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A new hybrid conjugate gradient method as a convex combination of HZ and FR and PRP methods

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Abstract. In this paper, we propose a new conjugate gradient method for solving unconstrained optimization problem, whom is convex combination of the Hager-Zhan, Fletcher-Reeves and Polak-Ribére-Polyak algorithms. The search direction satisfies the sufficient descent condition and guarantees global convergence under the strong Wolfe line search conditions. Moreover, several numerical experiments on standard test functions are presented to illustrate that the proposed method is efficient and competitive compared to existing conjugate gradient methods.

1. Introduction

The unconstrained nonlinear optimization problem can be formulated by

$$\min\left\{f(x), x \in \mathbb{R}^n\right\}. \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ continuously differentiable and bounded below. The conjugate gradient methods are used for solving (1) which is an iterative method given by

$$x_0 \epsilon \mathbb{R}^n, \quad x_{k+1} = x_k + \alpha_k d_k \quad k \epsilon \mathbb{N}. \tag{2}$$

where the scalar $\alpha_k > 0$ is the step size.

The search direction d_k is defined by

$$d_k = \begin{cases} -g_0 & \text{for } k = 0 \\ -g_k + \beta_{k-1} s_{k-1} & \text{for } k \geq 1. \end{cases} \tag{3}$$

where $s_k = x_{k+1} - x_k = \alpha_k d_k.$, and $\beta_k \epsilon \mathbb{R}$ is the conjugate gradient coefficient. The different choices for β_k correspond to different conjugate gradient methods.

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Some classical conjugate gradient methods are: Hestenes-Stiefel method [16], Fletcher-Reeves method [10], Polyak-Polak-Ribére method [19, 21], Liu-Storey method [18], Dai-Yaun method [6] and Hager-Zhan method [11] which are given, respectively, by

$$\beta_k^{HS} = \frac{g_{k+1}^T y_k}{s_k^T y_k}, \tag{4}$$

$$\beta_k^{FR} = \frac{\left\|g_{k+1}\right\|^2}{\left\|g_k\right\|^2}, \tag{5}$$

$$\beta_k^{PRP} = \frac{g_{k+1}^T y_k}{\left\|g_k\right\|^2}, \tag{6}$$

$$\beta_k^{LS} = -\frac{g_{k+1}^T y_k}{s_k^T g_k}, \tag{7}$$

$$\beta_k^{DY} = \frac{\left\|g_{k+1}\right\|^2}{s_k^T y_k}, \tag{8}$$

$$\beta_k^{HZ} = \frac{1}{s_k^T y_k}\left(y_k^T - 2s_k^T \frac{\left\|y_k\right\|^2}{s_k^T y_k}\right)^T g_{k+1}, \tag{9}$$

where $y_k = g_{k+1} - g_k$, $g_k = \nabla f(x_k)$ the gradient of f and $\|.\|$ denotes the Euclidean norm.

The methods FR, DY, and HZ are globally convergent but have poor practical performance. In contrast, HS, PRP and LS algorithms are more efficient than FR, DY and HZ methods, but their global convergence cannot be demonstrated without changes. The hybrid conjugate gradient method is one of the most useful CG methods, which is a combination of different gradient conjugate algorithms, it is more efficient than the classical conjugate gradient methods for global convergence properties and excellent numerical performance.

Some well known hybrid CG methods

Touati Ahmed Storey [23] proposed the first hybrid conjugate gradient method, which β_k is calculated as

$$\beta_k^{\mathrm{TS}} = \begin{cases} \beta_k^{PRP} & \text{if } 0 \le \beta_k^{PRP} \le \beta_k^{FR} \\ \beta_k^{FR} & \text{else} \end{cases} \tag{10}$$

Andrie [3], [4]presented another two hybrid conjugate gradient methods, in which

$$\beta_k^{\text{HS-DY}} = (1-\theta_k)\,\beta_k^{HS} + \theta_k \beta_k^{DY}, \tag{11}$$

$$\beta_k^{\text{NDOMB}} = (1-\theta_k)\,\beta_k^{PRP} + \theta_k \beta_k^{DY}. \tag{12}$$

Djordjevic [7], [8] proposed a family of conjugate methods, where

$$\beta_k^{\text{FRPRPCC}} = (1-\theta_k)\,\beta_k^{PRP} + \theta_k \beta_k^{FR}, \tag{13}$$

$$\beta_k^{\text{LSCDCC}} = (1-\theta_k)\,\beta_k^{LS} + \theta_k \beta_k^{CD}. \tag{14}$$

Sellami et al. [23] proposed a family of conjugate methods where

$$\beta_k^{*} = \frac{(1-\lambda_k)\left\|g_{k+1}\right\|^2 + \lambda_k s_{k+1}^T g_{k+1}}{(1-\lambda_k-\mu_k)\left\|g_k\right\|^2 + (\lambda_k+\mu_k)\, s_k^T g_k}. \tag{15}$$

Hallal et al. [13] proposed a family of conjugate methods where

$$\beta_k^{\text{HDYCDHS}} = \lambda_k \beta_k^{DY} + \theta_k \beta_k^{CD} + (1 - \lambda_k - \theta_k) \beta_k^{HS}. \tag{16}$$

More related researches are detailed in the references [5], [12], and [12].

In this research, our main motivation is to improve the efficiency and robustness of nonlinear conjugate gradient (CG) methods by combining the advantages of existing approaches. Specifically, we integrate the FR, HZ, and PRP conjugate gradient algorithms to develop a new hybrid nonlinear CG method. In Section 2, we present the detailed algorithm of the proposed method. In Section 3, we prove that the new method satisfies the sufficient descent condition and achieves global convergence under an inexact line search. Section 4 provides several numerical experiments to illustrate the effectiveness and performance of the proposed algorithm.

2. New Conjugate Gradient Coefficient

In this article, motivated by the convex combination of conjugate gradient methods defined in [7] and [13], we give a new hybrid conjugate gradient formula for β_k know as β_k^{New} is defined by

$$\beta_k^{\text{New}} = \theta_k \beta_k^{HZ} + \eta_k \beta_k^{FR} + (1 - \theta_k - \eta_k) \beta_k^{PRP}, \tag{17}$$

where $0 < \theta_k < 1$, $0 < \eta_k < 1$, and $0 < \theta_k + \eta_k < 1$.

The new search direction d_k is defined as as

$$d_0 = -g_0, \; d_{k+1}^{\text{New}} = -g_{k+1} + \beta_k^{\text{New}} s_k. \tag{18}$$

The step size α_k is determinated according to the following strong Wolfe conditions

$$f(x_k + \alpha_k d_k) - f(x_k) \leq \delta \alpha_k \nabla f(x_k)^T d_k, \tag{19}$$

$$\sigma \nabla f(x_k)^T d_k \leq \nabla f(x + \alpha_k d_k)^T d_k \leq -\sigma \nabla f(x_k)^T d_k, \tag{20}$$

where $0 < \sigma < \frac{1}{2}$ and $0 < \delta < 1$.

We choose θ_k in a way that d_{k+1} satisfies the next conjugacy condition

$$0 = y_k^T d_{k+1}^{\text{New}}. \tag{21}$$

By substituting equation (18) into equation ((21), we obtain

$$0 = -y_k^T g_{k+1} + \theta_k \left(\beta_k^{HZ} - \beta_k^{PRP} \right) y_k^T s_k + \eta_k \left(\beta_k^{FR} - \beta_k^{PRP} \right) y_k^T s_k + \beta_k^{PRP} y_k^T s_k.$$

After some algebra, we get

$$\theta_k = \frac{\eta_k \left(\frac{-g_{k+1}^T g_k}{\|g_k\|^2} \right) y_k^T s_k + y_k^T g_{k+1} - \frac{g_{k+1}^T y_k}{\|g_k\|^2} y_k^T s_k}{\left(\frac{g_{k+1}^T y_k}{\|g_k\|^2} s_k^T y_k - y_k^T g_{k+1} + 2 \frac{\|y_k\|^2}{s_k^T y_k} s_k^T g_{k+1} \right)}. \tag{22}$$

The values of θ_k can be outside the interval $[0, 1]$. The next rule is represented as

If $\theta_k \geq 1$ put $\theta_k = 1$, if $\theta_k \leq 0$ put $\theta_k = 0$, and if $\theta_k + \eta_k \geq 1$ put $\theta_k + \eta_k = 1$.

Based on the above analysis, the new algorithm can be presented as follows

Algorithm 2.1

Step 1: Choose $x_0 \in \mathbb{R}^n$,and $\epsilon > 0$, set $d_0 = -g_0$, $\alpha_0 = \frac{1}{\|g_0\|^2}$, and $g_k = \nabla f(x_k)$, set $k = 0$.

Step 2: If $\|g_k\| < \epsilon$ then stop.

Step 3: Compute α_k using the Wolfe line sezrch (19) and (20).

Step 4: Compute $x_{k+1} = x_k + \alpha_k d_k$, $s_k = x_{k+1} - x_k$, g_{k+1}, $y_k = g_{k+1} - g_k$.

Step 5: If $\left(\frac{g_{k+1}^T y_k}{\|g_k\|^2} s_k^T y_k - y_k^T g_{k+1} + 2\frac{\|y_k\|^2}{s_k^T y_k} s_k^T g_{k+1}\right) = 0$, then $\theta_k = 0$ else compute θ_k as in (22).

Step 6: Compute β_k

If $\theta_k = 1, \eta_k = 0 :\ \beta_k^{\text{New}} = \beta_k^{HZ}$, if $\theta_k = 0, \eta_k = 1 :\ \beta_k^{\text{New}} = \beta_k^{FR}$, if $\theta_k = 0, \eta_k = 0 :\ \beta_k^{\text{New}} = \beta_k^{PRP}$, if $\theta_k \epsilon]0,1[$, $\eta_k = 0 : \beta_k^{\text{New}} = (1-\theta_k)\beta_k^{PRP} + \theta_k \beta_k^{HZ}$, if $\theta_k = 0, \eta_k \epsilon]0,1[: \beta_k^{\text{New}} = (1-\eta_k)\beta_k^{PRP} + \eta_k \beta_k^{FR}$, if $\theta_k \epsilon]0,1[, \eta_k \epsilon]0,1[$ and $\theta_k + \eta_k = 1 : \beta_k^{\text{New}} = \theta_k \beta_k^{HZ} + \eta_k \beta_k^{FR}$, if $\theta_k, \eta_k \epsilon]0,1[$, and $0 < \theta_k + \eta_k < 1 : \beta_k^{\text{New}} = \theta_k \beta_k^{HZ} + \eta_k \beta_k^{FR} + (1-\theta_k-\eta_k)\beta_k^{PRP}$.

Step 7: Generate $d = -g_{k+1} + \beta_k^{\text{New}} s_k$, if the restart criterion of Powell condition

$$\left|g_{k+1}^T g_k\right| \geq 0.2 \left\|g_{k+1}\right\|^2, \tag{23}$$

is ful lled, then $d_{k+1}^{\text{New}} = -g_{k+1}$ else $d_{k+1}^{\text{New}} = d^{\text{New}}$.

Step 8: Put $k = k+1$ and go to step 2.

3. Global Convergence Analysis

In this section, we will study the sufficient descent property and global convergence of the new algorithm. For that we suppose that f satisfies the hypotheses (**i**) and (**ii**).

(**i**). The level set $\mathcal{F} = \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$ is bounded.

(**ii**). In a neighborhood $\mathcal{V}$ of $\mathcal{G}$ the function f is continuously differentiable and its gradient is Lipschitz continuous, for each $x, y \in \mathcal{V}$ ie there $\exists\, L$ non negative such that

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x-y\|, \tag{24}$$

under (**i**) and (**ii**) there $\exists\ \Psi \geq 0$ where

$$\|\nabla f(x)\| \leq \Psi, \text{ for all } x \in \mathcal{G}. \tag{25}$$

3.1. *Sufficient descent condition*

The next lemma show that the sufficient descent condition possesses for the new method.

Theorem 3.1. *Suppose d_k génirated by Algorithm 2.1. Then we have for $k \geq 0$*

$$g_k^T d_k \leq C\left\|g_k\right\|^2 \qquad C > 0. \tag{26}$$

Proof. The demostration is with induction.

From (18) we have $d_0 = -g_0$ then $g_0^T d_0 = -\left\|g_0\right\|^2 < 0$, for $k = 0$ condition holds.

We will show that holds also for $k \geq 1$.

If the restart criterion of Powell condition is satisfied then $g_{k+1}^T d_{k+1}^{\text{New}} = g_{k+1}^T(-g_{k+1}) = -\left\|g_{k+1}\right\|^2 < 0$, the search direction satisfies the sufficient direction condition.

If the restart criterion of Powell condition doesn't hold in that case and from (18) with some arithmetic operation we have

$$\begin{aligned} d_{k+1}^{\text{New}} &= -g_{k+1} + \theta_k \beta_k^{HZ} s_k + \eta_k \beta_k^{FR} s_k + (1-\theta_k-\eta_k)\beta_k^{PRP} s_k, \\ &= -\theta_k g_{k+1} - \eta_k g_{k+1} - (1-\theta_k-\eta_k) g_{k+1} + \theta_k \beta_k^{HZ} s_k \\ &\quad + \eta_k \beta_k^{FR} s_k + (1-\theta_k-\eta_k)\beta_k^{PRP} s_k, \end{aligned}$$

hence

$$d_{k+1}^{\text{New}} = \theta_k d_{k+1}^{\text{HZ}} + \eta_k d_{k+1}^{\text{FR}} + (1-\eta_k-\eta_k)\, d_{k+1}^{\text{PRP}}, \tag{27}$$

after multiplying the above equation by g_{k+1}^T we get

$$g_{k+1}^T d_{k+1}^{\mathrm{New}} = \theta_k g_{k+1}^T d_{k+1}^{\mathrm{HZ}} + \eta_k g_{k+1}^T d_{k+1}^{\mathrm{FR}} + (1-\theta_k-\eta_k)\, g_{k+1}^T d_{k+1}^{\mathrm{PRP}}.$$

If $\theta_k = 1, \eta_k = 0 :\ g_{k+1}^T d_{k+1}^{\mathrm{New}} = g_{k+1}^T d_{k+1}^{\mathrm{HZ}}$
Harger and Zhan in [11] proved that there exists $m_1 = \frac{7}{8}$ such that

$$g_{k+1}^T d_{k+1}^{\mathrm{HZ}} \leq -m_1 \left\|g_{k+1}\right\|^2, \tag{28}$$

where $0 < \sigma < \frac{1}{2}$.

If $\theta_k = 0, \eta_k = 1$: $g_{k+1}^T d_{k+1}^{\mathrm{New}} = g_{k+1}^T d_{k+1}^{\mathrm{FR}}$,
in this case was also proven that the direction fulfills the condition In [1], by Al-Baali using the strong Wolfe line search for $0 < \sigma < \frac{1}{2}$, ie $\exists m_2 > 0$ where

$$g_{k+1}^T d_{k+1}^{\mathrm{FR}} \leq -m_2 \left\|g_{k+1}\right\|^2. \tag{29}$$

If $\theta_k = 0, \eta_k = 0 :\ g_{k+1}^T d_{k+1}^{\mathrm{New}} = g_{k+1}^T d_{k+1}^{\mathrm{PRP}}$

It was also established that the direction satisfies the condition undre the strong Wolfe line search conditions, is mentioned in [8] for $0 < \sigma < \frac{1}{2}$, i.e $\exists m_3 > 0$

$$g_{k+1}^T d_{k+1}^{\mathrm{PRP}} \leq -m_3 \left\|g_{k+1}\right\|^2, \quad m_3 > 0. \tag{30}$$

If $\theta_k \epsilon\,]0,1[\,, \eta_k = 0 : g_{k+1}^T d_{k+1}^{\mathrm{New}} = (1-\theta_k)\, \beta_k^{PRP} + \theta_k \beta_k^{HZ}$, then
$\exists \gamma_1, \gamma_2 \epsilon : 0 < \gamma_1 < \theta_k < \gamma_2$, such that

$$\begin{aligned} g_{k+1}^T d_{k+1}^{\mathrm{New}} &= -\left\|g_{k+1}\right\|^2 + \left((1-\theta_k)\, \beta_k^{PRP} + \theta_k \beta_k^{HZ}\right) g_{k+1}^T s_k \\ &= (1-\theta_k)\, g_{k+1}^T d_{k+1}^{\mathrm{PRP}} + \theta_k g_{k+1}^T d_{k+1}^{\mathrm{HZ}} \\ &\leq -\left(m_3 (1-\gamma_2) + m_1 \gamma_1\right) \left\|g_{k+1}\right\|^2. \end{aligned}$$

so, we get

$$g_{k+1}^T d_{k+1}^{\mathrm{New}} \leq -m_4 \left\|g_{k+1}\right\|^2, \quad m_4 = m_3 (1-\gamma_2) + m_1 \gamma_1 > 0. \tag{31}$$

If $\eta_k \epsilon\,]0,1[\,, \theta_k = 0 : g_{k+1}^T d_{k+1}^{\mathrm{New}} = (1-\theta_k)\, \beta_k^{PRP} + \theta_k \beta_k^{FR}$, then
$\exists l_1, l_2 \epsilon : 0 < l_1 < \eta_k < l_2$, such that

$$\begin{aligned} g_{k+1}^T d_{k+1}^{\mathrm{New}} &= -\left\|g_{k+1}\right\|^2 + \left((1-\eta_k)\, \beta_k^{PRP} + \eta_k \beta_k^{FR}\right) g_{k+1}^T s_k \\ &= (1-\eta_k)\, g_{k+1}^T d_{k+1}^{\mathrm{PRP}} + \eta_k g_{k+1}^T d_{k+1}^{\mathrm{FR}} \\ &\leq -\left(m_3 (1-l_1) + m_2 l_2\right) \left\|g_{k+1}\right\|^2, \end{aligned}$$

hence

$$g_{k+1}^T d_{k+1}^{\mathrm{New}} \leq -m_5 \left\|g_{k+1}\right\|^2, \quad m_5 = m_3 (1-l_1) + m_2 l_2 > 0. \tag{32}$$

If $\theta_k, \eta_k \epsilon]0,1[$, where $\theta_k + \eta_k = 1$: $g_{k+1}^T d_{k+1}^{\mathrm{New}} = (1-\theta_k)\, \beta_k^{HZ} + \theta_k \beta_k^{FR}$, then

$\exists u_1, u_2 \epsilon : 0 < u_1 < \eta_k < u_2$

$$\begin{aligned} g_{k+1}^T d_{k+1}^{\mathrm{New}} &= -\left\|g_{k+1}\right\|^2 + \left((1-\eta_k)\, \beta_k^{HZ} + \eta_k \beta_k^{FR}\right) g_{k+1}^T s_k \\ &= (1-\eta_k)\, g_{k+1}^T d_{k+1}^{\mathrm{HZ}} + \eta_k g_{k+1}^T d_{k+1}^{\mathrm{FR}} \\ &\leq -\left(m_1 (1-u_2) + m_2 u_1\right) \left\|g_{k+1}\right\|^2. \end{aligned}$$

Implies

$$g_{k+1}^T d_{k+1}^{\text{New}} \leq -m_6 \left\|g_{k+1}\right\|^2, \quad m_6 = m_1 (1 - u_2) + m_2 u_1 > 0. \tag{33}$$

If $\theta_k, \eta_k \epsilon]0, 1[$, where $0 < \theta_k + \eta_k < 1$, then
$\exists \xi_1, \xi_2, \xi_3, \xi_4 \epsilon : 0 < \xi_1 < \theta_k < \xi_2, 0 < \xi_3 < \eta_k < \xi_4$.

$$g_{k+1}^T d_{k+1}^{\text{New}} = \theta_k g_{k+1}^T d_{k+1}^{\text{HZ}} + \eta_k g_{k+1}^T d_{k+1}^{\text{FR}} + (1 - \theta_k - \eta_k) g_{k+1}^T d_{k+1}^{\text{PRP}}. \tag{34}$$

From (28) , (29) and (30), we conclude that

$$g_{k+1}^T d_{k+1}^{\text{New}} \leq -(\xi_2 m_1 + \xi_4 m_2 + (1 - \xi_1 - \xi_3) m_3) \left\|g_{k+1}\right\|^2, \tag{35}$$

where $C = \xi_2 m_1 + \xi_4 m_2 + (1 - \xi_1 - \xi_3) m_3 > 0$.

Under the strong Wolfe conditions, the sufficient descent condition holds for the New algorithm. The proof is finished. □

3.2. Global convergence properties

For the global convergence also we need the following lemma which is famous.

Lemma 3.2. *[8] Consider the conjugate gradient method generated by* (2) *and* (18)*, with d_k satisfies the sufficient descent and α_k computed with* (19) *and* (20)*. If*

$$\sum_{k \geq 1} \frac{1}{\|d_k\|^2} = \infty,$$

then

$$\lim_{k \to \infty} \inf \left\|g_k\right\| = 0.$$

The following global convergence theorem is the result of Lemma 3.2.

Theorem 3.3. *Assume that the hypotheses (**i**) and (**ii**) hold.*
Consider CG method given by (2) *and* (3)*, and let $d_k = d_{k+1}^{New}$ is descent direction, and α_k iis chosen to satisfy, then*

$$\lim_{k \to \infty} \inf \left\|g_k\right\| = 0. \tag{36}$$

Proof. Suppose by contradiction that $\lim_{k \to \infty} \inf \left\|g_k\right\| = 0$ does not hold. ie $g_k \neq 0 \ \forall \ k$, then there exists a constant $\exists \Psi > 0$ such that

$$\left\|g_k\right\| \geq \bar{\Psi} \ \text{ for all } k = 0, \tag{37}$$

$D = x_{k+1} - x_k = s_k$ be the diameter of the level set $\mathcal{G}$.

$$\| y_k \| \leq L \| s_k \| = L \| x_{k+1} - x_k \| \leq LD. \tag{38}$$

Therefore

$$\|d_{k+1}\| \leq \| g_{k+1} \| + \left|\beta_k^{\text{New}}\right| \| s_k . \| .$$

In the other hand

$$\left|\beta_k^{\text{New}}\right| \leq \left|\beta_k^{HZ}\right| + \left|\beta_k^{FR}\right| + \left|\beta_k^{PRP} .\right| .$$

From (19) and (20), we get

$$s_k^T y_k = s_k^T (g_{k+1} - g_k) \geq -(1-\sigma) s_k^T g_k \Rightarrow \frac{1}{s_k^T y_k} \leq \frac{1}{-(1-\sigma) s_k^T g_k}. \quad (39)$$

According to (25), (26), (37), (38) and (39), we have

$$\left|\beta_k^{HZ}\right| \leq \frac{\| g_{k+1} \| \| y_k \|}{C(1-\sigma)\left\|g_k\right\|^2} + 2\frac{\|s_k\| \left\|g_{k+1}\right\| \left\|y_k\right\|^2}{(C(1-\sigma))^2 \left\|g_k\right\|^4}$$

$$\left|\beta_k^{HZ}\right| \leq \frac{\Psi LD}{C(1-\sigma)\bar{\Psi}^2} + 2\frac{L^2 D^3 \Psi}{(C(1-\sigma))^2 \bar{\Psi}^4} = M_1. \quad (40)$$

And

$$\left|\beta_k^{FR}\right| \leq \frac{\Psi^2}{\bar{\Psi}^2} = M_2. \quad (41)$$

And

$$\left|\beta_k^{PRP}\right| \leq \frac{\| g_{k+1} \| \| y_k \|}{\left\|g_k\right\|^2} \leq \frac{\Psi LD}{\bar{\Psi}^2} = M_3. \quad (42)$$

Then

$$\left|\beta_k^{\text{New}}\right| \leq M_1 + M_2 + M_3 = M.$$

Now, we get

$$\|d_{k+1}\| \leq \Psi + MLD.$$

Hence

$$\frac{1}{\|d_{k+1}\|^2} \geq \frac{1}{(\Psi + MLD)^2}.$$

Therfore

$$\sum_{k\geq 0} \frac{1}{\|d_{k+1}\|^2} \geq \sum_{k\geq 0} \frac{1}{(\Psi + MLD)^2} = \infty. \quad (43)$$

This is contradiction. So the proof is complete. □

4. Results and Discussion

In this section, the authors demonstrate the computational effectiveness of the new algorithm on a set of test problems [3, 17].

The parameters used include $\delta = 0.0001$, $\sigma = 0.1$, with varying initial points x_0 and dimensions. The algorithm terminates when $\left\|g_k\right\|_\infty < \varepsilon = 10^{-6}$ is reached. For this problem, we compare the performance of two methods. Let f_i^1 and f_i^2 represent the optimal values obtained by the first and second methods, respectively. The first method is considered superior if $\left\|f_i^1 - f_i^2\right\| < 10^{-3}$, and it involves fewer CPU time or iterations compared to the second method. We evaluated the performance of the New algorithm against the FRPRPCC[7], and LSCDCC [8] algorithms using the Dolan-Moré profiles [9].

The performance profiles in Figures 1, 2, 3, and 4 illustrate the comparison of different methods based on the CPU Time, number of iteration (NI), number of function evaluation (NF), and number of gradient (NG).

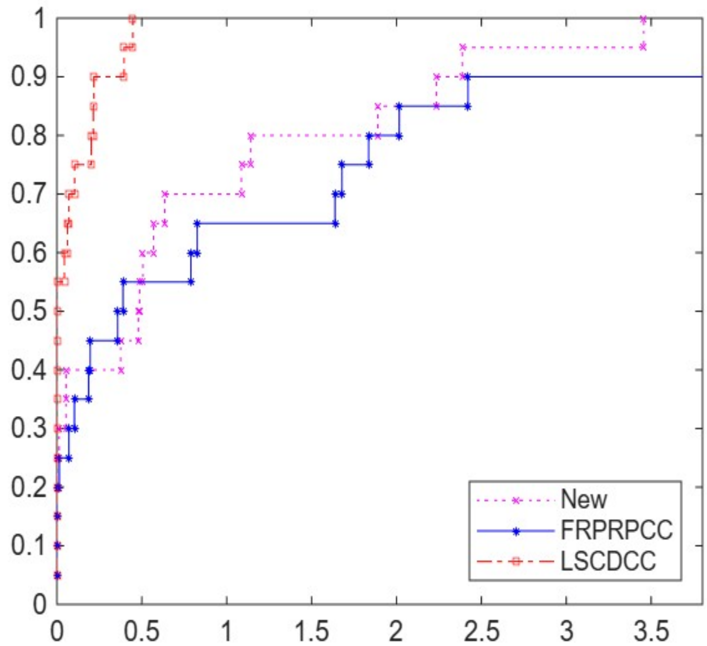

Figure 1: Results using the CPU time.

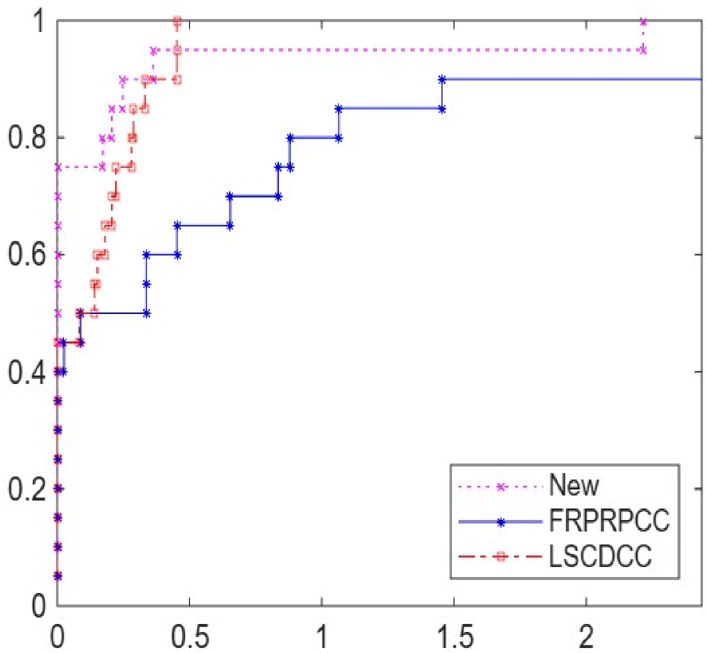

Figure 2: Results using the number iteration.

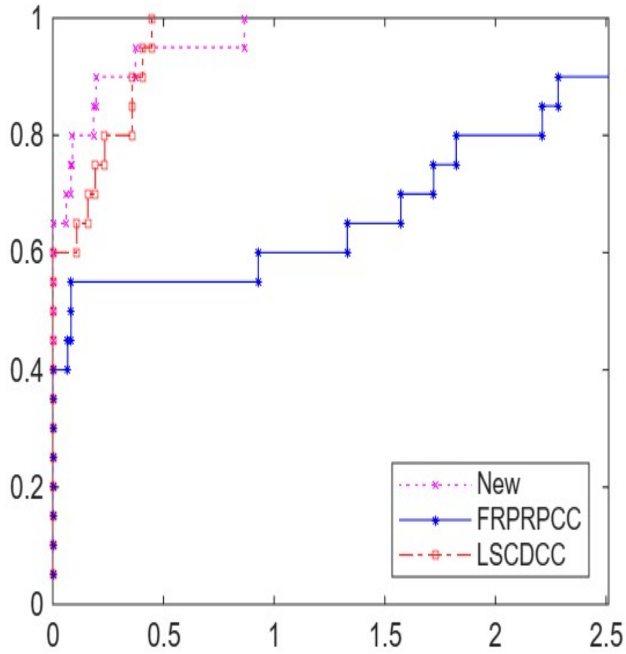

Figure 3: Results using the number function.

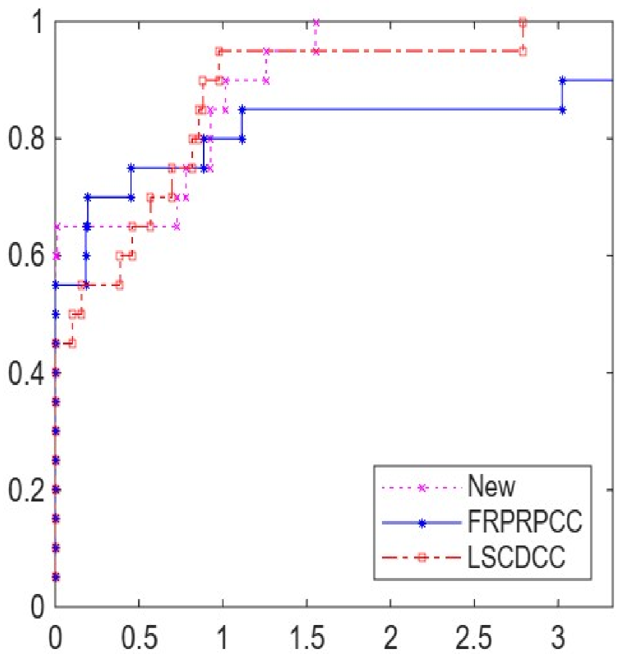

Figure 4: Results using the number gradient.

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