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The largest α-spectral radius of the k-uniform unicyclic hypergraphs with perfect matchings

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Abstract. Let G be a k-uniform hypergraph with $k \geqslant 2$ and $0 \leqslant \alpha < 1$. The α-spectral radius of G is the largest modulus of all the eigenvalues of $\mathcal{A}_\alpha(G)$, where $\mathcal{A}_\alpha(G) = \alpha\mathcal{D}(G) + (1-\alpha)\mathcal{A}(G)$ is the convex linear combination of $\mathcal{D}(G)$ and $\mathcal{A}(G)$ with $\mathcal{D}(G)$ and $\mathcal{A}(G)$ being the degree diagonal tensor and the adjacency tensor of G, respectively. Let $\mathcal{U}(n,k)$ be the set of the k-uniform unicyclic hypergraphs having perfect matchings with n vertices, where $n \geqslant k(k-1)$ and $k \geqslant 3$. By using a creative method of the α-Perron vector and several techniques for studying the α-spectral radii of hypergraphs, such as the well-known Perron–Frobenius theorem, the moving-edge operation, and the 2-switch transformation, the hypergraph with the largest α-spectral radius is characterized among $\mathcal{U}(n,k)$, where $n \geqslant k(k-1)$ and $k \geqslant 3$.

1. Introduction

Let $G = (V(G), E(G))$ be a simple (i.e., no loops or multiple edges) hypergraph with n vertices and a edges, where $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{e_1, e_2, \cdots, e_a\}$ are the sets of the vertices and the edges of G, respectively. If $|e_i| = k$ for $1 \leqslant i \leqslant a$, then G is called a k-uniform hypergraph. A k-uniform hypergraph G is linear if any two edges of $E(G)$ share at most one vertex. Let $u, v \in V(G)$ and $e \in E(G)$. If $\{u, v\} \subseteq e$, then u and v are adjacent, and u is incident with e. The degree of v, denoted by $d_G(v)$, is the number of the edges of G incident with v. Without confusion, $d_G(v)$ is simplified as d_v. If $d_v = 1$, then v is a core vertex. For an edge $e = \{v_1, v_2, \cdots, v_k\} \in E(G)$, if $d_{v_1} \geqslant 2$ and $d_{v_i} = 1$ for $2 \leqslant i \leqslant k$, then e is a pendant edge at v_1.

A path between u and v is denoted by $P = (v_1, e_1, v_2, \ldots, v_p, e_p, v_{p+1})$, where $v_1 = u$, $v_{p+1} = v$, all v_i and all e_i are distinct, and $v_i, v_{i+1} \in e_i$ for $1 \leqslant i \leqslant p$. For $p \geqslant 2$, if we identify v_1 with v_{p+1} in P together, then we get a cycle of length p. In G, if every pair of vertices has a path connecting them, then we say that G is connected.

For a k-uniform hypergraph G, if $a(k-1) - n + \omega(G) = r(G)$, then we call G an $r(G)$-cyclic hypergraph [3], where $\omega(G)$ and $r(G)$ are the number of components and the cyclomatics number of G, respectively. If

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$\omega(G) = 1$ and $r(G) = 0$, then G is a supertree [9]. Namely, a supertree is a k-uniform hypergraph which is connected and acyclic. If $\omega(G) = r(G) = 1$, then G is a k-uniform connected unicyclic hypergraph. Let H be a simple ordinary graph. The k-th power of H is obtained from H by inserting $(k-2)$ new vertices into each edge (a 2-set) of H, where $k \geqslant 3$. A hypertree is the k-th power of an ordinary tree. Obviously, a hypertree is a supertree.

A perfect matching of G is $S_1 \cup S_2 \cup \cdots \cup S_h$, where $h \geqslant 1, S_1, S_2, \cdots, S_h \in E(G)$, $S_i \cap S_j = \emptyset\ (1 \leqslant i < j \leqslant h)$, and $S_1 \cup S_2 \cup \cdots \cup S_h = V(G)$. It is known that the hypergraphs with perfect matchings have many applications in graph theory. For the results about some properties of the hypergraphs with perfect matchings, one can refer to Refs. [7, 8, 18].

Let $\mathbb{R}$ and $\mathbb{C}$ be the sets of real and complex numbers, respectively. A k-ordered and n-dimensional real tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$ over $\mathbb{R}$ is a multi-dimensional array with n^k entries, where $a_{i_1 i_2 \cdots i_k} \in \mathbb{R}$ with $i_1, i_2, \cdots, i_k \in [n] = \{1, 2, \cdots, n\}$. The concept of tensor eigenvalues and the spectra of tensors are independently introduced by Qi [15] and Lim [10] as follows. If there exist a number $\lambda \in \mathbb{C}$ and an eigenvector $\boldsymbol{x} = \{x_1, x_2, \cdots, x_n\}^{\mathrm{T}} \in \mathbb{C}^n$ satisfying

$$\sum_{i_2, \cdots, i_k = 1}^{n} a_{i i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} = \lambda x_i^{k-1}, \text{ for any } 1 \leqslant i \leqslant n, \tag{1}$$

then λ is called an eigenvalue of $\mathcal{A}$ and $\boldsymbol{x}$ an eigenvector of $\mathcal{A}$ corresponding to λ. The spectral radius of $\mathcal{A}$ is the largest modulus of the eigenvalues of $\mathcal{A}$, i.e., $\rho(\mathcal{A}) = \max\{|\lambda| \big| \lambda \text{ is an eigenvalue of } \mathcal{A}\}$.

For a k-uniform hypergraph G with $n \geqslant 2$ vertices, the adjacency tensor of G is $\mathcal{A}(G) = (a_{i_1 i_2 \cdots i_k})$, where $a_{i_1 i_2 \cdots i_k} = \frac{1}{(k-1)!}$ if $\{v_{i_1}, v_{i_2}, \cdots, v_{i_k}\} \in E(G)$ and $a_{i_1 i_2 \cdots i_k} = 0$ otherwise [1]. The degree diagonal tensor of G is $\mathcal{D}(G) = (d_{i_1 i_2 \cdots i_k})$, where $d_{i_1 i_2 \cdots i_k} = d_{v_i}$ for any $v_i \in V(H)$ if $i_1 = i_2 = \cdots = i_k = i$ with $i \in [n]$ and $d_{i_1 i_2 \cdots i_k} = 0$ otherwise.

Let $0 \leqslant \alpha < 1$. Nikiforov [13] proposed to merge the spectral properties of the adjacency matrix and the signless Laplacian matrix of a graph. Motivated by the work of Nikiforov [13], Lin et al. [11] introduced the convex linear combination of $\mathcal{D}(G)$ and $\mathcal{A}(G)$ for a k-uniform hypergraph G as follows: $\mathcal{A}_\alpha(G) = \alpha\mathcal{D}(G) + (1-\alpha)\mathcal{A}(G)$.

The α-spectral radius of G, denoted by $\rho_\alpha(G)$, is the spectral radius of $\mathcal{A}_\alpha(G)$. When $\alpha = 0$, $\mathcal{A}_\alpha(G)$ is $\mathcal{A}(G)$ and $\rho_\alpha(G)$ is the spectral radius of G. When $\alpha = \frac{1}{2}$, $2\mathcal{A}_\alpha(G)$ is the signless Laplacian tensor of G and $2\rho_\alpha(G)$ is the signless Laplacian spectral radius of G.

For a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ of dimension n and a subset $W \subseteq [n]$, we define $x_W = \prod_{i \in W} x_i$. We have

$$\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}(G)\boldsymbol{x}) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} \cdots x_{i_k} = k \sum_{e \in E(G)} x_e, \tag{2}$$

$$\boldsymbol{x}^{\mathrm{T}}(\mathcal{D}(G)\boldsymbol{x}) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} d_{i_1 i_2 \ldots i_k} x_{i_1} \cdots x_{i_k} = \sum_{i=1}^{n} d_{v_i} x_i^k, \tag{3}$$

$$\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}_\alpha(G)\boldsymbol{x}) = \alpha \boldsymbol{x}^{\mathrm{T}}(\mathcal{D}(G)\boldsymbol{x}) + (1-\alpha)\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}(G)\boldsymbol{x}). \tag{4}$$

Since the studies on the α-spectral radii of hypergraphs are of practical significance, some results about the hypergraphs with the extremal α-spectral radii have been obtained. For the k-uniform supertrees, the supertrees with the first to the third largest α-spectral radii were characterized [23] and the supertrees with the fourth to the eighth largest α-spectral radii were determined [20]. For the k-uniform non-caterpillar hypergraphs with a given diameter, the supertrees with the first and the second largest α-spectral radii were derived [19]. The hypergraphs with the largest α-spectral radii were also characterized respectively among the hypergraphs with a given number of pendant edges [12], among the unicyclic hypergraphs [12], among the k-uniform unicyclic hypergraphs with a fixed diameter [6], and among the k-uniform unicyclic hypergraphs with a given number of pendant edges [6]. For the results about the upper bounds of the α-spectral radii for hypergraphs, one can refer to Refs. [2, 5, 12].

We denote by $\mathcal{U}(n,k)$ the set of the k-uniform connected unicyclic hypergraphs having perfect matchings with n vertices, where $n \geqslant k(k-1)$ and $k \geqslant 3$. Let G be an arbitrary hypergraph in $\mathcal{U}(n,k)$. We use $M(G)$ to denote a perfect matching of G. By Property 2.10 (as shown in Section 2), we know that $M(G)$ is unique. An edge of $M(G)$ is called a perfect matching edge of G. If a vertex of G is incident with a perfect matching edge, then it is saturated. Let $Q(G) = E(G) - M(G)$ and $\widehat{G}$ be the hypergraph induced by $Q(G)$, that is, $\widehat{G} = G - M(G) - S_0$, where S_0 is the set of the isolated vertices in $G - M(G)$. We call $\widehat{G}$ the capped hypergraph of G and G the original hypergraph of $\widehat{G}$.

Let $|M(G)|$ and $|Q(G)|$ be the numbers of the edges in $M(G)$ and $Q(G)$, respectively. Since each vertex of G is saturated, we have $|M(G)| = \frac{n}{k}$, where n is divisible by k and $k \geqslant 3$. Thus, it follows from $n = |E(G)|(k-1)$ that $|Q(G)| = |E(G)| - \frac{n}{k} = \frac{n}{k(k-1)}$, where n is divisible by $k(k-1)$. For simplicity, let $|Q(G)| = m$. Namely, m is the number of the edges of $\widehat{G}$. Thus, in $\mathcal{U}(n,k)$, we get $n = mk(k-1)$, where $m \geqslant 1$ and $k \geqslant 3$.

In $\mathcal{U}(n,k)$, Sun et al. [17] obtained the hypergraph with the largest spectral radius. Motivated by the preceding results on the hypergraphs with the extremal α-spectral radii, the aim of this article is to characterize the hypergraph with the largest α-spectral radius among $\mathcal{U}(n,k)$, where $n \geqslant k(k-1)$ and $k \geqslant 3$.

This paper is organized as follows. In Section 2, relevant notations and some necessary lemmas are introduced. In Section 3, to obtain our results, we first introduce Lemmas 3.1–3.9. We will develop a creative method of the α-Perron vector (as shown in Lemma 3.7 and Lemma 3.8) and apply several useful methods, such as the well-known Perron–Frobenius theorem [4, 21], the moving-edge operation, and the 2-switch transformation introduced by Guo and Zhou [5], etc. The hypergraph with the largest α-spectral radius is derived among $\mathcal{U}(n,k)$, where $n \geqslant k(k-1)$ and $k \geqslant 3$, which is shown in Theorem 3.11.

2. Preliminary

In this section, we will introduce some notations and quote some necessary lemmas for subsequent proofs.

The nonnegative weakly irreducible tensor was defined by Friedland et al. [4] and Yang et al. [22] represented it as follows.

Definition 2.1. *[4, 22] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_k})$ be a nonnegative tensor of order k and dimension n. For any nonempty proper index subset $I \subset [n]$, if there is at least one entry $a_{i_1 i_2 \cdots i_k} > 0$, where $i_1 \in I$ and at least one $i_j \in [n] \setminus I$ for $j = 2, 3, \cdots, k$, then $\mathcal{A}$ is called a nonnegative weakly irreducible tensor.*

Pearson and Zhang [14] proved that a k-uniform hypergraph G is connected if and only if its adjacency tensor $\mathcal{A}(G)$ is weakly irreducible. Therefore, if G is connected, then $\mathcal{A}(G)$ and $\mathcal{A}_\alpha(G)$ are all weakly irreducible.

Let $\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n \mid x_i \geqslant 0,\ \forall i \in [n]\}$ and $\mathbb{R}^n_{++} = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} \in \mathbb{R}^n \mid x_i > 0,\ \forall i \in [n]\}$.

Lemma 2.2. *[4, 21] (The Perron–Frobenius theorem for nonnegative tensors). Let $\mathcal{A}$ be a nonnegative tensor of order k and dimension n, where $k \geqslant 2$. Then we have the following statements.*

(i). $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a nonnegative eigenvector $\boldsymbol{x} \in \mathbb{R}^n_+$ corresponding to it.

(ii). If $\mathcal{A}$ is weakly irreducible, then $\rho(\mathcal{A})$ is the unique eigenvalue of $\mathcal{A}$ with a positive eigenvector $\boldsymbol{x} \in \mathbb{R}^n_{++}$, and $\boldsymbol{x}$ is unique up to a positive scaling coefficient.

Lemma 2.3. *[16] Let $\mathcal{A}$ be a nonnegative symmetric tensor of order k and dimension n. Then we have*

$$\rho(\mathcal{A}) = \max\left\{\boldsymbol{x}^T(\mathcal{A}\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n_+,\ \|\boldsymbol{x}\|_k^k = 1\right\}.$$

Furthermore, $\boldsymbol{x} \in \mathbb{R}^n_+$ with $\|\boldsymbol{x}\|_k^k = 1$ is an optimal solution of the above optimization problem if and only if $\boldsymbol{x}$ is an eigenvector of $\mathcal{A}$ corresponding to the eigenvalue $\rho(\mathcal{A})$.

By Lemma 2.2, for a k-uniform connected hypergraph G, there exists the unique positive eigenvector $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ of G corresponding to $\rho_\alpha(G)$, where $\|\boldsymbol{x}\|_k^k = 1$. This vector $\boldsymbol{x}$ is referred to as the α-Perron vector of G and it plays an important role in studying $\rho_\alpha(G)$. By Lemma 2.3, we obtain Lemma 2.4 as follows.

Lemma 2.4. *Let G be a k-uniform connected hypergraph, where $k \geqslant 2$. Then we have*

$$\rho_\alpha(G) = \max\left\{\boldsymbol{x}^T(\mathcal{A}_\alpha(G)\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}_+^n,\ \|\boldsymbol{x}\|_k^k = 1\right\}.$$

Furthermore, $\boldsymbol{x} \in \mathbb{R}_+^n$ with $\|\boldsymbol{x}\|_k^k = 1$ is an optimal solution of the above optimization problem if and only if it is an eigenvector of G corresponding to the eigenvalue $\rho_\alpha(G)$.

In studying the spectral radius and the α-spectral radius of hypergraphs, the method of transformation is a key tool. In Definition 2.5, Li et al. [9] introduced the definition of the moving-edge operation for the spectral radii of hypergraphs. In Lemma 2.6, Guo and Zhou [5] generalized it to the α-spectral radii of hypergraphs.

Definition 2.5. *[9] Let $G = (V(G), E(G))$ be a hypergraph with $u \in V(G)$ and $e_1, \cdots, e_r \in E(G)$, where $u \notin e_i$ for any $i \in [r]$ with $r \geqslant 1$. Suppose that $v_i \in e_i$ and $e_i' = (e_i \backslash \{v_i\}) \cup \{u\}$, where $i \in [r]$. Let $G' = (V(G'), E(G'))$ be the hypergraph with $E(G') = (E(G) \backslash \{e_1, \cdots, e_r\}) \cup \left\{e_1', \cdots, e_r'\right\}$. Then we say that G' is obtained from G by moving edges $(e_1, \cdots, e_r)$ from $(v_1, \cdots, v_r)$ to u.*

Lemma 2.6. *[5] Let $G = (V(G), E(G))$ be a k-uniform hypergraph with $u, v_1, \cdots, v_r \in V(G)$ and $e_1, \cdots, e_r \in E(G)$, where $k \geqslant 2$ and $r \geqslant 1$. Suppose that $u \notin e_i$ and $v_i \in e_i$ for any $i \in [r]$, where $v_1, \cdots, v_r$ are not necessarily distinct. Let $e_i' = (e_i \backslash \{v_i\}) \cup \{u\}$, where $i \in [r]$. Suppose that $e_i' \notin E(G)$ for any $i \in [r]$. Let G' be obtained from G by moving edges $(e_1, \cdots, e_r)$ from $(v_1, \cdots, v_r)$ to u. Let $\boldsymbol{x}$ be the α-Perron vector of G. If $x_u \geqslant \max\{x_{v_1}, \cdots, x_{v_r}\}$, then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leqslant \alpha < 1$.*

Li et al. [9] proposed the edge-releasing operation for the k-uniform linear hypergraphs. In Definition 2.7, we generalize the edge-releasing operation to the α-spectral radius of k-uniform hypergraphs.

Definition 2.7. *Let G be a k-uniform connected hypergraph with $k \geqslant 3$. Let $e \in E(G)$ be a non-pendant edge, and let $\{e_1, \ldots, e_r\} \subseteq E(G)$ be the set of all the edges that share exactly with e at one common vertex. Let v_i be the unique vertex in $e \cap e_i$, where $1 \leqslant i \leqslant r$. Fix an arbitrary vertex (denoted by u) in e. Let G' be a hypergraph obtained from G by replacing each e_i with $e_i' = (e_i \setminus \{v_i\}) \cup \{u\}$ $(i = 1, \ldots, r)$ and keeping all other edges of G unchanged (in particular, if $v_i = u$, then $e_i' = e_i$). Then G' is said to be obtained from G by the edge-releasing operation on e at u.*

Lemma 2.8. *Let G and G' be the two k-uniform connected hypergraphs as defined in Definition 2.7, where $k \geqslant 3$. Then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leqslant \alpha < 1$.*

Proof. Let G and G' be the two hypergraphs as defined in Definition 2.7. Since e is a non-pendant edge of G, there exist some vertices in e which have degrees not less than 2. We denote these vertices by $v_1, \cdots, v_r$, where $2 \leqslant r \leqslant k$ and $k \geqslant 3$. As G is a connected hypergraph, let $\boldsymbol{x}$ be the α-Perron vector of G. Without loss of generality, we assume $x_{v_1} \geqslant \max\{x_{v_2}, \cdots, x_{v_r}\}$. Let G'' be the hypergraph obtained from G by moving edges $(e_1, \cdots, e_r)$ (except for all the edges which are incident with v_1) from $(v_2, \cdots, v_r)$ (except for v_1) to v_1. Obviously, G'' is connected. By Lemma 2.6, for $0 \leqslant \alpha < 1$, we have $\rho_\alpha(G'') > \rho_\alpha(G)$. Since $|e_i \cap e| = 1$ for $1 \leqslant i \leqslant r$, G'' is the hypergraph G' in Lemma 2.6. Thus, Lemma 2.8 holds. □

Let $G = (V(G), E(G))$ be a hypergraph. For $E' \subseteq E(G)$, let $G - E'$ be the hypergraph obtained from G by deleting all the edges in E'. If E' is a set of subsets of $V(G)$ and no elements of E' is an edge of G, then let $G + E'$ be the hypergraph obtained from G by adding all the elements in E'. The 2-switch transformation for the α-spectral radii of hypergraphs was proposed by Guo and Zhou in Lemma 2.9.

Lemma 2.9. *[5] Let G be a k-uniform connected hypergraph with $k \geqslant 2$, and e, f be two edges of G with $e \cap f = \varnothing$. Let $\boldsymbol{x}$ be the α-Perron vector of G. Let $U \subset e$ and $V \subset f$ with $1 \leqslant |U| = |V| \leqslant k-1$. Let $e' = U \cup (f \backslash V)$ and $f' = V \cup (e \backslash U)$. Suppose that $e', f' \notin E(G)$. Let $G' = G - \{e, f\} + \{e', f'\}$. If $x_U \geqslant x_V$, $x_{f\backslash V} \geqslant x_{e\backslash U}$ and at least one inequality holds, then $\rho_\alpha(G') > \rho_\alpha(G)$ for $0 \leqslant \alpha < 1$.*

For a k-uniform connected unicyclic hypergraph having perfect matchings, we have Property 2.10.

Property 2.10. *[17] Let $G \in \mathcal{U}(n,k)$, where $n \geqslant k(k-1)$ and $k \geqslant 3$. Then $M(G)$ of G is unique.*

3. The hypergraph with the largest α-spectral radius among $\mathcal{U}(n,k)$

In this section, we will deduce the hypergraph with the largest α-spectral radius among $\mathcal{U}(n,k)$, where $n = mk(k-1)$, $m \geqslant 1$ and $k \geqslant 3$. To obtain our result (as shown in Theorem 3.11), we firstly introduce some definitions and Lemmas 3.1–3.9.

Let $\mathcal{U}(n,k,l)$ be a subset of $\mathcal{U}(n,k)$ in which each hypergraph has a cycle C_l, where l is an integer with $l \geqslant 2$. Let $G \in \mathcal{U}(n,k,l)$. Then G contains a cycle $C_l = v_1e_1v_2e_2v_3 \cdots v_le_lv_1$, where $e_i = \{v_i, v_{i,1}, \ldots, v_{i,k-2}, v_{i+1}\}$ with $1 \leqslant i \leqslant l-1$ and $e_l = \{v_l, v_{l,1}, \ldots, v_{l,k-2}, v_1\}$. According to the fact whether C_l of G contains at least one perfect matching edge or not, we classify $\mathcal{U}(n,k,l)$ into two subsets which are denoted by $\mathcal{U}_1(n,k,l)$ and $\mathcal{U}_2(n,k,l)$, where $\mathcal{U}_1(n,k,l)$ (respectively $\mathcal{U}_2(n,k,l)$) satisfies that each hypergraph G in it has no perfect matching edges on C_l (respectively at least one perfect matching edge on C_l). Obviously, $\mathcal{U}(n,k) = \bigcup_{l\geq 2}\left(\mathcal{U}_1(n,k,l) \cup \mathcal{U}_2(n,k,l)\right)$.

Let $\overline{\mathcal{U}}_1(n,k,2)$ be a subset of $\mathcal{U}_1(n,k,2)$ in which each hypergraph satisfies two conditions: (1) each vertex in C_2 must be incident with a pendant edge; and (2) at most one of the vertices in $e_1 \cup e_2$ of C_2 is attached by a k-uniform supertree which has at least k edges, where $k \geqslant 3$.

Let $\overline{\mathcal{U}}_2(n,k,2)$ be a subset of $\mathcal{U}_2(n,k,2)$ in which each hypergraph satisfies two conditions: (1) each vertex in $e_2 \setminus \{v_1, v_2\}$ of C_2 must be incident with a pendant edge; and (2) at most one of the vertices in $e_1 \cup e_2$ of C_2 is attached by a k-uniform supertree which has at least k edges, where $k \geqslant 3$.

We use S_m to denote a star with m edges and let u_0 be the center vertex of S_m with degree m, where $m \geqslant 1$. Let S_m^k be the hypertree obtained from S_m by inserting $k-2$ new vertices into each edge of S_m, where $m \geqslant 1$ and $k \geqslant 3$. Obviously, all the edges of S_m^k share a common vertex u_0. Let G and H be two hypergraphs whose vertex sets are disjoint with $v \in V(G)$ and $w \in V(H)$. We use $G(v,w)H$ to denote the hypergraph obtained by identifying the vertices v and w. For example, $C_2(v_1,u_0)S_{m-1}^k$ is shown in Fig. 1, where $C_2 = v_1e_1v_2e_2v_1$ is a cycle of length 2.

We use $E_{n,k}$ (respectively $D_{n,k}$) to denote the hypergraph obtained from $C_2(v_{2,1},u_0)S_{m-1}^k$ (respectively $C_2(v_1,u_0)S_{m-1}^k$) by attaching a pendant edge at each vertex (except for the vertices in e_1) of $C_2(v_{2,1},u_0)S_{m-1}^k$ (respectively $C_2(v_1,u_0)S_{m-1}^k$), where $n = mk(k-1)$, $m \geqslant 1$, and $k \geqslant 3$. $E_{n,k}$ and $D_{n,k}$ are shown in Figs. 2 and 3, respectively. When $n = k(k-1)$, it is obvious $D_{n,k} \cong E_{n,k}$. Let $F_{n,k}$ be the hypergraph obtained from $C_2(v_1,u_0)S_{m-2}^k$ by attaching a pendant edge at each vertex of $C_2(v_1,u_0)S_{m-2}^k$, where $n = mk(k-1)$, $m \geqslant 2$, and $k \geqslant 3$. $F_{n,k}$ is shown in Fig. 5.

Obviously, $D_{n,k}, E_{n,k} \in \overline{\mathcal{U}}_2(n,k,2)$ and $F_{n,k} \in \overline{\mathcal{U}}_1(n,k,2)$.

Lemma 3.1. *Let $G \in \mathcal{U}(n,k,l)$, where $n \geqslant 3k(k-1)$ and $k, l \geqslant 3$. Let e be a perfect matching edge of G and e is not a pendant edge. Let G_0 be the hypergraph obtained from G by applying the edge-releasing operation on e at an arbitrary vertex of e such that e of G_0 is a pendant edge. Then $\rho_\alpha(G_0) > \rho_\alpha(G)$, where $G_0 \in \mathcal{U}(n,k)$ and $0 \leqslant \alpha < 1$.*

Proof. Let G and G_0 be the two hypergraphs as defined in Lemma 3.1. In G, since e is a perfect matching edge and e is not a pendant edge, all the edges of G adjacent to e belong to $Q(G)$. By applying the edge-releasing operation on e at an arbitrary vertex in e, we get $\rho_\alpha(G_0) > \rho_\alpha(G)$ (by Lemma 2.8), where G_0 has the perfect matching $M(G)$ and $G_0 \in \mathcal{U}(n,k)$. By the definition of the edge-releasing operation, in G_0, e is a pendant edge. Thus, we get Lemma 3.1. □

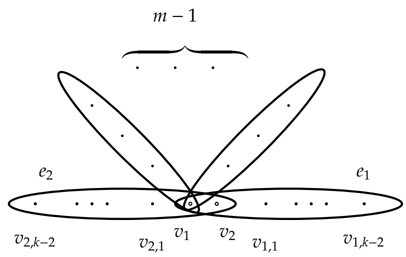

Figure 1: $C_2(v_1, u_0)S^k_{m-1}$

Lemma 3.2. *Let $G \in \mathcal{U}(n,k,l)$, where $n \geqslant 2k(k-1)$ and $k, l \geqslant 3$. There exists a hypergraph $G' \in \mathcal{U}_1(n,k,2)$ such that $\rho_\alpha(G') > \rho_\alpha(G)$, where $0 \leqslant \alpha < 1$.*

Proof. Let $G \in \mathcal{U}(n,k,l)$, where $n \geqslant 2k(k-1)$ and $k, l \geqslant 3$. Let $0 \leqslant \alpha < 1$. The cycle contained in G is denoted by $C_l = v_1e_1 \ldots v_le_lv_1$. Since $\mathcal{U}(n,k,l) = \mathcal{U}_1(n,k,l) \cup \mathcal{U}_2(n,k,l)$, two cases are considered.

Case (i). $G \in \mathcal{U}_1(n,k,l)$.

According to the definition of $\mathcal{U}_1(n,k,l)$, $e_i \notin M(G)$ and each vertex of e_i is incident with an edge of $M(G)$, where $1 \leqslant i \leqslant l$. Let $\boldsymbol{x}$ be the α-Perron vector of G. Without loss of generality, we suppose that $x_{v_1} \geqslant x_{v_2}$. Let G_1 be the hypergraph obtained from G by removing e_2 from v_2 to v_1. Since all the edges incident with v_2 (except for the edge e_2) remain unchanged, $M(G)$ is the perfect matching of G_1 and G_1 contains C_{l-1}. Thus, $G_1 \in \mathcal{U}_1(n,k,l-1)$. By Lemma 2.6, we get $\rho_\alpha(G_1) > \rho_\alpha(G)$. By repeatedly using the same operation, we finally get a hypergraph $G' \in \mathcal{U}_1(n,k,2)$ such that $\rho_\alpha(G') > \rho_\alpha(G)$.

Case (ii). $G \in \mathcal{U}_2(n,k,l)$.

According to the definition of $\mathcal{U}_2(n,k,l)$, in C_l of G, there exists one edge (denoted by e) such that $e \in M(G)$. Let v be an arbitrary vertex in e. Let G_2 be the hypergraph obtained from G by applying the edge-releasing operation on e at v. By Lemma 3.1, we have $\rho_\alpha(G_2) > \rho_\alpha(G)$, where G_2 satisfies that e is a pendant edge, and the number of the perfect matching edges in the cycle of G_2 is one less than $|M(C_l)|$ of G. Obviously, $G_2 \in \mathcal{U}(n,k,l-1)$. By repeatedly using the same operation as above and as in Case (i), we finally get a hypergraph $G' \in \mathcal{U}_1(n,k,2)$ such that $\rho_\alpha(G') > \rho_\alpha(G)$. Thus, we get Lemma 3.2. □

Lemma 3.3. *Let $G \in \mathcal{U}_1(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. There exists a hypergraph $G'' \in \overline{\mathcal{U}_1}(n,k,2)$ such that $\rho_\alpha(G'') \geqslant \rho_\alpha(G)$ with the equality if and only if $G \cong G''$, where $0 \leqslant \alpha < 1$.*

Proof. Let $n = mk(k-1)$, $m \geqslant 2$, and $k \geqslant 3$. When $m = 2$, obviously, Lemma 3.3 holds. Next, let $m \geqslant 3$. Let $G \in \mathcal{U}_1(n,k,2)$. According to the definition of $\mathcal{U}_1(n,k,2)$, each vertex in C_2 of G is incident with an edge in $M(G)$ which does not belong to $E(C_2)$. Let e be an arbitrary edge of $M(G)$ which is incident with a vertex in C_2 of G, where $e \notin E(C_2)$. By applying the edge-releasing operation on e at a vertex of e and using the methods similar to those for the proofs of Lemma 3.1, we can get a hypergraph (denoted by G_3) such that $\rho_\alpha(G_3) \geqslant \rho_\alpha(G)$ for $0 \leqslant \alpha < 1$, with the equality if and only if $G \cong G_3$, where G_3 satisfies: (1). each vertex in C_2 of G_3 is incident with a pendant edge; and (2). each vertex in C_2 of G_3 may be attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$.

Since $m \geqslant 3$, there exists at least a vertex in C_2 of G_3 which is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. Let $\boldsymbol{x}$ be the α-Perron vector of G_3. In G_3, let V_1 be a subset of $V(C_2)$ such that each vertex in V_1 satisfies that it is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. For all the components in $\boldsymbol{x}$ corresponding to the vertices in V_1, we can choose a maximum value among them. Let $w \in V_1$ be such a vertex having the maximum value and x_w be the component corresponding to w among $\boldsymbol{x}$. Let G and G' in Lemma 2.6 be G_3 and G_4, respectively, where G_4 is obtained from G_3 by moving all the k-uniform supertrees which are attached at all the vertices (except for w) in V_1 to w, and G_4 satisfies: (1). each vertex in C_2 of G_4 is incident with a pendant edge; and (2). only one vertex (namely w) in C_2 of G_4 is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. Obviously, $G_4 \in \overline{\mathcal{U}_1}(n,k,2)$.

By Lemma 2.6, we get $\rho_\alpha(G_4) \geqslant \rho_\alpha(G_3)$ for $0 \leqslant \alpha < 1$, with the equality if and only if $G_3 \cong G_4$. Therefore, $\rho_\alpha(G_4) \geqslant \rho_\alpha(G)$, with the equality if and only if $G \cong G_4$. Thus, we get Lemma 3.3. □

Lemma 3.4. *Let $G \in \overline{\mathcal{U}}_1(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. Then $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$ with the equality if and only if $G \cong F_{n,k}$, where $0 \leqslant \alpha < 1$.*

Proof. Let G° be the hypergraph with the largest α-spectral radius among $\overline{\mathcal{U}}_1(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. Let $0 \leqslant \alpha < 1$. Let $C_2 = v_1e_1v_2e_2v_1$ be the cycle of G°, where $e_i = \{v_1, v_{i,1}, \ldots, v_{i,k-2}, v_2\}$ with $i = 1, 2$. When $n = 2k(k-1)$, we have $\overline{\mathcal{U}}_1(n,k,2) = \{F_{n,k}\}$. Thus, Lemma 3.4 holds. Let $n \geqslant 3k(k-1)$. According to the definition of $\overline{\mathcal{U}}_1(n,k,2)$, there exists a vertex (denoted by v^*) in C_2 of G° which is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. Without loss of generality, we suppose that $v^* \in e_1$. Let $\boldsymbol{x}$ be the α-Perron vector of G°.

Claim 1. $v^* = v_1$ or $v^* = v_2$.

We prove Claim 1 by contradiction. Without loss of generality, we suppose that $v^* = v_{1,1}$. Let $M(G^\circ)$ be the perfect matching of G°. By the definition of $\overline{\mathcal{U}}_1(n,k,2)$, $e_2 \notin M(G^\circ)$ and each vertex in C_2 of G° is incident with an edge in $M(G^\circ)$ which is a pendant edge. If $x_{v_1} \geqslant x_{v_{1,1}}$, then let G_5 be the hypergraph obtained from G° by removing all the edges which are incident with $v_{1,1}$ (except for e_1 and the pendant edge incident with $v_{1,1}$) from $v_{1,1}$ to v_1. Obviously, $G_5 \in \overline{\mathcal{U}}_1(n,k,2)$. By Lemma 2.6, $\rho_\alpha(G_5) > \rho_\alpha(G^\circ)$. This is a contradiction. Therefore, we have $x_{v_1} < x_{v_{1,1}}$. Let G_6 be the hypergraph obtained from G° by removing e_2 from v_1 to $v_{1,1}$. Obviously, $G_6 \in \overline{\mathcal{U}}_1(n,k,2)$. By Lemma 2.6, $\rho_\alpha(G_6) > \rho_\alpha(G^\circ)$. This is a contradiction. Therefore, when $n \geqslant 3k(k-1)$, each vertex in C_2 of G° is incident with a pendant edge, and only v_1 in C_2 of G° is attached by a k-uniform supertree (denoted by T) containing at least k edges, where $k \geqslant 3$. Thus, we get Claim 1.

Next, we will prove that each edge in $E(T) \cap M(G^\circ)$ is a pendant edge. Otherwise, we suppose that there exists an edge (denoted by e) in $E(T) \cap M(G^\circ)$ such that e is not a pendant edge. By applying the edge-releasing operation on e at a vertex of e and using the methods similar to those for the proofs of Lemma 3.1, we can get a hypergraph G_7 such that $\rho_\alpha(G_7) > \rho_\alpha(G^\circ)$, where G_7 satisfies: (1). each vertex in C_2 of G_7 is incident with a pendant edge; (2). only v_1 in C_2 of G_7 is attached by a k-uniform supertree (denoted by T_1) containing at least k edges, where $k \geqslant 3$; and (3). e of G_7 is a pendant edge. Thus, $G_7 \in \overline{\mathcal{U}}_1(n,k,2)$. Obviously, the inequality $\rho_\alpha(G_7) > \rho_\alpha(G^\circ)$ contradicts the definition of G°. Thus, each edge in $E(T) \cap M(G^\circ)$ is a pendant edge. Therefore, When $n = 3k(k-1)$, Lemma 3.4 holds since $G^\circ \cong F_{n,k}$. Let $n \geqslant 4k(k-1)$. We will prove $G^\circ \cong F_{n,k}$ by contradiction. Otherwise, we suppose that $G^\circ \ncong F_{n,k}$. Then T of G° contains at least two edges of $Q(G^\circ)$, and there exists an edge (denoted by $g = \{w_1, \ldots, w_k\}$) in T of G° such that three conditions are satisfied: (1). $v_1 \notin g$; (2). v_1 is adjacent to w_1; and (3). g is not a pendant edge. Two cases are considered.

Case (i). $x_{v_1} \geqslant x_{w_1}$.

Let G_8 be the hypergraph obtained from G° by removing g from w_1 to v_1. Obviously, $G_8 \in \overline{\mathcal{U}}_1(n,k,2)$. By Lemma 2.6, $\rho_\alpha(G_8) > \rho_\alpha(G^\circ)$. This is a contradiction.

Case (ii). $x_{v_1} < x_{w_1}$.

Let G_9 be the hypergraph obtained from G° by removing (e_1, e_2) from v_1 to w_1. Let $e'_i = \{w_1, v_{i,1}, \ldots, v_{i,k-2}, v_2\}$, where $i = 1, 2$. Obviously, $w_1e'_1v_2e'_2w_1$ is a cycle of G_9 and $G_9 \in \overline{\mathcal{U}}_1(n,k,2)$. By Lemma 2.6, $\rho_\alpha(G_9) > \rho_\alpha(G^\circ)$. This is a contradiction.

By combining the proofs of Cases (i) and (ii), when $n \geqslant 4k(k-1)$, we have $G^\circ \cong F_{n,k}$. Thus, we get Lemma 3.4. □

Corollary 3.5. *Let $G \in \mathcal{U}(n,k) \backslash \mathcal{U}_2(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. For $0 \leqslant \alpha < 1$, we have $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$ with the equality if and only if $G \cong F_{n,k}$.*

Proof. Since $\mathcal{U}(n,k) \setminus \mathcal{U}_2(n,k,2) = \bigcup_{l \geqslant 3} \mathcal{U}(n,k,l) \cup \mathcal{U}_1(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$, by Lemmas 3.2–3.4, we get Corollary 3.5. □

Lemma 3.6. *Let $G \in \mathcal{U}_2(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. For $0 \leqslant \alpha < 1$, there exists a hypergraph $G''' \in \overline{\mathcal{U}}_2(n,k,2)$ such that $\rho_\alpha(G''') \geqslant \rho_\alpha(G)$ with the equality if and only if $G \cong G'''$.*

Proof. Let $n \geqslant 2k(k-1)$, $k \geqslant 3$ and $0 \leqslant \alpha < 1$. Let $G \in \mathcal{U}_2(n,k,2)$ and $C_2 = v_1e_1v_2e_2v_1$ be the cycle of G. By the definition of $\mathcal{U}_2(n,k,2)$, there exists one edge in C_2 of G which belongs to $M(G)$. Without loss of generality, we suppose that $e_1 \in M(G)$. Then e_2 in C_2 of G is an edge in $Q(G)$. Thus, each vertex in $e_2 \setminus \{v_1, v_2\}$ is incident with an edge in $M(G)$. By the methods similar to those for the proofs of Lemma 3.1, we get a hypergraph G_{10} such that $\rho_\alpha(G_{10}) \geqslant \rho_\alpha(G)$, with the equality if and only if $G \cong G_{10}$, where G_{10} satisfies: (1). each vertex in $e_2 \setminus \{v_1, v_2\}$ of C_2 of G_{10} is incident with a pendant edge; and (2) each vertex in $e_1 \cup e_2$ of C_2 of G_{10} may be attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$.

Let $\boldsymbol{x}$ be the α-Perron vector of G_{10}. In G_{10}, let V_2 be a subset of $V(C_2)$ such that each vertex in V_2 satisfies that it is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. For all the components in $\boldsymbol{x}$ corresponding to the vertices in V_2, we can choose a maximum value among them. Let $w' \in V_2$ be such a vertex having the maximum value and $x_{w'}$ be the component corresponding to w' among $\boldsymbol{x}$. Let G and G' in Lemma 2.6 be G_{10} and G_{11}, respectively, where G_{11} is obtained from G_{10} by moving all the k-uniform supertrees which are attached at all the vertices (except for w') in V_2 to w', and G_{11} satisfies: (1). each vertex in $e_2 \setminus \{v_1, v_2\}$ of C_2 of G_{11} is incident with a pendant edge; and (2). only one vertex (namely w') in C_2 of G_{11} is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. Obviously, $G_{11} \in \overline{\mathcal{U}}_2(n,k,2)$. By Lemma 2.6, we get $\rho_\alpha(G_{11}) \geqslant \rho_\alpha(G_{10})$, with the equality if and only if $G_{11} \cong G_{10}$. Thus, $\rho_\alpha(G_{11}) \geqslant \rho_\alpha(G)$, with the equality if and only if $G \cong G_{11}$. Therefore, we get Lemma 3.6. □

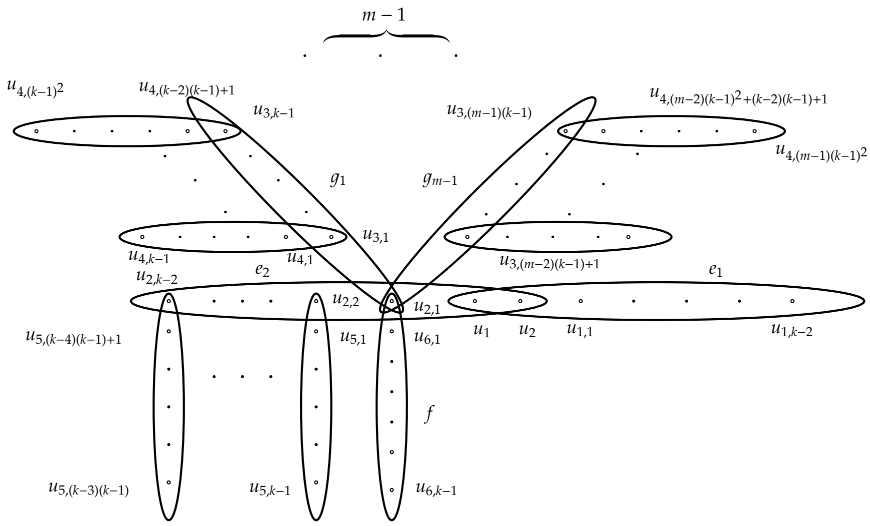

Figure 2: $E_{n,k}$

Lemma 3.7. *Let $n \geqslant 2k(k-1)$, where $k \geqslant 3$. We have $\rho_\alpha(D_{n,k}) > \rho_\alpha(E_{n,k})$, where $0 \leqslant \alpha < 1$.*

Proof. Let $E_{n,k}$ be the hypergraph as shown in Fig. 2. Let $0 \leqslant \alpha < 1$. We divide $V(E_{n,k})$ into eight subsets as follows. Let $V_0 = \{u_1, u_2\}$, $V_1 = \{u_{1,1}, \cdots, u_{1,k-2}\}$, $V_2 = \{u_{2,2}, \cdots, u_{2,k-2}\}$, $V_3 = \left\{u_{3,1}, \cdots, u_{3,(m-1)(k-1)}\right\}$, $V_4 = \left\{u_{4,1}, \cdots, u_{4,(m-1)(k-1)^2}\right\}$, $V_5 = \left\{u_{5,1}, \cdots, u_{5,(k-3)(k-1)}\right\}$, $V_6 = \{u_{6,1}, \cdots, u_{6,k-1}\}$, and $V_7 = \{u_{2,1}\}$, where $d_{E_{n,k}}(u_{2,1}) = m+1$, each vertex in V_i ($i = 0, 2, 3$) has degree 2 and each vertex in V_i ($i = 1, 4, 5, 6$) has degree 1. Furthermore, V_i with $1 \leqslant i \leqslant 6$ satisfies the following conditions: (1). each vertex in V_1 is adjacent to u_1 and u_2 in V_0 simultaneously; (2). each vertex in V_2 is adjacent to $u_{2,1}$, u_1 and u_2 in V_0, and $k-1$ core vertices in V_5 which are incident with a pendant edge; (3). each vertex in V_3 is adjacent to $u_{2,1}$ and $k-1$ core vertices in V_4 which are incident with a pendant edge; (4). each vertex in V_4 is adjacent to a vertex in V_3; (5). each vertex in V_5 is adjacent to a vertex in V_2; and (6). each vertex in V_6 is adjacent to $u_{2,1}$. Obviously, we have $V(E_{n,k}) = \bigcup_{i=0}^{7} V_i$ and $V_i \cap V_j = \varnothing$ for $0 \leqslant i < j \leqslant 7$. All the vertices of $E_{n,k}$ are shown in Fig. 2.

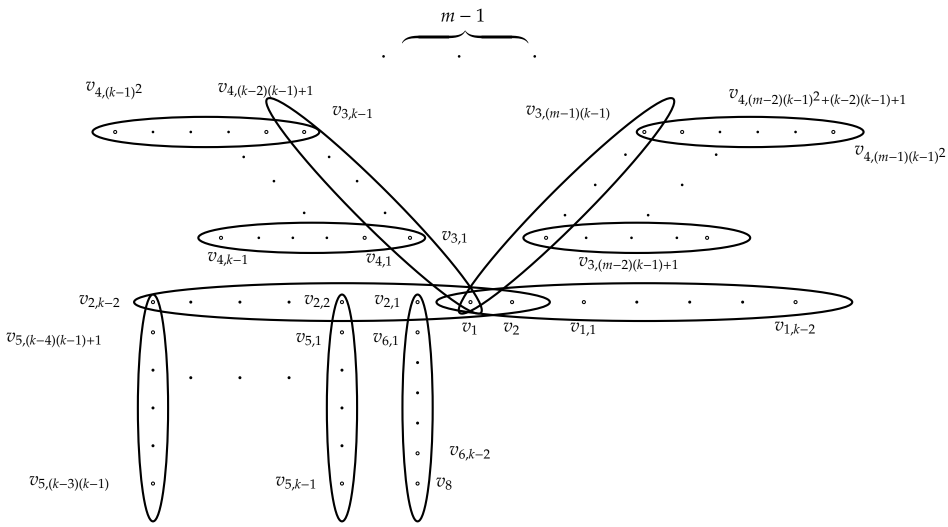

Figure 3: $D_{n,k}$

Let $\boldsymbol{x}$ be the α-Perron vector of $E_{n,k}$. Namely, we have $\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}_\alpha(E_{n,k})\boldsymbol{x}) = \rho_\alpha(E_{n,k})$, where $\|\boldsymbol{x}\|_k^k = 1$ and $\boldsymbol{x} \in \mathbb{R}^n_{++}$. By the symmetry, all the vertices in V_i have the same component in $\boldsymbol{x}$, where $0 \leqslant i \leqslant 6$. We use x_i to denote the component in $\boldsymbol{x}$ which corresponds to the vertices in $V_i\,(E_{n,k})$, where $0 \leqslant i \leqslant 7$. In $E_{n,k}$, let $e_i = \{u_1, u_{i,1}, \cdots, u_{i,k-2}, u_2\}$ with $i = 1, 2$ and $g_1, \cdots, g_{m-1}$ be the $m-1$ edges which are incident with $u_{2,1}$ and $k-1$ core vertices in V_4, where $m \geqslant 2$.

If $x_0 \geqslant x_7$, then let G_1' be the hypergraph obtained from $E_{n,k}$ by removing $(g_1, \cdots, g_{m-1})$ from $u_{2,1}$ to u_1. Obviously, $G_1' \cong D_{n,k}$. By Lemma 2.6, we get $\rho_\alpha(D_{n,k}) > \rho_\alpha(E_{n,k})$.

Next, let $x_7 > x_0$. Let $f = \{u_{2,1}, u_{6,1}, \cdots, u_{6,k-1}\}$. Two cases are considered as follows.

Case (i) $x_{e_1\backslash\{u_1\}} \geqslant x_{f\backslash\{u_{2,1}\}}$.

Let $U = \{u_{1,1}, \cdots, u_{1,k-2}, u_2\}$ and $V = \{u_{6,1}, \cdots u_{6,k-1}\}$. Then $e_1 = U \cup \{u_1\}$ and $f = V \cup \{u_{2,1}\}$. Let $e_1' = U \cup (f\backslash V) = \{u_{1,1}, \cdots, u_{1,k-2}, u_2, u_{2,1}\}$ and $f' = V \cup (e_1\backslash U) = \{u_{6,1}, \cdots u_{6,k-1}, u_1\}$. Let $G_2' = E_{n,k} - \{e_1, f\} + \left\{e_1', f'\right\}$. Obviously, $u_{2,1}e_2u_2e_1'u_{2,1}$ is the cycle of G_2', u_1 is incident with the pendant edge f' in G_2', and all the perfect matching edges of G_2' are pendant edges. Therefore, $G_2' \cong D_{n,k}$. In this case, we have $x_U = x_{e_1\backslash\{u_1\}} \geqslant x_{f\backslash\{u_{2,1}\}} = x_V$. Furthermore, we get $x_{f\backslash V} = x_{u_{2,1}} = x_7 > x_0 = x_{u_1} = x_{e_1\backslash U}$. By Lemma 2.9, we obtain $\rho_\alpha(D_{n,k}) > \rho_\alpha(E_{n,k})$.

Case (ii). $x_{f\backslash\{u_{2,1}\}} > x_{e_1\backslash\{u_1\}}$.

Two subcases are considered.

Subcase (ii.i). $x_0 \geqslant x_6$.

Let $U = \{u_{6,1}\}$ and $V = \{u_1\}$. Let $e_1'' = U \cup (e_1\backslash V) = \{u_{6,1}, u_{1,1}, \cdots u_{1,k-2}, u_2\}$ and $f'' = V \cup (f\backslash U) = \{u_1, u_{2,1}, u_{6,2} \cdots u_{6,k-1}\}$. Let $G_3' = E_{n,k} - \{e_1, f\} + \left\{e_1'', f''\right\}$. Obviously, $u_{2,1}e_2u_1f''u_{2,1}$ is the cycle of G_3', u_2 is incident with the pendant edge e_1'' in G_3', and all the perfect matching edges of G_3' are pendant edges. Therefore, $G_3' \cong D_{n,k}$. Since $x_0 \geqslant x_6$, $x_{f\backslash\{u_{2,1}\}} = x_6^{k-1} > x_{e_1\backslash\{u_1\}} = x_0x_1^{k-2}$, and $\boldsymbol{x} \in \mathbb{R}^n_{++}$, we get $x_6^{k-2} > x_1^{k-2}$. Since $x_{f\backslash U} = x_{f\backslash\{u_{6,1}\}} = x_7x_6^{k-2}$ and $x_{e_1\backslash V} = x_{e_1\backslash\{u_1\}} = x_0x_1^{k-2}$, it follows from $x_7 > x_0$ and $x_6^{k-2} > x_1^{k-2}$ that $x_{f\backslash U} > x_{e_1\backslash V}$. Since $x_V \geqslant x_U$ and $x_{f\backslash U} > x_{e_1\backslash V}$, by Lemma 2.9, we obtain $\rho_\alpha(D_{n,k}) > \rho_\alpha(E_{n,k})$.

Subcase (ii.ii). $x_6 > x_0$.

We divide $V\,(D_{n,k})$ into ten subsets as follows. Let $V_0' = \{v_{2,1}\}$, $V_1' = \{v_{6,1}, \cdots, v_{6,k-2}\}$, $V_2' = \{v_{2,2}, \cdots, v_{2,k-2}\}$, $V_3' = \left\{v_{3,1}, \cdots, v_{3,(m-1)(k-1)}\right\}$, $V_4' = \left\{v_{4,1}, \cdots, v_{4,(m-1)(k-1)^2}\right\}$, $V_5' = \left\{v_{5,1}, \cdots, v_{5,(k-3)(k-1)}\right\}$, $V_6' = \{v_{1,1}, \cdots, v_{1,k-2}\}$, $V_7' = \{v_1\}$, $V_8' = \{v_8\}$, and $V_9' = \{v_2\}$, where $d_{D_{n,k}}(v_1) = m+1$, each vertex in V_i' has degree 2 with $i = 0, 2, 3, 9$, and each vertex in V_i' has degree 1 with $1 \leqslant i \leqslant 8$ and $i \neq 2, 3, 7$. Furthermore, V_i' with $1 \leqslant i \leq 6$ satisfies the following conditions: (1). each vertex in V_1' is adjacent to $v_{2,1}$ and v_8 simultaneously; (2). each vertex in

V_2' is adjacent to $v_{2,1}$, v_1, v_2, and $k-1$ core vertices in V_5' which are incident with a pendant edge; (3). each vertex in V_3' is adjacent to v_1 and $k-1$ core vertices in V_4' which are incident with a pendant edge; (4). each vertex in V_4' is adjacent to a vertex in V_3'; (5). each vertex in V_5' is adjacent to a vertex in V_2'; and (6). each vertex in V_6' is adjacent to v_1 and v_2 simultaneously. Obviously, we have $V(D_{n,k}) = \bigcup_{i=0}^{9} V_i'$ and $V_i' \cap V_j' = \varnothing$ for $0 \leqslant i < j \leqslant 9$. The hypergraph $D_{n,k}$ and all the vertices of $D_{n,k}$ are shown in Fig. 3.

We construct a vector $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$ of $D_{n,k}$ as follows. By the symmetry, the components in $\boldsymbol{y}$ which correspond to the vertices in V_i' are the same, and we denote them by y_i, where $0 \leqslant i \leqslant 9$. Let $y_i = x_i$ for $0 \leqslant i \leqslant 7$, $y_0 = y_8 = x_0$ and $y_6 = y_9 = x_6$. Since $\|\boldsymbol{x}\|_k^k = 1$ and $\boldsymbol{x} \in \mathbb{R}_{++}^n$, we also have $\|\boldsymbol{y}\|_k^k = 1$ and $\boldsymbol{y} \in \mathbb{R}_{++}^n$. By (3), we have

$$\begin{aligned}\boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(E_{n,k})\boldsymbol{x}\right) = {} & 4x_0^k + (k-2)x_1^k + 2(k-3)x_2^k + 2(m-1)(k-1)x_3^k \\ & + (m-1)(k-1)^2x_4^k + (k-3)(k-1)x_5^k + (k-1)x_6^k + (m+1)x_7^k.\end{aligned} \tag{5}$$

Bearing in mind that $y_i = x_i$ with $0 \leqslant i \leqslant 7$, $y_0 = y_8 = x_0$, and $y_6 = y_9 = x_6$, by (3), we get

$$\begin{aligned}\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{y}\right) = {} & 3x_0^k + (k-2)x_1^k + 2(k-3)x_2^k + 2(m-1)(k-1)x_3^k \\ & + (m-1)(k-1)^2x_4^k + (k-3)(k-1)x_5^k + kx_6^k + (m+1)x_7^k.\end{aligned} \tag{6}$$

It follows from (5), (6) and $x_6 > x_0$ that

$$\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(E_{n,k})\boldsymbol{x}\right) = x_6^k - x_0^k > 0. \tag{7}$$

By (2), we have

$$\begin{aligned}\boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(E_{n,k})\boldsymbol{x}\right) &= k\sum_{e\in E(E_{n,k})} x_e \\ &= k\Big(x_0^2x_1^{k-2} + x_0^2x_7x_2^{k-3} + (k-3)x_2x_5^{k-1} + x_7x_6^{k-1} \\ &\quad + (m-1)x_7x_3^{k-1} + (m-1)(k-1)x_3x_4^{k-1}\Big).\end{aligned} \tag{8}$$

Since $y_i = x_i$ for $0 \leqslant i \leqslant 7$, $y_0 = y_8 = x_0$, and $y_6 = y_9 = x_6$, by (2), we obtain

$$\begin{aligned}\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{y}\right) &= k\sum_{e\in E(D_{n,k})} x_e \\ &= k\Big(x_0^2x_1^{k-2} + x_0x_6x_7x_2^{k-3} + (k-3)x_2x_5^{k-1} + x_7x_6^{k-1} \\ &\quad + (m-1)x_7x_3^{k-1} + (m-1)(k-1)x_3x_4^{k-1}\Big).\end{aligned} \tag{9}$$

It follows from (8), (9), $x_6 > x_0$, and $\boldsymbol{x} \in \mathbb{R}_{++}^n$ that

$$\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(E_{n,k})\boldsymbol{x}\right) = kx_0x_7x_2^{k-3}(x_6 - x_0) > 0. \tag{10}$$

By Lemma 2.3, (7) and (10), we obtain

$$\begin{aligned}\rho_\alpha(D_{n,k}) - \rho_\alpha(E_{n,k}) &\geqslant \boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}_\alpha(D_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}_\alpha(E_{n,k})\boldsymbol{x}\right) \\ &= \alpha\Big(\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(E_{n,k})\boldsymbol{x}\right)\Big) \\ &\quad + (1-\alpha)\Big(\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(E_{n,k})\boldsymbol{x}\right)\Big) > 0.\end{aligned} \tag{11}$$

By (11), we have $\rho_\alpha(D_{n,k}) > \rho_\alpha(E_{n,k})$, where $k \geqslant 3$ and $m \geqslant 2$. Thus, Lemma 3.7 holds. □

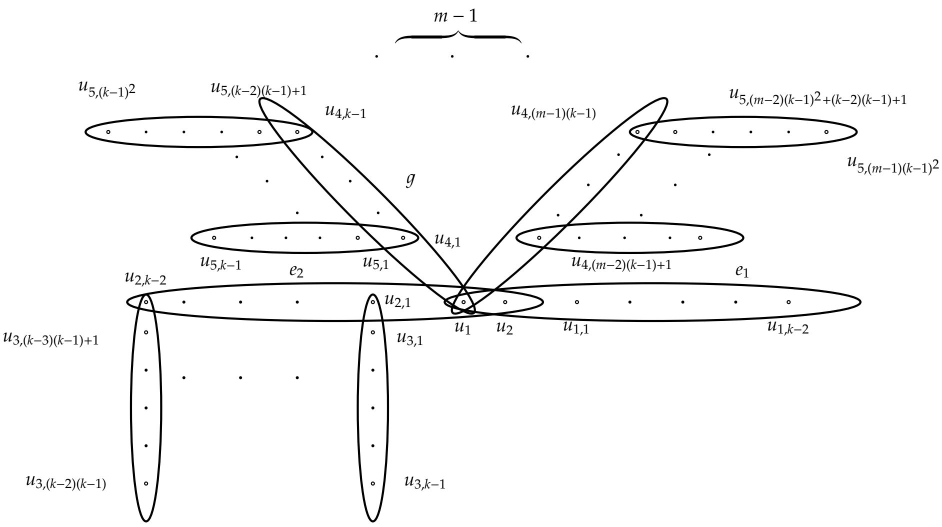

Figure 4: $D_{n,k}$

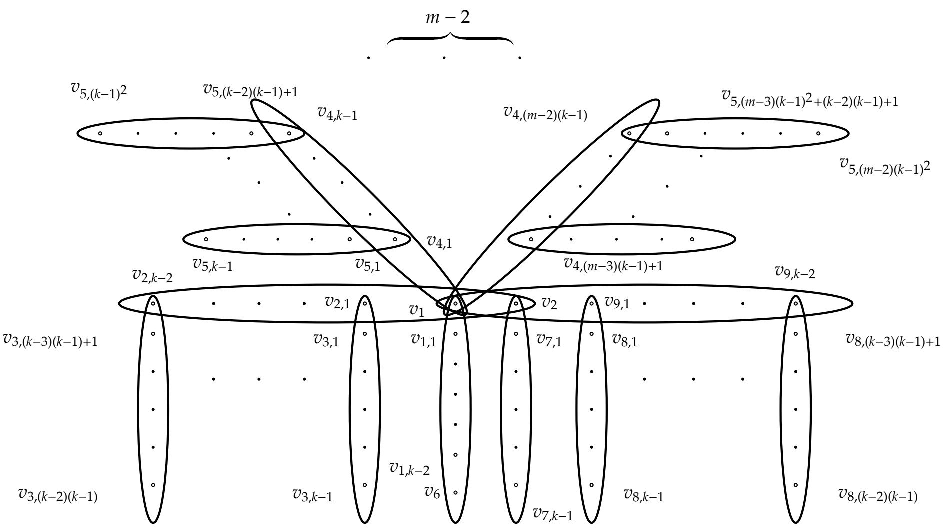

Figure 5: $F_{n,k}$

Lemma 3.8. *Let $n \geqslant 2k(k-1)$, where $k \geqslant 3$. We have $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(D_{n,k})$.*

Proof. We divide $V(D_{n,k})$ into seven subsets as follows. Let $V_0 = \{u_1\}$, $V_1 = \{u_{1,1}, \cdots, u_{1,k-2}\}$, $V_2 = \{u_{2,1}, \cdots, u_{2,k-2}\}$, $V_3 = \left\{u_{3,1}, \cdots, u_{3,(k-2)(k-1)}\right\}$, $V_4 = \left\{u_{4,1}, \cdots, u_{4,(m-1)(k-1)}\right\}$, $V_5 = \left\{u_{5,1}, \cdots, u_{5,(m-1)(k-1)^2}\right\}$, and $V_6 = \{u_2\}$, where $d_{D_{n,k}}(u_1) = m+1$, each vertex in V_i $(i = 2, 4, 6)$ has degree 2 and each vertex in V_i $(i = 1, 3, 5)$ has degree 1. Furthermore, V_i with $1 \leqslant i \leqslant 5$ satisfies the following conditions: (1). each vertex in V_1 is adjacent to u_1 and u_2 simultaneously; (2). each vertex in V_2 is adjacent to u_1, u_2, and $k-1$ core vertices in V_3 which are incident with a pendant edge; (3). each vertex in V_3 is adjacent to a vertex in V_2; (4). each vertex in V_4 is adjacent to u_1 and $k-1$ core vertices in V_5 which are incident with a pendant edge; and (5). each vertex in V_5 is adjacent to a vertex in V_4. Obviously, we have $V(D_{n,k}) = \bigcup_{i=0}^{6} V_i$ and $V_i \cap V_j = \varnothing$ for $0 \leqslant i < j \leqslant 6$. The hypergraph $D_{n,k}$ and all the vertices of $D_{n,k}$ are shown in Fig. 4.

Let $\boldsymbol{x}$ be the α-Perron vector of $D_{n,k}$. Namely, we have $\boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}_{\alpha}(D_{n,k})\boldsymbol{x}\right) = \rho_{\alpha}(D_{n,k})$, where $\|\boldsymbol{x}\|_k^k = 1$ and $\boldsymbol{x} \in \mathbb{R}_{++}^n$. By the symmetry, all the vertices in V_i have the same component in $\boldsymbol{x}$, where $0 \leqslant i \leqslant 6$. We use x_i to denote the component in $\boldsymbol{x}$ which corresponds to the vertices in $V_i\,(D_{n,k})$, where $0 \leqslant i \leqslant 6$. Let $e_i = \{u_1, u_{i,1}, \cdots, u_{i,k-2}, u_2\}$ with $i = 1, 2$ and $g = \{u_1, u_{4,1}, \cdots, u_{4,k-1}\}$.

If $x_1 \geqslant x_6$, then let G_4' be the hypergraph obtained from $D_{n,k}$ by removing e_2 from u_2 to $u_{1,1}$. Obviously, $G_4' \cong D_{n,k}$. By Lemma 2.6, we get $\rho_{\alpha}(D_{n,k}) > \rho_{\alpha}(D_{n,k})$, where $0 \leqslant \alpha < 1$. This is a contradiction. Therefore, $x_6 > x_1$. Two cases are considered as follows.

Case (i). $x_1 \geqslant x_4$.

Since $x_6 > x_1$ and $x_1 \geqslant x_4$, we get $x_6 > x_4$. Let G_5' be the hypergraph obtained from $D_{n,k}$ by removing the $k-1$ pendant edges which are adjacent to g from $u_{4,1}, u_{4,2}, \cdots, u_{4,k-1}$ to $u_2, u_{1,1}, \cdots, u_{1,k-2}$, respectively. Obviously, $G_5' \cong F_{n,k}$. By Lemma 2.6, we get $\rho_{\alpha}(F_{n,k}) > \rho_{\alpha}(D_{n,k})$, where $0 \leqslant \alpha < 1$.

Case (ii). $x_4 > x_1$.

We divide $V\,(F_{n,k})$ into eleven subsets as follows. Let $V_0' = \{v_1\}$, $V_1' = \{v_{1,1}, \cdots, v_{1,k-2}\}$, $V_2' = \{v_{2,1}, \cdots, v_{2,k-2}\}$, $V_3' = \left\{v_{3,1}, \cdots, v_{3,(k-2)(k-1)}\right\}$, $V_4' = \left\{v_{4,1}, \cdots, v_{4,(m-2)(k-1)}\right\}$, $V_5' = \left\{v_{5,1}, \cdots, v_{5,(m-2)(k-1)^2}\right\}$, $V_6' = \{v_6\}$, $V_7' = \{v_{7,1}, \cdots, v_{7,k-1}\}$, $V_8' = \left\{v_{8,1}, \cdots, v_{8,(k-2)(k-1)}\right\}$, $V_9' = \{v_{9,1}, \cdots, v_{9,k-2}\}$, and $V_{10}' = \{v_2\}$, where $d_{F_{n,k}}(v_1) = m+1$, $d_{F_{n,k}}(v_2) = 3$, each vertex in V_i' has degree 2 with $i = 2, 4, 9$, and each vertex in V_i' has degree 1 with $1 \leqslant i \leqslant 8$ and $i \neq 2, 4$. Furthermore, V_i' with $1 \leqslant i \leqslant 9$ $(i \neq 6)$ satisfies the following conditions: (1). each vertex in V_1' is adjacent to v_1 and v_6 simultaneously; (2). each vertex in V_2' is adjacent to v_1, v_2, and $k-1$ core vertices in V_3' which are incident with a pendant edge; (3). each vertex in V_3' is adjacent to a vertex in V_2'; (4). each vertex in V_4' is adjacent to v_1 and $k-1$ core vertices in V_5' which are incident with a pendant edge; (5). each vertex in V_5' is adjacent to a vertex in V_4'; (6). each vertex in V_7' is adjacent to v_2; (7). each vertex in V_8' is adjacent to a vertex in V_9'; and (8). each vertex in V_9' is adjacent to v_1, v_2, and $k-1$ core vertices in V_8' which are incident with a pendant edge. Obviously, we have $V\,(F_{n,k}) = \bigcup_{i=0}^{10} V_i'$ and $V_i' \cap V_j' = \varnothing$ for $0 \leqslant i < j \leqslant 10$. The hypergraph $F_{n,k}$ and all the vertices of $F_{n,k}$ are shown in Fig. 5.

Two cases are considered as follows.

Subcase (ii.i). $x_4 \geqslant x_6$.

We construct a vector $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$ for $F_{n,k}$ as follows. By the symmetry, the components in $\boldsymbol{y}$ which correspond to the vertices in V_i' are the same, and we denote them by y_i, where $0 \leqslant i \leqslant 10$. Let $y_i = x_i$ for $0 \leqslant i \leqslant 6$, $y_4 = y_9 = y_{10} = x_4$ and $y_5 = y_7 = y_8 = x_5$. Since $\|\boldsymbol{x}\|_k^k = 1$ and $\boldsymbol{x} \in \mathbb{R}_{++}^n$, we also have $\|\boldsymbol{y}\|_k^k = 1$ and $\boldsymbol{y} \in \mathbb{R}_{++}^n$. By (3), we have

$$\begin{aligned}\boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{x}\right) &= (m+1)x_0^k + (k-2)x_1^k + 2(k-2)x_2^k + (k-2)(k-1)x_3^k \\ &\quad + 2(m-1)(k-1)x_4^k + (m-1)(k-1)^2 x_5^k + 2x_6^k.\end{aligned} \tag{12}$$

Bearing in mind that $y_i = x_i$ with $0 \leqslant i \leqslant 6$, $y_4 = y_9 = y_{10} = x_4$, and $y_5 = y_7 = y_8 = x_5$, by (3), we get

$$\begin{aligned}\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{D}(F_{n,k})\boldsymbol{y}\right) &= (m+1)x_0^k + (k-2)x_1^k + 2(k-2)x_2^k + (k-2)(k-1)x_3^k \\ &\quad + (2(m-1)(k-1)+1)x_4^k + (m-1)(k-1)^2 x_5^k + x_6^k.\end{aligned} \tag{13}$$

It follows from (12), (13) and $x_4 \geqslant x_6$ that

$$\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{D}(F_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{x}\right) = x_4^k - x_6^k \geqslant 0. \tag{14}$$

By (2), we have

$$\begin{aligned}\boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{x}\right) &= k \sum_{e \in E(D_{n,k})} x_e \\ &= k\Big(x_0 x_6 x_1^{k-2} + x_0 x_6 x_2^{k-2} + (m-1)x_0 x_4^{k-1} + (k-2)x_2 x_3^{k-1} + (m-1)(k-1)x_4 x_5^{k-1}\Big).\end{aligned} \tag{15}$$

Since $y_i = x_i$ for $0 \leqslant i \leqslant 6$, $y_4 = y_9 = y_{10} = x_4$, and $y_5 = y_7 = y_8 = x_5$, by (2), we get

$$\begin{aligned} \boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}(F_{n,k})\boldsymbol{y}\right) &= k \sum_{e \in E(F_{n,k})} x_e \\ &= k\Big(x_0 x_6 x_1^{k-2} + x_0 x_4 x_2^{k-2} + (m-1)x_0 x_4^{k-1} + (k-2)x_2 x_3^{k-1} + (m-1)(k-1)x_4 x_5^{k-1}\Big). \end{aligned} \tag{16}$$

It follows from (15), (16), $x_4 \geqslant x_6$, and $\boldsymbol{x} \in \mathbb{R}_{++}^n$ that

$$\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}(F_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{x}\right) = k x_0 x_2^{k-2}(x_4 - x_6) \geqslant 0. \tag{17}$$

By Lemma 2.3, (14) and (17), we obtain

$$\begin{aligned} \rho_\alpha\left(F_{n,k}\right) - \rho_\alpha\left(D_{n,k}\right) &\geqslant \boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}_\alpha(F_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}_\alpha(D_{n,k})\boldsymbol{x}\right) \\ &= \alpha\Big(\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{D}(F_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{x}\right)\Big) \\ &\quad + (1-\alpha)\Big(\boldsymbol{y}^{\mathrm{T}}\left(\mathcal{A}(F_{n,k})\boldsymbol{y}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{x}\right)\Big) \geqslant 0. \end{aligned} \tag{18}$$

By (18), we obtain $\rho_\alpha\left(F_{n,k}\right) \geqslant \rho_\alpha\left(D_{n,k}\right)$ for $0 \leqslant \alpha < 1$, where $k \geqslant 3$ and $m \geqslant 2$.

Subcase (ii.ii). $x_6 > x_4$.

We construct a vector $\boldsymbol{z} = (z_1, z_2, \cdots, z_n)^{\mathrm{T}}$ for $F_{n,k}$ as follows. By the symmetry, the components in $\boldsymbol{z}$ which correspond to the vertices in V_i' are the same, and we denote them by z_i, where $0 \leqslant i \leqslant 10$. Let $z_i = x_i$ for $0 \leqslant i \leqslant 5$, $z_4 = z_6 = z_9 = x_4$, $z_5 = z_7 = z_8 = x_5$ and $z_{10} = x_6$. Since $\|\boldsymbol{x}\|_k^k = 1$ and $\boldsymbol{x} \in \mathbb{R}_{++}^n$, we also have $\|\boldsymbol{z}\|_k^k = 1$ and $\boldsymbol{z} \in \mathbb{R}_{++}^n$. By (3), we get

$$\begin{aligned} \boldsymbol{z}^{\mathrm{T}}\left(\mathcal{D}(F_{n,k})\boldsymbol{z}\right) &= (m+1)x_0^k + (k-2)x_1^k + 2(k-2)x_2^k + (k-2)(k-1)x_3^k \\ &\quad + \Big(2(m-1)(k-1) - 1\Big)x_4^k + (m-1)(k-1)^2 x_5^k + 3x_6^k. \end{aligned} \tag{19}$$

It follows from (12), (19) and $x_6 > x_4$ that

$$\boldsymbol{z}^{\mathrm{T}}\left(\mathcal{D}(F_{n,k})\boldsymbol{z}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{x}\right) = x_6^k - x_4^k > 0. \tag{20}$$

Since $z_i = x_i$ with $0 \leqslant i \leqslant 5$, $z_4 = z_6 = z_9 = x_4$, $z_5 = z_7 = z_8 = x_5$, and $z_{10} = x_6$, by (2), we get

$$\begin{aligned} \boldsymbol{z}^{\mathrm{T}}\left(\mathcal{A}(F_{n,k})\boldsymbol{z}\right) &= k \sum_{e \in E(F_{n,k})} x_e \\ &= k\Big(x_0 x_4 x_1^{k-2} + x_0 x_6 x_2^{k-2} + x_0 x_6 x_4^{k-2} + (m-2)x_0 x_4^{k-1} \\ &\quad + (k-2)x_2 x_3^{k-1} + \Big((m-1)(k-1) - 1\Big)x_4 x_5^{k-1} + x_6 x_5^{k-1}\Big). \end{aligned} \tag{21}$$

It follows from (15), (21), $x_6 > x_4$, $x_4 > x_1$, and $\boldsymbol{x} \in \mathbb{R}_{++}^n$ that

$$\begin{aligned} &\boldsymbol{z}^{\mathrm{T}}\left(\mathcal{A}(F_{n,k})\boldsymbol{z}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{x}\right) \\ &= k\Big(x_0(x_6 - x_4)(x_4^{k-2} - x_1^{k-2}) + x_5^{k-1}(x_6 - x_4)\Big) > 0. \end{aligned} \tag{22}$$

By Lemma 2.3, (20) and (22), we obtain

$$\begin{aligned} \rho_\alpha\left(F_{n,k}\right) - \rho_\alpha\left(D_{n,k}\right) &\geqslant \boldsymbol{z}^{\mathrm{T}}\left(\mathcal{A}_\alpha(F_{n,k})\boldsymbol{z}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}_\alpha(D_{n,k})\boldsymbol{x}\right) \\ &= \alpha\Big(\boldsymbol{z}^{\mathrm{T}}\left(\mathcal{D}(F_{n,k})\boldsymbol{z}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{D}(D_{n,k})\boldsymbol{x}\right)\Big) \\ &\quad + (1-\alpha)\Big(\boldsymbol{z}^{\mathrm{T}}\left(\mathcal{A}(F_{n,k})\boldsymbol{z}\right) - \boldsymbol{x}^{\mathrm{T}}\left(\mathcal{A}(D_{n,k})\boldsymbol{x}\right)\Big) > 0. \end{aligned} \tag{23}$$

By (23), we get $\rho_\alpha\left(F_{n,k}\right) > \rho_\alpha\left(D_{n,k}\right)$, where $0 \leqslant \alpha < 1$, $k \geqslant 3$ and $m \geqslant 2$. Thus, Lemma 3.8 holds. □

Lemma 3.9. *Let $G \in \overline{\mathcal{U}}_2(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. Then $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$, where $0 \leqslant \alpha < 1$.*

Proof. Let $n \geqslant 2k(k-1)$, $k \geqslant 3$, and $0 \leqslant \alpha < 1$. Let $G^\diamond$ be the hypergraph having the largest α-spectral radius among $\overline{\mathcal{U}}_2(n,k,2)$. Let $C_2 = v_1e_1v_2e_2v_1$ be the cycle of $G^\diamond$, where $e_i = \{v_1, v_{i,1}, \ldots, v_{i,k-2}, v_2\}$ with $i = 1, 2$. According to the definition of $\overline{\mathcal{U}}_2(n,k,2)$, there exists one perfect matching edge in C_2 of $G^\diamond$, and there exists a vertex (denoted by v^*) in $e_1 \cup e_2$ of $G^\diamond$ which is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. We suppose that e_1 in C_2 is the perfect matching edge. In $G^\diamond$, if $v^* \in e_1 \setminus \{v_1, v_2\}$, without loss of generality, we suppose that $v^* = v_{1,1}$. By the same methods similar to those for the proofs of Claim 1 in Lemma 3.4, we can get a hypergraph G' such that $\rho_\alpha(G') > \rho_\alpha(G^\diamond)$, where G' satisfies: (1). $G' \in \overline{\mathcal{U}}_2(n,k,2)$; and (2). there exists only one vertex in $e_1 \cap e_2$ of C_2 of G' which is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$. Obviously, the inequality $\rho_\alpha(G') > \rho_\alpha(G^\diamond)$ contradicts the definition of G°. Therefore, In $G^\diamond$, $v^* \in e_2$. Namely, $G^\diamond \in \overline{\mathcal{U}}_2(n,k,2)$ and only one vertex (namely v^*) in e_2 of C_2 of $G^\diamond$ is attached by a k-uniform supertree containing at least k edges, where $k \geqslant 3$.

Two cases are considered.

Case (i). $v^* \in \{v_1, v_2\}$.

We suppose $v^* = v_1$. When $n = 2k(k-1)$, by the definitions of $G^\diamond$ and $D_{n,k}$, we obtain $G^\diamond \cong D_{n,k}$. By Lemma 3.8, we get $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(D_{n,k})$. Thus, Lemma 3.9 holds. Next, let $n \geqslant 3k(k-1)$. By the definition of $G^\diamond$, v_1 of $G^\diamond$ is attached by a k-uniform supertree (denoted by $\widetilde{T}$) containing at least $2k$ edges, where $k \geqslant 3$. We will prove that each edge in $M(G^\diamond) \setminus \{e_1\}$ is a pendant edge. Otherwise, we suppose that there exists an edge (denoted by e) in $E(\widetilde{T}) \cap M(G^\diamond)$ which is not a pendant edge. By applying the edge-releasing operation on e at a vertex of e and using the methods similar to those for the proofs of Lemma 3.1, we can get a hypergraph (denoted by G_{12}) such that $\rho_\alpha(G_{12}) > \rho_\alpha(G^\diamond)$, where $G_{12} \in \overline{\mathcal{U}}_2(n,k,2)$ and e of G_{12} is a pendant edge. This is a contradiction. Thus, each edge in $E(\widetilde{T}) \cap M(G^\diamond)$ is a pendant edge. Namely, each edge in $M(G^\diamond) \setminus \{e_1\}$ is a pendant edge.

If $G^\diamond \cong D_{n,k}$, then by Lemma 3.8, $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G^\diamond)$. Namely, Lemma 3.9 holds. If $G^\diamond \ncong D_{n,k}$, then according to the definitions of $G^\diamond$ and $D_{n,k}$, there exists an edge (denoted by $g' = \{w_1', \ldots, w_k'\}$) in $\widetilde{T}$ of $G^\diamond$ satisfying: (1). $v_1 \notin g'$; (2). v_1 and w_1' are incident with a common edge in $\widetilde{T}$ (denoted by g_1); and (3). g' is not a pendant edge. Obviously, g_1 is not a perfect matching edge of $G^\diamond$. Let $\boldsymbol{x}$ be the α-Perron vector of $G^\diamond$. If $x_{v_1} \geqslant x_{w_1'}$, then let G_{13} be the hypergraph obtained from $G^\diamond$ by removing g' from w_1' to v. Obviously, $G_{13} \in \overline{\mathcal{U}}_2(n,k,2)$. By Lemma 2.6, $\rho_\alpha(G_{13}) > \rho_\alpha(G^\diamond)$. This is a contradiction. Thus, we have $x_{w_1'} > x_{v_1}$. Let G_{14} be the hypergraph obtained from $G^\diamond$ by removing e_2 from v_1 to w_1'. By Lemma 2.6, $\rho_\alpha(G_{14}) > \rho_\alpha(G^\diamond)$. Let $e_2^* = \{w_1', v_{2,1}, \ldots, v_{2,k-1}, v_2\}$. Obviously, $w_1'e_2^*v_2e_1v_1g_1w_1'$ is a cycle of length 3 in G_{14}. Since $G^\diamond \in \overline{\mathcal{U}}_2(n,k,2)$, by the definition of $\overline{\mathcal{U}}_2(n,k,2)$, $e_2 \notin M(G^\diamond)$. Thus, $M(G^\diamond)$ is the perfect matching of G_{14} and e_1 is a perfect matching edge of G_{14}. Therefore, $G_{14} \in \mathcal{U}_2(n,k,3)$.

Let G_{15} be the hypergraph obtained from G_{14} by applying the edge-releasing operation on e_1 at v_1. By Lemma 3.3, we get $\rho_\alpha(G_{15}) \geqslant \rho_\alpha(G_{14})$, with the equality if and only if $G_{14} \cong G_{15}$. Obviously, $w_1'e_2^*v_1g_1w_1'$ is a cycle of G_{15} and e_1 of G_{15} is a pendant edge. Since e_1 is a perfect matching edge of G_{14}, all the edges of G_{14} which are adjacent to e_1 are not perfect matching edges of G_{14}. Thus, $G_{15} \in \overline{\mathcal{U}}_1(n,k,2)$, where $n \geqslant 3k(k-1)$. By Lemma 3.4, $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G_{15})$, with the equality if and only if $G_{15} \cong F_{n,k}$. Thus, we get $\rho_\alpha(F_{n,k}) > \rho_\alpha(G^\diamond)$. Namely, Lemma 3.9 holds.

Case (ii). $v^* \in e_2 \setminus \{v_1, v_2\}$.

Suppose $v^* = v_{2,1}$. If $n = 2k(k-1)$, by the definitions of $G^\diamond$ and $E_{n,k}$, $G^\diamond \cong E_{n,k}$. By Lemmas 3.7 and 3.8, $\rho_\alpha(F_{n,k}) > \rho_\alpha(G^\diamond)$. Next, let $n \geqslant 3k(k-1)$. By the definition of $G^\diamond$, $v_{2,1}$ is attached by a k-uinform supertree (denoted by T') containing at least $2k$ edges, where $k \geqslant 3$. We will prove that each edge in $M(G^\diamond) \setminus \{e_1\}$ is a pendant edge. Otherwise, we suppose that there exists an edge (denoted by e) in $E(T') \cap M(G^\diamond)$ which is not a pendant edge. By applying the edge-releasing operation on e at a vertex of e and using the methods similar to those for the proofs of Lemma 3.1, we can get a hypergraph (denoted by G_{16}) such that $\rho_\alpha(G_{16}) > \rho_\alpha(G^\diamond)$, where $G_{16} \in \overline{\mathcal{U}}_2(n,k,2)$ and e of G_{16} is a pendant edge. This is a contradiction. Thus, each edge in $E(T') \cap M(G^\diamond)$ is a pendant edge. Namely, each edge in $M(G^\diamond) \setminus \{e_1\}$ is a pendant edge. If $G^\diamond \cong E_{n,k}$, by Lemmas 3.7 and 3.8, $\rho_\alpha(F_{n,k}) > \rho_\alpha(G^\diamond)$. If $G^\diamond \ncong E_{n,k}$, by the definition of $G^\diamond$, there exist two edges in $E(T')$ which belong to $Q(G^\diamond)$. By using the methods similar to those for the proofs of Cases (i) and (ii) in Lemma 3.4, we obtain $\rho_\alpha(E_{n,k}) \geqslant \rho_\alpha(G^\diamond)$, with the equality if and only if $G^\diamond \cong E_{n,k}$. Furthermore, by Lemmas 3.7 and 3.8, we get $\rho_\alpha(F_{n,k}) > \rho_\alpha(G^\diamond)$.

By combining the above proofs, we get $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$, where $0 \leqslant \alpha < 1$ and $G \in \overline{\mathcal{U}}_2(n,k,2)$. □

By Lemmas 3.6 and 3.9, we get Corollary 3.10.

Corollary 3.10. *Let $G \in \mathcal{U}_2(n,k,2)$, where $n \geqslant 2k(k-1)$ and $k \geqslant 3$. Then $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$, where $0 \leqslant \alpha < 1$.*

Theorem 3.11. *Let $G \in \mathcal{U}(n,k)$, where $n \geqslant k(k-1)$ and $k \geqslant 3$.*
(i). When $n = k(k-1)$, $G \cong D_{n,k}$.
(ii). When $n \geqslant 2k(k-1)$, for $0 \leqslant \alpha < 1$, we have $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$ with the equality if and only if $G \cong F_{n,k}$.

Proof. When $n = k(k-1)$, by the definition of $D_{n,k}$, we have Theorem 3.11(i). Let $n \geqslant 2k(k-1)$ and $0 \leqslant \alpha < 1$. If $G \in \mathcal{U}(n,k) \setminus \mathcal{U}_2(n,k,2)$, then by Corollary 3.5, we have $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$, with the equality if and only if $G \cong F_{n,k}$. If $G \in \mathcal{U}_2(n,k,2)$, then by Corollary 3.10, we get $\rho_\alpha(F_{n,k}) \geqslant \rho_\alpha(G)$. Since $\mathcal{U}(n,k) = \big(\mathcal{U}(n,k) \setminus \mathcal{U}_2(n,k,2)\big) \cup \mathcal{U}_2(n,k,2)$, we obtain Theorem 3.11(ii). □

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