat{\alpha}(C_1).$$

It is pertinent to mention that only one function $\psi(x)$ can generate a wavelet frame for the space $W_0$ provided $\mathcal{P}_1^{(12)} = \emptyset$. This condition has also been thoroughly investigated in [22].

We now demonstrate our theory with the aid of some illustrative examples on the Euclidean group $\mathbb{R}$ and $\mathbb{R}^2$. Example 4.1 deals in itself with multiple cases, where in some cases, only one function is required to generate wavelet frames, while in others, two functions are required to generate the wavelet frame.

**Example 4.1.** *Let $G = \mathbb{R}$ be the group of real numbers with Haar measure*

$$\mu_G(\mathcal{B}) = \int_{\mathcal{B}} d\mu_G(x),$$

where $\mathcal{B}$ is any Borel set in $G$ and $d\mu_G(x) = dx$. Let $\Lambda = \mathbb{Z}$ be the uniform lattice in $\mathbb{R}$ and the map $\alpha : x \mapsto x^2$ as a dilative automorphism on $\mathbb{R}$. Then, we observe that the map $x \mapsto e^{2\pi i x\omega}$, $x,\omega\in\mathbb{R}$ acts as a continuous character on the group $\mathbb{R}$, and hence the dual group $\widehat{\mathbb{R}}$ of $\mathbb{R}$ can be identified with $\mathbb{R}$ itself. As a consequence, we have $\Lambda^\perp = \Lambda$ and $\hat{\alpha} = \alpha$. Besides, the set $\mathcal{S} = [0,1)$ with $\mu_G(\mathcal{S}) = 1$, acts as a self similar fundamental domain for both $\mathbb{R}$ and $\widehat{\mathbb{R}}$. Therefore, a suitable representation of the quotient group $\Lambda/\alpha(\Lambda)$ can be obtained via

$$\Lambda/\alpha(\Lambda) = \Lambda^\perp/\hat{\alpha}(\Lambda^\perp) = \mathbb{Z}/2\mathbb{Z} = \Big\{2\mathbb{Z}, 1+2\mathbb{Z}\Big\}.$$

For the set $K = [-y, y)$, $y\in\mathbb{R}$, we discuss the following cases:

**Case 1:** Assume that $1/3 < y < 1/2$. Then, we observe that

$$K \subseteq \Big((0+\mathcal{S})\cup(-1+\mathcal{S})\Big), \text{ with } \mathcal{S} = [0,1).$$

Clearly, $\hat{\alpha}(K) = [-2y, 2y)$ and hence, $K \subseteq \hat{\alpha}(K)$. Also, we see that $P(\hat{\alpha}^{-1}(K)) \cap P(K\backslash\hat{\alpha}^{-1}(K)) = \emptyset$. Note that $\hat{\phi}(\omega) \neq 0$ on any neighbourhood of $0\in\mathbb{R}$, so if we define the subspaces $V_j$, $j\in\mathbb{Z}$ via (23), then we shall get a function $\phi(x)$ of the form (22) which generates an FMRA for $L^2(G)$. Subsequently, the function $\Phi(\omega)$ and the associated two-scale symbol $m_0(\omega)$ takes the form

$$\Phi(\omega) = \sum_{n\in\mathbb{Z}} \chi_{(n+K)}(\omega), \quad m_0(\omega) = \chi_{P(\hat{\alpha}^{-1}(K))}(\omega),\ \forall\ \omega\in\widehat{G}.$$

We now partition the set $\widehat{G} = \mathbb{R}$ as discussed above to obtain

$$\mathcal{P}^{(1)} = \bigcup_{n\in\mathbb{Z}}\Big(\Big[n, n+\frac{1}{2}-y\Big)\cup\Big[n+\frac{1}{2}+y, n+1\Big)\Big), \quad \mathcal{P}^{(2)} = \bigcup_{n\in\mathbb{Z}}\left(n+[y,1-y)\right),$$

and

$$\mathcal{P}_1^{(12)} = \bigcup_{n\in\mathbb{Z}}\Big(\Big[n+\frac{y}{2}, n+\frac{1}{2}-\frac{y}{2}\Big)\cup\Big[n+\frac{1}{2}+\frac{y}{2}, n+1-\frac{y}{2}\Big)\Big),$$
$$\mathcal{P}_2^{(12)} = \bigcup_{n\in\mathbb{Z}}\Big(n+\Big(\Big[\frac{1}{2}-y,\frac{y}{2}\Big)\cup\Big[\frac{1}{2}-\frac{y}{2}, y\Big)\cup\Big[1-y,\frac{1}{2}+\frac{y}{2}\Big)\cup\Big[1-\frac{y}{2},\frac{1}{2}+y\Big)\Big)\Big).$$

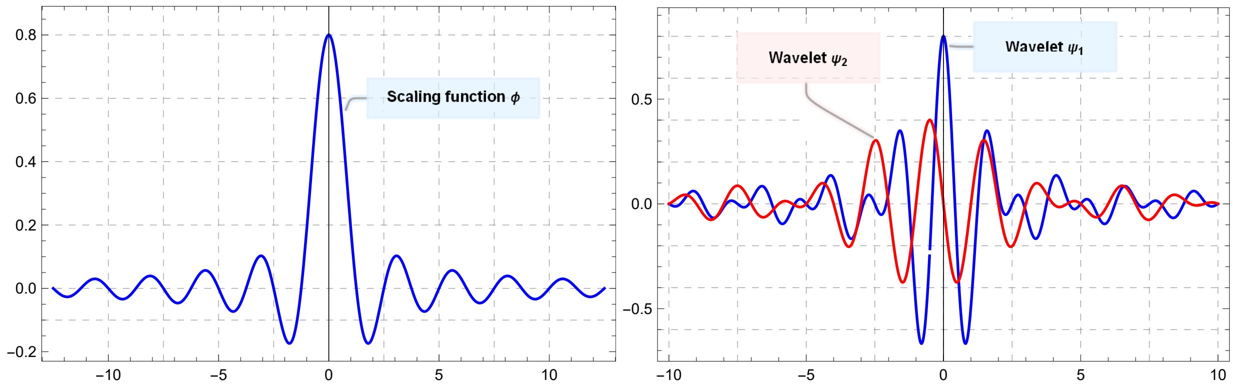


Figure 1: Scaling function $\phi$ and the corresponding wavelet functions $\psi_1$ and $\psi_2$, when $y = 0.4$ (Case 1).

Finally, we define the functions $F_1, F_2 \in L^2(\mathbb{T})$ via the relations (32) and (33) to get

$$\psi_1(\omega) = \chi_{B_1}(\omega), \quad \psi_2(\omega) = e^{\pi i \omega}\chi_{B_2}(\omega)$$

where $B_1 = [-2y, -y) \cup [y, 2y)$ and $B_2 = [-1 + y, -y) \cup [y, 1 - y)$.

**Case 2:** Let $1/4 < y \leq 1/3$. This can be dealt with in a similar manner as that of Case 1 with a slight difference in the representation of the intervals involved.

**Case 3:** For $y \leq \frac{1}{4}$, the set $\mathcal{P}_1^{(12)}$ becomes a null set, and hence, we require only one function $\psi$ to generate a wavelet frame for $L^2(\mathbb{R})$. An explicit representation of such function $\psi$ is given by

$$\hat{\psi}(\omega) = \chi_{B_1}(\omega), \text{ where } B_1 = [-2y, -y) \cup [y, 2y).$$

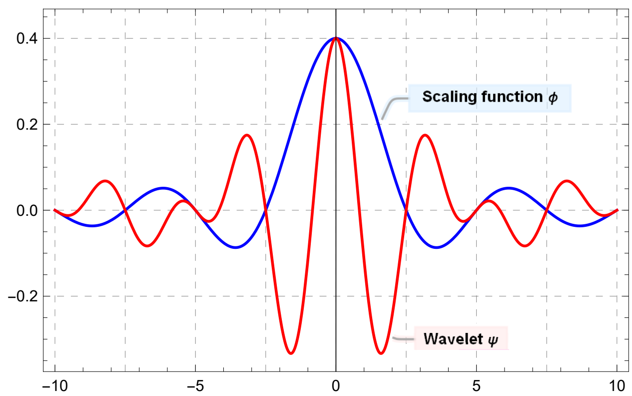


Figure 2: Scaling function $\phi$ and the corresponding wavelet functions $\psi_1$ and $\psi_2$ when $y = 0.2$ (Case 3).

**Case 4:** For the case $y = 1/2$, we shall obtain only one wavelet $\psi$ of the form

$$\hat{\psi}(\omega) = e^{\pi i \omega}\chi_{B_3}(\omega); \text{ where } B_3 = \left[-1, \frac{-1}{2}\right) \bigcup \left[\frac{1}{2}, 1\right).$$

**Case 5:** For the case $y > 2/3$, $P(\hat{\alpha}^{-1}(K)) \cap P(K \backslash \hat{\alpha}^{-1}(K)) \neq \emptyset$. Therefore, we shall not get an FMRA for $L^2(\mathbb{R})$ via the procedure discussed above.

**Remark 4.2.** *It is worth noticing that for the Haar wavelet, we always choose the scaling function $\phi$ with compact support, whereas, in our above example, we have taken the Fourier transform $\hat{\phi}(\omega)$ to be of compact support. Besides, the Haar wavelet can't be considered as a wavelet in Paley Weiner spaces.*

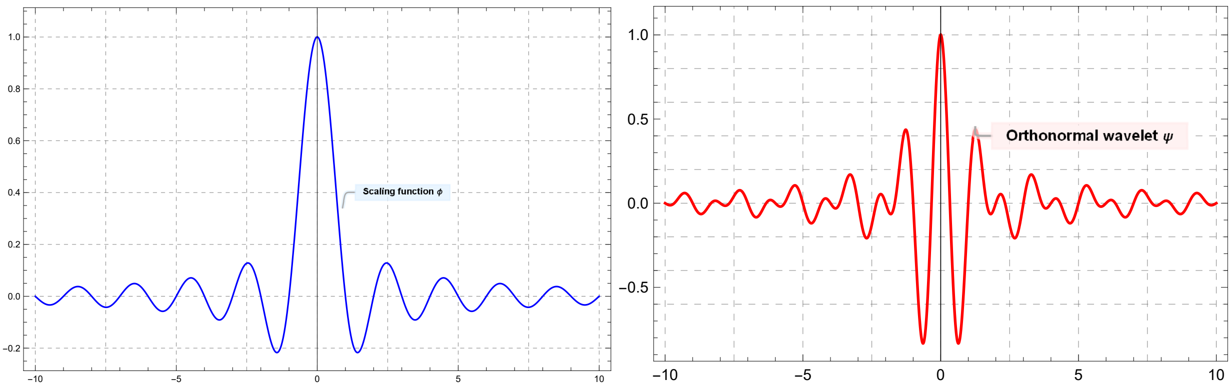


Figure 3: Scaling function $\phi$ and the corresponding wavelet $\psi$, when $y = 0.5$ (Case 4).

**Example 4.3.** *Let $G = \mathbb{R}^2$ be an LCA group with the standard Haar measure*

$$\mu(\mathcal{B}) = \iint_{\mathcal{B}} dx_1\, dx_2,$$

where $\mathcal{B} \subseteq \mathbb{R}^2$ is a Borel set. Define a map

$$\alpha\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{where} \quad M = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}.$$

As the eigenvalues of the matrix $M$ are strictly greater than 1, so it is easy to conclude that the automorphism $\alpha$ on $G$ is dilative with $\Delta(\alpha) = 2$. Since the dual group $\widehat{G}$ of $G$ is $\mathbb{R}^2$ with the same measure $\mu$. Therefore, the action of a character $\omega \in \mathbb{R}^2$ on an element $x \in \mathbb{R}^2$ can be defined by $(\omega, x) = e^{2\pi i \omega \cdot x}$, where $\omega \cdot x$ represents the usual dot product in $\mathbb{R}^2$. It is worth noticing that with the choice of the uniform lattice $\Lambda = \mathbb{Z} \times \mathbb{Z}$, both the uniform lattices $\Lambda$ and $\Lambda^\perp$ become equal. Consequently, the dilative automorphism $\hat{\alpha}$ on $\widehat{G}$ takes the form

$$\hat{\alpha}\left(\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}\right) = \tilde{M} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \quad \text{where} \quad \tilde{M} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

We observe that $\tilde{M} = M^T$, the transpose of the matrix $M$. Assume that the quotient group $\Lambda^\perp / \hat{\alpha}(\Lambda^\perp)$ has the following representation:

$$\frac{\Lambda^\perp}{\hat{\alpha}(\Lambda^\perp)} = \left\{ \hat{\alpha}(\Lambda^\perp), \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \hat{\alpha}(\Lambda^\perp) \right\}.$$

Then, the set $\mathcal{S} = \left[-\frac{2}{3}, \frac{1}{3}\right] \times \left[-\frac{1}{3}, \frac{2}{3}\right]$ will act as a self-similar tile for $\widehat{G}$. Define a set $K \subset \mathbb{R}^2$ by

$$K = \left[-\frac{1}{2}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{8}\right],$$

and subsequently define a function $\phi \in L^2(G)$ via the relation (23). Then, we have the following observations:

(i). Equation (24) holds for $K \subset \mathcal{S}$ with $L = 1$.

(ii). If $\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \in K$, then $\hat{\alpha}^{-1}(\omega) = \tilde{M}^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} -\omega_2 \\ \omega_1/2 \end{bmatrix}$. Subsequently, we have

$$\hat{\alpha}^{-1}(K) = \left[-\frac{1}{8}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{8}\right] \subset K.$$

Note that $(\omega+\hat{\alpha}^{-1}(K))\cap(\tilde{\omega}+\hat{\alpha}^{-1}(K))=\emptyset$, whenever $\omega\neq\tilde{\omega}$. A similar result hold for the set $K\backslash\hat{\alpha}^{-1}(K)$. Thus, we conclude that $P\Big(\hat{\alpha}^{-1}(K)\Big)\cap P\Big(K\backslash\hat{\alpha}^{-1}(K)\Big)=\emptyset$.

(iii). The function $\hat{\phi}$ does not vanish on any neighbourhood of $0\in\widehat{G}$.

Therefore, if we define the subspaces $V_j$, $j\in\mathbb{Z}$ via (23), then we shall obtain the function $\phi$ appearing in (iii) to generates an FMRA for $L^2(\mathbb{R}^2)$. Moreover, an application of Theorem 3.7 implies that, for each $j\in\mathbb{Z}$, $V_j=PW_G(\hat{\alpha}^j(K))$. Finally, we divide the space $\widehat{G}=\mathbb{R}^2$ into disjoint subspaces as

$$\mathcal{P}^{(0)}=\bigcup_{m,n\in\mathbb{Z}}\left(\begin{bmatrix}m\\n\end{bmatrix}+\mathcal{S}\backslash\left(K\cup\left(\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}+K\right)\right)\right),\quad \mathcal{P}^{(1)}=\bigcup_{m,n\in\mathbb{Z}}\left(\begin{bmatrix}m\\n\end{bmatrix}+K\right),$$
$$\mathcal{P}^{(2)}=\bigcup_{m,n\in\mathbb{Z}}\left(\begin{bmatrix}m\\n\end{bmatrix}+\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}+K\right)\quad\text{and}\quad \mathcal{P}^{(12)}=\emptyset.$$

Evidently, after some straightforward calculations, we obtain

$$\hat{\psi}(\omega)=\chi_{B_4}(\omega),\quad\text{where } B_4=\left[-\frac{1}{2},\frac{1}{4}\right]\times\left[\frac{1}{8},\frac{1}{2}\right].$$

### *4.2. FMRA Wavelet Frames with Arbitrary Dilations*

This subsection is entirely devoted to the construction of FMRA-based wavelet frames associated with arbitrary dilations. Although, we have studied such constructions in our recent work [22, 23], however, here our intention is to develop the theory for band-limited functions on locally compact abelian groups. More precisely, our aim is to show that the family

$$\left\{D^jT_\lambda\psi_\ell: j\in\mathbb{Z},\lambda\in\Lambda,1\leq\ell\leq\delta_\alpha\right\}\tag{34}$$

associated with the band-limited scaling function $\phi(x)$ defined by (22) forms a frame for $L^2(G)$.

We consider the functions $\psi_\ell$ in the Fourier domain as

$$\hat{\psi}_\ell(\omega)=m_\ell(\omega)\,\hat{\phi}(\omega),\quad\forall\,\omega\in\widehat{G},\ 1\leq\ell\leq\delta_\alpha,\tag{35}$$

where $m_\ell\in L^2(\widehat{G}/\Lambda)$. Then, our motive shall be to find such functions $m_\ell(\omega)$ so that $\hat{\phi}=\chi_K$. In analogy with the precious case, we begin with the decomposition of the space $\widehat{G}$ as

$$\mathcal{P}_N=\left(\bigcup_{j=0}^{\delta_\alpha-1}P(\omega_j+\hat{\alpha}^{-1}(K))\right)^c\quad\text{and}\quad\widetilde{\mathcal{P}_N}=\bigcup_{j=0}^{\delta_\alpha-1}P\left(\omega_j+\hat{\alpha}^{-1}(K)\right);$$

We further decompose the sets $\mathcal{P}_N$ and $\widetilde{\mathcal{P}_N}$ into even smaller and disjoint sets as

$$\mathcal{P}_N^0=\left(\bigcap_{j=0}^{\delta_\alpha-1}P(\omega_j+K)^c\right)\bigcap\mathcal{P}_N,\quad\mathcal{P}_N^f=\left(\bigcap_{j=0}^{\delta_\alpha-1}P(\omega_j+K)\right)\bigcap\mathcal{P}_N,$$
$$\mathcal{P}_N^{j_1j_2\cdots j_\ell}=\left(\bigcap_{i=1}^{\ell}P(\omega_{j_i}+K)\right)\bigcap\left(\bigcap_{j\neq j_i}P(\omega_j+K)^c\right)\bigcap\mathcal{P}_N,\quad\ell\leq\delta_\alpha-1,$$
$$\widetilde{\mathcal{P}_N}^{j_1j_2\cdots j_\ell}=\left(\bigcap_{i=1}^{\ell}P(\omega_{j_i}+K)\right)\bigcap\left(\bigcap_{j\neq j_i}P(\omega_j+K)^c\right)\bigcap\widetilde{\mathcal{P}_N},\quad\ell\leq\delta_\alpha,$$
$${}_m\widetilde{\mathcal{P}_N}^{j_1j_2\cdots j_\ell}=P(\omega_{j_m}+\hat{\alpha}^{-1}(K))\cap\left(\bigcap_{i=1}^{\ell}P(\omega_{j_i}+K)\right)\cap\left(\bigcap_{j\neq j_i}P(\omega_j+K)^c\right),\quad 1\leq m\leq\ell.$$

As is known that the choice of the functions $m_\ell, 1 \leq \ell \leq \delta_\alpha - 1$ is of utmost importance. As pointed in [23], the choice of the functions $m_\ell$'s is quite trivial if any one of the following conditions is satisfied:

(i). $\omega \in \mathcal{P}_N^0$

(ii). $\omega \in \mathcal{S}_N^{j_1 j_2 \cdots j_\ell}, \quad j_1, j_2, \cdots, j_\ell \neq 0$

(iii). $\omega \in \widetilde{\mathcal{S}_N}^{j_1 j_2 \cdots j_l}, \quad j_1, j_2, \cdots, j_\ell \neq 0.$

For each case, we consider $m_1 = m_2 = \cdots = m_{\delta_\alpha} = 0$, and discuss the remaining cases one by one. For the case $\omega \in \mathcal{P}_N^0$, we choose

$$m_\ell(\omega) = (\omega, \lambda_{\ell-1}), \;\; 1 \leq \ell \leq \delta_\alpha.$$

Similarly, for the case $\omega \in \mathcal{P}_N^{j_1 j_2 \cdots j_\ell}$, we take $j_m = 0$, for some $1 \leq m \leq \ell$. Since the values of the functions $m_\ell$, $1 \leq \ell \leq \delta_\alpha - 1$ are interdependent, as such we have listed these values in the form of a Table 1 given below.

| | $\mathcal{P}_N^{j_1 j_2 \cdots j_l}$ | $-\omega_{j_2} + \mathcal{P}_N^{j_1 j_2 \cdots j_l}$ | $\cdots$ | $\cdots$ | $-\omega_{j_l} + \mathcal{P}_N^{j_1 j_2 \cdots j_l}$ |
|---|---|---|---|---|---|
| $m_1$ | 1 | 0 | $\cdots$ | $\cdots$ | 0 |
| $m_2$ | 0 | 1 | $\cdots$ | $\cdots$ | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m_l$ | 0 | 0 | $\cdots$ | $\cdots$ | 1 |
| $m_{l+1}$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m_{\delta_\alpha}$ | 0 | 0 | $\cdots$ | $\cdots$ | 0 |

Table 1: Choice of variable for the case $\omega \in \mathcal{P}_N^{j_1 j_2 \cdots j_l}$, $j_1 = 0$.

Likewise, the possible choice of the functions $m_\ell$, $1 \leq \ell \leq \delta_\alpha$, when ${}_1\omega \in \widetilde{\mathcal{P}_N}^{j_1 j_2 \cdots j_\ell}$, $j_1 = 0$ is listed in Table 2.

| | ${}_1\widetilde{\mathcal{P}_N}^{j_1 j_2 \cdots j_\ell}$ | $-\omega_{j_2} +_1 \widetilde{\mathcal{P}_N}^{j_1 j_2 \cdots j_\ell}$ | $-\omega_{j_3} +_1 \widetilde{\mathcal{P}_N}^{j_1 j_2 \cdots j_\ell}$ | $\cdots$ | $\cdots$ | $-\omega_{j_l} +_1 \widetilde{\mathcal{P}_N}^{j_1 j_2 \cdots j_\ell}$ |
|---|---|---|---|---|---|---|
| $m_1$ | $B_{j_2}$ | 1 | 0 | $\cdots$ | $\cdots$ | 0 |
| $m_2$ | $B_{j_3}$ | 0 | 1 | $\cdots$ | $\cdots$ | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m_{\ell-1}$ | $B_{j_\ell}$ | 0 | 0 | $\cdots$ | $\cdots$ | 1 |
| $m_\ell$ | 0 | 0 | 0 | $\cdots$ | $\cdots$ | 0 |
| $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ | $\cdots$ |
| $m_{\delta_\alpha}$ | 0 | 0 | 0 | $\cdots$ | $\cdots$ | 0 |

Table 2: Choice of $m_j's$ on the set ${}_1\widetilde{\mathcal{P}_N}^{j_1 j_2 \cdots j\ell}$, $j_1 = 0$

For each $1 \leq i \leq \ell - 1$, the terms $B_{j_{i+1}}$, appearing in Table 2, are given via

$$B_{j_{i+1}} = -\chi_{C_{i+1}}(\omega) \left( \sum_{\gamma \in \Lambda^\perp} \chi_{(\gamma + \omega_{j_{i+1}} + K)}(\omega) \right) \left( \sum_{\gamma \in \Lambda^\perp} \chi_{(\gamma + K)}(\omega) \right)^{-1},$$

$$C_{i+1} = P(\hat{\alpha}^{-1}(K)) \cap P(\omega_{j_{i+1}} + \hat{\alpha}^{-1}(K)).$$

Thus, we conclude that the wavelet functions $\psi_i$, $1 \le i \le \delta_\alpha$ can now be defined via (35).

**Example 4.4.** *Let $G = \mathbb{R}$ be the group of real numbers with $\Lambda = \mathbb{Z}$ as a uniform lattice and let $\alpha : x \mapsto 3x$ be the dilative automorphism on $G$. Then, we observe that $\widehat{\mathbb{R}} = \mathbb{R}, \Lambda^{\perp} = \Lambda = \mathbb{Z}$ and $\hat{\alpha} = \alpha$. We choose $\mathcal{S} = [0, 1)$ as the fundamental domain associated with the lattice $\mathbb{Z}$ in $\mathbb{R}$. Then, $\mathcal{S} = [0, 1)$ can be also represented as*

$$\mathcal{S} = \bigcup_{j=0}^{2} \left( \frac{j}{3} + \left[0, \frac{1}{3}\right) \right)$$

Clearly, $\mathcal{S}$ is self-similar with respect to $\alpha$ and $\mathbb{Z}$. Assume that $K = P_1 \cup (-P_1)$, where $P_1 = \frac{1}{9} \bigcup_{n=0}^{1} [2n, 2n+1]$ and $-P_1 = \{-x : x \in P_1\}$. Define $\phi(x) \in L^2(G)$ via the relation (22) and the subspace $V_j = PW_G(\alpha^j(K)), j \in \mathbb{Z}$. Then, we observe that

(i). $K \subset \mathcal{S} \cup (-1 + \mathcal{S})$;

(ii). $\hat{\alpha}^{-1}(K) = P_2 \cup (-P_2)$; where $27P_2 = \bigcup_{n=0}^{1} [2n, 2n+1]$, and hence, $\hat{\alpha}^{-1}(K) \subset K$;

(iii). $P(\hat{\alpha}^{-1}(K)) \cap P(K \backslash \hat{\alpha}^{-1}(K)) = \emptyset$;

(iv). Any open neighbourhood $U$ of 0 satisfying $9U \subseteq [-1, 1]$, also satisfies $U \subseteq K$.

Thus, we conclude that the set $K$ satisfies all the properties listed in Theorem 3.6 and therefore, we can claim that the function $\phi$ generates an FMRA for $L^2(\mathbb{R})$.

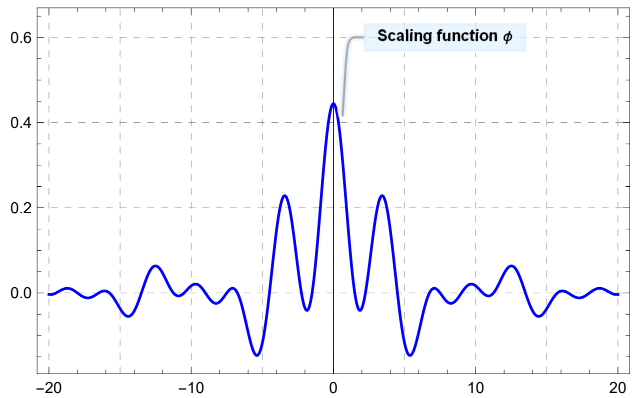


Figure 4: The scaling function $\phi$.

Note that the partitioning of set $\widehat{G}$ will lead us to know that:

(i). $\mathcal{P}_N^f$ is an empty set;

(ii). $\mathcal{P}_N^0 = \bigcup_{n \in \mathbb{Z}} \left( n + \left( \frac{1}{9} \bigcup_{n=0}^{2} [3n+1, 3n+2] \right) \right)$;

(iii). Further partitioning is possible

$$\mathcal{P}_N{}^{01} = \bigcup_{n\in\mathbb{Z}}\left(n + \left(\bigcup_{n=0}^{1}\left[\frac{6n+1}{27}, \frac{6n+2}{27}\right]\right)\right), \quad \widetilde{\mathcal{P}_N}^{01} = \bigcup_{n\in\mathbb{Z}}\left(n + \left(\bigcup_{\substack{n=0\\ n\neq 2}}^{4}\left[\frac{2n}{27}, \frac{2n+1}{27}\right]\right)\right),$$

$$\mathcal{P}_N{}^{02} = \bigcup_{n\in\mathbb{Z}}\left(n + \left(\bigcup_{n=3}^{4}\left[\frac{6n+1}{27}, \frac{6n+2}{27}\right]\right)\right), \quad \widetilde{\mathcal{P}_N}^{02} = \bigcup_{n\in\mathbb{Z}}\left(n + \left(\bigcup_{\substack{n=9\\ n\neq 11}}^{13}\left[\frac{2n}{27}, \frac{2n+1}{27}\right]\right)\right).$$

Moreover, the intervals corresponding to $n = 0, 1$ for $\widetilde{\mathcal{P}_N}^{01}$ are ${}_0\widetilde{\mathcal{P}_N}^{01}$ whereas the intervals corresponding to $n = 9, 10$ for $\widetilde{\mathcal{P}_N}^{02}$ are ${}_2\widetilde{\mathcal{P}_N}^{02}$. Besides, they have the following relationship:

$${}_0\widetilde{\mathcal{P}_N}^{01} = -\frac{1}{3} + \widetilde{\mathcal{P}_N}^{02}, \quad {}_0\widetilde{\mathcal{P}_N}^{02} = -\frac{2}{3} + {}_1\widetilde{\mathcal{P}_N}^{01}$$

Since the set $\mathcal{P}_N^f = \emptyset$, so only two functions $\psi_1$ and $\psi_2$ are enough to generate a wavelet frame for $L^2(\mathbb{R})$. As we know that the periodic functions $m_1(\omega)$ and $m_2(\omega)$ are essential for defining the generators $\psi_1(x)$ and $\psi_2(x)$ which can be derived by making use of Tables 1 and 2. Explicitly, the restriction of $m_1(\omega)$ and $m_2(\omega)$ on $\mathcal{S} = [0, 1)$ is given by

$$m_1(\omega) = \chi_{B_1}(\omega), \quad m_2(\omega) = \chi_{B_2}(\omega),$$

where $B_1 = \frac{1}{27}\big([1, 2] \cup [6, 9] \cup [18, 19] \cup [20, 21]\big)$, $\quad B_2 = \frac{1}{27}\big([19, 20] \cup [25, 26]\big)$.

Consequently, the wavelet functions $\psi_1(x)$ and $\psi_2(x)$ in the Fourier domain takes the form

$$\hat{\psi_1}(\omega) = \chi_{C_1}(\omega), \quad \hat{\psi_2}(\omega) = \chi_{C_2}(\omega),$$

where

$$C_1 = \frac{1}{9}\big([-9, -8] \cup [-7, -6] \cup [1, 2] \cup [6, 9]\big), \quad C_2 = \frac{1}{9}\big([-8, -7] \cup [-2, -1]\big).$$

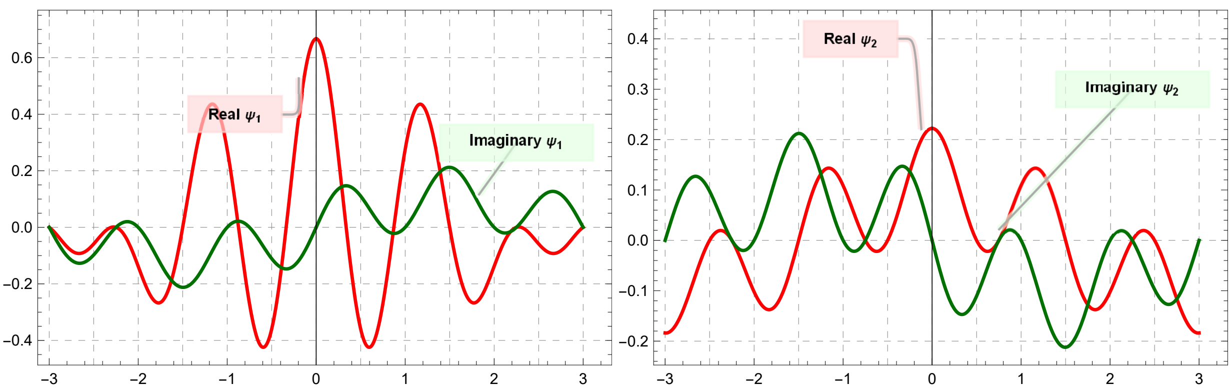


Figure 5: Wavelet functions $\psi_1$ and $\psi_2$.

*Data Availability:* Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

*Conflict of interest*: The authors declare that they have no competing interests.

## References


[1] L. Debnath and F.A. Shah, *Wavelet Transforms and Their Applications*, Birkhäuser, New York, 2015.
[2] F.A. Shah and A.Y. Tantary, *Wavelet Transforms: Kith and Kin*, CRC Press, Boca Raton, 2023.
[3] S. Mallat, Multiresolution approximations and wavelet orthonormal basis of L2(R), Trans. Amer. Math. Soc. **315** (1989), 69-87.
[4] M. Bownik, Riesz wavelets and generalized multiresolution analyses, Appl. Comput. Harmon. Anal. **14**(3) (2003), 181-194.
[5] R.A. Zalik, On MRA Riesz wavelets, Proc. Amer. Math. Soc. **135**(3) (2007), 787-793.
[6] F.A. Shah and Abdullah, Nonuniform multiresolution analysis on local fields of positive characteristic. Complex Anal. Opert. Theory. **9** (2015), 1589-1608.
[7] H.M. Srivastava, F.A. Shah and W.Z. Lone, Fractional nonuniform multiresolution analysis in $L^2(\mathbb{R})$, Math. Methods Appl. Sci. **44**(11) (2021), 9351-9372.
[8] J.J. Benedetto and S. Li, The theory of multiresolution analysis frames and applications to filter banks, Appl. Comput. Harmon. Anal. **5** (1998), 398-427.
[9] X. Yu, "Semiorthogonal multiresolution analysis frames in higher dimensions, Acta Applicandae Mathematicae, **111**(3) (2010), 257-286.
[10] S. Dahlke, Multiresolution analysis and wavelets on locally compact Abelian groups, in *Wavelets, Images, and Surface Fitting (1993)*, 141–156, A.K. Peters, Wellesley, MA.
[11] R.A. Kamyabi-Gol and R.R. Tousi, Some equivalent multiresolution conditions on locally compact Abelian groups, Proc. Indian Acad. Sci. Math. Sci. **120**(3) (2010), 317-331.
[12] R.A. Kamyabi-Gol and R.R. Tousi, The structure of shift-invariant spaces on a locally compact Abelian group, J. Math. Anal. Appl. **340** (2008), 219-225.
[13] Q. Yang and K.F. Taylor, Multiresolution analysis and Haar-like wavelet bases on locally compact groups, J. Appl. Funct. Anal. **7**(4) (2012), 413-439.
[14] M. Bownik and Q. Jahan, Wavelets on compact Abelian groups, Appl. Comput. Harmon. Anal. **49**(2) (2020), 471-494.
[15] R. Kumar and Satyapriya, Construction of a frame multiresolution analysis on locally compact Abelian groups, Aust. J. Math. Anal. Appl. **18** (2021), Article 5. 19 pages.
[16] W. Chen and S.S. Goh, Band-limited refinable functions for wavelets and framelets, Appl. Comput. Harmon. Anal. **28** (2010) 338-345.
[17] I. Daubechies, B. Han, A. Ron and Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. **14** (2003), 1-46.
[18] B. Han, *Framelets and Wavelets*, Birkhäuser, Basel, 2017.
[19] Z. Zhang, Framelet sets and associated scaling sets, Mathematics. **9** (2021), 2824.
[20] N. Atreasa, N. Karantzasb, M. Papadakis and T. Stavropoulosc, On the design of multi-dimensional compactly supported Parseval framelets with directional characteristics, Linear Algebra Appl. **582** (2019), 1-36.
[21] Z. Zhang, Characterization of frequency domains of band-limited frame multiresolution analysis, Mathematics. **9** (2021), 1050.
[22] R. Kumar, Satyapriya and Firdous A. Shah, Explicit construction of wavelet frames on locally compact Abelian groups, Anal. Math. Physics. **12** (2022), 83.
[23] R. Kumar, Satyapriya and Firdous A. Shah, Riesz multiresolution analysis on locally compact Abelian groups: Construction and exceptions, J. Contemp. Math. Anal.**58**(2) (2023), 125-135.
[24] G.B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, Florida, 1995.
[25] O. Christensen, *An introduction to Frames and Riesz Bases*, Birkhäuser, Boston, 2016.
[26] C. Cabrelli and V. Paternostro, Shift-invariant spaces on LCA groups, J. Funct. Anal. **258**(6) (2010), 2034–2059.
[27] O. Christensen and S.S. Goh, The unitary extension principle on locally compact Abelian groups, Appl. Comput. Harmon. Anal. **47** (2019), 1-29.
[28] A. Ron and Z. Shen, Frames and stable bases for shift-invariant subspaces of $L^2(\mathbb{R}^d)$, Canadian J. Math. **47**(5) (1995), 1051-1094.