Filomat 40:1 (2026), 121–131
https://doi.org/10.2298/FIL2601121P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On conditions for the Levin-Stečkin inequality and applications

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Abstract. Using an identity involving the Green function, we give a generalization of the Levin-Stečkin inequality as a result concerning difference of integral arithmetic means with different measures. The results are given with usual conditions for the Levin-Stečkin type inequalities. As a special case, generalization of the Levin-Stečkin inequality is given. We also prove that result for general measures can be easily reduced to one single condition. Comparing the two approaches we get some surprisingly good estimations.

1. Introduction

V. I. Levin and S. B. Stečkin proved in [5] the following theorem (see also [1], [3, pages 414-415], [6]).

Theorem 1.1. *Let f be defined on $[0,1]$ satisying the conditions:*

$$f(x) \text{ is nondecreasing for } 0 \le x \le \frac{1}{2}, \tag{1}$$

and

$$f(x) = f(1-x), x \in [0,1]. \tag{2}$$

Then for any convex function ϕ we have

$$\int_0^1 f(x)\phi(x)dx \le \int_0^1 f(x)dx \int_0^1 \phi(x)dx. \tag{3}$$

In 1988. J. Pečarić and S. S. Dragomir in [2] gave the following generalization of the Levin-Stečkin inequality.

Theorem 1.2. *Let $\mu\colon [0,1] \to \mathbb{R}$ be an increasing function such that $\mu(x) = -\mu(1-x)$, and let $f\colon [0,1] \to \mathbb{R}$ be an integrable function with respect to μ such that* (1) *and* (2) *hold. Then for any continuous convex function ϕ we have*

$$\int_0^1 d\mu(x) \int_0^1 f(x)\phi(x)d\mu(x) \le \int_0^1 f(x)d\mu(x) \int_0^1 \phi(x)d\mu(x). \tag{4}$$

2020 *Mathematics Subject Classification*. Primary 26D15; Secondary 26A51
Keywords. Levin-Stečkin inequality, Green function, convexity, Lebesgue-Stieltjes measure
Received: 05 September 2024; Accepted: 01 October 2025
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Consider the Green function $G\colon [a,b]\times[a,b]\to\mathbb{R}$ defined by

$$G(t,s)=\begin{cases}\frac{(t-b)(s-a)}{b-a}, & a\le s\le t,\\ \frac{(s-b)(t-a)}{b-a}, & t\le s\le b.\end{cases} \tag{5}$$

The function G is convex and continuous with respect to both s and t. It is also symmetric.

We will use the following lemma (see [4], [9]).

Lemma 1.3. *For every function $f\colon [a,b]\to\mathbb{R}$, $f\in C^2([a,b])$, the following identity holds:*

$$f(x)=\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)+\int_a^b G(x,s)f''(s)ds, \tag{6}$$

where the function G is defined as in (5).

Using Lemma 1.3 we obtain some interesting results concerning difference of the integral arithmetic means

$$\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)}-\frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}, \tag{7}$$

where expressions are well-defined, and $\lambda_1,\lambda_2\colon [a,b]\to\mathbb{R}$ are suitable functions (see rest of the paper).
Lemma 1.3 enables us to reduce proofs of inequalities for convex (or concave) functions to proofs of analogous inequalities for the Green function. It appears that this method works quite elegantly even in the cases in which is known that the proofs are rather involved.
Using this method we give a simple geometric condition for the Levin-Stečkin inequality in the case of measures $d\lambda$ and $wd\lambda$ and a simple analytic condition in the case of measures $d\lambda_1$ and $d\lambda_2$. We also show that the former one implies the latter one, but not vice versa (see Example 3.4).

2. Conditions for the Levin-Stečkin inequality via the Green function

The following equivalence is used throughout the paper.

Theorem 2.1. *Let $\lambda_i\colon [a,b]\to\mathbb{R}$, $i=1,2$ be continuous functions of bounded variation such that $\lambda_i(a)\neq\lambda_i(b)$, $i=1,2$. Let the function G be defined as in* (5)*. Then the following two statements are equivalent:*

1. *For every continuous convex function $f\colon [a,b]\to\mathbb{R}$ holds*

$$\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)}-\frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)} \ge \frac{f(b)-f(a)}{b-a}\left[\frac{\int_a^b xd\lambda_2(x)}{\int_a^b d\lambda_2(x)}-\frac{\int_a^b xd\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right]. \tag{8}$$

2. *For every $s\in[a,b]$ holds*

$$\frac{\int_a^b G(x,s)d\lambda_2(x)}{\int_a^b d\lambda_2(x)}-\frac{\int_a^b G(x,s)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\ge 0. \tag{9}$$

Proof. (1) ⇒ (2) Let (1) holds. As the function $G(\cdot,s)$, for $s\in[a,b]$ is also continuous and convex on $[a,b]$, the inequality (8) also holds for this function, and we use $G(b,s)=G(a,s)=0$.

(2) ⇒ (1) Let (2) holds. From Lemma 1.3 we have that every function f defined on interval $[a, b]$ such that continuous f'' exists, can be represented in the form (5). Now we have

$$\begin{aligned}
&\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)} = \int_a^b f(x)\left[\frac{d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right] \\
&= \int_a^b \left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b) + \int_a^b G(x,s)f''(s)ds\right)\left[\frac{d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right] \\
&= \int_a^b \left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)\right)\left[\frac{d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right] \\
&+ \int_a^b \left(\int_a^b G(x,s)f''(s)ds\right)\left[\frac{d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right]. \qquad (10)
\end{aligned}$$

Using the Fubini theorem on the last term in (10), and the obvious identity

$$\begin{aligned}
&\int_a^b \left(\frac{b-x}{b-a}f(a) + \frac{x-a}{b-a}f(b)\right)\left[\frac{d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right] \\
&= \frac{f(b)-f(a)}{b-a}\left[\frac{\int_a^b x d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b x d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right],
\end{aligned}$$

(10) becomes

$$\begin{aligned}
&\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)} \qquad (11) \\
&= \frac{f(b)-f(a)}{b-a}\left[\frac{\int_a^b x d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b x d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right] \\
&+ \int_a^b f''(s)\left[\frac{\int_a^b G(x,s)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b G(x,s)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}\right]ds.
\end{aligned}$$

In addition since f is convex, it follows that $f''(s) \geq 0$ for all $s \in [a, b]$. Therefore, if for every s in $[a, b]$ the inequality (9) is valid, then for every continuous convex $f : [a, b] \to \mathbb{R}$, such that continuos f'' exists, inequality (8) holds.

Furthermore, it should be noticed that it is not necessary to demand the existence of the second derivative of the function f (see [8, page 172] and references therein). The differentiability condition can be eliminated by using the fact that it is possible to approximate uniformly a continuous convex function by convex polynomials. □

In Theorem 2.1 we have a statement for convex functions. The similar conclusion holds for concave functions.

Remark 2.2. *Suppose that the assumptions as in Theorem 2.1 are fulfilled. Then the following statements are equivalent:*

(1') *For every continuous concave function* $f : [a, b] \to \mathbb{R}$ *the reversed inequality in* (8) *holds,*

(2') *For every* $s \in [a, b]$ *the inequality in* (9) *holds.*

Results in Theorem 2.1 are given for two general measures $d\lambda_1$ and $d\lambda_2$. Inequalities of the Levin-Stečkin type are usually given for functions λ_1, λ_2 (that is, measures $d\lambda_1, d\lambda_2$) satisying the following conditions:

1. λ is increasing on $[a, b]$,
2. $\lambda(x) = -\lambda(a + b - x)$, for all x in $[a, b]$.

Under this furher requirements, we have the following theorem.

Theorem 2.3. *Let $\lambda_1, \lambda_2 \colon [a, b] \to \mathbb{R}$ be continuous functions, increasing on interval $[a, b]$, such that $\lambda_i(a) \neq \lambda_i(b)$ and $\lambda_i(x) = -\lambda_i(a + b - x)$, for all x in $[a, b]$, $i = 1, 2$. Let the function G be defined as in* (5).
Then the following two statements are equivalent:

1. *For every continuous convex function $f \colon [a, b] \to \mathbb{R}$ holds*

$$\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} \geq \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}. \tag{12}$$

2. *For all $s \in [a, b]$ holds*

$$\frac{\int_a^b G(x, s)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} \geq \frac{\int_a^b G(x, s)d\lambda_1(x)}{\int_a^b d\lambda_1(x)}. \tag{13}$$

Proof. Under aditional conditions on the weights and the measure, we have

$$\int_a^b x d\lambda(x) = (a + b) \int_a^{\frac{a+b}{2}} d\lambda(x),$$

$$\int_a^b d\lambda(x) = 2 \int_a^{\frac{a+b}{2}} d\lambda(x),$$

which gives

$$\frac{\int_a^b x d\lambda(x)}{\int_a^b d\lambda(x)} = \frac{a + b}{2},$$

where $d\lambda$ is $d\lambda_1$ or $d\lambda_2$, and the claims follow from Theorem 2.1. □

Remark 2.4. *Suppose that the assumptions as in Theorem 2.3 are fulfilled. Then the following statements are equivalent:*

(1′) *For every continuous concave function $f \colon [a, b] \to \mathbb{R}$ the reversed inequality in* (12) *holds,*
(2′) *For every $s \in [a, b]$ the inequality in* (13) *holds.*

We now show that using Theorem 2.3 we can elegantly prove Theorem 1.2, and by that the Levin-Stečkin inequality (3).
Since the Levin-Stečkin inequality is given on the interval $[0, 1]$, we give the following two results on that interval. The analogous result for the interval $[a, b]$ easily follows.

Theorem 2.5. *Let $\lambda \colon [0, 1] \to \mathbb{R}$ be an continuous increasing function such that $\lambda(0) \neq \lambda(1)$ and $\lambda(x) = -\lambda(1 - x)$, for $x \in [0, 1]$. Let w be an increasing function on interval $[0, \frac{1}{2}]$ such that $w(x) = w(1 - x)$, for $x \in [0, 1]$ and $\int_0^1 w(x)d\lambda(x) \neq 0$. Then the inequality*

$$\frac{\int_0^1 f(x)d\lambda(x)}{\int_0^1 d\lambda(x)} \geq \frac{\int_0^1 w(x)f(x)d\lambda(x)}{\int_0^1 w(x)d\lambda(x)} \tag{14}$$

holds for every continuous convex function $f \colon [0, 1] \to \mathbb{R}$.

Proof. From Theorem 2.3, with substitutions $d\lambda_1(x) = w(x)d\lambda(x), d\lambda_2(x) = d\lambda(x)$, to prove (14), it is enough to prove

$$\int_0^1 G(x,s)w(x)d\lambda(x) \int_0^1 d\lambda(x) \le \int_0^1 G(x,s)d\lambda(x) \int_0^1 w(x)d\lambda(x) \tag{15}$$

for every $s \in [0,1]$.
Set

$$F(s) := \int_0^1 G(x,s)d\lambda(x) \int_0^1 w(x)d\lambda(x) - \int_0^1 G(x,s)w(x)d\lambda(x) \int_0^1 d\lambda(x). \tag{16}$$

The claim is that $F(s) \ge 0$ for every $s \in [0,1]$.
Using the definition of the Green function G, we get:

$$\begin{aligned} F(s) &= \left[(s-1)\int_0^s x d\lambda(x) + s\int_s^1 (x-1)d\lambda(x)\right]\int_0^1 w(x)d\lambda(x) \\ &\quad -\left[(s-1)\int_0^s xw(x)d\lambda(x) + s\int_s^1 (x-1)w(x)d\lambda(x)\right]\int_0^1 d\lambda(x). \end{aligned}$$

Using obvious substitutions it is easy to prove $F(s) = F(1-s)$ for every $s \in [0,1]$, and $F(0) = F(1) = 0$.
Differentiation gives:

$$\begin{aligned} F'(s) &= \left[\int_0^s x d\lambda(x) + \int_s^1 (x-1)d\lambda(x)\right]\int_0^1 w(x)d\lambda(x) \\ &\quad -\left[\int_0^s xw(x)d\lambda(x) + \int_s^1 (x-1)w(x)d\lambda(x)\right]\int_0^1 d\lambda(x). \end{aligned} \tag{17}$$

Let's prove that $F'(s) \ge 0$ for $0 \le s \le \frac{1}{2}$. Using some elementary calculus, we get

$$F'(s) = 2\int_0^s d\lambda(x) \cdot \int_0^{\frac{1}{2}} w(x)d\lambda(x) - 2\int_0^s w(x)d\lambda(x) \cdot \int_0^{\frac{1}{2}} d\lambda(x).$$

In this way the claim $F'(s) \ge 0$ is equivalent to the inequality

$$\frac{\int_0^s w(x)d\lambda(x)}{\int_0^s d\lambda(x)} \le \frac{\int_0^{1/2} w(x)d\lambda(x)}{\int_0^{1/2} d\lambda(x)},$$

which is a simple consequence of the increasing property of w on $[0, \frac{1}{2}]$.
It follows that $F'(s) \ge 0$ for $s \in [0, \frac{1}{2}]$, and $F(s) \ge 0$ for $s \in [0, \frac{1}{2}]$. Due to the symmetry of F this concludes the proof. □

The next result gives us conditions on real Lebesgue-Stieltjes measures $d\lambda_1$ and $d\lambda_2$ so that for continuos convex functions the inequality (13) holds.

Theorem 2.6. *Let $\lambda_i \colon [0,1] \to \mathbb{R}, i = 1,2$ be continuous increasing functions such that $\lambda_i(0) \ne \lambda_i(1)$, and $\lambda_i(x) = -\lambda_i(1-x)$, $i = 1,2$, for all x in $[0,1]$. Let the function G be as defined in* (5). *If*

$$\frac{\lambda_1(s)}{\lambda_1(1)} \le \frac{\lambda_2(s)}{\lambda_2(1)} \tag{18}$$

holds for every s in $[0,1/2]$, then

$$\frac{\int_0^1 G(x,s)d\lambda_2(x)}{\int_0^1 d\lambda_2(x)} - \frac{\int_0^1 G(x,s)d\lambda_1(x)}{\int_0^1 d\lambda_1(x)} \ge 0$$

holds for every s in $[0, 1]$.

Proof. We search for conditions under which inequality

$$\int_0^1 G(x,s)d\lambda_1(x)\int_0^1 d\lambda_2(x) \le \int_0^1 G(x,s)d\lambda_2(x)\int_0^1 d\lambda_1(x), \tag{19}$$

holds.
Define the function $F\colon [0,1] \to \mathbb{R}$ by

$$F(s) := \int_0^1 G(x,s)d\lambda_2(x)\int_0^1 d\lambda_1(x) - \int_0^1 G(x,s)d\lambda_1(x)\int_0^1 d\lambda_2(x).$$

Using the definition of the Green function, we get

$$\begin{aligned} F(s) &= \left[(s-1)\int_0^s x d\lambda_2(x) + s\int_s^1 (x-1)d\lambda_2(x)\right]\int_0^1 d\lambda_1(x) \\ &\quad -\left[(s-1)\int_0^s x d\lambda_1(x) + s\int_s^1 (x-1)d\lambda_1(x)\right]\int_0^1 d\lambda_2(x). \end{aligned}$$

Using obvious substitutions it is easy to prove $F(s) = F(1-s)$ for every $s \in [0,1]$, and $F(0) = F(1) = 0$.
The proof reduces to $F'(s) \ge 0$, $s \in [0, 1/2]$.
Differentiation gives

$$\begin{aligned} F'(s) &= \left[\int_0^s x d\lambda_2(x) + \int_s^1 (x-1)d\lambda_2(x)\right]\int_0^1 d\lambda_1(x) \\ &\quad -\left[\int_0^s x d\lambda_1(x) + \int_s^1 (x-1)d\lambda_1(x)\right]\int_0^1 d\lambda_2(x). \end{aligned}$$

Using some elementary calculus we get

$$\begin{aligned} &F'(s) = \left(\int_0^1 x d\lambda_2(x) - \int_s^1 d\lambda_2(x)\right)\int_0^1 d\lambda_1(x) \\ &-\left(\int_0^1 x d\lambda_1(x) - \int_s^1 d\lambda_1(x)\right)\int_0^1 d\lambda_2(x) \\ &= -2\cdot\int_s^{\frac{1}{2}} d\lambda_2(x)\cdot\int_0^{\frac{1}{2}} d\lambda_1(x) + 2\cdot\int_s^{\frac{1}{2}} d\lambda_1(x)\cdot\int_0^{\frac{1}{2}} d\lambda_2(x). \end{aligned} \tag{20}$$

Now $F'(s) \ge 0$ for $s \in [0, 1/2]$ is equivalent to

$$\frac{\int_s^{1/2} d\lambda_2(x)}{\int_0^{1/2} d\lambda_2(x)} \le \frac{\int_s^{1/2} d\lambda_1(x)}{\int_0^{1/2} d\lambda_1(x)}, s \in [0, 1/2]$$

which simply gives

$$\frac{\lambda_2(s)}{\lambda_2(0)} \le \frac{\lambda_1(s)}{\lambda_1(0)},$$

which is in some sense misleading (compare to (19), note that $\lambda_i(0) < 0$, $i = 1, 2$). So more accordingly to (19) it can be written as

$$\frac{\lambda_1(s)}{\lambda_1(1)} \le \frac{\lambda_2(s)}{\lambda_2(1)}.$$

□

From Theorems 2.1 and 2.6 the next result easily follows.

Corollary 2.7. *Let $\lambda_i\colon [a,b] \to \mathbb{R}, i = 1,2$ be continuous increasing functions such that $\lambda_i(a) \neq \lambda_i(b)$, and $\lambda_i(x) = -\lambda_i(a+b-x)$, $i = 1,2$, for all x in $[a,b]$. If*

$$\frac{\lambda_1(s)}{\lambda_1(b)} \leq \frac{\lambda_2(s)}{\lambda_2(b)}$$

holds for every s in $[a, \frac{a+b}{2}]$, then the inequality

$$\frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)} - \frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)} \geq 0$$

holds for every continuous convex function $f\colon [a,b] \to \mathbb{R}$.

Remark 2.8. *In this remark we want to emphasize the important connections between Theorem 2.5 and Theorem 2.6. We can write the inequality (9) in the following form*

$$\int_0^1 G(x,s)d\lambda_1(x)\int_0^1 d\lambda_2(x) \leq \int_0^1 G(x,s)d\lambda_2(x)\int_0^1 d\lambda_1(x), \tag{21}$$

where λ_1 and λ_2 are continuous increasing with $\lambda_i(1-x) = -\lambda_i(x)$, $i = 1,2$. As it is shown in Theorem 2.6, $F'(s) \geq 0$ for $s \in [0,1/2]$, in this case, is equivalent to

$$\frac{\lambda_1(s)}{\lambda_1(1)} \leq \frac{\lambda_2(s)}{\lambda_2(1)},\ s \in [0,1/2].$$

Let $w\colon [0,1] \to \mathbb{R}$ be such that w is increasing on $[0, \frac{1}{2}]$ and $w(x) = w(1-x)$, for all x in $[0,1]$, and λ_2 be continuous increasing function with $\lambda_2(1-x) = -\lambda_2(x)$.
Now, let λ_1 be such that $d\lambda_1(s) = w(s)d\lambda_2(s)$. Using (20) we have

$$F'(s) = 2\cdot\int_s^{\frac{1}{2}} w(x)d\lambda_2(x)\cdot\int_0^{\frac{1}{2}} d\lambda_2(x) - 2\cdot\int_s^{\frac{1}{2}} d\lambda_2(x)\cdot\int_0^{\frac{1}{2}} w(x)d\lambda_2(x).$$

Since w is increasing on $[0, \frac{1}{2}]$, we have

$$\frac{\int_s^{\frac{1}{2}} w(x)d\lambda_2(x)}{\int_s^{\frac{1}{2}} d\lambda_2(x)} \geq \frac{\int_0^{\frac{1}{2}} w(x)d\lambda_2(x)}{\int_0^{\frac{1}{2}} d\lambda_2(x)},$$

that is $F'(s) \geq 0$ on $[0,1/2]$, which gives $\frac{\lambda_1(s)}{\lambda_1(1)} \leq \frac{\lambda_2(s)}{\lambda_2(1)}$ on $[0,1/2]$.
Our proof of the above implication is indirect. It could be of some interest to give a direct proof of the above implication, or more explicitly, the proof of the following claim:
Suppose that $\lambda : [0,1] \to \mathbb{R}$ is an increasing and antisymmetric with respect to $1/2$. If $w : [0,1] \to [0,\infty)$ is increasing on $[0,1/2]$ and $w(1-x) = w(x)$ on $[0,1]$, then

$$\frac{\lambda_1(s)}{\lambda_1(1)} \leq \frac{\lambda_2(s)}{\lambda_2(1)},\ s \in [0,1/2],$$

where $d\lambda_1(x) := w(x)d\lambda(x)$ and $d\lambda_2(x) := d\lambda(x)$.

To complete the picture we give the following general result. It uses the same technique as the proof of Theorem 1.2. Compare with similar results given in [7].
It is based on following theorem proved in [3, Chap. XI, Theorem 5.1].

Theorem 2.9. *Let $d\mu$ be a signed measure on $(a, b) \subseteq \mathbb{R}$. Then*

$$\int_a^b f(x)d\mu(x) \geq 0$$

for every convex function f on (a, b) if and only if

$$\int_a^b d\mu(x) = 0, \quad \int_a^b x d\mu(x) = 0, \quad \int_a^t (t-x)d\mu(x) \geq 0 \text{ for every } t \in [a,b].$$

Theorem 2.10. *For $i = 1, 2$ let $\lambda_i \colon [a, b] \to \mathbb{R}$ be continuous increasing functions such that $\lambda_i(a) \neq \lambda_i(b)$, and $\lambda_i(x) = -\lambda_i(a + b - x)$ for all x in $[a, b]$. Then the following two statements are equivalent:*

1. *For every convex function $f \colon (a, b) \to \mathbb{R}$ holds*

$$\frac{\int_a^b f(x)d\lambda_1(x)}{\int_a^b d\lambda_1(x)} \leq \frac{\int_a^b f(x)d\lambda_2(x)}{\int_a^b d\lambda_2(x)}. \tag{22}$$

2. *For every $x \in [a, (a + b)/2]$ holds*

$$\frac{\int_a^x \lambda_1(s)ds}{\lambda_1(b)} \leq \frac{\int_a^x \lambda_2(s)ds}{\lambda_2(b)}. \tag{23}$$

Proof. Set for $s \in [a, (a + b)/2]$:

$$\widetilde{f}(x) = \begin{cases} s - x & , x \in [a,s] \\ 0 & , x \in [s, a+b-s] \\ x - (a+b-s) & , x \in [a+b-s, b] \end{cases}$$

Then, using symmetry and integration by parts, we get for $i = 1, 2$:

$$\begin{aligned} \frac{\int_a^b \widetilde{f}(x)d\lambda_i(x)}{\int_a^b d\lambda_i(x)} &= \frac{2\int_a^s (s-x)d\lambda_i(x)}{\lambda_i(b)-\lambda_i(a)} \\ &= -\frac{(s-a)\lambda_i(a)}{\lambda_i(b)} + \frac{\int_a^s \lambda_i(x)dx}{\lambda_i(b)} = s - a + \frac{\int_a^s \lambda_i(x)dx}{\lambda_i(b)}. \end{aligned} \tag{24}$$

Suppose that (22) holds for any convex f on (a, b). Since $\widetilde{f}$ is obviously convex, using (24) in (22) and by simple rearranging we get (23).
Assume that (23) holds. Set

$$d\mu := \frac{d\lambda_2}{\lambda_2(b)} - \frac{d\lambda_1}{\lambda_1(b)}.$$

Obviously $d\mu$ is a signed measure. Note that $\int_a^b f(x)d\mu(x)$ is well-defined (although may be infinite) for any convex f on (a, b). The first two conditions are trivially satisfied. The formula (24) and the assumption (23) give:

$$\int_a^s (s-x)d\mu(x) = \frac{\int_a^s \lambda_2(x)dx}{\lambda_2(b)} - \frac{\int_a^s \lambda_1(x)dx}{\lambda_1(b)} \geq 0.$$

From Theorem 2.9 follows that (22) holds for any convex f on (a, b). □

We remark that Theorem 2.10 implies both Corollary 2.7 and Theorem 2.5, but note that our proofs are based on applications of Lemma 1.3 (the Green function), and this technique allows us to establish the connection of conditions given in these results as explained in Remark 2.8 (see also the following section).

3. Applications

We can write the inequality (9) in the following form

$$\int_0^1 G(x,s)d\lambda_1(x)\int_0^1 d\lambda_2(x) \le \int_0^1 G(x,s)d\lambda_2(x)\int_0^1 d\lambda_1(x), \tag{25}$$

where λ_1 and λ_2 are increasing with $\lambda_i(1-x) = -\lambda_i(x)$, $i = 1,2$. As it is shown, $F'(s) \ge 0$ for $s \in [0,1/2]$ is, in this case, equivalent to

$$\frac{\lambda_1(s)}{\lambda_1(1)} \le \frac{\lambda_2(s)}{\lambda_2(1)},\ s \in [0,1/2].$$

If we let λ_1, λ_2 be such that $\frac{d\lambda_1(s)}{d\lambda_2(s)} = w(s)$, where $w\colon [0,1] \to \mathbb{R}$ is increasing on $[0,\frac{1}{2}]$ and $w(x) = w(1-x)$, for all x in $[0,1]$, we have shown in the Remark 2.8 that condition $\frac{\lambda_1(s)}{\lambda_1(1)} \le \frac{\lambda_2(s)}{\lambda_2(1)}$ on $[0,1/2]$ is fulfilled.

Example 3.1. *Suppose that $\lambda_2(x) = -\cos(\pi x)$ and $w_1(x) = x(1-x)$. From $\lambda_1'(x) = w_1(xs)\lambda_2'(x)$, we easily get:*

$$\lambda_1(x) = \int_{1/2}^x w_1(t)\lambda_2'(t)dt = \frac{1}{\pi^2}\Big(\pi(1-2x)\sin(\pi x) + \Big(\pi^2(x-1)x - 2\Big)\cos(\pi x)\Big).$$

Notice $\lambda_1(1) = 2/\pi^2$. Rearranging $\lambda_1(s)/\lambda_1(1) \le \lambda_2(s)/\lambda_2(1)$ it follows

$$\tan(\pi s) \le \frac{\pi s(1-s)}{1-2s},\ s \in [0,1/2).$$

Although weaker than the Becker-Stark inequality

$$\tan(\pi s) \le \frac{\pi s}{1-4s^2},\ s \in [0,1/2),$$

it is still interesting, especially in view how it is obtained.

Let's look at behaviour for the general case for the weight functions $w_n(x) = x^n(1-x)^n$, where n in $\mathbb{N}$. With $A_n(s)$ we denote the upper estimate for $\tan(\pi s)$ obtained by this technique, and with $A(s)$ denote $\frac{\pi s}{1-4s^2}$ the upper estimate in the Becker-Stark inequality.
The following results are obtained using Wolfram Mathematica.

For $n = 1$, that is for $w_1(x) = x(1-x)$, $\lambda_2(x) = -\cos(\pi x)$, we saw previously

$$A_1(s) = \frac{\pi s(1-s)}{1-2s},$$

and it is easy to see that $A(s) < A_1(s)$ on $(0,\frac{1}{2})$, $A(0) = A_1(0) = 0$.
For $n = 2$, in the similar fashion, we get

$$A_2(s) = \frac{A_1(s)}{2}\frac{\pi^2 s^2 - \pi^2 s - 12}{\pi^2 s^2 - \pi^2 s - 6},$$

$A_1(s) \ge A_2(s) \ge \tan(\pi s)$, *for $s \in [0,\frac{1}{2})$, and for $s_2 \approx 0.217925$, we have*

$$A(s) < A_2(s),\ \text{for } s \in (0,s_2),$$
$$A(s) > A_2(s),\ \text{for } s \in (s_2,\frac{1}{2}).$$

We skip the case $n = 3$. For $n = 4$, we get

$$A_4(s) = \frac{A_1(s)}{4} \cdot \frac{\pi^6(s-1)^3 s^3 + 240\pi^2\Big(7s^2 - 7s + 2\Big) - 4\pi^4\Big(14s^3 - 28s^2 + 17s - 3\Big)s - 20160}{\pi^6(s-1)^3 s^3 + 120\pi^2(7s^2 - 7s + 1) - 6\pi^4(7s^3 - 14s^2 + 8s - 1)s - 5040},$$

$A_2(s) \geq A_4(s) \geq \tan(\pi s)$, *for* $s \in [0, 1/2)$, *and for* $s_4 \approx 0.000964534$, *we have*

$$A(s) < A_4(s), \text{ for } s \in (0, s_4),$$
$$A(s) > A_4(s), \text{ for } s \in (s_4, \frac{1}{2}).$$

From this examples, one could make the following hypotheses:

1. $$A_k(s) \geq A_{k+1}(s) \text{ for } s \in [0, \frac{1}{2}), k \in \mathbb{N}.$$
2. $$s_k > s_{k+1}, \text{ for } k \in \mathbb{N}, k \geq 2,$$
   where s_n *is the solution of the equation* $A_n(x) = A(x)$ *on the interval* $(0, \frac{1}{2})$.
3. *for* $k \in \mathbb{N}$, $k \geq 2$
   $$A(s) < A_k(s), \text{ for } s \in (0, s_k),$$
   $$A(s) > A_k(s), \text{ for } s \in (s_k, \frac{1}{2}).$$
4. $$\lim_{k\to\infty} A_k(s) = \tan(\pi s).$$

Example 3.2. *Set* $\lambda_2(x) = \sinh(x - 1/2)$, $w(x) = x(1-x)$. *Using* $\lambda_1'(x) = w(x)\lambda_2'(x)$, *and* $\lambda_1(x) = \int_{1/2}^{x} w(t)\lambda_2'(t)dt$ *we can express* $\lambda_2(s)/\lambda_2(1) - \lambda_1(s)/\lambda_1(1)$ *in two different ways (using addition theorems or not):*

$$\frac{\lambda_2(s)}{\lambda_2(1)} - \frac{\lambda_1(s)}{\lambda_1(1)} = \frac{\operatorname{csch}\left(\frac{1}{2}\right)\left((1-2s)\cosh\left(\frac{1}{2}-s\right) - \sinh\left(\frac{1}{2}-s\right)\left(s^2 - s + \coth\left(\frac{1}{2}\right)\right)\right)}{\coth\left(\frac{1}{2}\right) - 2} \tag{26}$$

$$\frac{\lambda_2(s)}{\lambda_2(1)} - \frac{\lambda_1(s)}{\lambda_1(1)} = \frac{(e-1)s(es - s + e + 3)\cosh(s) - \left(-4e(s-1) - (s-3)s + e^2 s(s+1)\right)\sinh(s)}{(e-3)(e-1)}. \tag{27}$$

What significantly differs these two formulas is that $\frac{\operatorname{csch}(\frac{1}{2})}{\coth(\frac{1}{2})-2} > 0$, *but* $(e-3)(e-1) < 0$. *The condition*

$$\frac{\lambda_2(s)}{\lambda_2(1)} - \frac{\lambda_1(s)}{\lambda_1(1)} \geq 0, \; s \in [0, 1/2],$$

and (26), rearranging give

$$\tanh\left(\frac{1}{2} - s\right) \leq \frac{1-2s}{s^2 - s + \coth\frac{1}{2}}, \; s \in [0, 1/2],$$

or

$$\tanh s \leq \frac{2s}{s^2 - \frac{1}{4} + \coth\frac{1}{2}}, \; s \in [0, 1/2].$$

On the other hand, the same condition and (27) imply:

$$\tanh s \geq \frac{(e-1)s(es - s + e + 3)}{4e(1-s) + (3-s)s + e^2 s(s+1)}, \; s \in [0, 1/2].$$

Both estimations have error ≈ 0.008. *These are rational approximations (with irrational coefficients) of* tanh. *We couldn't find similar estimations to compare with.*

It remains to resolve if the following implication is valid:

$$\frac{\lambda_1(s)}{\lambda_1(1)} \leq \frac{\lambda_2(s)}{\lambda_2(1)},\ s \in [0,1/2] \Rightarrow w(s) = \frac{\lambda_1'(s)}{\lambda_2'(s)} \text{ is an increasing function on } [0,1/2). \tag{28}$$

Note that for $w(s) = \frac{\lambda_1'(s)}{\lambda_2'(s)}$ trivially holds $w(s) = w(1-s)$, $s \in [0,1]$ (this obviously holds for λ_1' and λ_2').

Example 3.3. *Let* $\lambda_1(x) = -\cos(\pi x)$, $\lambda_2(x) = (2x-1)^3$. *Obviously* $\lambda_1(s) \leq \lambda_2(s)$, $s \in [0,1/2]$ ($\lambda_1(0) = \lambda_2(0) = -1$, $\lambda_1(1/2) = \lambda_2(1/2) = 0$ *and* λ_1 *is convex,* λ_2 *is concave on* $[0,1/2]$). *It is easy to see that* $w(s) = \lambda_1'(s)/\lambda_2'(s) = \frac{\pi \sin(\pi s)}{6(2s-1)^2}$ *is an increasing function on* $[0,1/2)$. *In this case the inequality*

$$\int_0^1 f(x)d\lambda_1(x) \int_0^1 d\lambda_2(x) \leq \int_0^1 f(x)d\lambda_2(x) \int_0^1 d\lambda_1(x), \tag{29}$$

where f *is convex on* $[0,1]$, *holds using both arguments. Of course,* $\lambda_1(s)/\lambda_1(1) \leq \lambda_2(s)/\lambda_2(1)$, $s \in [0,1/2]$, *is easier to check.*

In the next example we show that the implication (28) generally doesn't hold.

Example 3.4. *Let* $\lambda_1(x) = \frac{1}{2}(4x-1)^3 - \frac{1}{2}$, *for* $x \in [0,1/2]$ *and* $\lambda_1(x) = -\lambda_1(1-x)$ *for* $x \in [1/2,1]$. *Obviously* $\lambda_1(1) = 1$, $\lambda_1(1/2) = 0$, λ_1 *increasing. Let* $\lambda_2(x) = (2x-1)^3$. *Straightforwardly*

$$\lambda_1(s) \leq \lambda_2(s),\ s \in [0,1/2],$$

hence the inequality (29) holds by the second condition.
On the other hand

$$w(s) = \frac{\lambda_1'(s)}{\lambda_2'(s)} = \frac{(4s-1)^2}{(2s-1)^2}$$

obviously is not an increasing function on $[0,1/2)$ ($w(0) = 1$, $w(1/4) = 0$). *Hence the first condition cannot be applied on the inequality (29).*

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