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Summation of hyperharmonic series in Banach algebras and Banach bimodules

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Abstract. By employing the Laplace transform for Banach-space-valued functions, in this paper we evaluate the sums of some hyperharmonic-like series in Banach algebras and modules. We discuss the cases when the general terms of the given series are invertible in the respective algebras, and when they are invertible in the Drazin-Koliha sense, or the Mary-Patrício sense. Afterwards, we extend our results to the multilateral modular series of the form

$$\sum_{k=1}^{\infty}(a_1+k)^{-n_1}c_1(a_2+k)^{-n_2}c_2\cdot\ldots(a_{m-1}+k)^{-n_{m-1}}c_{m-1}(a_m+k)^{-n_m},$$

where a_i belong to possibly different Banach algebras, c_j belong to possibly different Banach bimodules, and $n_1,\ldots,n_m$ are positive integers. As an application, we obtain a new necessary solvability condition for the Sylvester equation $ax-xb=c$ in Banach bimodules.

1. Introduction

In this paper we evaluate the sums of "hyperharmonic" series in the setting of Banach algebras and Banach modules. Precisely, for a given (complex unital) Banach algebra $\mathcal{A}$ and a fixed element $a\in\mathcal{A}$ we study the series of the form

$$\sum_{k=1}^{\infty}(a+k)^{-n},\quad \sum_{k=1}^{\infty}(-1)^k(a+k)^{-n}, \tag{1}$$

where $n>1$ and $(a+k)$ are assumed to be invertible in $\mathcal{A}$. Afterwards, we proceed to study the hyperharmonic series defined with respect to a generalized invertibility, that is, we evaluate the sums of

$$\sum_{k=1}^{\infty}\left((a+k)^d\right)^n,\quad \sum_{k=1}^{\infty}\left((a+k)^{-(1-\pi_k)}\right)^n, \tag{2}$$

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where d represents respectively the Drazin-Koliha inverse, while $-(1-\pi_k)$ stands for the inverse along the spectral idempotent $1-\pi_k$ (see e. g. [26], [27], and [28]). In the end, we extend our results to bilateral and multilateral hyperharmonic series in Banach (bi)modules, i.e, to the series of the form

$$\sum_{k=1}^{\infty}(a_1+k)^{-n_1}c_1(a_2+k)^{-n_2}c_2\cdot\ldots(a_{m-1}+k)^{-n_{m-1}}c_{m-1}(a_m+k)^{-n_m} \tag{3}$$

where a_i belong to possibly different Banach algebras such that (a_i+k) are invertible in the respective algebras, c_j belong to possibly different Banach bimodules, and $n_1,\ldots,n_m$ are positive integers.

All these series represent natural adaptations of the numerical hyperharmonic series. Therefore, some identities from the scalar case transfer to Banach algebras automatically: for instance, we have shown that the identities involving the polygamma functions, the Hurwitz function, and the hypergeometric functions ${}_2F_1$ and ${}_3F_2$, hold in Banach algebras as well.

However, once the regular invertibility is replaced with a generalized invertibility, the analogy to the scalar case is lost. Furthermore, it is shown that once the commutativity is omitted, and the multilateral series like (3) are studied, then new representations emerge. As an application of the obtained results, we have demonstrated how our findings provide a new necessary solvability condition for the Sylvester equation $ax-xb=c$ in Banach algebras and modules.

1.1. Preliminaries

Throughout this paper, all Banach algebras are assumed to be unital and complex. They are denoted as $\mathcal{A}$, or as $\mathcal{A}_i$ for some natural index i. If the algebra is commutative, it will be emphasized as $\mathcal{A}_{comm}$.

The spectrum of a given element a within the respective algebra $\mathcal{A}\ni a$ is denoted as $\sigma_{\mathcal{A}}(a)$, or simply as $\sigma(a)$, if there is no chance of confusion. The spectrum of $a\in\mathcal{A}$ is always a nonempty and compact set, therefore we introduce $\operatorname{acc}\sigma(a)$, the set of accumulation points in $\sigma(a)$, and $\operatorname{iso}\sigma(a)$, the set of isolated points in $\sigma(a)$. The set $\rho_{\mathcal{A}}(a):=\mathbb{C}\setminus\sigma_{\mathcal{A}}(a)$ is a nonempty unbounded set, and denotes the resolvent set of a. Specially, for a complex Banach space V, the Banach algebra of bounded linear operators in V is denoted as $\mathbb{B}(V)$, where the algebra unit is the identity operator I_V on V. For two different Banach spaces V_1 and V_2, the Banach space of bounded linear operators from V_1 to V_2 is denoted as $\mathbb{B}(V_1,V_2)$.

Since the results of this paper rely on where the spectrum is positioned in the complex plane, we introduce the following notation. For a given $x\in\mathbb{R}$, we denote by $\mathbb{H}^r_x$ the open right complex half-plane, defined as $\mathbb{H}^r_x=\{z\in\mathbb{C}:\operatorname{Re}z>x\}$. Respectively, $\mathbb{H}^\ell_x:=\{z\in\mathbb{C}:\operatorname{Re}z<x\}$, and $x+i\mathbb{R}:=\mathbb{C}\setminus\left(\mathbb{H}^r_x\cup\mathbb{H}^\ell_x\right)$.

The hyperharmonic series mentioned in (1) and (3) are evaluated via the Laplace transform method in Banach spaces (see below). Consequently, the resulting sums are expressed as improper Bochner integrals with respect to the scalar Lebesgue measure. Therefore, it is convenient to consider the following measure space. Let $\Omega:=[0,\infty)$, and let $\mathcal{F}$ be the smallest $\sigma-$algebra which contains all Borel subsets of Ω. Let μ be the Lebesgue measure defined in $[0,\infty)$. Then, for a given Banach space $(V,\|\cdot\|)$, by $L_1(\Omega,\mu;V)$ we denote the space of $V-$valued functions defined in Ω, which are Bochner integrable with respect to the measure μ, where we equate the functions which are equal $\mu-$almost everywhere, see [7], [16], [29], [33], or [34] for a detailed construction. From this point on, every $V-$valued integral defined over a Borel subset of Ω is regarded as the Bochner integral with respect to the Lebesgue measure. We recall the Dominated Convergence Theorem (DCT) for the Bochner integral, see e.g. [16] or [29].

Theorem 1.1 (Dominated Convergence Theorem). *Let $g\in L_1(\Omega,\mu;\mathbb{R})$ and let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L_1(\Omega,\mu;V)$ such that:*

(a) $|f_n|\le g$ for every $n\in\mathbb{N}$ and μ-a.e. on Ω;

(b) $\lim_{n\to\infty}f_n=f$ μ-a.e. on Ω for some function $f:\Omega\to V$.

Then, $f\in L_1(\Omega,\mu;V)$, and the following equalities hold:

$$\lim_{n\to\infty}\int_\Omega\|f_n-f\|\,d\mu=0,\quad \textit{and}\quad \lim_{n\to\infty}\int_\Omega f_n\,d\mu=\int_\Omega f\,d\mu.$$

2. Laplace Transform Method in Banach Spaces

The inspiration for our summation technique stems from the scalar technique developed for the classical numerical series summation, called the Laplace Transform Method (LTM) for series summation.

The LTM was introduced by Gautschi and Milovanović, see [18], [20], survey [30], and [31]. It states that, for the given numerical series $\sum_{k=1}^{\infty} f(k)$, one ought to find a Laplace transformable complex function g, such that its Laplace transform $\mathcal{L}g$ at the point k satisfies $\mathcal{L}g(k) = f(k)$. Respectively, the input series is rewritten as

$$\sum_{k=1}^{\infty} f(k) = \sum_{k=1}^{\infty} \mathcal{L}g(k) = \int_0^{\infty} \sum_{k=1}^{\infty} \mathrm{e}^{-ks}\, g(s)ds = \int_0^{\infty} (e^s - 1)^{-1}\, g(s)ds,$$

where the summation and integration "switch places" due to the Dominated convergence theorem (for more details see [18]). The advantage of this procedure lies in the fact that the convergence rate of the right-hand-side integral is faster than the convergence rate of the given series. Because of this, the LTM is quite suitable for evaluating the sums of slowly convergent series, provided that the original function g is easily obtainable.

However, it is a known fact (see [2], [3], [17], [21], [22], [25], and [35]) that the theory of Laplace transforms cannot be directly transferred to the Banach space setting. For a function $g \in L_1([0,\infty),\mu;V)$, its forward Laplace transform, denoted as $\mathcal{L}g$, is heuristically given by the improper integral

$$\mathcal{L}g(\lambda) := \int_0^{\infty} e^{-t\lambda} g(t)dt, \quad \lambda \in \mathcal{D}_{\mathcal{L}g}. \tag{4}$$

Provided that the above integral converges, the function g is said to be Laplace transformable. Necessary and sufficient conditions for the given measurable function g to be Laplace transformable, as well as the domain for $\mathcal{L}g$, were provided by Hennig and Neubrander in Theorem 2.1 below (see also [15]):

Theorem 2.1. *[22, Proposition 1.1.] A Bochner measurable function $g \in L_1([0,\infty),\mu;V)$ is Laplace transformable if and only if there are constants $M,\omega > 0$ such that $\| \int_0^r g(t)dt\| \le Me^{\omega r}$ for all $r \ge 0$.*

With respect to the notation from the previous theorem, the domain $\mathcal{D}_{\mathcal{L}g}$ is determined as in the scalar case (see [15] and [22]): $\mathcal{D}_{\mathcal{L}g} \supset \mathbb{H}^r_\omega$. Conversely, for the $V-$valued function f, if there exists an $x \in \mathbb{R}^+$, such that f can be represented as

$$f(\lambda) = \int_0^{\infty} e^{-\lambda t} g(t)dt, \quad \lambda \in \mathbb{H}^r_x,$$

for some Bochner integrable function $g \in L_1([0,\infty),\mu;V)$, then f is well-defined and analytic (precisely, has an analytic extension) in all $\mathbb{H}^r_x$, and the function g is its original (the inverse Laplace transform of f), satisfying

$$(\forall r \ge 0) \quad \left\| \int_0^r g(t)dt \right\| \le Me^{xr}$$

for some $M > 0$. In that case, the original g is unique in the following sense (see [22, Corollary 1.4.]): if

$$f(\lambda) = \int_0^{\infty} e^{-\lambda t} g(t)dt = \int_0^{\infty} e^{-\lambda t} g_1(t)dt, \quad \lambda \in \mathbb{H}^r_x,$$

where $g, g_1 \in L_1([0,\infty),\mu;V)$ are Bochner integrable functions, then for every $r \ge 0$:

$$\int_0^r g(t)dt = \int_0^r g_1(t)dt. \tag{5}$$

Below we proceed to prove that the LTM is viable in Banach spaces as well, provided that all the expressions converge absolutely.

Recall the Einstein's function ε, defined as

$$\varepsilon(s) = s(e^s - 1)^{-1}, \quad s \geq 0, \tag{6}$$

and the Fermi's function φ, defined as

$$\varphi(s) = (1 + e^s)^{-1}, \quad s \geq 0. \tag{7}$$

We begin with the following result:

Lemma 2.2. *Let V be a Banach space, and let $f : \mathbb{N} \to V$ be a bounded vanishing sequence in V. Assume there exists a Bochner integrable, Laplace transformable $V-$valued function g, such that*

$$f(k) = \int_0^\infty e^{-ks} g(s) ds \tag{8}$$

holds for every $k \in \mathbb{N}$. If the series $\sum\limits_{k=1}^{\infty} f(k)$ is absolutely convergent, then the following statements are true:

(a) For the function and $\varphi(s)$ given via (7), the equality holds:

$$\sum_{k=1}^{\infty} (-1)^{k+1} f(k) = \int\limits_0^\infty \varphi(s) g(s) ds. \tag{9}$$

(b) Moreover, if the function $s \mapsto \|s^{-1}\varepsilon(s)g(s)\|$ is Lebesgue integrable in $[0, \infty)$, where $\varepsilon(s)$ is given by (6), then the equality holds:

$$\sum_{k=1}^{\infty} f(k) = \int\limits_0^\infty \frac{\varepsilon(s)}{s} g(s) ds. \tag{10}$$

Proof. (a) The function $s \mapsto \varphi(s)g(s)$ is absolutely integrable in $[0, \infty)$, therefore the function $s \mapsto \|\varphi(s)g(s)\|$ can be used as the integral dominant for DCT. By premise, the sum $\sum\limits_{k=1}^{\infty} (-1)^{k+1} f(k)$ converges absolutely as well, therefore

$$\sum_{k=1}^{\infty} (-1)^{k+1} f(k) = -\int_0^\infty \sum_{k=1}^{\infty} (-e^s)^k g(s) ds = \int\limits_0^\infty \varphi(s) g(s) ds.$$

(b) Similarly, assume that the function $s \mapsto \|s^{-1}\varepsilon(s)g(s)\|$ is Lebesgue integrable in $[0, \infty)$. Then by Theorem 1.1:

$$\sum_{k=1}^{\infty} f(k) = \int_0^\infty \sum_{k=1}^{\infty} e^{-ks} g(s) ds = \int_0^\infty \frac{\varepsilon(s)}{s} g(s) ds.$$

□

3. Unilateral series in Banach algebras

For a given unital complex Banach algebra $\mathcal{A}$, let $a \in \mathcal{A}$ be such that $\sigma(a) \subset \mathbb{H}^r_{-1}$. It is straightforward to see that the integral

$$\mathcal{L}(e^{-sa})(k) = \int_0^\infty e^{-ks} e^{-sa} ds = \int_0^\infty e^{-s(k+a)} ds \tag{11}$$

converges for every $k \in \mathbb{N}$, and is evaluated as

$$\mathcal{L}(e^{-sa})(k) = (a+k)^{-1}. \tag{12}$$

Differentiating the latter n times with respect to k, one obtains the known formula:

$$(a+k)^{-n} = \mathcal{L}\left(\frac{1}{(n-1)!}s^{n-1}e^{-as}\right)(k), \quad n \in \mathbb{N}. \tag{13}$$

In analogy to the scalar case, for a fixed $a \in \mathcal{A}$ with $\sigma(a) \subset \mathbb{H}^r_{-1}$, and for any positive integer $n \in \mathbb{N}$, the $\mathcal{A}$−valued expression

$$f_n(k;a) := (a+k)^{-n} \tag{14}$$

forms a unilateral hyperharmonic sequence $(f_n(k;a))_{k\in\mathbb{N}}$ in $\mathcal{A}$ (or a unilateral harmonic sequence when $n = 1$). With respect to (13)–(14), whenever $a \in \mathcal{A}$ is such that $\sigma(a) \subset \mathbb{H}^r_{-1}$, for an arbitrary $n \in \mathbb{N}$ we define the function $g_n(\cdot;a)$ as

$$g_n(s;a) := \frac{1}{(n-1)!}s^{n-1}e^{-sa}, \quad s \geq 0. \tag{15}$$

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and let $a \in \mathcal{A}$ be such that $\sigma(a) \in \mathbb{H}^r_{-1}$. Then, for every $n \in \mathbb{N} \setminus \{1\}$ the following identities hold:*

$$\sum_{k=1}^{\infty}(a+k)^{-n} = \frac{1}{(n-1)!}\int_0^{\infty}\frac{s^{n-1}}{e^s-1}e^{-sa}ds, \tag{16}$$

and

$$\sum_{k=1}^{\infty}(-1)^{k+1}(a+k)^{-n} = \frac{1}{(n-1)!}\int_0^{\infty}\frac{s^{n-1}}{e^s+1}e^{-sa}ds. \tag{17}$$

Proof. For $n > 1$, it follows that the scalar functions

$$s \mapsto \frac{s^{n-1}}{e^s-1}, \quad s \mapsto \frac{s^{n-1}}{e^s+1} \quad s \geq 0$$

are bounded and integrable in $[0,\infty)$, while the function $s \mapsto e^{-sa}$ is absolutely integrable in $[0,\infty)$. Therefore, the integrals (see [33])

$$\int_0^{\infty}\frac{s^{n-1}}{e^s-1}e^{-sa}ds, \quad \int_0^{\infty}\frac{s^{n-1}}{e^s+1}e^{-sa}ds$$

converge absolutely and regularly in $\mathcal{A}$. On the other hand, the series

$$\sum_{k=1}^{\infty}(a+k)^{-n} \quad \text{and} \quad \sum_{k=1}^{n}(-1)^{k+1}(a+k)^{-n}$$

are absolutely convergent as well, therefore the DCT can be applied, which gives

$$\sum_{k=1}^{\infty}(a+k)^{-n} = \frac{1}{(n-1)!}\int_0^{\infty}\sum_{k=1}^{\infty}e^{-ks}s^{n-1}e^{-sa}ds = \frac{1}{(n-1)!}\int_0^{\infty}\frac{s^{n-1}}{e^s-1}e^{-sa}ds,$$

and, similarly,

$$\sum_{k=1}^{\infty}(-1)^{k+1}(a+k)^{-n} = \frac{1}{(n-1)!}\int_0^{\infty}\frac{s^{n-1}}{e^s+1}e^{-sa}ds.$$

□

The formulae (16)–(17) were to be expected, since the same type of identities holds in the scalar case, see e.g. [24] and the references therein. This scalar analogue can be extended further in the following manner.

For a commutative unital complex Banach algebra $\mathcal{A}_{comm}$, let a, b, c, d, and $e \in \mathcal{A}_{comm}$ be such that

$$-\mathbb{N}_0 \subset \rho(c) \cap \rho(d).$$

For any $w \in \{a, b, c, d, e\}$, recall the $\mathcal{A}_{comm}$−valued Pochhammer symbol $(w)_k$ defined as in the scalar case:

$$\begin{cases}(w)_0 = 1_{\mathcal{A}_{comm}},\\ (w)_k = w \cdot (w + 1_{\mathcal{A}_{comm}}) \cdot (w + 2 \cdot 1_{\mathcal{A}_{comm}}) \cdot \ldots \cdot (w + (k-1) \cdot 1_{\mathcal{A}_{comm}}),\ k \geq 1.\end{cases}$$

Let U be the open unit disc in the complex plane. Then for any $z \in U$ the hypergeometric function $z \mapsto {}_2F_1(a, b; c; z)$ is defined as

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} (a)_k \cdot (b)_k \cdot ((c)_k)^{-1} \frac{z^k}{k!}, \tag{18}$$

while the hypergeometric function $z \mapsto {}_3F_2(a, b, e; c, d; z)$ is defined as

$$ {}_3F_2(a, b, e; c, d; z) = \sum_{k=0}^{\infty} (a)_k \cdot (b)_k \cdot (e)_k \cdot ((c)_k)^{-1} \cdot ((d)_k)^{-1} \frac{z^k}{k!}. \tag{19}$$

Theorem 3.2. *Let $a \in \mathcal{A}$ be such that $\sigma(a) \subset \mathbb{H}^r_{-1} \setminus \{0\}$. Then for every $t > 0$ the following equalities are true:*

(a)

$$\begin{aligned}\sum_{k=1}^{\infty} (a+k)^{-2} = &\int_0^t \frac{s}{e^s - 1} e^{-sa} ds + \\ +a^{-2}e^{-at} &\left[{}_3F_2\left(1, a, a; 1+a, 1+a; e^{-t}\right) + ta\ {}_2F_1\left(1, a; 1+a; e^{-t}\right) - (1+at)\right].\end{aligned} \tag{20}$$

(b)

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^k (k+a)^{-2} = &\int_0^t \frac{s}{(e^s + 1)} e^{-sa} ds + \\ +a^{-2}e^{-at} &\left[{}_3F_2\left(1, a, a; 1+a, 1+a; -e^{-t}\right) - ta\ {}_2F_1(1, a; 1+a; -e^{-t}) - (1-at)\right].\end{aligned} \tag{21}$$

Proof. Note that in both statements, all arguments of the hypergeometric functions ${}_2F_1$ and ${}_3F_2$ belong to the smallest commutative algebra $\langle a \rangle_{\mathcal{A}}$ generated by a. Therefore $\mathcal{A}_{comm} = \langle a \rangle_{\mathcal{A}}$. We proceed as following:

(a) From (18) and (19) we have

$$\begin{aligned}&{}_3F_2\left(1, a, a; 1+a, 1+a; e^{-t}\right) + ta\ {}_2F_1\left(1, a; 1+a; e^{-t}\right) = \\ =&1 + \sum_{k=1}^{\infty} a^2 (a+k)^{-2} e^{-tk} + ta + \sum_{k=1}^{\infty} ta^2 (a+k)^{-1} e^{-tk} = \\ =&(1+ta) + \sum_{k=1}^{\infty} e^{-tk} (a+k)^{-2} a^2 (1 + t(a+k)),\end{aligned}$$

therefore

$$\begin{aligned}&a^{-2}e^{-at}\left[{}_3F_2\left(1,a,a;1+a,1+a;e^{-t}\right)+t a{}_2F_1\left(1,a;1+a;e^{-t}\right)\right]=\\&=a^{-2}e^{-at}(1+ta)+\sum_{k=1}^{\infty}e^{-t(k+a)}(a+k)^{-2}(1+t(a+k)).\end{aligned}\tag{22}$$

On the other hand, the integral expression in (20) gives

$$\begin{aligned}&\int_0^t se^{-as}(e^s-1)^{-1}ds=\int_0^t s\sum_{k=0}^{\infty}e^{-s(k+1+a)}ds=\\&=\sum_{k=0}^{\infty}\left[-e^{-s(k+1+a)}s(k+a+1)^{-1}\Big|_0^t+\int_0^t e^{-s(k+a+1)}(k+a+1)^{-1}ds\right]=\\&=\sum_{k=1}^{\infty}\left[(k+a)^{-2}(-1-(k+a)t)e^{-t(k+a)}+(a+k)^{-2}\right].\end{aligned}\tag{23}$$

Since the series $\sum(a+k)^{-2}$ converges absolutely and unconditionally, it follows from (23) that

$$\sum_{k=1}^{\infty}(a+k)^{-2}=\int_0^t se^{-as}(e^s-1)^{-1}ds+\sum_{k=1}^{\infty}e^{-t(k+a)}(1+(k+a)t)(k+a)^{-2}.\tag{24}$$

Finally, from (22) and (24) we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}(a+k)^{-2}=\int_0^t(e^s-1)^{-1}se^{-sa}ds+\\&+a^{-2}e^{-at}\left[{}_3F_2\left(1,a,a;1+a,1+a;e^{-t}\right)+ta{}_2F_1\left(1,a;1+a;e^{-t}\right)-(1+at)\right].\end{aligned}$$

(b) Similarly as in part (a) we have

$$\begin{aligned}&\int_0^t\frac{s}{(e^s+1)}e^{-sa}ds=\int_0^t s\sum_{k=1}^{\infty}(-1)^{k-1}e^{-s(k+a)}ds=\\&=\sum_{k=1}^{\infty}\left[(-1)^k e^{-s(k+a)}s(k+a)^{-1}\Big|_0^t+\int_0^t(-1)^k e^{-s(k+a)}(k+a)^{-1}ds\right]=\\&=\sum_{k=1}^{\infty}\left[(-1)^k e^{-t(k+a)}t(k+a)^{-1}+(-1)^{k-1}e^{-s(k+a)}(k+a)^{-2}\Big|_{s=0}^{s=t}\right]=\\&=\sum_{k=1}^{\infty}(-1)^{k-1}e^{-t(k+a)}(k+a)^{-2}\left(1-(k+a)t\right)-\sum_{k=1}^{\infty}(-1)^{k-1}(k+a)^{-2}.\end{aligned}$$

On the other hand, we see that

$$\sum_{k=1}^{\infty}(-1)^{k-1}e^{-t(k+a)}(k+a)^{-2}\left(1-(k+a)t\right)=$$
$$=\sum_{k=1}^{\infty}(-1)^{k-1}e^{-t(k+a)}(k+a)^{-2}-\sum_{k=1}^{\infty}(-1)^{k-1}e^{-t(k+a)}(k+a)^{-1}t=$$
$$-a^{-2}e^{-at}\left(\sum_{k=1}^{\infty}(-1)^{k}e^{-tk}a^{2}(k+a)^{-2}-at\sum_{k=1}^{\infty}(-1)^{k}e^{-kt}(k+a)^{-1}a\right)=$$
$$=\left(-a^{-2}e^{-at}\right)\cdot$$
$$\cdot\left[{}_3F_2\left(1,a,a;1+a,1+a;-e^{-t}\right)-at{}_2F_1(1,a;1+a;-e^{-t})-(1-at)\right].$$

□

3.1. Drazin-Koliha hyperharmonic series: The sum $\sum_{k=1}^{\infty}\left((a+k)^{d}\right)^{n}$

In this subsection we study the Banach-algebra-valued hyperharmonic series which is defined with respect to a fixed regularity (generalized invertibility), rather than the actual invertibility. Precisely, we will weaken the premise that $\sigma(a)\subset\mathbb{H}^{r}_{-1}$ to a certain extent.

Let $a\in\mathcal{A}$ be a nonzero element. If there exists a nonzero element $b\in\mathcal{A}$ such that

$$bab=b\quad ab=ba,\quad \sigma(a(1-ab))=\{0\},$$

then b is the Drazin-Koliha inverse of a (or the generalized Drazin inverse of a), and is traditionally denoted as a^{d}. It is a known fact that a^{d} exists if and only if $0\notin\operatorname{acc}\sigma(a)$, see [12], [13], [14], and the references therein. In that case, a^{d} is uniquely determined by a. Specially, when a is a regular element in $\mathcal{A}$, then $a^{d}=a^{-1}$. Moreover, when a^{d} exists, then the element $1-aa^{d}$ defines the spectral idempotent for a corresponding to the set $\{0\}$. The idempotent $1-aa^{d}$ is denoted as a^{π}. Since a and a^{d} commute, it follows that a and a^{π} commute as well. Also, the element $(a+a^{\pi})$ is invertible since the spectral mapping theorem implies that $\sigma(a+a^{\pi})=\sigma(a)\setminus\{0\}$. Therefore,

$$a^{d}=(a+a^{\pi})^{-1}(1-a^{\pi}).\tag{25}$$

Intuitively, because of $aa^{d}=a^{d}a$, we write $a^{nd}=\left(a^{d}\right)^{n}$, for every $n\in\mathbb{N}$. Then

$$a^{nd}a^{n}=(a^{d}a)^{n}=(1-a^{\pi})^{n}=(1-a^{\pi}).$$

The following corollary (Corollary 3.3 below) follows immediately. The proof is omitted since it is analogous to the proof of Theorem 3.2:

Corollary 3.3. *Let $a\in\mathcal{A}$ be such that $\sigma(a)\subset\mathbb{H}^{r}_{-1}$, and assume that $0\notin\operatorname{acc}\sigma(a)$. Then, for every $t>0$ the following equalities are true:*

(a)

$$\sum_{k=1}^{\infty}(a+k)^{-2}(1-a^{\pi})=\int_{0}^{t}(e^{s}-1)^{-1}se^{-sa}(1-a^{\pi})ds+$$
$$+a^{2d}e^{-at}\left[{}_3F_2\left(1,a,a;1+a,1+a;e^{-t}\right)+ta\,{}_2F_1\left(1,a;1+a;e^{-t}\right)-(1+at)\right].\tag{26}$$

(b)

$$\sum_{k=1}^{\infty}(-1)^k(k+a)^{-2}(1-a^{\pi}) = \int_0^t \frac{s}{(e^s+1)}e^{-sa}(1-a^{\pi})ds+ \\ +a^{2d}e^{-at}\left[{}_3F_2\left(1,a,a;1+a,1+a;-e^t\right) - ta\,{}_2F_1(1,a;1+a;-e^t) - (1-at)\right]. \tag{27}$$

More generally, let $k \in \mathbb{N}$ be arbitrary. Then, the spectral mapping theorem implies that the expression $(a+k)$ is Drazin-Koliha invertible in $\mathcal{A}$, if and only if $-k \notin \operatorname{acc}\sigma(a)$, or, equivalently, if and only if there exists a spectral idempotent $(a+k)^{\pi} \in \mathcal{A}$ for $(a+k)$, such that $(a+k)$ and $(a+k)^{\pi}$ commute, $(a+k)(1-(a+k)(a+k)^{\pi})$ is quasinilpotent while $(a+k+(a+k)^{\pi})$ is invertible in $\mathcal{A}$. In that case, the Drazin-Koliha inverse of $(a+k)$ is unique, and can be represented as

$$(a+k)^d = (a+k+(a+k)^{\pi})^{-1}(1-(a+k)^{\pi}). \tag{28}$$

Similarly, the spectral projector $(a+k)^{\pi}$ is uniquely determined, and is equal to $(a+k)^{\pi} = 1-(a+k)(a+k)^d$. Moreover, if the element $(a+k)$ is invertible, then $(a+k)^d = (a+k)^{-1}$, therefore we can without the loss of generality treat the hyperharmonic expressions $(a+k)^{-n}$ as the respective powers of $(a+k)^d$:

$$(a+k)^{-n} = \left((a+k)^d\right)^n, \quad -k \in \rho(a),\ n \in \mathbb{N}.$$

Before we proceed with Theorem 3.5 below, we recall one more result (see [13] and [14]):

Theorem 3.4. *Let $a \in \mathcal{A}$ be such that $0 \notin \operatorname{acc}\sigma(a)$. Then, in the punctured neighborhood of $\lambda_0 = 0$, the following is true:*

$$\lim_{\lambda\to 0}(\lambda + a^s)^{-l}(a^d)^p = (a^d)^{sl+p},$$

for any positive integers s, l, and p.

Theorem 3.5. *Let $a \in \mathcal{A}$. Assume there exists an $M \in \mathbb{N}$, such that*

$$\sigma(a) \subset \{-M, -M+1, \dots, -1\} \cup \sigma^r_{-1}(a), \tag{29}$$

where $\sigma^r_{-1}(a)$ is a compact subset of $\mathbb{H}^r_{-1}$. Let $n \in \mathbb{N}$ be such that $n > 1$, and let s, l, p be natural numbers for which $sl + p = n$. Then

$$\sum_{k=1}^{\infty}\left((a+k)^d\right)^n = \\ = \frac{1}{(n-1)!}\int_0^{\infty}(e^t-1)^{-1}t^{n-1}e^{-t(a+M)}dt + \lim_{\lambda\to 0}\sum_{k=1}^{M}(\lambda+(a+k)^s)^{-l}\left((a+k)^d\right)^p. \tag{30}$$

Proof. Since $(a+k)^d = (a+k)^{-1}$ whenever $-k \in \rho(a)$, we have

$$\sum_{k=1}^{\infty}\left((a+k)^d\right)^n = \sum_{k=1}^{M}\left((a+k)^d\right)^n + \sum_{k=1}^{\infty}((a+M+k))^{-n} = \\ = \sum_{k=1}^{M}\left((a+k)^d\right)^n + \frac{1}{(n-1)!}\int_0^{\infty}(e^t-1)^{-1}t^{n-1}e^{-t(a+M)}dt.$$

Applying Theorem 3.4 completes the proof. □

The formula (30) reduces the problem in the sense of reducing the powers of the summands $(a + k)^d$. However, the expressions $(\lambda + (a + k)^s)^{-l}$ themselves cannot be treated as members of a unilateral hyperharmonic series, because when $M > k > 1$, we have

$$\sigma(a + k + (a + k)^\pi) = \sigma(a + k) \setminus \{-k\} \subset \{-M, \dots, -k - 1, -k + 1, \dots, -1\} \cup \sigma^r_{-1}(a),$$

and the integral (13) will diverge for $j \in \{-M, \dots, -k-1\}$. In Subsection 3.2 below we overcome this problem, by considering the inverse along an idempotent, rather than the Drazin-Koliha inverse.

3.2. *Hyperharmonic series with respect to an idempotent: The sum* $\sum_{k=1}^{\infty} \left((a + k)^{-(1-\pi_k)}\right)^n$

The inverse along an idempotent, or, more generaly, along an element, was introduced by Mary in [26], see also [27] and [28]. It naturally generalizes the Drazin-Koliha inverse in the following manner.

Let $\mathcal{S}$ be a given semigroup, and let $\mathcal{S}^1$ be the monoid generated by $\mathcal{S}$. For $a, b \in \mathcal{S}$, Green had introduced the preorder $\leq$ in $\mathcal{S}$ as (see [19]):

$$a \leq b \Leftrightarrow (\exists x, y \in \mathcal{S}^1)\ a = xb = by.$$

Then, for $a, e \in \mathcal{S}$, a is said to be invertible along e in $\mathcal{S}$, if there exists $b \in \mathcal{S}$ such that $b \leq e$ and

$$e = eab = bae.$$

If such an element exists, then it is unique and is denoted as a^{-e} (see [26], [27], and [28]).

Returning to our problem, let $a \in \mathcal{A}$ be a given nonzero element in the Banach algebra $\mathcal{A}$. Assume there exists an $M \in \mathbb{N}$, such that

$$\sigma(a) \subset \{-M, -M + 1, \dots, -1\} \cup \sigma^r_{-1}(a), \tag{31}$$

where $\sigma^r_{-1}(a)$ is a compact subset of $\mathbb{H}^r_{-1}$. By construction, the expressions $a + k$ are Drazin-Koliha invertible for every $k \in \{1, \dots, M\}$ and there exist the spectral idempotents $(a + k)^\pi$ for $(a + k)$ which corresponds to 0 for each k. We define the expressions π_k as:

$$\pi_k := \sum_{j=k}^{M} (a + j)^\pi.$$

Then for every $m > k$, it follows that $\pi_m \pi_k = \pi_k \pi_m = \pi_m$, i.e., the idempotents $\{\pi_k\}_{k \in \{1,\dots,M\}}$ exhibit the standard partial order. Furthermore, due to the spectral mapping theorem, the elements $a + k + \pi_k$ are invertible for every k, and

$$\sigma_{\mathcal{A}}(a + k + \pi_k) = \sigma_{\mathcal{A}}(a + k) \setminus \{-M + k, \dots, -1, 0\} \subset \mathbb{H}^r_{-1} \setminus \{0\}.$$

Moreover, we have

$$(1 - \pi_k)(a + k)(a + k + \pi_k)^{-1}(1 - \pi_k) = (1 - \pi_k)$$

and

$$(a + k + \pi_k)^{-1}(1 - \pi_k)(a + k)(1 - \pi_k) = (1 - \pi_k),$$

therefore the expression $(a + k + \pi_k)^{-1}(1 - \pi_k)$ is the unique inverse for $(a + k)$ along $1 - \pi_k$, and we denote it as

$$(a + k)^{-(1-\pi_k)} := (a + k + \pi_k)^{-1}(1 - \pi_k). \tag{32}$$

Now, Theorem 3.1 and Theorem 3.5 can be generalized in the following sense:

Theorem 3.6. *Let $a \in \mathcal{A}$. Assume there exists an $M \in \mathbb{N}$, such that* (31) *holds. For every $n \in \mathbb{N} \setminus \{1\}$, the series $\sum_{k=1}^{\infty} \left((a+k)^{-(1-\pi_k)}\right)^n$ converges, and*

$$\begin{aligned}&\sum_{k=1}^{\infty} \left((a+k)^{-(1-\pi_k)}\right)^n = \\ &= \frac{1}{(n-1)!} \int_0^{\infty} t^{n-1} \left[(e^t - 1)^{-1} e^{-t(a+M)} + \sum_{k=1}^{M} e^{-(a+k+\pi_k)t} (1-\pi_k) \right] dt.\end{aligned} \tag{33}$$

Proof. Since $(a+k)^{-(1-\pi_k)} = (a+k)^{-1}$ whenever $-k \in \rho(a)$, we have

$$\sum_{k=1}^{\infty} \left((a+k)^{-(1-\pi_k)}\right)^n = \sum_{k=1}^{\infty} ((a+M+k))^{-n} + \sum_{k=1}^{M} \left((a+k)^{-(1-\pi_k)}\right)^n =$$
$$\frac{1}{(n-1)!} \left(\int_0^{\infty} (e^t-1)^{-1} t^{n-1} e^{-t(a+M)} dt + \sum_{k=1}^{M} \int_0^{\infty} t^{n-1} e^{-(a+k+\pi_k)t} (1-\pi_k) dt \right).$$

Combining the latter two summands gives (33). □

4. Hyperharmonic series in Banach bimodules

For given Banach algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a Banach space $\mathfrak{M}$ is a Banach $(\mathcal{A}_1, \mathcal{A}_2)$−bimodule, if it is simultaneously a left Banach module for $\mathcal{A}_1$, and a right Banach module for $\mathcal{A}_2$. It is inherently understood that the sub-multiplicativity is preserved within such structures, that is,

$$\|a_1 \cdot_1 c \cdot_2 a_2\|_{\mathfrak{M}} \le \|a_1\|_{\mathcal{A}_1} \cdot \|c\|_{\mathfrak{M}} \cdot \|a_2\|_{\mathcal{A}_2}.$$

The reason why we are interested in such structures is because they envelop several very important classes of algebraic structures associated with operator algebras (see e. g. [1], [5], [16], [32]):

- Every Banach algebra $\mathcal{A}$ defines a Banach $(\mathcal{A}, \mathcal{A})$−bimodule over itself. In that case, $\mathcal{A}_1 = \mathcal{A}_2 = \mathfrak{M} = \mathcal{A}$.
- Let $\mathcal{A}$ be a Banach algebra and let $\mathfrak{M}$ be a left (right) Banach $\mathcal{A}$−module. Then $\mathfrak{M}$ is a Banach $(\mathcal{A}, \mathbb{C})$−bimodule
(respectively, $\mathfrak{M}$ is a Banach $(\mathbb{C}, \mathcal{A})$−bimodule).
- Let $\mathcal{A}$ be a Banach algebra and let $\mathfrak{M}$ be a Banach $\mathcal{A}$−bimodule. Then $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ and $\mathfrak{M}$ is a Banach $(\mathcal{A}, \mathcal{A})$−bimodule.
- Let V_1 and V_2 be complex Banach spaces, and let $\mathcal{B}_i$ be the Banach algebras of bounded linear operators in V_i, $\mathcal{B}_i = \mathbb{B}(V_i)$, $i \in \{1, 2\}$. Let $\mathcal{A}_i$ be any closed Banach subalgebra of $\mathcal{B}_i$, such that it contains the unit $1_{\mathcal{B}_i} \equiv I_{V_i}$. Moreover, let $\mathbb{B}(V_2, V_1)$ be the (Banach) space of bounded linear operators from V_2 to V_1, and let $\mathfrak{M}$ be any closed Banach subspace of $\mathbb{B}(V_2, V_1)$, which is invariant under the actions from $\mathcal{A}_i$. Then $\mathfrak{M}$ is a Banach $(\mathcal{A}_1, \mathcal{A}_2)$−bimodule.

In this bimodule setting we lose the resemblance with the scalar case. However, some new applications emerge, and a connection with Sylvester operator equations is established.

4.1. Bilateral hyperharmonic series

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital complex Banach algebras, and let $\mathfrak{M}$ be a Banach $(\mathcal{A}_1, \mathcal{A}_2)$−bimodule. For $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$, and $c \in \mathfrak{M}$, such that $\rho_{\mathcal{A}_1}(a) \cap \rho_{\mathcal{A}_2}(b) \supset -\mathbb{N}$, the expression

$$F^c(k; a, b) := (a + k)^{-1} c(b + k)^{-1} \tag{34}$$

defines a bilateral hyperharmonic sequence in $\mathfrak{M}$. The term "hyperharmonic" refers to the decay rate of the sequence $F^c(k; a, b)$, which is at least k^{-2}: it is bounded in norm by the scalar hyperharmonic sequence

$$\|F^c(k; a, b)\| \le \|c\| \max\{\|(a + k)^{-1}\|^2, \|(b + k)^{-1}\|^2\}.$$

More generally, for arbitrary n_1, $n_2 \in \mathbb{N}$, the expression

$$F^c_{(n_1,n_2)}(k; a, b) := (a + k)^{-n_1} c(b + k)^{-n_2} \tag{35}$$

defines an arbitrary-order bilateral hyperharmonic sequence in $\mathfrak{M}$. Within this section, we obtain the summation formulae for the seires of the form

$$\sum_{k=1}^{\infty} F^c(k; a, b), \quad \sum_{k=1}^{\infty} F^c_{(n_1,n_2)}(k; a, b),$$

where $F^c(k; a, b)$ and $F^c_{(n_1,n_2)}(k; a, b)$ are provided by (34) and (35), respectively.

Recall that one defines the convolution for the scalar functions ϕ and ψ as

$$(\phi * \psi)(s) := \int_0^s \phi(t)\psi(s - t)dt.$$

The convolution is a commutative operation in the space of integrable functions with numerous convenient properties, out of which one in particular is suitable for the problem studied in this paper: if ϕ and ψ are Laplace transformable over $\mathbb{R}^+_0$, then $\phi * \psi$ is Laplace transformable over $\mathbb{R}^+_0$ as well and

$$\mathcal{L}(\phi * \psi)(s) = \mathcal{L}\phi(s) \cdot \mathcal{L}\psi(s), \quad s \ge 0.$$

Motivated by this property, we prove the following result.

Theorem 4.1. *Let $\mathcal{A}_{comm}$ be a unital commutative Banach algebra, a, b and $c \in \mathcal{A}$, such that $\sigma(a) \cup \sigma(b) \subset \mathbb{H}^r_{-1}$, and $(k, n_1, n_2) \in \mathbb{N}^3$. Then:*

(a) The unilateral (hyper)harmonic sequence

$$f^c_{n_1}(k; a) := c(a + k)^{-n_1} \tag{36}$$

has the inverse Laplace transform $g_{n_1}(t)$ given as

$$g^c_{n_1}(t; a) = \frac{1}{(n_1 - 1)!} t^{n_1 - 1} c e^{-at}, \quad t \ge 0. \tag{37}$$

(b) The arbitrary-order bilateral hyperharmonic sequence

$$F^c_{(n_1,n_2)}(k; a, b) = (a + k)^{-n_1} c(b + k)^{-n_2} \tag{38}$$

has the inverse Laplace transform $G^c_{(n_1,n_2)}(s; a, b)$ given as

$$G^c_{(n_1,n_2)}(s; a, b) = \frac{1}{(n_1 - 1)!(n_2 - 1)!} \int_0^s t^{n_1}(s - t)^{n_2} e^{-at} c e^{-b(s-t)} dt, \quad s \ge 0. \tag{39}$$

Proof. For the given Bochner-integrable $\mathcal{A}_{comm}-$valued functions h_1 and h_2, their convolution $h_1 * h_2$ is defined as:

$$(h_1 * h_2)(s) = \int_0^s h_1(s-t)h_2(t)dt, \quad s \geq 0.$$

It is easy to conclude (see e.g. [22]) that the convolution is well-defined this way. Obviously, $h_1 * h_2 = h_2 * h_1$ in such algebra. Furthermore, we have

$$\mathcal{L}(h_1 * h_2)(p) = \int_0^\infty e^{-ps} \left[\int_0^s h_1(s-t)h_2(t)dt \right] ds =$$

$$= \int_0^\infty e^{-ps} \left[\int_0^\infty h_1(s-t)h_2(t)\chi_{[0,\infty)}(s-t)dt \right] ds.$$

By Fubini theorem (see [4], [29], or [33]), the last expression is equal to

$$\int_0^\infty h_2(t) \left[\int_t^\infty e^{-ps} h_1(s-t)ds \right] dt,$$

which, after the regular substitution $s \mapsto s + t$, gives

$$\mathcal{L}(h_1 * h_2)(p) = \int_0^\infty e^{-pt} h_2(t)dt \int_0^\infty e^{-ps} h_1(s)ds = \mathcal{L}(h_1)(p) \cdot \mathcal{L}(h_2)(p).$$

(a) Proof is straightforward by utilizing (13).

(b) Analogously, by (13) and the convolution property discussed above, we have

$$\begin{aligned}
&\mathcal{L}^{-1}\left((a+k)^{-n_1} c (b+k)^{-n_2}\right)(s) = \\
&= \mathcal{L}^{-1}\left\{ \mathcal{L}\left(\frac{1}{(n_1-1)!} t^{n_1-1} e^{-at} c \right)(k) \cdot \mathcal{L}\left(\frac{1}{(n_2-1)!} t^{n_2-1} c e^{-bt} \right)(k) \right\}(s) = \\
&= \left(\frac{1}{(n_1-1)!} t^{n_1-1} e^{-at} c \right) * \left(\frac{1}{(n_2-1)!} t^{n_2-1} c e^{-bt} \right)(s) = \\
&= \frac{1}{(n_1-1)!(n_2-1)!} \int_0^s t^{n_1-1}(s-t)^{n_2-1} e^{-at} c e^{-b(s-t)} dt.
\end{aligned}$$

□

Theorem 4.2. *Let $(a, c, b) \in \mathcal{A}_1 \times \mathfrak{M} \times \mathcal{A}_2$ be such that $\sigma(a) \cup \sigma(b) \subset \mathbb{H}^r_{-1}$. Then for every $(k, n_1, n_2) \in \mathbb{N}^3$, the arbitrary-order bilateral hyperharmonic sequence*

$$F^c_{(n_1,n_2)}(k; a, b) = (a+k)^{-n_1} c (b+k)^{-n_2}$$

has the inverse Laplace transform given as

$$G^c_{(n_1,n_2)}(s; a, b) = \frac{1}{(n_1-1)!(n_2-1)!} \int_0^s t^{n_1-1}(s-t)^{n_2-1} e^{-at} c e^{-b(s-t)} dt, \quad s \geq 0. \tag{40}$$

Consequently, the following formula holds:

$$\begin{aligned}
&\sum_{k=1}^\infty (a+k)^{-n_1} c (b+k)^{-n_2} = \\
&= \frac{1}{(n_1-1)!(n_2-1)!} \int_0^\infty (e^s - 1)^{-1} \int_0^s t^{n_1-1}(s-t)^{n_2-1} e^{-at} c e^{-b(s-t)} dt ds.
\end{aligned} \tag{41}$$

Proof. Observe the Banach algebra $\mathbb{B}(\mathfrak{M})$, and its elements $A, B, C \in \mathbb{B}(\mathfrak{M})$, defined as

$$Ax := ax, \quad Bx := xb, \quad Cx := I_{\mathfrak{M}}x = x, \quad x \in \mathfrak{M}.$$

Then A, B, and C commute in $\mathbb{B}(\mathfrak{M})$. Denote by $\mathcal{A}$ the largest commutative algebra in $\mathbb{B}(\mathfrak{M})$, generated by A and B. Clearly $C \in \mathcal{A}$. Then we have

$$\sigma_{\mathcal{A}}(A) = \sigma_{\mathbb{B}(\mathfrak{M})}(A) \subset \sigma_{\mathcal{A}_1}(a) \subset \mathbb{H}^r_{-1}$$

and

$$\sigma_{\mathcal{A}}(B) = \sigma_{\mathbb{B}(\mathfrak{M})}(B) \subset \sigma_{\mathcal{A}_2}(b) \subset \mathbb{H}^r_{-1}.$$

Let $k, n_1, n_2 \in \mathbb{N}$ be arbitrary. Then

$$(a+k)^{-n_1} c (b+k)^{-n_2} = \left((A+k)^{-n_1} C (B+k)^{-n_2}\right)(c),$$

therefore the inverse Laplace transform of the left-hand side exists, and is equal to the inverse Laplace transform of the right-hand side. By Theorem 4.1 we have

$$\mathcal{L}^{-1}\left(\left((A+k)^{-n_1} C (B+k)^{-n_2}\right)(c)\right)(t)c =$$
$$\left(\frac{1}{(n_1-1)!} t^{n_1-1} e^{-At} C\right) * \left(\frac{1}{(n_2-1)!} t^{n_2-1} e^{-Bt}\right)(s)(c) =$$
$$\frac{1}{(n_1-1)!(n_2-1)!} \int_0^s t^{n_1-1} (s-t)^{n_2-1} e^{-at} c e^{-b(s-t)} dt.$$

Specially, by Lemma 2.2 the formula (41) holds. □

4.2. Applications to algebraic Sylvester equations

We demonstrate how the obtained results can be used in the consistency analysis of Sylvester equations.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Banach algebras, and let $\mathfrak{M}$ be a Banach $(\mathcal{A}_1, \mathcal{A}_2)$–bimodule. For given $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$, and $c \in \mathfrak{M}$, the algebraic equation of the form

$$ax - xb = c \tag{42}$$

is called the algebraic $\mathfrak{M}$-Sylvester equation, with the unknown $x \in \mathfrak{M}$. If (42) is solvable, then the ordered triplet (a, b, c) is said to be Sylvester-consistent, while it is inconsistent otherwise. Some sufficient conditions for (42) to be solvable in operators and matrices have been obtained in [8], [9]–[11]; see also [6], [10] and the references therein.

Below it is shown how our results can serve as a necessary solvability condition for the equation (42).

Assume that (a, b, c) is Sylvester-consistent. Then all solutions x satisfy the following identity obtained by Hu and Cheng in [23]:

$$a^n x - x b^n = \sum_{j=0}^{n-1} a^{n-1-j} c b^j, \quad \forall n \in \mathbb{N}. \tag{43}$$

The above identity was proved in the matrix setting, however, the proof itself is purely algebraical and trivially holds in the Banach bimodule $\mathfrak{M}$ as well. Respectively, we denote as in [23]:

$$\eta(n, a, c, b) := \sum_{j=0}^{n} a^{n-j} c b^j, \quad (n, a, c, b) \in \mathbb{N} \times \mathcal{A}_1 \times \mathfrak{M} \times \mathcal{A}_2. \tag{44}$$

Theorem 4.3. *Let $a \in \mathcal{A}_1$, $c \in \mathfrak{M}$, and $b \in \mathcal{A}_2$, be such that $\sigma(a) \cup \sigma(b) \subset \mathbb{H}^r_{-1}$. Assume that the triplet (a, b, c) is Sylvester-consistent in $\mathfrak{M}$. Then:*

(a) *The sequence*

$$F^c(k;a,b) := (a+k)^{-1}c(b+k)^{-1}, \quad k \in \mathbb{N}$$

has the inverse Laplace transform which can be expressed as

$$G^c_\eta(t;a,b) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{n!}\eta(n-1,a,c,b), \quad t \geq 0, \tag{45}$$

where η *is given via* (44).

(b) *Moreover, for every* $r \geq 0$ *the following equality is true:*

$$\int_0^r \int_0^s e^{-at}ce^{-b(s-t)}dtds = \int_0^r \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{n!}\eta(n-1,a,c,b)\right)dt. \tag{46}$$

Proof. Assume that (a,b,c) is Sylvester-consistent. Then there exists at least one $x \in \mathfrak{M}$ such that $ax - xb = c$, i.e., $(a+k)x - x(b+k) = c$, for every $k \in \mathbb{N}_0$. In that sense,

$$x(b+k)^{-1} - x(a+k)^{-1} = (a+k)^{-1}c(b+k)^{-1} = F^c(k;a,b), \quad k \in \mathbb{N}.$$

Combined with (11), it follows that

$$\begin{aligned} F^c(k;a,b) &= \int_0^\infty e^{-kt}\left(xe^{-bt} - e^{-at}x\right)dt = \\ &= \int_0^\infty e^{-kt}\left(-\sum_{n=1}^{\infty} \frac{(-t)^n}{n!}\left(a^n x - xb^n\right)\right)dt = \\ &= \int_0^\infty e^{-kt}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}t^n}{n!}\eta(n-1,a,c,b)\right)dt. \end{aligned} \tag{47}$$

Comparing (45) with (40), by the virtue of (5), it is obvious that the equality (46) holds. □

4.3. *Extension to multilateral hyperharmonic series*

Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be unital complex Banach algebras, $m \in \mathbb{N}$, $m > 1$, and let $\mathfrak{M}_1, \dots, \mathfrak{M}_{m-1}$ be Banach spaces. We say that the chain $(\mathfrak{M}_1, \dots, \mathfrak{M}_{m-1})$ is compatible with the chain $(\mathcal{A}_1, \dots \mathcal{A}_m)$ if for every $j \in \{1, \dots, m-1\}$:

- The space $\mathfrak{M}_j$ is a Banach $(\mathcal{A}_j, \mathcal{A}_{j+1})$–bimodule.
- The tensor product $\mathfrak{M}_j \otimes \mathfrak{M}_{j+1}$ of any two consecutive modules $\mathfrak{M}_j$ and $\mathfrak{M}_{j+1}$, defined pointwise as

  $$c_j \otimes c_{j+1} := c_j \cdot 1_{\mathcal{A}_{j+1}} \cdot c_{j+1} \in \mathfrak{M}_j \otimes \mathfrak{M}_{j+1}, \quad \forall c_j \in \mathfrak{M}_j,$$

  is:

  – Consistent with the associative law defined for the spaces $\mathfrak{M}_j$, $\mathcal{A}_{j+1}$, and $\mathfrak{M}_{j+1}$:

  $$c_j \otimes (a_{j+1} \cdot c_{j+1}) = (c_j \cdot a_{j+1}) \otimes c_{j+1} = c_j a_{j+1} c_{j+1},$$

  for every $c_j \in \mathfrak{M}_j$, $c_{j+1} \in \mathfrak{M}_{j+1}$, and $a_{j+1} \in \mathcal{A}_{j+1}$.

  – It (the tensor product) subjects to the standard sub-multiplicativity:

  $$\|c_j \otimes c_{j+1}\|_{\mathfrak{M}_j \otimes \mathfrak{M}_{j+1}} \leq \|c_j 1_{\mathcal{A}_{j+1}}\|_{\mathfrak{M}_j} \cdot \|1_{\mathcal{A}_{j+1}} c_{j+1}\|_{\mathfrak{M}_{j+1}} = \|c_j\|_{\mathfrak{M}_j} \cdot \|c_{j+1}\|_{\mathfrak{M}_{j+1}}.$$

- The previous two points can be directly extended to the arbitrary-length tensor product of the consecutive modules $\mathfrak{M}_j, \dots, \mathfrak{M}_{j+k}$, where $1 \leq j < j+k \leq m-1$.

As mentioned before, any Banach algebra $\mathcal{A}$ defines a Banach $(\mathcal{A},\mathcal{A})-$bimodule, therefore the results obtained in this subsection hold in the special case when

$$\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_m = \mathfrak{M}_1 = \ldots = \mathfrak{M}_{m-1} = \mathcal{A}.$$

Also note that the above compatibility is naturally obtained via operator compositions: if V_i are Banach spaces, $i \in \{1.\ldots,m\}$, let $\mathcal{B}_i$ be the Banach algebras $\mathcal{B}_i = \mathbb{B}(V_i)$, and let $\mathcal{A}_i$ be closed Banach subalgebras of $\mathcal{B}_i$, such that $I_{V_i} = 1_{\mathcal{B}_i} \in \mathcal{A}_i$. Furthermore, let $\mathfrak{M}_j \subset \mathcal{B}(V_{j+1}, V_j)$, for $j \in \{1,\ldots,m-1\}$, be closed Banach subspaces of $\mathcal{B}(V_{j+1}, V_j)$, provided in the manner that they are invariant under the actions of $\mathcal{A}_j$ and $\mathcal{A}_{j+1}$. Then for any $A_i \in \mathcal{A}_i$, and for any $C_j \in \mathfrak{M}_j$, $j \in \{1,\ldots,m-1\}$, $i \in \{1,\ldots,m\}$, the composition

$$A_1C_1A_2C_2\ldots A_{m-1}C_{m-1}A_m$$

exists, and belongs to the space

$$\mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \ldots \otimes \mathfrak{M}_{m-1}.$$

Before we proceed with Theorem 4.4 below, we recall the following notation: for any $j \in \{1,\ldots,m-1\}$, let $a \in \mathcal{A}_j$ be such that $\sigma(a) \subset \mathbb{H}^r_{-1}$, and let $c \in \mathfrak{M}_j$ be arbitrary. For $n \in \mathbb{N}$, we denote by

$$g^c_n(t;a) := \frac{1}{(n-1)!}t^{n-1}e^{-at}c \in \mathfrak{M}_j, \quad t \geq 0. \tag{48}$$

This expression resembles the ones from (15) and (37), however, as opposed to the latter, the order of factors in (48) is important because $\mathfrak{M}_j$ is a left Banach $\mathcal{A}_j-$module. Finally, when $a_m \in \mathcal{A}_m$ is given with the same condition that $\sigma(a_m) \subset \mathbb{H}^r_{-1}$, we define as before

$$g_n(t;a_m) := \frac{1}{(n-1)!}t^{n-1}e^{-a_mt} \in \mathcal{A}_m, \quad t \geq 0. \tag{49}$$

Theorem 4.4. *For $m > 1$, let $(\mathfrak{M}_1,\ldots,\mathfrak{M}_{m-1})$ be compatible with $(\mathcal{A}_1,\ldots,\mathcal{A}_m)$. Assume the vector*

$$\mathbf{a} := (a_1,\ldots,a_m) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_m$$

to be chosen in the manner that

$$\sigma(\mathbf{a}) := \bigcup_{j=1}^{m} \sigma_{\mathcal{A}_j}(a_j) \subset \mathbb{H}^r_{-1}.$$

Then for every $k \in \mathbb{N}$, any

$$\mathbf{n} = (n_1,\ldots,n_m) \in \mathbb{N}^m,$$

and any

$$\mathbf{c} = (c_1,\ldots,c_{m-1}) \in \mathfrak{M}_1 \times \ldots \times \mathfrak{M}_{m-1},$$

the expressions

$$F^{\mathbf{c}}_{\mathbf{n}}(k;\mathbf{a}) := (a_1+k)^{-n_1}c_1(a_2+k)^{-n_2}c_2 \cdot \ldots (a_{m-1}+k)^{-n_{m-1}}c_{m-1}(a_m+k)^{-n_m} \tag{50}$$

define a multilateral hyperharmonic sequence in $\mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \ldots \otimes \mathfrak{M}_{m-1}$. The function $k \mapsto F^{\mathbf{c}}_{\mathbf{n}}(k;\mathbf{a})$ has the inverse Laplace transform $G^{\mathbf{c}}_{\mathbf{n}}(s;\mathbf{a})$ for $s \geq 0$, and the following representation is true:

$$G^{\mathbf{c}}_{\mathbf{n}}(s;\mathbf{a}) = \\ = \int_0^s \left(\int_0^{s_{m-1}} \ldots \int_0^{s_3} \left(\int_0^{s_2} \Psi^{\mathbf{c}}_{(m,\mathbf{n})}(s_1,\ldots,s_{m-1},s;\mathbf{a}) ds_1 \right) ds_2 \ldots ds_{m-2} \right) ds_{m-1}, \tag{51}$$

where $0 \leq s_1 \leq s_2 \leq \ldots \leq s_{m-1} \leq s$, and $\Psi^{\mathbf{c}}_{(m,\mathbf{n})}(s_1,\ldots,s_{m-1},s;\mathbf{a})$ is the ordered tensor product defined in terms of (48)–(49) *as*

$$\Psi^{\mathbf{c}}_{(m,\mathbf{n})}(s_1,\ldots,s_{m-1},s;\mathbf{a}) := g^{c_1}_{n_1}(s_1;a_1) \bigotimes_{j=2}^{m-1} g^{c_j}_{n_j}(s_j - s_{j-1};a_j) \otimes g_{n_m}(s - s_{m-1};a_m). \tag{52}$$

Consequently, the following representation holds

$$\sum_{k=1}^{\infty} F_{\mathbf{n}}^{\mathbf{c}}(k;\mathbf{a}) = \int_0^{\infty} s^{-1}\varepsilon(s)G_{\mathbf{n}}^{\mathbf{c}}(s;\mathbf{a})ds. \tag{53}$$

Proof. The proof is completely analogous to the one of Theorem 4.2. We prove the statement for $m=3$, and point out the obvious generalization.

Observe the Banach algebra $\mathbb{B}(\mathfrak{M}_1 \otimes \mathfrak{M}_2)$, and its elements A_1, A_2, and $A_3 \in \mathbb{B}(\mathfrak{M}_1 \otimes \mathfrak{M}_2)$, defined as

$$A_1(x_1 \otimes x_2) := a_1x_1x_2, \quad A_2(x_1 \otimes x_2) := x_1a_2x_2, \quad A_3(x_1 \otimes x_2) := x_1x_2a_3,$$

for $x_1x_2 \in \mathfrak{M}_1 \otimes \mathfrak{M}_2$. Then A_1, A_2, and A_3 commute in $\mathbb{B}(\mathfrak{M}_1 \otimes \mathfrak{M}_2)$. Denote by $\mathcal{B}_{(123)}$ the largest commutative algebra in $\mathbb{B}(\mathfrak{M}_1 \otimes \mathfrak{M}_2)$, generated by A_1, A_2, and A_3. Then we have

$$\sigma_{\mathcal{B}_{(123)}}(A_j) = \sigma_{\mathbb{B}(\mathfrak{M}_1\otimes\mathfrak{M}_2)}(A_j) \subset \sigma_{\mathcal{A}_j}(a_j) \subset \mathbb{H}_{-1}^r, \quad j \in \{1,2,3\}.$$

Let $k, n_1, n_2, n_3 \in \mathbb{N}$ be arbitrary. Then

$$\begin{aligned}&(a_1+k)^{-n_1}c_1(a_2+k)^{-n_2}c_2(a_3+k)^{-n_3} = \\ &= \left((A_1+k)^{-n_1}(A_2+k)^{-n_2}(A_3+k)^{-n_3}\right)c_1c_2.\end{aligned}$$

On the other hand, by Theorem 4.1 and Theorem 4.2 we have

$$\begin{aligned}&\left(\prod_{j=1}^{3}(n_j-1)!\right)\cdot(A_1+k)^{-n_1}(A_2+k)^{-n_2}(A_3+k)^{-n_3} = \\ &\mathcal{L}\left(\int_0^{s_2} s_1^{n_1-1}(s_2-s_1)^{n_2-1}e^{-A_1s_1}e^{-A_2(s_2-s_1)}ds_1\right)(k)\cdot\mathcal{L}\left(t^{n_3-1}e^{-A_3t}\right)(k) = \\ &\mathcal{L}\left\{\left[\left(\int_0^{s_2} s_1^{n_1-1}(s_2-s_1)^{n_2-1}e^{-A_1s_1}e^{-A_2(s_2-s_1)}ds_1\right)*\left(t^{n_3-1}e^{-A_3t}\right)\right](s)\right\}(k),\end{aligned}$$

therefore

$$\begin{aligned}&\left(\prod_{j=1}^{3}(n_j-1)!\right)\mathcal{L}^{-1}\left((A_1+k)^{-n_1}(A_2+k)^{-n_2}(A_3+k)^{-n_3}\right)(s)(c_1c_2) = \\ &\int_0^{s}\int_0^{s_2} s_1^{n_1-1}(s_2-s_1)^{n_2-1}(s_3-s_2)^{n_3-1}e^{-a_1s_1}c_1e^{-a_2(s_2-s_1)}c_2e^{-a_3(s-s_2)}ds_1ds_2.\end{aligned}$$

□

Declarations

Conflict of interest. The authors declare that there is no conflict of interest in publishing the findings obtained in this article.

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