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Topological and graph analysis of L-fuzzy proximity spaces via rough sets

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Abstract. Čech L-fuzzy rough proximity spaces build upon proximities under the wing of the L-fuzzy notion and extend it using the Čech completion process. The Čech completion is a technique commonly used in topology to turn a given topological space into a complete space. This approach provides a more refined understanding of closeness and connectivity in spaces with fuzzy or uncertain relationships between points, contributing to the development of fuzzy topology and its applications in various fields. The primary objective of this study is to explore the junction between the categories of Čech proximity (and closure) spaces, specifically in relation to L-fuzzy rough sets, where L is a complete distributive lattice. Additionally, we will discuss their properties. Further, we introduce L-fuzzy rough ideal creation and the mutual relation between ideals and proximity spaces according to the L-fuzzy rough notion. Finally, we apply the results in a model of a fuzzy topological graph, yielding valid observations.

Nomenclature

Symbols

R	L-fuzzy relation.
L-FAPS	L-fuzzy approximation spaces.
CL-FRPRX	Čech L-fuzzy rough proximity spaces.
ACL-FRPRX	Alexandrov CL-FRPRX.
CL-FRCS	Čech L-fuzzy rough closure spaces.
ACL-FRCS	Alexandrov CL-FRCS.
L-FRIS	L-fuzzy rough ideal space.
AL-FRIS	Alexandrov$L$-FRIS.

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1. Introduction

Relations are fundamental concepts for expressing preferences, but the two-valued concept is not useful for expressing the complexity of real-life preferences. Pawlak [31, 32] introduced rough set theory, which is an excellent and helpful tool for processing uncertainty and incomplete information. Axiomatic and constructive approaches are continuously driving the development of theoretical research and practical applications in rough set theory. Researchers in literature [26, 51] generalized the commitment of rough sets, recommending other forms of arbitrary relation instead of the equivalence one. To overcome this limitation, fuzzy relations are generally used. Dubois and Prade [12] used fuzzy relations to bring up a fuzzy version of rough sets rather than crisp ones.

Recently, merging fuzzy sets with rough sets was served by fuzzy logic with binary fuzzy relation in [13, 30, 33], where fuzzy implications [29, 34, 35, 47, 49] make a major change in the extensions of fuzzy rough sets, at which L-fuzzy topological structures are an extremely important part of it [14, 15, 22, 44, 48, 53].

Respectively, Zhou [22, 52] discussed the most important features of closure spaces in the L-fuzzy notion (see [5]), while Bělohlávek and Höhle widely investigated the category aspects. Using closure spaces to induce new topologies according to the L-fuzzy concept attracted the attention of many researchers (Fang [13–15], Pang [30]). There were several other contributions by many authors ([40, 41, 50, 52, 54]) that discussed some properties.

Proximity structure is another topological construct that has found various applications in pattern recognition, feature selection, digital image classification, data analysis, cluster analysis, multidimensional scaling, concept analysis, computational biology, and many other fields [21]. Fuzzy proximity structures in a completely distributive lattice were introduced by Katsaras [23, 24]. Bayoumi [3] extends L-fuzzy proximity structures, Katsaras's definition, in a slightly different sense than Čimoka and Šostak [11]. Kim [25] and Ramadan [36–38] gave valuable efforts in this area. In other words, the nearness in L-fuzzy between the topological structures that are respected by two sets can also help clarify the nearness between the sets. Čech closure spaces [10] and proximity spaces [42, 43, 45] are closely related and come from the same field. They are both topological spaces that have been extended. The Čech closure operator [10] can be induced by every basic proximity structure. The theory of filters is also related to proximity spaces [39].

Čech closure spaces and Čech proximity spaces are closely related and come from the same field. They are both topological spaces that have been extended. A Čech closure operator can be induced by every basic proximity structure. The relationship of Čech rough proximity spaces and Čech closure spaces was studied in [27, 45]. We further find the relationship between Čech L-fuzzy rough proximity, closure, and ideals.

Graph theory, such an important mathematical tool discussed by Chartrand et al. [9], has several applications in many fields, including civil engineering, networking problems, mechanism analysis, electric engineering, graphics, medical, genetics, etc. Because of its involvement in the same fields of work as topology, many researchers are inclined to mix them in various applications. Nada et al. [28] studied the concept of topological structures via a graph based on neighborhoods. Recently, Atef et al. [2] initially introduced fuzzy topological structures via fuzzy graphs, which included very useful applications in real-life health problems.

This research will be divided into five sections as follows: the basic definitions and important results will be stated in Section 2. Section 3 used to introduce the Čech L-fuzzy rough proximity and studies its relation with a Čech fuzzy rough closure space. In Section 4, we define rough ideals under the notion of L-fuzzy with a discussion of their connection toward Čech fuzzy rough proximity spaces. In Section 5, we apply the results to a 3-vertex fuzzy topological graph. Section 6 is a conclusion and future work.

2. Preliminaries

In our quest, L indicates $(L, \leq, \wedge, \vee)$ as a complete lattice, where each subfamily $A \subseteq L$ contains its own joins (suprema) and meets (infima). In particular, $\top \neq \bot$ where $\top$ is a top and $\bot$ is a bottom elements in L. In lattice, we use $\bigvee$ and $\bigwedge$ for the case of finite arbitrary families of elements. If L satisfied the first infinite distributive law, it would be completely distributive [8]. In other words, $a \wedge (\bigvee_{\iota\in\gamma} b_\iota) = \bigvee_{\iota\in\gamma}(a \wedge b_\iota)$, $\forall\, a \in L$, $\{b_\iota\}_{\iota\in\gamma} \subseteq L$. The implication binary operation can also induced by $\wedge$ as: $a \wedge b \leq c \Leftrightarrow a \leq b \to c\ \forall\, a, b, c \in L$. Here, suppose * will be define by $a \vee b = (a^* \wedge b^*)^*$, $a^* = a \to \bot$ as an order reversing involution whom made $(L, \leq, \wedge, ^*)$ as complete lattice.

The proposition below is considered as a collection of some basic characteristics of $\wedge$ and the implication $\to$ that may be found in [4, 17, 18, 20, 22, 46].

Proposition 2.1. *For $a, b, c, a_\iota, b_\iota, w \in L$ and $\iota \in \gamma$, we have the following properties:*

(1) $a \to b = \bigvee\{c \in L | a \wedge c \leq b\}$,

(2) $\top \to a = a, \bot \wedge a = \bot$,

(3) *If* $b \leq c$, *then* $a \to b \leq a \to c$ *and* $c \to a \leq b \to a$,

(4) $a \leq b \Leftrightarrow a \to b = \top$,

(5) $(\bigwedge_\iota b_\iota)^* = \bigvee_\iota b_\iota^*$ *and* $(\bigvee_\iota b_\iota)^* = \bigwedge_\iota b_\iota^*$,

(6) $a \to (\bigvee_\iota b_\iota) \geq \bigvee_\iota(a \to b_\iota)$ *and* $(\bigwedge_\iota a_\iota) \to b \geq \bigvee_\iota(a_\iota \to b)$,

(7) $a \to (\bigwedge_\iota b_\iota) = \bigwedge_\iota(a \to b_\iota)$ *and* $(\bigvee_\iota a_\iota) \to b = \bigwedge_\iota(a_\iota \to b)$,

(8) $\bigvee_\iota a_\iota \to \bigvee_\iota b_\iota \geq \bigwedge_\iota(a_\iota \to b_\iota)$ *and* $\bigwedge_\iota a_\iota \to \bigwedge_\iota b_\iota \geq \bigwedge_\iota(a_\iota \to b_\iota)$.

(9) $a \wedge b = (a \to b^*)^*$, $a \vee b = a^* \to b$ *and* $a \to b = b^* \to a^*$,

(10) $(a \to b) \wedge (c \to w) \leq (a \wedge c) \to (b \wedge w)$,

(11) $a \to b \leq (a \wedge c) \to (b \wedge c)$ *and* $(a \to b) \wedge (b \to c) \leq a \to c$,

(12) $(a \wedge b) \wedge (c \vee w) \leq (a \wedge c) \vee (b \wedge w)$,

(13) $(a \to b) \wedge (c \to w) \leq (a \vee c) \to (b \vee w)$,

(14) $(a \to b) \vee (c \to w) \leq (a \wedge c) \to (b \vee w)$,

(15) $a \wedge (a \to b) \leq b$, $b \leq a \to (a \wedge b)$ *and* $(a \to b) \to b \geq a$,

(16) $a \wedge (b \to c) \leq b \to (a \wedge c)$ *and* $a \wedge (b \to c) \leq (a \to b) \to c$,

(17) $c \to a \leq (a \to b) \to (c \to b)$ *and* $b \to c \leq (a \to b) \to (a \to c)$,

(18) $(a \wedge b) \to c = a \to (b \to c) = b \to (a \to c)$ *and* $a \leq (a \to b) \to b$,

L^X is used to pointing to all L-fuzzy sets [19] which defined on a universal set X. In addition, $\top_X$ and $\bot_X$ are denoted by sets in a fuzzy L that given as $\top_X(a) = \top$ and $\bot_X(a) = \bot$, $\forall\, a \in X$, are called the universal greatest and smallest bound (upper and lower, respectively) in L^X.

We do not differentiate between $\alpha \in L$ as an element and $\alpha : X \to L$ as a constant function with $\alpha(a) = \alpha$, $\forall$ a element on X. Every algebraic operation on L^X is considered as the extension of that in L. Such for any $\top_a, f, g$ sets in fuzzy lattice, $a \in X$, $\alpha \in L$, for f enough to be less than or equal to g if $f(a) \leq g(a)$, moreover $(f \wedge g)(a) = f(a) \wedge g(a)$, and $(f \to g)(a) = f(a) \to g(a)$,

$$\top_a(b) = \begin{cases} \top, & \text{if } b = a, \\ \bot, & \text{o.w.}, \end{cases} \quad \top_a^*(b) = \begin{cases} \bot, & \text{if } b = a, \\ \top, & \text{o.w.} \end{cases}$$

In this paper, all the categories are concrete.

Definition 2.2. ([1]) Given $\mathcal{X}$ as a category, the functor $\mathcal{M}$ is faithful, where $\mathcal{M} : \mathcal{X} \to \mathbf{Set}$. Then, $(\mathcal{X}, \mathcal{M})$ is a concrete category, write $\mathcal{X}$ (for short) if it is clear. If a set A is considered as $\mathcal{X}$-object, the underlying set of A is $\mathcal{M}(A)$. Take $(\mathcal{X}, \mathcal{M})$ and $(\mathcal{Y}, \mathcal{N})$ as concrete categories. Thus, the functor between them is $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ with $\mathcal{M} = \mathcal{N} \circ \mathcal{G}$, which refers to the sets changes w. r. to $\mathcal{G}$. The necessary two conditions to define $\mathcal{G} : \mathcal{X} \to \mathcal{Y}$ as a concrete functor are stated as follows, firstly, we consider the set A for each $\mathcal{X}$-object and $\mathcal{G}(A)$ for $\mathcal{Y}$-object where $\mathcal{N}(\mathcal{G}(A)) = \mathcal{M}(A)$. Secondly, we make sure that if $\Omega : \mathcal{M}(A) \to \mathcal{M}(B)$ be a $\mathcal{X}$-morphism mapping $A \to B$, then also be $\mathcal{Y}$-morphism $\mathcal{G}(A) \to \mathcal{G}(B)$.

Definition 2.3. ([4]) If R defined on a nonempty set X, where $R(a,b)$ means the junction of a and b out of degree tenet, then $\forall\, a,b,c \in X$ the relation R is said to be

(i) reflexive if $R(a,a) = \top$,

(ii) symmetric if $R(b,a) = R(a,b)$,

(iii) transitive if $R(b,c) \wedge R(a,b) \leq R(a,c)$.

Moreover, it is an L-fuzzy preorder if it achieves (i), (iii), tolerance if it achieves (i), (ii), and equivalence if R satisfies all previous conditions.

Suppose X is a universe with R. Then, (X,R) be **L-FAPS**. Upper (lower) approximation that is recalled in the definition below, used by many researchers [33, 35, 36, 48, 49, 53].

Definition 2.4. Assume that $R : L^X \to L^X$ defined on X. A function $\underline{\mathfrak{L}}, \overline{\mathfrak{U}} : L^X \to L^X$ for $f,g \in L^X$,"a " element in X defined as:

$\underline{\mathfrak{L}}(f)(a) = \bigwedge_{b\in X} (R(a,b) \to f(b))$ and $\overline{\mathfrak{U}}(f)(a) = \bigvee_{b\in X} (f(b) \wedge R(a,b))$.

$\underline{\mathfrak{L}}$($\overline{\mathfrak{U}}$) are called lower (upper) **L-FAP** operators, respectively. Moreover $(\underline{\mathfrak{L}}(f), \overline{\mathfrak{U}}(f))$ consider as L-fuzzy rough of f out of (X,R).

The distinctive features of upper (lower) approximation in L-fuzzy notion, that collected below, are from [4, 12, 33, 35, 36, 48, 49, 53].

Proposition 2.5. *If (X,R) be **L-FAPS** and $\overline{\mathfrak{U}}(\underline{\mathfrak{L}})$, be textbfL-FAP(upper, lower. respectively) approximation operator on X. Then, $\forall\, f, f_\iota \in L^X$, $a \in X$ and $\alpha \in L$, we receive that :*

(1) being, R, reflexive lead to $\overline{\mathfrak{U}}(\top_a) = \top_a$, $\overline{\mathfrak{U}}(\bot_X) = \bot_X$, $\overline{\mathfrak{U}}(\top_X) = \top_X$,

(2) being, R, reflexive, then $\underline{\mathfrak{L}}(\bot_X) = \bot_X$, $\underline{\mathfrak{L}}(\top_X) = \top_X$ *and* $\underline{\mathfrak{L}}(f) \leq f \leq \overline{\mathfrak{U}}(f)$,

(3) $S_d(f,g) \leq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$ *and* $S_d(f,g) \leq S_d(\underline{\mathfrak{L}}(f), \underline{\mathfrak{L}}(g))$,

(4) $\overline{\mathfrak{U}}(f \vee g) = \overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g)$ *and* $\overline{\mathfrak{U}}(f \wedge g) \leq \overline{\mathfrak{U}}(f) \wedge \overline{\mathfrak{U}}(g)$,

(5) $\underline{\mathfrak{L}}(f \wedge g) = \underline{\mathfrak{L}}(f) \wedge \underline{\mathfrak{L}}(g)$ *and* $\underline{\mathfrak{L}}(f \vee g) \geq \underline{\mathfrak{L}}(f) \vee \underline{\mathfrak{L}}(g)$,

(6) $\underline{\mathfrak{L}}(\bigwedge_{\iota\in\gamma} f_\iota) = \bigwedge_{\iota\in\gamma} \underline{\mathfrak{L}}(f_\iota)$ *and* $\overline{\mathfrak{U}}(\bigvee_{\iota\in\gamma} f_\iota) = \bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(f_\iota)$,

(7) $\underline{\mathfrak{L}}(\alpha \to f) = \alpha \to \underline{\mathfrak{L}}(f)$ *and* $\overline{\mathfrak{U}}(\alpha \wedge f) = \alpha \wedge \overline{\mathfrak{U}}(f)$,

(8) $\overline{\mathfrak{U}}(\alpha \to f) \leq \alpha \to \overline{\mathfrak{U}}(f)$ *and* $\underline{\mathfrak{L}}(\alpha \wedge f) \geq \alpha \wedge \underline{\mathfrak{L}}(f)$,

(9) If R is transitive, then $\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)) \leq \overline{\mathfrak{U}}(f)$ *and* $\underline{\mathfrak{L}}(f) \leq \underline{\mathfrak{L}}(\underline{\mathfrak{L}}(f))$,

(10) If R is reflexive then, $\underline{\mathfrak{L}}(\underline{\mathfrak{L}}(f)) \leq \underline{\mathfrak{L}}(f) \leq f$ *and* $f \leq \overline{\mathfrak{U}}(f) \leq \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))$.

One may notice that:

(i) If $f \leq g$, *then* $\overline{\mathfrak{U}}(f) \leq \overline{\mathfrak{U}}(g))$ *and* $\underline{\mathfrak{L}}(f) \leq \underline{\mathfrak{L}}(g))$,

(ii) If R is reflexive and transitive, then $\underline{\mathfrak{L}}(\underline{\mathfrak{L}}(f)) = \underline{\mathfrak{L}}(f)$ *and* $\overline{\mathfrak{U}}(f) = \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))$,

(iii) $\overline{\mathfrak{U}}^*(f) = \underline{\mathfrak{L}}(f^*)$ *and* $\underline{\mathfrak{L}}^*(f) = \overline{\mathfrak{U}}(f^*)$.

For any L-FAPS (X, R_X) and (Y, R_Y). **L-FAP** map is the function $\eta : X \to Y$ with $\eta^{\leftarrow}(\underline{\mathfrak{L}}_Y(f)) \leq \underline{\mathfrak{L}}_X(\eta^{\leftarrow}(f))$ and $\eta^{\leftarrow}(\overline{\mathfrak{U}}_Y(f)) \geq \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f))$, $\forall\, f \in L^X$.

Definition 2.6. ([4, 53]) For L-fuzzy sets f,g, the maps $S_d, N_d : L^X \times L^X \to L$ are, respectively, defined by

(i) The subsethood degree of f,g and it define by $S_d(f,g) = \bigwedge_{a\in X} (f(a) \to g(a))$.

(ii) The degree of intersection of f, g and it define by $N_d(f, g) = \bigvee_{a \in X} f(a) \wedge g(a)$.

Subsethood (intersection) degree properties are collected, [4, 14, 30, 47, 49, 53], in Proposition 2.7..

Proposition 2.7. *For $f, g, h, k, f_\iota(\iota \in \gamma) \in L^X$ and $\alpha \in L$, we have*

(S1) $S_d(f, g) = \top \Leftrightarrow g \geq f$,

(S2) $S_d(f \wedge k, g \wedge h) \geq S_d(f, g) \wedge S_d(k, h)$, $S_d(f \vee k, g \vee h) \geq S_d(f, g) \wedge S_d(k, h)$,

(S3) $S_d(h, f) \to S_d(h, g) \geq S_d(f, g)$, $S_d(g, h) \to S_d(f, h) \geq S_d(f, g)$,

(S4) $S_d(f, g) \geq S_d(f, g) \wedge S_d(g, h)$, $S_d(f, g) \to g \geq f$ *and* $g \geq S_d(f, g) \wedge f$,

(S5) $S_d(f, \alpha \wedge g) \geq \alpha \wedge S_d(f, g)$ *and* $S_d(f, g) = S_d(g^*, f^*)$,

(N1) $N_d(\perp_X, \top_X) = N_d(\top_X, \perp_X)$,

(N2) $N_d(g, f) = N_d(f, g)$,

(N3) $f \leq g$ *implies* $N_d(h, f) \leq N_d(h, g)$, $N_d(f, h) \leq N_d(g, h)$,

(N4) $N_d(f, \bigvee_{\iota \in \gamma} f_\iota) = \bigvee_{\iota \in \gamma} N_d(f, f_\iota)$,

(N5) $\alpha \wedge N_d(f, k) = N_d(f, \alpha \wedge k)$,

(N6) $\alpha \to N_d(f, k) \geq N_d(f, \alpha \to k)$,

(N7) $N_d(f, k) \to \alpha = S_d(f, k \to \alpha)$,

(N8) *if* $\eta : X \to Y$ *be a function, then* $N_d(h, k) \geq N_d(\eta^{\leftarrow}(h), \eta^{\leftarrow}(k))$, *for* $h, k \in L^Y$.

Definition 2.8. ([6]) A fuzzy subset is defined as $\sigma : X \to [0, 1]$ on X, and the fuzzy relation on X is a fuzzy subset of $X \times X$. Then, for $f, g \in [0, 1]^X$, $f \vee g$ (join) is defined as $(f \vee g)(a) = max(f(a), g(a))$ for every a in X. Moreover $f \wedge g$ (meet) given by $(f \wedge g)(a) = min(f(a), g(a))$ per a in X.

Definition 2.9. ([7]) A fuzzy graph is a triplet $G = (X, \sigma, R)$, in which $X$is a universe, σ is a fuzzy subset of X with R on σ satisfying $R(a, b) \leq \sigma(a) \wedge \sigma(b)$, for all $a, b \in X$. The fuzzy set σ and R are called the fuzzy vertex and fuzzy edge set of G, respectively.

3. Čech *L*-fuzzy rough proximities and Čech *L*-fuzzy closure operators

Here, we axiomatize Čech *L*-fuzzy rough proximity spaces. Some basic results on Čech *L*-fuzzy rough proximity space are proved. Throughout this paper, (X, R) is **L-FAPS**, with equivalence R on X.

Definition 3.1. Let (X, R) is **L-FAPS**. **CL-FRPRX** is a function $\delta_R : L^X \times L^X \to L$ if $\forall f, g, h, k \in L^X$ satisfies:

(P1) $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(\perp_X)) = \delta_R(\overline{\mathfrak{U}}(\perp_X), \overline{\mathfrak{U}}(f)) = \perp$,

(P2) $\delta_R(f, k) \geq N_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(k))$,

(P3) $S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(k)) \leq \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \to \delta_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(g))$,
$S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(k)) \leq \delta_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f))) \to \delta_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(k))$,

(P4) $\delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g)) \leq \delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \vee \delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g))$ and
$\delta_R(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(h)) \leq \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(h)) \vee \delta_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(h))$.

Then the pair (X, δ_R) is **CL-FRPRX**. Furthermore, it is **ACL-FRPRX** if satisfies:

(AL) for each family $\{f_\iota, g_\iota : \iota \in \gamma\}$ subset or equal of L^X,
$\delta_R(\bigvee_{\iota \in \gamma} \overline{\mathfrak{U}}(f_\iota), \overline{\mathfrak{U}}(g)) \leq \bigvee_{\iota \in \gamma} \delta_R(\overline{\mathfrak{U}}(f_\iota), \overline{\mathfrak{U}}(g))$ and $\delta_R(\overline{\mathfrak{U}}(f), \bigvee_{\iota \in \gamma} \overline{\mathfrak{U}}(g_\iota)) \leq \bigvee_{\iota \in \gamma} \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g_\iota))$.

A binary relation δ_R is called a basic **CL-FRPRX**, if it a Čech proximity and additionally satisfies the following axioms:

(P) $\delta^s_R = \delta_R$, where $\delta^s_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) = \delta_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f))$.

A binary relation δ_R is **CL-FRPRX**, if it is Čech L-fuzzy basic rough proximity and additionally satisfies the following axioms:

(P5) $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \geq \bigwedge_{h \in L^X} \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(h)) \vee \delta_R(\overline{\mathfrak{U}}(h)^*, \overline{\mathfrak{U}}(g))$. In this study, we will not consider the axiom(P5).

A function $\eta : (X, \delta_{R_X}) \to (Y, \delta_{R_Y})$ between two **CL-FRPRXS** is LF-proximity map if $\delta_{R_X}(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)), \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(k))) \leq \delta_{R_Y}(\overline{\mathfrak{U}}_Y(f), \overline{\mathfrak{U}}_Y(k))\ \forall\ f, k \in L^Y$.

Remark 3.2. If (X, δ_R) be a **CL-FRPRX** , then for all $\alpha \in L$ and f, g, h subsets in L^X. We have from (P3):

(1) If $\overline{\mathfrak{U}}(f) \leq \overline{\mathfrak{U}}(g)$, then $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(h)) \leq \delta_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(h))$ and $\delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \leq \delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g))$.

(2) $\delta_R(\alpha \wedge \overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \geq \alpha \wedge \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \Leftrightarrow \delta_R(\alpha \to \overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \leq \alpha \to \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$,

(3) $\delta_R(\overline{\mathfrak{U}}(f), \alpha \wedge \overline{\mathfrak{U}}(g)) \geq \alpha \wedge \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \Leftrightarrow \delta_R(\overline{\mathfrak{U}}(f), \alpha \to \overline{\mathfrak{U}}(g)) \leq \alpha \to \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(4) If δ_R is Alexandrov, then by axiom (P3), we have
$\delta_R(\overline{\mathfrak{U}}(f), \bigvee_{\iota \in \gamma} \overline{\mathfrak{U}}(g_\iota)) = \bigvee_{\iota \in \gamma} \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g_\iota))$ and $\delta_R(\bigvee_{\iota \in \gamma} \overline{\mathfrak{U}}(f_\iota), \overline{\mathfrak{U}}(g)) = \bigvee_{\iota \in \gamma} \delta_R(\overline{\mathfrak{U}}(f_\iota), \overline{\mathfrak{U}}(g))$.

Definition 3.3. Given (X, R) as **L-FAPS**, a function $C_R : L^X \to L^X$ is said to be Čech L-fuzzy rough closure operator on X if $\forall\ f, g, f_\iota \in L^X$, it satisfies:

(LC1) $C_R(\perp_X) = \perp_X$,

(LC2) $C_R(f) \geq \overline{\mathfrak{U}}(f)$,

(LC3) $S_d(f, g) \leq S_d(C_R(f), C_R(g))$,

(LC4) $C_R(f \vee g) \leq C_R(f) \vee C_R(g)$.

Then the pair (X, C_R) is **CL-FRCS**. Furthermore, it is **ACL-FRCS** if satisfies

(AL) $C_R(\bigvee_{\iota \in \gamma} f_\iota) = \bigvee_{\iota \in \gamma} C_R(f_\iota)$. in addition, called topological **CL-FRCS** if it satisfies

(T) $C_R(C_R(f)) \leq C_R(f)$.

An LF-closure map is a function $\eta : (X, C_{R_X}) \to (Y, C_{R_Y})$ satisfing
$\eta^{\leftarrow}(C_{R_Y}(f)) \geq C_{R_X}(\eta^{\leftarrow}(f)), \forall\ f \in L^Y$.

Remark 3.4. Let (X, C_R) be an **CL-FRCS**. Then, by (LC3), we have

(1) $f \leq g$ implies that $C_R(f) \leq C_R(g)$,

(2) $C_R(\alpha \wedge f) \geq \alpha \wedge C_R(f)$, equivalently, $C_R(\alpha \to f) \leq \alpha \to C_R(f)$.

Theorem 3.5. *Let δ_R be **CL-FRPRX**. Define a function $C_{\delta_R} : L^X \to L^X$ as $C_{\delta_R}(f)(a) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f))$. Then, we have*

*(1) (X, C_{δ_R}) is **CL-FRCS** with $C_{\delta_R}(\alpha \wedge f) \geq \alpha \wedge C_{\delta_R}(f)$,*

(2) if δ_R is Alexandrov, then C_{δ_R} is so.

*(3) if $\delta_R(\overline{\mathfrak{U}}(f), \alpha \wedge \overline{\mathfrak{U}}(g)) = \alpha \wedge \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$ and δ_R is **ACL-FRPRX**, then $C_{\delta_R}(f)(a) = \bigvee_{b \in X} f(b) \wedge \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\top_b))$.*

Proof. To prove (1), it must satisfy the following conditions:

(LC1) Since $\delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\perp_X)) = \perp$, then $C_{\delta_R}(\perp_X)(a) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\perp_X) = \perp$.

(LC2) $C_{\delta_R}(f)(a) \geq \bigvee_{a \in X} \overline{\mathfrak{U}}(\top_a)(a) \wedge \overline{\mathfrak{U}}(f)(a) \geq \bigvee_{a \in X} \top_a(a) \wedge \overline{\mathfrak{U}}(f)(a) \geq \overline{\mathfrak{U}}(f)(a)$.

(LC3) $S_d(C_{\delta_R}(f), C_{\delta_R}(g)) = \bigwedge_{a \in X} (C_{\delta_R}(f)(a) \to C_{\delta_R}(g)(a)) = \bigwedge_{a \in X} (\delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f)) \to \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(g))) \geq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \geq S_d(f, g)$.

(LC4) $C_{\delta_R}(f)(a) \vee C_{\delta_R}(g)(a) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f)) \vee \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(g)) \geq \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g)) \geq \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f \vee g)) = C_{\delta_R}(f \vee h)(a)$.

$C_{\delta_R}(\alpha \wedge f)(a) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\alpha \wedge f)) = \delta_R(\overline{\mathfrak{U}}(\top_a), \alpha \wedge \overline{\mathfrak{U}}(f)) \geq \alpha \wedge \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f)) = \alpha \wedge C_{\delta_R}(f)(a)$.

(2) If δ_R is Alexandrov, then $C_{\delta_R}(\bigvee_{\iota\in\gamma} f_\iota)(a) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\bigvee_{\iota\in\gamma} f_\iota)) = \delta_R(\overline{\mathfrak{U}}(\top_a), \bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(f_\iota)) = \bigvee_{\iota\in\gamma} \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f_\iota)) = \bigvee_{\iota\in\gamma} C_{\delta_R}(f_\iota)(a)$.

(3) Since $f = \bigvee_{b\in X}(f(b) \wedge \top_b)$. Then, we get $C_{\delta_R}(f)(a) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f)) = \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\bigvee_{b\in X}(f(b) \wedge \top_b))) = \delta_R(\overline{\mathfrak{U}}(\top_a), \bigvee_{b\in X} \overline{\mathfrak{U}}(f(b) \wedge \top_b)) = \delta_R(\overline{\mathfrak{U}}(\top_a), \bigvee_{b\in X} f(b) \wedge \overline{\mathfrak{U}}(\top_b))) = \bigvee_{b\in X} f(b) \wedge \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(\top_b))$. □

Remark 3.6. Let (X, R) be an **L-FAPS** and (X, C_R) be a **CL-FRCS**. Then,

(1) If $\delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(C_{\delta_R}(f))) \leq \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f))$, then C_{δ_R} is a a fuzzy topology,

(2) $S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \leq C_{\delta_R}(f) \to C_{\delta_R}(g)$.

From Theorem 3.7, a Čech L-fuzzy rough proximity induced by a Čech L-fuzzy rough closure operator is obtained.

Theorem 3.7. *Given (X, R) as **L-FAPS** and (X, C_R) as **CL-FRCS**. Define a function $\delta_{C_R} : L^X \times L^X \to L$ as $\delta_{C_R}(f, g) = N_d(C_R(g), \overline{\mathfrak{U}}(f))\ \forall\ f, g \in L^X$. We obtain the following:*

*(1) δ_{C_R} is a **CL-FRPRX** on X with $\delta_{C_R}(\overline{\mathfrak{U}}(f), \alpha \wedge \overline{\mathfrak{U}}(g)) \geq \alpha \wedge \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$ and $\delta_{C_R}(\alpha \to \overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \leq \alpha \to \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.*

(2) if C_R is Alexandrov, then δ_{C_R} is Alexandrov.

(3) $C_{\delta_{C_R}} \geq C_R$ and if $\delta_R(f, g) = \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a)$, then $\delta_{C_{\delta_R}} \leq \delta_R$.

Proof. (1) It is sufficient to prove the following conditions

(P1) Since $C_R(\overline{\mathfrak{U}}(\bot_X)) = \bot_X$ and $\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\top_X)) = \top_X$, getting $\delta_{C_R}(\overline{\mathfrak{U}}(\top_X), \overline{\mathfrak{U}}(\bot_X)) = N_d(C_R(\overline{\mathfrak{U}}\ (\bot_X)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\top_X))) = \bot, \delta_{C_R}(\overline{\mathfrak{U}}(\bot_X), \overline{\mathfrak{U}}(\top_a)) = N_d(C_R\ (\overline{\mathfrak{U}}(\top_a)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\bot_X))) = \bot$.

(P2) Since $C_R(g) \geq \overline{\mathfrak{U}}(g)$, we obtain $\delta_{C_R}(f, g) = N_d(C_R(g), \overline{\mathfrak{U}}(f)) \geq N_d(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) = N_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(P3) Since R is reflexive and transitive, we receive $\delta_{C_R}\ (\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(h)) \to \delta_{C_R}\ (\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(h)) = N_d(C_R\ (\overline{\mathfrak{U}}(h)), \overline{\mathfrak{U}}\ (\overline{\mathfrak{U}}(f))) \to N_d(C_R(\overline{\mathfrak{U}}(h)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))) \geq \Big(\bigvee_{a\in X} C_R(\overline{\mathfrak{U}}(h))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a)\Big) \to \Big(\bigvee_{b\in X} C_R\ (\overline{\mathfrak{U}}(h))(b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b)\Big) \geq \bigwedge_{a\in X}(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \to \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(a)) = S_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))) \geq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$,

$\delta_{C_R}(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \to \delta_{C_R}\ (\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) = N_d(C_R(\overline{\mathfrak{U}}(f)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))) \to N_d(C_R(\overline{\mathfrak{U}}(g)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))) \geq \Big(\bigvee_{a\in X} C_R(\overline{\mathfrak{U}}(f))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))(a)\Big) \to \Big(\bigvee_{a\in X} C_R(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g)))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))(a)\Big) \geq \bigwedge_{a\in X}(C_R(\overline{\mathfrak{U}}(f))(a) \to C_R(\overline{\mathfrak{U}}(g))(a)) = S_d(C_R(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g))) \geq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(P4) $\delta_{C_R}\ (\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \vee \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(h)) = N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g))) \vee N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R\ (\overline{\mathfrak{U}}(h))) \geq N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g)) \vee C_R(\overline{\mathfrak{U}}(h))) \geq N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g) \vee \overline{\mathfrak{U}}(h))) \geq N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g \vee h))) = \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g \vee h)))$. By similarity, the other case is proved. Finally, From Remark 3.2, we have $\delta_{C_R}(\overline{\mathfrak{U}}(f), \alpha \wedge \overline{\mathfrak{U}}(g)) = N_d(\overline{\mathfrak{U}}(f), C_R(\alpha \wedge \overline{\mathfrak{U}}(g))) \geq N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), \alpha \wedge C_R(\overline{\mathfrak{U}}(g))) = \alpha \wedge N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g))) = \alpha \wedge \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$,

$\delta_{C_R}(\alpha \to \overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) = N_d(C(\overline{\mathfrak{U}}(g)), \overline{\mathfrak{U}}(\alpha \to \overline{\mathfrak{U}}(f))) \leq \alpha \to N_d(C(\overline{\mathfrak{U}}(f)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))) = \alpha \to \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$. Hence, $\delta_{C_R}(\overline{\mathfrak{U}}(f), \alpha \wedge \overline{\mathfrak{U}}(g)) \geq \alpha \wedge \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$ and $\delta_{C_R}(\alpha \to \overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \leq \alpha \to \delta_{C_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(2) By Proposition 2.7 (($N1$) and ($N4$)), we have $\delta_{C_R}\ (\bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(f_\iota), \overline{\mathfrak{U}}(g)) = N_d(C_R(\overline{\mathfrak{U}}(g)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\bigvee_{\iota\in\gamma} f_\iota))) = \bigvee_{\iota\in\gamma} N_d(C_R\ (\overline{\mathfrak{U}}(g)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f_\iota))) = \bigvee_{\iota\in\gamma} \delta_{C_R}\ (\overline{\mathfrak{U}}(f_\iota), \overline{\mathfrak{U}}(g))$,

$\delta_{C_R}(\overline{\mathfrak{U}}(f), \bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(g_\iota)) = N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(g_\iota))) \leq \bigvee_{\iota\in\gamma} N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)), C_R(\overline{\mathfrak{U}}(g_\iota))) = \bigvee_{\iota\in\gamma} \delta_{C_R}\ (\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g_\iota))$,

(3) $C_{\delta_{C_R}}(f)(a) = \delta_{C_R}(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f)) = N_d(\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\top_a)), C_R(\overline{\mathfrak{U}}(f))) \geq N_d(\overline{\mathfrak{U}}(\top_a), C_R(f)) = \bigvee_{a\in X} \overline{\mathfrak{U}}(\top_a)(a) \wedge C_R(f)(a) \geq \top \wedge C_R(f)(a) = C_R(f)(a)$. Finally, $\delta_{C_{\delta_R}}(f,g) = \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge C_{\delta_R}(g)(a) = \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \bigvee_{a\in X} \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(g)) \leq \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \bigvee_{a\in X} \overline{\mathfrak{U}}(g)(a) \leq \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a) = \delta_R(f,g)$. Hence, $\delta_{C_{\delta_R}} \leq \delta_R$. □

Corollary 3.8. *Given (X,R) as **L-FAPS**. Let $\delta_{C_R} : L^X \times L^X \to L$ be a function identified by $\delta_{C_R}(f,g) = N_d(\overline{\mathfrak{U}}(g), C_R(f))$. Then, δ_{C_R} is a **CL-FRPRX** on X.*

We can easily prove Theorem 3.9 is the same as Theorem 3.7. So, it will be omitted.

Theorem 3.9. *Given (X,R) as **L-FAPS**. Let (X,C_R) be an **CL-FRCS**and $\delta_{C_R} : L^X \times L^X \to L$ be a function defined by $\delta_{C_R}(f,g) = N_d(C_R(g), C_R(f))\ \forall\ f,g \in L^X$. Which is coming are hold*

(1) δ_{C_R} is an L-fuzzy rough basic proximity on X,

(2) if C_R is Alexandrov, then δ_{C_R} is Alexandrov.

Theorem 3.10 shows **L-FAPS**'s category can be incorporated into the category of **L-FPRXS**.

Theorem 3.10. *If (X,R) is an **L-FAPS** and $\delta_R : L^X \times L^X \to L$ be a function defined for all $f,g \in L^X$ by $\delta_R(f,g) = N_d(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f))$. Then,*

(1) (X,δ_R) present as a L-fuzzy rough basic proximity space,

(2) $\delta_{C_{\delta_R}} \geq \delta_R$,

(3) if $\eta : (X,R_X) \to (Y,R_Y)$ be a LF-approximation function. Then, $\eta : (X,\delta_{R_X}) \to (Y,\delta_{R_Y})$ is a LF-proximity function.

Proof.

(1) It is sufficient to prove 4 conditions as in the proof of Theorem 3.7(1).

(2) $\delta_{C_{\delta_R}}(f,g) = \bigvee_{a\in X} C_{\delta_R}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a) = \bigvee_{a\in X} \delta_R(\overline{\mathfrak{U}}(\top_a), \overline{\mathfrak{U}}(f)) \wedge \overline{\mathfrak{U}}(g)(a) = \bigvee_{a\in X} \Big(\bigvee_{a\in X} \overline{\mathfrak{U}}(\top_a)(a) \wedge \overline{\mathfrak{U}}(f)(a)\Big) \wedge \overline{\mathfrak{U}}(g)(a) \geq \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a) = \delta_R(f,g)$.

(3) The proof is given in the same manner. □

We have the concrete functor Φ : **L-FAPS** → **CL-FRPRX**, according to Theorems 3, 4 by the following $\Phi(X,R_X) = (X,\delta_{R_X})$, $\Phi(\eta) = \eta$.

Example 3.11. If the universal fuzzy set given as $X = \{(a_1,0.4), (a_2,0.3), (a_3,0.5), (a_4,0.7)\}$ with a fuzzy topology $\tau = \{f = 0, e = \{(a_1,0.4)\}, d = \{(a_1,0.4), (a_3,0.5)\}, c = \{(a_1,0.4), (a_2,0.3)\}, b = \{(a_1,0.4), (a_2,0.3), (a_3,0.5)\}, a = \{(a_1,0.4), (a_2,0.3), (a_3,0.5), (a_4,0.7)\}\}$ and shown in Fig. 1. Define $R : X \times X \to [0,1]$ as

$$R = \begin{pmatrix} 1 & 0.9 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0.9 & 0.8 & 1 & 1 \\ 0.7 & 0.6 & 0.8 & 1 \end{pmatrix}$$

Where $\overline{\mathfrak{U}}(f)(a_\iota) = \bigvee_{a_j\in X} R(a_\iota, a_j) \wedge f(a_j)$. Then, we have for each $a_\iota, a_j \in X$, $i,j \in \{1,2,3,4\}$

$\overline{\mathfrak{U}}(f)(a_1) = 0$ $\overline{\mathfrak{U}}(f)(a_2) = 0$ $\overline{\mathfrak{U}}(f)(a_3) = 0$ $\overline{\mathfrak{U}}(f)(a_4) = 0$;
$\overline{\mathfrak{U}}(a)(a_1) = 0.7$ $\overline{\mathfrak{U}}(a)(a_2) = 0.7$ $\overline{\mathfrak{U}}(a)(a_3) = 0.7$ $\overline{\mathfrak{U}}(a)(a_4) = 0.7$;
$\overline{\mathfrak{U}}(b)(a_1) = 0.5$ $\overline{\mathfrak{U}}(b)(a_2) = 0.5$ $\overline{\mathfrak{U}}(b)(a_3) = 0.5$ $\overline{\mathfrak{U}}(b)(a_4) = 0.5$;
$\overline{\mathfrak{U}}(c)(a_1) = 0.4$ $\overline{\mathfrak{U}}(c)(a_2) = 0.4$ $\overline{\mathfrak{U}}(c)(a_3) = 0.4$ $\overline{\mathfrak{U}}(c)(a_4) = 0.4$;
$\overline{\mathfrak{U}}(d)(a_1) = 0.5$ $\overline{\mathfrak{U}}(d)(a_2) = 0.5$ $\overline{\mathfrak{U}}(d)(a_3) = 0.5$ $\overline{\mathfrak{U}}(d)(a_4) = 0.5$;
$\overline{\mathfrak{U}}(e)(a_1) = 0.4$ $\overline{\mathfrak{U}}(e)(a_2) = 0.4$ $\overline{\mathfrak{U}}(e)(a_3) = 0.4$ $\overline{\mathfrak{U}}(e)(a_4) = 0.4$.

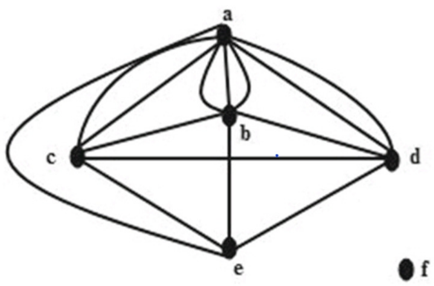

Figure 1: A fuzzy topological graph

Define $\delta_R(a,b) = N(\overline{\mathfrak{U}}(a), \overline{\mathfrak{U}}(b)$. For each $a, b \in \tau$, we have

$\delta_R(e,c) = 0.4$	$\delta_R(f,e) = 0$	$\delta_R(e,d) = 0.4$	$\delta_R(f,c) = 0$
$\delta_R(e,b) = 0.4$	$\delta_R(f,d) = 0$	$\delta_R(a,e) = 0.4$	$\delta_R(f,d) = 0$
$\delta_R(c,d) = 0.4$	$\delta_R(f,b) = 0$	$\delta_R(c,b) = 0.4$	$\delta_R(a,f) = 0$
$\delta_R(a,c) = 0.4$	$\delta_R(a,b) = 0.5$	$\delta_R(d,b) = 0.5$	$\delta_R(a,d) = 0.5$

Define $C_R(f)(a) = \bigvee_{a_j \in X} R(a_\iota, a_j) \wedge f(a_j)$

$C_R(f)(a_1) = 0$	$C_R(f)(a_2) = 0$	$C_R(f)(a_3) = 0$	$C_R(f)(a_4) = 0;$
$C_R(a)(a_1) = 0.7$	$C_R(a)(a_2) = 0.7$	$C_R(a)(a_3) = 0.7$	$C_R(a_4) = 0.7;$
$C_R(b)(a_1) = 0.5$	$C_R(b)(a_2) = 0.5$	$C_R(b)(a_3) = 0.5$	$C_R(b)(a_4) = 0.5;$
$C_R(c)(a_1) = 0.4$	$C_R(c)(a_2) = 0.4$	$C_R(c)(a_3) = 0.4$	$C_R(c)(a_4) = 0.4;$
$C_R(d)(a_1) = 0.5$	$C_R(d)(a_2) = 0.5$	$C_R(d)(a_3) = 0.5$	$C_R(d)(a_4) = 0.5;$
$C_R(e)(a_1) = 0.4$	$C_R(e)(a_2) = 0.4$	$C_R(e)(a_3) = 0.4$	$C_R(e)(a_4) = 0.4.$

Where $\delta_{C_R}(f,g) = N_d(C_R(g), \overline{\mathfrak{U}}(f))$. Then, we have

$\delta_{C_R}(e,c) = 0.4$	$\delta_{C_R}(f,e) = 0$	$\delta_{C_R}(e,d) = 0.4$	$\delta_{C_R}(f,c) = 0$
$\delta_{C_R}(e,b) = 0.4$	$\delta_{C_R}(f,d) = 0$	$\delta_{C_R}(e,a) = 0.4$	$\delta_{C_R}(f,d) = 0$
$\delta_{C_R}(c,d) = 0.4$	$\delta_{C_R}(f,b) = 0$	$\delta_{C_R}(c,b) = 0.4$	$\delta_{C_R}(f,a) = 0$
$\delta_{C_R}(c,a) = 0.4$	$\delta_{C_R}(b,a) = 0.7$	$\delta_{C_R}(d,b) = 0.5$	$\delta_{C_R}(d,a) = 0.5$

Theorem 3.12. *If (X, R) is an **L-FAPS** and $\eta : (X, \delta_{R_X}) \to (Y, \delta_{R_Y})$ be LF-proximity function, then $\eta : (X, C_{\delta_{R_X}}) \to (Y, C_{\delta_{R_Y}})$ is LF-closure function.*

Proof. For each $f \in L^Y$,

$$\begin{aligned} \eta^{\leftarrow}(C_{\delta_{R_Y}}(f))(a) &= C_{\delta_{R_Y}}(f)(\eta(a)) = \delta_{R_Y}(\overline{\mathfrak{U}}_Y(\top_{\eta(a)}), \overline{\mathfrak{U}}_Y(f)) \\ &\geq \delta_{R_X}(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(\top_{\eta(a)})), \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f))) \\ &\geq \delta_{R_X}(\overline{\mathfrak{U}}_X(\top_a), \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f))) = C_{\delta_{R_X}}(\eta^{\leftarrow}(f))(a). \end{aligned}$$

Getting the concrete functor Δ : **CL-FRPRX** $\to$ **CL-FRCS**, according to Theorems 3.5 and 3.12, by $\Delta(X, \delta_{R_X}) = (X, C_{\delta_{R_X}})$, $\Delta(\eta) = \eta$. If the functor Δ : **CL-FRPRX** $\to$ **CL-FRCS** to the category **ACL-FRPRX** is still written, then by Theorem 3.5, Δ : **ACL-FRPROX** $\to$ **ACL-FRCS** considered as a concrete functor. □

Theorem 3.13. *If (X, R) is an **L-FAPS** and $\eta : (X, C_{R_X}) \to (Y, C_{R_Y})$ be a LF-closure function, then $\eta : (X, \delta_{C_{R_X}}) \to (Y, \delta_{C_{R_Y}})$ is a LF-proximity function.*

Proof. Since $C_{R_X}(\eta^{\leftarrow}(g)) \leq \eta^{\leftarrow}(C_{R_Y}(g))$,and by $(N6)$ in Proposition 2.5, then we have,

$$\delta_{C_{R_X}}(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)), \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(g))) = N\Big(C_{R_X}(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(g))), \overline{\mathfrak{U}}_X(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f))))\Big)$$
$$\leq N\Big(C_{R_X}(\eta^{\leftarrow}(\overline{\mathfrak{U}}_Y(g))), \eta^{\leftarrow}(\overline{\mathfrak{U}}_Y(f))\Big) \leq N\Big(\eta^{\leftarrow}(C_{R_Y}(g)), \eta^{\leftarrow}(\overline{\mathfrak{U}}_Y(f))\Big)$$
$$\leq N\Big(C_{R_Y}(g), \overline{\mathfrak{U}}_Y(f)\Big) \leq N\Big(C_{R_Y}(\overline{\mathfrak{U}}_Y(g), \overline{\mathfrak{U}}_Y(\overline{\mathfrak{U}}_Y(f))\Big) = \delta_{C_{R_Y}}(\overline{\mathfrak{U}}_Y(f), \overline{\mathfrak{U}}_Y(g)).$$

□

By Theorems 3.5 and 3.13, we obtain a concrete functor Υ : **CL-FRCS** → **CL-FRPRX** by $\Upsilon(X, C_{R_X}) = (X, \delta_{C_{R_X}})$, $\Upsilon(\eta) = \eta$. By Theorem 3.5(3), we have $\Delta(\Upsilon(X, C_R)) = (X, C_{\delta_{C_R}}) \geq (X, C_R)$. Thus, Δ is a left inverse of Υ. If the restriction of the functor Υ : **CL-FRCS** → **CL-FRPROX** to the full subcategory **ACL-FRCS** is still written, then by Theorem 3.5, Υ : **ACL-FRCS** → **ACL-FRPRX** forms a concrete functor.

The proof of Theorem 3.14 is obvious. Then, the proof is omitted.

Theorem 3.14. (Δ, Υ) *forms a Galois connection between the category* ***CL-FRPRX*** *and the category* ***CL-FRCS****.*

Proposition 3.15. *If $R \in L^{X\times X}$ is defined on a set X, and a function $\delta_R : L^X \times L^X \to L$ is defined by $\delta_R(f, g)) = \bigvee_{a,b\in X} R(a, b) \wedge \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(b)$. Then,*

(1) δ_{R_X} is ***ACL-FRPRX*** *with $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(\alpha\wedge g)) = \alpha\wedge\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$ and $\delta_R(\overline{\mathfrak{U}}(\alpha \to f), \overline{\mathfrak{U}}(g)) \leq \alpha \to \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$,*

(2) if $\eta : (X, R_X) \to (Y, R_Y)$ is an order preserving function, this lead to $\eta : (X, \delta_{R_X}) \to (Y, \delta_{R_Y})$ is LF-proximity function.

Proof. (1) The conditions **(P1)**, **(P3)**, and **(AL)** can be proved easily. So, we omit it.

(P2): Since R is reflexive, then $\delta_R(f, g) = \bigvee_{a,b\in X} R(a, b) \wedge \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(b) \geq \bigvee_{a\in X} R(a, a) \wedge \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a) = N_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(P4) Since R is transitive, then $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \vee \delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) = \Big(\bigvee_{a,b\in X}(R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b))\Big) \vee \Big(\bigvee_{b,c\in X}(R(b, c) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))(b)\Big) \wedge \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(g))(c)\Big) \leq \bigvee_{a,c\in X}\Big(\bigvee_{b\in X}(R(a, b) \wedge R(b, c) \wedge \overline{\mathfrak{U}}(g)(b) \wedge \overline{\mathfrak{U}}(g)(c) \wedge (\overline{\mathfrak{U}}(f)(a) \vee \overline{\mathfrak{U}}(h)(b)))\Big) \leq \bigvee_{a,c\in X}\Big(R(a, c) \wedge \overline{\mathfrak{U}}(g)(c) \wedge(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(h))(a)\Big) = \delta_R(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g))$.

Next, $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(\alpha \wedge g)) = \bigvee_{a,b\in X} R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\alpha \wedge g))(b) = \bigvee_{a,b\in X} R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge\alpha \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b) = \alpha\wedge \bigvee_{a\in X} R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b) = \alpha \wedge \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

$\delta_R(\overline{\mathfrak{U}}(\alpha \to f), \overline{\mathfrak{U}}(g)) = \bigvee_{a,b\in X} R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\alpha \to f))(a)\wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b) \leq \bigvee_{a,b\in X} R(a, b) \wedge (\alpha \to \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b) \leq \alpha \to \delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(P) Since R is symmetric, then we have $\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) = \bigvee_{a,b\in X} R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b) = \bigvee_{b,a\in X} R(b, a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) = \delta_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f))$,

(P5) Since R is reflexive and transitive. Then, $\bigvee_{b\in X} R(b, c) \wedge R(a, b) = R(a, c)$. For all $f, g \in L^X$, let $\overline{\mathfrak{U}}^*(h)(b) = \bigvee_{a\in X} R(a, b)\wedge \overline{\mathfrak{U}}(f)(a)$.

$\bigwedge_{h\in L^X} (\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(h)) \vee\delta_R(\overline{\mathfrak{U}}^*(h), \overline{\mathfrak{U}}(g))) = \bigwedge_{h\in L^X}\Big((\bigvee_{a,b\in X}(R(a, b) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))(b))) \vee (\bigvee_{b,c\in X}(R(b, c) \wedge \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}^*(h))(b) \wedge \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(g))(c)))\Big) \leq \bigwedge_{h\in L^X}\Big((\bigvee_{a,b\in X}(R(a, b) \wedge \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(h)(b))) \vee (\bigvee_{b,c\in X}(R(b, c) \wedge \overline{\mathfrak{U}}^*(h)(b) \wedge \overline{\mathfrak{U}}(g)(c)))\Big) \leq \bigwedge_{h\in L^X}(\bigvee_{b\in X}(\overline{\mathfrak{U}}^*(h)(b)$

$\wedge\, \overline{\mathfrak{U}}(h)(b)))\vee\Big(\bigvee_{a,b,c\in X}(R(b,c)\wedge(R(a,b)\wedge\overline{\mathfrak{U}}(f)(a)\wedge\overline{\mathfrak{U}}(g)(c))))\Big)=\bot\vee\Big(\bigvee_{a,c\in X}R(a,c)\wedge\overline{\mathfrak{U}}(f)(a)\wedge\overline{\mathfrak{U}}(g)(c))\Big)=\bigvee_{a,c\in X}R(a,c)$ $\wedge\, \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a)\wedge\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))(c)=\delta_R(\overline{\mathfrak{U}}(f),\overline{\mathfrak{U}}(g))$.

(2) Since R is reflexive and transitive, then $\delta_{R_X}(\overline{\mathfrak{U}}_X\,(\eta^{\leftarrow}(f)),\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(g)))=\bigvee_{a,b\in X}\Big(R_X(a,b)\wedge\overline{\mathfrak{U}}_X\,(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)))(a)$ $\wedge\,\overline{\mathfrak{U}}_X(\overline{\mathfrak{U}}_X\,(\eta^{\leftarrow}(g)))(b)\Big)\leq\bigvee_{a,b\in X}\Big(R_X(a,b)\wedge\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f))(a)\wedge\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(g))(b)\Big)\leq\bigvee_{a,b\in X}\Big(R_X(a,b)\wedge\eta^{\leftarrow}(\overline{\mathfrak{U}}_Y(f))(a)\wedge$ $\eta^{\leftarrow}(\overline{\mathfrak{U}}_R(g))(b)\Big)\leq\bigvee_{a,b\in X}\Big(R_Y(\eta(a),\eta(b))\wedge\overline{\mathfrak{U}}_Y(f)(\eta(a))\wedge\overline{\mathfrak{U}}_Y(g)(\eta(b))\Big)\leq\bigvee_{c,w\in Y}\Big(R_Y(c,w)\wedge\overline{\mathfrak{U}}_Y(\overline{\mathfrak{U}}_Y(f))(c)\wedge\overline{\mathfrak{U}}_Y(\overline{\mathfrak{U}}_Y(g))(w)\Big)=$ $\delta_{R_Y}\,(\overline{\mathfrak{U}}_Y(f),\overline{\mathfrak{U}}_Y(g))$. □

From Proposition 3.15 , we get a concrete functor Ω : **CL-FRR** $\rightarrow$ **ACL-FRPRX** is a functor defined by $\Omega(X,R_X)=(X,\delta_{R_X})$, $\Omega(\eta)=\eta$.

Example 3.16. Let X be a set and $R\in L^{X\times X}$. Define a function $\delta_R:L^X\times L^X\rightarrow L$ as in Proposition 3.15. By Theorem 3.5, we obtain L-fuzzy rough closure operator $C_\delta:L^X\rightarrow L^X$ as follows

$C_{\delta_R}(f)(a)=\delta_R(\overline{\mathfrak{U}}(\top_a),\overline{\mathfrak{U}}(f))=\bigvee_{a,b\in X}R(a,b)\wedge\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(\top_a))(a)\wedge\overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)))(b)=\bigvee_{b\in X}R(a,b)\wedge\overline{\mathfrak{U}}(f)(b)=\bigvee_{b\in X}R(a,b)\wedge$ $\overline{\mathfrak{U}}(\bigvee(f(b)\wedge\top_b))(b)=\bigvee_{b\in X}R(a,b)\wedge f(b)\wedge\overline{\mathfrak{U}}(\top_b)(b)=\bigvee_{b\in X}R(a,b)\wedge f(b)=\overline{\mathfrak{U}}(f)(a)$.

(1) If $R=\top_{X\times X}$ is given. From Proportion 3.15, we have $\delta_1(f,g)=\bigvee_{a,b\in X}\overline{\mathfrak{U}}(f)(a)\wedge\overline{\mathfrak{U}}(g)(b)$. Hence, δ_1 is **CL-FRPRX**. And from Theorem 3.5, we obtain the operator **CL-FRC** as $C_\delta:L^X\rightarrow L^X$ as follows $C_{\delta_R}(f)(a)=\delta_R(\overline{\mathfrak{U}}(\top_a),\overline{\mathfrak{U}}(f))\geq\overline{\mathfrak{U}}(f)(a)$.

(2) If $R=\triangle_{X\times X}$ is given by

$$\triangle_{X\times X}(a,b)=\begin{cases}\top, & \text{if } b=a,\\ \bot, & \text{o.w.}\end{cases}$$

Then, $\delta_2(f,g)=\bigvee_{a\in X}(\overline{\mathfrak{U}}(f)(a)\wedge\overline{\mathfrak{U}}(g)(a))$. Hence, δ_2 is a **CL-RPRX**.

For each reflexive relation R, we obtain Alexandrov L-fuzzy rough closure operator as $C_R:L^X\rightarrow L^X$ as $C_R(f)(a)=\bigvee_{b\in X}(R(a,b)\wedge\overline{\mathfrak{U}}(f)(b)$. By Theorem 3.5, we obtain Alexandrov L-fuzzy rough proximity δ_{C_R} as $\delta_{C_R}(f,g)=\bigvee_{a\in X}(\overline{\mathfrak{U}}(f)(a)\wedge C_R(g))(a))=\bigvee_{a\in X}(\overline{\mathfrak{U}}(f)(a)\wedge\bigvee_{b\in X}(R(a,b)\wedge\overline{\mathfrak{U}}(g)(b))=\bigvee_{a,b\in X}(R(a,b)\wedge\overline{\mathfrak{U}}(f)(a)\wedge\overline{\mathfrak{U}}(g)(b))=$ $\delta_R(f,g)$.

4. The relationships between L-fuzzy rough proximities and L-fuzzy rough ideals

The implication between Čech L-fuzzy rough proximity spaces and L-fuzzy rough ideal spaces is provided in the following section.

Definition 4.1. Given (X,R) as an **L-FAPS**. **L-FRIS** is the function $\mathcal{D}_R:L^X\rightarrow L$ that satisfies:

(I1) $\mathcal{D}_R(\bot_X)=\top$ and $\mathcal{D}_R(\top_X)=\bot$,

(I2) $S_d(\overline{\mathfrak{U}}(f),\overline{\mathfrak{U}}(g))\leq\mathcal{D}_R(g)\rightarrow\mathcal{D}_R(f)\ \forall\, f,g\in L^X$,

(I3) $\mathcal{D}_R(\overline{\mathfrak{U}}(f)\vee\overline{\mathfrak{U}}(g))\geq\mathcal{D}_R(\overline{\mathfrak{U}}(f))\wedge\mathcal{D}_R(\overline{\mathfrak{U}}(g))\ \forall\, f,g\in L^X$.

L-FRIS with Alexandrov condition (AL) will be **AL-FRIS** where

(AL) $\mathcal{D}_R(\bigvee_{\iota\in\gamma}\overline{\mathfrak{U}}(f_\iota))\geq\bigwedge_{\iota\in\gamma}\mathcal{D}_R(\overline{\mathfrak{U}}(f_\iota))\ \forall\,\{f_\iota:\iota\in\gamma\}\subseteq L^X$,

Take $(X,\mathcal{D}_{R_X})$ and $(Y,\mathcal{D}_{R_Y})$ as **L-FRIS**. A function $\eta:X\rightarrow Y$ is LF-ideal function for $f\in L^Y$ iff $\mathcal{D}_{R_Y}(\overline{\mathfrak{U}}_Y(f))\leq\mathcal{D}_{R_X}(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)))$.

Theorem 4.2. *Given (X, R) as an L-**FAPS**, (X, δ_R) as a **CL-FRPRX**. Define $\mathcal{D}^k_{\delta_R} : L^X \longrightarrow L$ a function is defined as follows*

$$\mathcal{D}^k_{\delta_R}(f) = \begin{cases} \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)), & \text{if } f \neq \top_X \\ \top, & \text{if } f = \top_X. \end{cases}$$

Where $\delta^*_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) = (\delta_R(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)))^* = (\bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a))^* = \bigwedge_{a\in X} \overline{\mathfrak{U}}^*(f)(a) \vee \overline{\mathfrak{U}}^*(g)(a)$. Then, $\mathcal{D}^k_{\delta_R}$ is L-fuzzy rough ideal. Moreover, if δ_R is Alexandrov, then $\mathcal{D}^k_{\delta_R}$ is so.

Proof. (I1) By dentition $\mathcal{D}^k_{\delta_R}(\bot_X) = \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(\bot_X)) = \top$ and $\mathcal{D}^k_{\delta_R}(\top_X) = \top$.

(I2) For any two fuzzy sets f, g,

[**Case 1**:] if $f = \top_X$, $g = \top_X$, then $\mathcal{D}^k_{\delta_R}(g) \to \mathcal{D}^k_{\delta_R}(f) = \top \geq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

[**Case 2**:] if $f \neq \top_X$ and $g \neq \top_X$, then $\mathcal{D}^k_{\delta_R}(g) \to \mathcal{D}^k_{\delta_R}(f) = \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(g)) \to \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)) = \delta_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)) \to \delta_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(g)) \geq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(I3) For any $f, g \in L^X$, we have

[**Case 1**:] if $f \vee g = \top_X$, then $\mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g)) = \top \geq \mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(f)) \wedge \mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(g))$.

[**Case 2**:] if $f \vee g \neq \top_X$, then $f \neq \top_X$ and $g \neq \top_X$. So, $\mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(f)) \wedge \mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(g)) = \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))) \wedge \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))) \leq \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f)) \vee \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))) = \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g))) = \mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g)))$.

(AL) For each family $\{f_\iota : \iota \in \gamma\}$, we have

[**Case 1**:] if $\bigvee_{\iota\in\gamma} f_\iota = \top_X$, then $\mathcal{D}^k_{\delta_R}(\bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(f_\iota)) = \top \geq \bigwedge_{\iota\in\gamma} \mathcal{D}^k_{\delta_R}(\overline{\mathfrak{U}}(f_\iota))$.

[**Case 2**:] if $\bigvee_{\iota\in\gamma} f_\iota \neq \top_X$, then $f_\iota \neq \top_X$ for each $i \in \gamma$. So, $\mathcal{D}^k_{\delta_R}(\bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(f_\iota)) = \delta^*_R(\overline{\mathfrak{U}}(k), \bigvee_{\iota\in\gamma} \overline{\mathfrak{U}}(f_\iota)) \geq \bigwedge_{\iota\in\gamma} \delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f_\iota)) = \bigwedge_{\iota\in\gamma} \mathcal{D}^k_{\delta_R}(f_\iota)$. □

Now, let $\mathcal{D}(X)$ be the family of all L-fuzzy rough ideals and $\mathcal{P}(X)$ be the family of all L-fuzzy rough proximities on X.

Theorem 4.3. *If (X, R) is an **L-FAPS** and $\mathcal{H} : \mathcal{P}(X) \times \mathcal{D}(X) \to \mathcal{D}(X)$ is a function defined as*

$$\mathcal{H}(\delta_R, \mathcal{D}_R)(f) = \bigvee_{g\in L^X} \Big(\delta^*_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) \wedge \mathcal{D}_R(f)\Big).$$

Then,

(1) $\mathcal{H}(\delta_R, \mathcal{D}_R) \in \mathcal{D}(X)$,

(2) $\mathcal{H}(\delta_R, \mathcal{D}^k_{\delta_R}) = \mathcal{D}^k_{\delta_R}$.

Proof. (1) We prove the following conditions by considering $f, g \in L^X$.

(I1) $\mathcal{H}(\delta_R, \mathcal{D}_R)(\bot_X) = \bigvee_{g\in L^X} \Big(\delta^*_R(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(\bot_X)) \wedge \mathcal{D}_R(\bot_X)\Big) = \top$.

(I2) $\mathcal{H}(\delta_R, \mathcal{D}_R)(g) \to \mathcal{H}(\delta_R, \mathcal{D}_R)(f) = \bigvee_{h\in L^X} \Big(\delta^*_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) \wedge \mathcal{D}_R(g)\Big) \to \bigvee_{k\in L^X} \Big(\delta^*_R(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)) \wedge \mathcal{D}_R(f))\Big) \geq \bigvee_{h\in L^X} \Big(\delta^*_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) \wedge \mathcal{D}_R(g)\Big) \to \Big(\delta^*_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \wedge \mathcal{D}_R(f)\Big) \geq \bigvee_{h\in L^X} \Big(\delta^*_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) \to \delta^*_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f))\Big) \wedge \Big(\mathcal{D}_R(g) \to \mathcal{D}_R(f)\Big) = \bigvee_{h\in L^X} \Big(\delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \to \delta_R(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) \wedge \Big(\mathcal{D}_R(g) \to \mathcal{D}_R(f)\Big) \geq S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) \wedge S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g)) = S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$.

(I3) Let $f, h \in L^X$. Then, $\mathcal{H}(\delta_R, \mathcal{D}_R)\,(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(h)) = \bigvee_{g \in L^X} \Big(\delta^*_R\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(h)) \wedge \mathcal{D}_R\,(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(h))\Big) \geq \bigvee_{g \in L^X} \Big((\delta^*\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) \wedge \delta^*\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(h)) \wedge (\mathcal{D}_R(\overline{\mathfrak{U}}(f) \wedge \mathcal{D}_R(\overline{\mathfrak{U}}(h))\Big) = \bigvee_{g \in L^X} \Big(\delta^*\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) \wedge \mathcal{D}_R(\overline{\mathfrak{U}}(f))\Big) \wedge \bigvee_{g \in L^X} \Big(\delta^*\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(h))) \wedge \mathcal{D}_R\,(\overline{\mathfrak{U}}(h))\Big) = \mathcal{H}(\delta_R, \mathcal{D}_R)\,(\overline{\mathfrak{U}}(f)) \wedge \mathcal{H}(\delta_R, \mathcal{D}_R)\,(\overline{\mathfrak{U}}(h))$.

(2) Let $f \in L^X$, then $\mathcal{H}(\delta_R, \mathcal{D}^k_{\delta_R})(f) = \bigvee_{g \in L^X} \Big(\delta^*_R\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) \wedge \mathcal{D}^k_{\delta_R}(f)\Big) \leq \top \wedge \mathcal{D}^k_{\delta}(f) = \mathcal{D}^k_{\delta}(f)$. Conversely, $\mathcal{H}(\delta_R, \mathcal{D}^k_{\delta_R})(f) = \bigvee_{g \in L^X} \Big(\delta^*_R\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) \wedge \mathcal{D}^k_{\delta_R}(f)\Big) = \bigvee_{g \in L^X} \Big(\delta^*_R\,(\overline{\mathfrak{U}}(g), \overline{\mathfrak{U}}(f)) \wedge \delta^*_R\,(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f))\Big) \geq \delta^*_R\,(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)) \wedge \delta^*_R\,(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)) = \delta^*_R\,(\overline{\mathfrak{U}}(k), \overline{\mathfrak{U}}(f)) = \mathcal{D}^k_{\delta_R}(f)$. Hence, $\mathcal{H}(\delta_R, \mathcal{D}^k_{\delta_R}) = \mathcal{D}^k_{\delta_R}$. □

Theorem 4.4. *Given (X, R) as an **L-FAPS**, $(X, \mathcal{D}_R)$ as a **L-FRIS** such that $\mathcal{D}_R(g) \leq \overline{\mathfrak{U}}^*(g)(a)$, $\forall\, a \in X$ and $g \in L^X$. Define a function $\delta_{\mathcal{D}_R} : L^X \times L^X \to L$ by $\delta_{\mathcal{D}_R}(f, g) = \bigvee_{a \in X} \Big(\overline{\mathfrak{U}}(f)(a) \wedge \mathcal{D}^*_R(g)\Big)$. Then, $\delta_{\mathcal{D}_R}$ be a **CL-FRPRX** on X. Moreover, if $\mathcal{D}_R$ is Alexandrov , then $\delta_{\mathcal{D}_R}$ is so.*

Proof. The following conditions will be proved under $f, g, h \in L^X$.

(P1) $\mathcal{D}_R(\perp_X) = \top \Rightarrow \delta_{\mathcal{D}_R}(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(\perp_X)) = \bigvee_{a \in X} \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(f))(a) \wedge \mathcal{D}^*_R(\overline{\mathfrak{U}}(\perp_X)) = \perp$, $\delta_{\mathcal{D}_R}\,(\overline{\mathfrak{U}}(\perp_X), \overline{\mathfrak{U}}(f)) = \bigvee_{a \in X} \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(\perp_X))(a) \wedge \mathcal{D}^*_R(\overline{\mathfrak{U}}(f)) = \perp$.

(P2) $\mathcal{D}_R(g) \leq \overline{\mathfrak{U}}^*(g)(a) \Rightarrow \delta_{\mathcal{D}_R}(f, g) = \bigvee_{a \in X} \Big(\overline{\mathfrak{U}}(f)(a) \wedge \mathcal{D}^*_R(g))\Big) \geq \bigvee_{a \in X} \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a)$.

(P3) $\delta_{\mathcal{D}_R}(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \to \delta_{\mathcal{D}_R}(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g)) = \Big(\bigvee_{a \in X} \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))(a) \wedge \mathcal{D}^*_R(\overline{\mathfrak{U}}(f))\Big) \to \Big(\bigvee_{a \in X} \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(h))(a) \wedge \mathcal{D}^*_R\,(\overline{\mathfrak{U}}(g))\Big) \geq \Big(\bigwedge_{a \in X} (\overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(h))(a) \to \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(h))(a)\Big) \wedge \Big(\mathcal{D}^*_R\,(\overline{\mathfrak{U}}(f)) \to \mathcal{D}^*_R\,(\overline{\mathfrak{U}}(g))\Big) \geq S_d(\overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(f)), \overline{\mathfrak{U}}(\overline{\mathfrak{U}}(g))) = S_d(\overline{\mathfrak{U}}(f), \overline{\mathfrak{U}}(g))$. The other case is proved similarly.

(P4) $\delta_{\mathcal{D}_R}(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g)) = \bigvee_{a \in X} \Big(\overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(h))(a) \wedge \mathcal{D}^*_R\,(\overline{\mathfrak{U}}(f) \vee \overline{\mathfrak{U}}(g))\Big) \leq \bigvee_{a \in X} \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(h))(a) \wedge \Big(\mathcal{D}^*_R\,(\overline{\mathfrak{U}}(f)) \vee \mathcal{D}^*_R\,(\overline{\mathfrak{U}}(g))\Big) \leq \Big(\bigvee_{a \in X} \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(h))(a) \wedge \mathcal{D}^*_R\,(\overline{\mathfrak{U}}(f))\Big) \vee \Big(\bigvee_{a \in X} \overline{\mathfrak{U}}\,(\overline{\mathfrak{U}}(h))(a) \wedge \mathcal{D}^*_R\,(\overline{\mathfrak{U}}(g))\Big) = \delta_{\mathcal{D}_R}(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(f)) \vee \delta_{\mathcal{D}_R}\,(\overline{\mathfrak{U}}(h), \overline{\mathfrak{U}}(g))$. □

Theorem 4.5. *Given $(X, \mathcal{D}_{R_X})$ and $(Y, \mathcal{D}_{R_Y})$ be **L-FRIS** and $\eta : (X, \mathcal{D}_{R_X}) \to (Y, \mathcal{D}_{R_Y})$ be an LF-ideal function, it follows that $\eta : (X, \delta_{\mathcal{D}_{R_X}}) \to (Y, \delta_{\mathcal{D}_{R_Y}})$ is a LF-proximity function.*

Proof. For every $f, g \in L^Y$, we have $\delta_{\mathcal{D}_{R_X}}\,(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)), \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(g))) = \bigvee_{a \in X} \Big(\overline{\mathfrak{U}}_X\,(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)))(a) \wedge \mathcal{D}^*_{R_X}\,(\overline{\mathfrak{U}}_X\,(\eta^{\leftarrow}(g)))\Big) \leq \bigvee_{a \in X} \Big(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f))(a) \wedge \mathcal{D}^*_{R_Y}(\overline{\mathfrak{U}}_Y(g))\Big) \leq \bigvee_{a \in X} \Big(\overline{\mathfrak{U}}_Y(f)\,(\eta(a)) \wedge \mathcal{D}^*_{R_Y}\,(\overline{\mathfrak{U}}_Y(g))\Big) \leq \bigvee_{y \in Y} \overline{\mathfrak{U}}_Y(\overline{\mathfrak{U}}_Y(f))(b) \wedge \mathcal{D}^*_{R_Y}\,(\overline{\mathfrak{U}}_Y(g)) = \delta_{\mathcal{D}_{R_Y}}(\overline{\mathfrak{U}}_Y(f), \overline{\mathfrak{U}}_Y(g))$. □

Theorem 4.6. *If $\eta : (X, \delta_{R_X}) \to (Y, \delta_{R_Y})$ is a LF-proximity function, then $\eta : (X, \mathcal{D}_{\delta_{R_X}}) \to (Y, \mathcal{D}_{\delta_{R_Y}})$ is an LF-ideal function.*

Proof. For every $f \in L^Y$, we have $\mathcal{D}^k_{\delta_{R_Y}}(\overline{\mathfrak{U}}_Y(f)) = \delta^*_{R_Y}\,(\overline{\mathfrak{U}}_Y(k), \overline{\mathfrak{U}}_Y\,(\overline{\mathfrak{U}}_Y(f))) = \delta^*_{R_Y}\,(\overline{\mathfrak{U}}_Y(k), \overline{\mathfrak{U}}_Y(f)) \leq \delta^*_{R_X}\,(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(k))), \overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)))) = \delta^*_{R_X}\,(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(k))), \overline{\mathfrak{U}}_X(\overline{\mathfrak{U}}_X\,(\eta^{\leftarrow}(f)))) = \mathcal{D}^{\eta^{\leftarrow}(k))}_{\delta_{R_X}}\,(\overline{\mathfrak{U}}_X(\eta^{\leftarrow}(f)))$. □

From Theorems 4.3 and 4.5, we obtain a concrete functor Υ : **L-FRIS** → **CL-FRPRX**. Also from Theorems 4.4 and 4.6, we obtain a concrete functor Ω : **CL-FRPRX** → **L-FRIS**.

Example 4.7. (1) Define $\mathcal{D}_1 : L^X \to L$ as $\mathcal{D}_1(f) = \bigwedge_{a \in X} \overline{\mathfrak{U}}^*(f)(a)$, $\mathcal{D}_1$ is **AL-FRIS**. By Theorem 4.4, we have $\delta_{\mathcal{D}_1}(f, g) = \bigvee_{a \in X} \overline{\mathfrak{U}}(f)(a) \wedge \mathcal{D}^*_1(g) = \bigvee_{a \in X} \overline{\mathfrak{U}}(f)(a) \wedge \bigvee_{b \in X} \overline{\mathfrak{U}}(g)(b)$.

(2) Define $\mathcal{D}_2 : L^X \to L$ as $\mathcal{D}_2(f) = \overline{\mathfrak{U}}^*(f)(a)$. Hence, $\mathcal{D}_2$ is **AL-FRIS**. By Theorem 4.4, we have $\delta_{\mathcal{D}_2}(f,g) = \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \mathcal{D}_2^*(g) = \bigvee_{a\in X} \overline{\mathfrak{U}}(f)(a) \wedge \overline{\mathfrak{U}}(g)(a)$.

5. Application(proximity induced by fuzzy graph)

A fuzzy topological graph, as defined by El-Atik et al., is a fuzzy topological structure created from a fuzzy set graph that provides useful models for three or more sets of topological spaces.

Algorithm. Input: a Universal set (X), a Fuzzy topological space (τ), a Fuzzy relation (R).

Output: $\overline{\mathfrak{U}}, \delta_R, C_R, \mathcal{D}_R, C_{\delta_R}, \delta_{C_R}, C_{\delta_{C_R}}, \delta_{C_{\delta_R}}, \mathcal{D}^b_{\delta_R}, \delta_{\mathcal{D}_R}$.

$\forall a,b,c \in \tau, a_\iota, a_j \in X$ and $\iota, j \in \{1,2,3\}$

Step 1: $\overline{\mathfrak{U}}(a)(a_\iota) = \bigvee_{a_j\in X} R(a_\iota, a_j) \wedge a(a_j)$.

Step 2: $\delta_R(a,b) = \bigvee_{a_\iota\in X} \overline{\mathfrak{U}}(a)(a_\iota) \wedge \overline{\mathfrak{U}}(b)(a_\iota)$.

Step 3: $C_R(a)(a_\iota) = \bigvee_{a_j\in X} R(a_\iota, a_j) \wedge a(a_j)$.

Step 4: $\mathcal{D}_R(a) = \bigwedge_{a_j\in X} R^*(a_\iota, a_j) \vee a^*(a_j)$.

Step 5:$C_{\delta_R}(a)(a_\iota) = \delta_R(\overline{\mathfrak{U}}(\top_{a_\iota}), \overline{\mathfrak{U}}(a))$.

Step 6:$\delta_{C_R}(a,b) = \bigvee_{a_\iota\in X} C_R(b)(a_\iota) \wedge \overline{\mathfrak{U}}(a)(a_\iota)$.

Step 7: $C_{\delta_{C_R}}(a)(a_\iota) = \delta_{C_R}(\overline{\mathfrak{U}}(\top_{a_\iota}), \overline{\mathfrak{U}}(a))$.

Step 8:$\delta_{C_{\delta_R}}(a,b) = \bigvee_{a_\iota\in X} C_{\delta_R}(b)(a_\iota) \wedge \overline{\mathfrak{U}}(a)(a_\iota)$.

Step 9: $\mathcal{D}^b_{\delta_R}(a) = \begin{cases} \delta_R^*(\overline{\mathfrak{U}}(b), \overline{\mathfrak{U}}(a)), & \text{if } a \neq \top_X \\ \top, & \text{if } a = \top_X. \end{cases}$

Step 10: $\delta_{\mathcal{D}_R}(a,b) = \bigvee_{a_\iota\in X} \overline{\mathfrak{U}}(a)(a_\iota) \wedge \mathcal{D}_R^*(b)$.

Example 5.1. Given a universal fuzzy set $X = \{(a_1, 0.4), (a_2, 0.6), (a_3, 0.2)\}$, and $\tau = \{d = 0, a = \{(a_1, 0.4)\}, b = \{(a_1, 0.4), (a_2, 0.6)\}, c = \{(a_1, 0.4), (a_2, 0.6), (a_3, 0.2)\}\}$ is a fuzzy topological graph on X as presented in Fig. 2. If $(L = [0,1], \wedge, \to, *, 0, 1)$ is a complete residuated lattice with for every $a_\iota, a_j \in X$, $a_\iota \to a_j = min\{1 - a_\iota + a_j, 1\}$. Define R as $R(a_\iota, a_j) = a_\iota \to a_j$,. Then, $R = \{((a_1,a_1),1), ((a_1,a_2),1), ((a_1,a_3),0.8), ((a_2,a_1),0.8), ((a_2,a_2),1), ((a_2,a_3),0.6), ((a_3,a_1),1), ((a_3,a_2),1), ((a_3,a_3),1)\}$.

To evaluate δ_R between any two fuzzy subsets, we calculate the upper approximation of eachelement "a " in τ, where $\overline{\mathfrak{U}}(a)(a_\iota) = \bigvee_{a_j\in X} R(a_\iota, a_j) \wedge a(a_j)$. Then, we have

$\overline{\mathfrak{U}}(a)(a_1) = 0.4 \quad \overline{\mathfrak{U}}(a)(a_2) = 0.4 \quad \overline{\mathfrak{U}}(a)(a_3) = 0.4;$
$\overline{\mathfrak{U}}(b)(a_1) = 0.6 \quad \overline{\mathfrak{U}}(b)(a_2) = 0.6 \quad \overline{\mathfrak{U}}(b)(a_3) = 0.6;$
$\overline{\mathfrak{U}}(c)(a_1) = 0.6 \quad \overline{\mathfrak{U}}(c)(a_2) = 0.6 \quad \overline{\mathfrak{U}}(c)(a_3) = 0.6;$
$\overline{\mathfrak{U}}(d)(a_1) = 0 \quad \overline{\mathfrak{U}}(d)(a_2) = 0 \quad \overline{\mathfrak{U}}(d)(a_3) = 0.$

Define $\delta_R(a,b) = \bigvee_{a_\iota\in X} \overline{\mathfrak{U}}(a)(a_\iota) \wedge \overline{\mathfrak{U}}(b)(a_\iota)$. For $a, b \in \tau$, we have

$\delta_R(a,b) = 0.4 \quad \delta_R(a,c) = 0.4 \quad \delta_R(a,d) = 0$
$\delta_R(b,c) = 0.6 \quad \delta_R(b,d) = 0 \quad \delta_R(c,d) = 0$

Define $C_R(a)(a_\iota) = \bigvee_{a_j\in X} R(a_\iota, a_j) \wedge a(a_j)$,

$C_R(a)(a_1) = 0.4 \quad C_R(a)(a_2) = 0.4 \quad C_R(a)(a_3) = 0.4;$
$C_R(b)(a_1) = 0.6 \quad C_R(b)(a_2) = 0.6 \quad C_R(b)(a_3) = 0.6;$
$C_R(c)(a_1) = 0.6 \quad C_R(c)(a_2) = 0.6 \quad C_R(c)(a_3) = 0.6;$
$C_R(d)(a_1) = 0 \quad C_R(d)(a_2) = 0 \quad C_R(d)(a_3) = 0.$

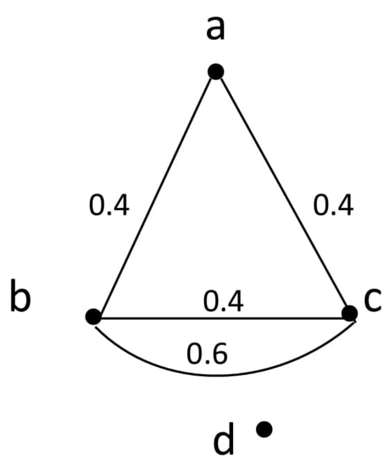

Figure 2: A fuzzy topological graph for 3 points

Define $\mathcal{D}_R(a) = \bigwedge_{a_j \in X} R^*(a_\iota, a_j) \vee a^*(a_j)$,

$\mathcal{D}_R(a)(a_1) = 0.6 \quad \mathcal{D}_R(a)(a_2) = 0.6 \quad \mathcal{D}_R(a)(a_3) = 0.6;$
$\mathcal{D}_R(b)(a_1) = 0.4 \quad \mathcal{D}_R(b)(a_2) = 0.4 \quad \mathcal{D}_R(b)(a_3) = 0.4;$
$\mathcal{D}_R(c)(a_1) = 0.4 \quad \mathcal{D}_R(c)(a_2) = 0.4 \quad \mathcal{D}_R(c)(a_3) = 0.4;$
$\mathcal{D}_R(d)(a_1) = 1 \quad \mathcal{D}_R(d)(a_2) = 1 \quad \mathcal{D}_R(d)(a_3) = 1.$

As $C_{\delta_R}(a) = \delta_R(\overline{\mathfrak{U}}(\top_{a_\iota}), \overline{\mathfrak{U}}(a))$ and $\top_{a_\iota}(a_j) = \begin{cases} 1, & \text{if } a_j = a_\iota, \\ 0, & \text{O.W.,} \end{cases}$. Then,

$C_{\delta_R}(a)(a_1) = 0.4 \quad C_{\delta_R}(a)(a_2) = 0.4 \quad C_{\delta_R}(a)(a_3) = 0.4;$
$C_{\delta_R}(b)(a_1) = 0.6 \quad C_{\delta_R}(b)(a_2) = 0.6 \quad C_{\delta_R}(b)(a_3) = 0.6;$
$C_{\delta_R}(c)(a_1) = 0.6 \quad C_{\delta_R}(c)(a_2) = 0.6 \quad C_{\delta_R}(c)(a_3) = 0.6;$
$C_{\delta_R}(d)(a_1) = 0 \quad C_{\delta_R}(d)(a_2) = 0 \quad C_{\delta_R}(d)(a_3) = 0.$

Where $\delta_{C_R}(a, b) = N_d(C_R(b), \overline{\mathfrak{U}}(a))$, where $N_d(a, b) = \bigvee_{a_\iota \in X} a(a_\iota) \wedge b(a_\iota)$. Then, we have

$\delta_{C_R}(a, b) = 0.4 \quad \delta_{C_R}(a, c) = 0.4 \quad \delta_{C_R}(a, d) = 0$
$\delta_{C_R}(b, c) = 0.6 \quad \delta_{C_R}(b, d) = 0 \quad \delta_{C_R}(c, d) = 0$

Now, we find that
$\delta_{C_{\delta_R}}(a, b) = 0.4 \quad \delta_{C_{\delta_R}}(a, c) = 0.4 \quad \delta_{C_{\delta_R}}(a, d) = 0$
$\delta_{C_{\delta_R}}(b, c) = 0.6 \quad \delta_{C_{\delta_R}}(b, d) = 0 \quad \delta_{C_{\delta_R}}(c, d) = 0$

Consequently, we also find that
$C_{\delta_{C_R}}(a)(a_1) = 0.4 \quad C_{\delta_{C_R}}(a)(a_2) = 0.4 \quad C_{\delta_{C_R}}(a)(a_3) = 0.4;$
$C_{\delta_{C_R}}(b)(a_1) = 0.6 \quad C_{\delta_{C_R}}(b)(a_2) = 0.6 \quad C_{\delta_{C_R}}(b)(a_3) = 0.6;$
$C_{\delta_{C_R}}(c)(a_1) = 0.6 \quad C_{\delta_{C_R}}(c)(a_2) = 0.6 \quad C_{\delta_{C_R}}(c)(a_3) = 0.6;$
$C_{\delta_{C_R}}(d)(a_1) = 0 \quad C_{\delta_{C_R}}(d)(a_2) = 0 \quad C_{\delta_{C_R}}(d)(a_3) = 0.$

we find out that, $C_{\delta_{C_R}} \geq C_R$ and $\delta_R \geq \delta_{C_{\delta_R}}$.

According to Theorem 4.3 , we define

$$\mathcal{D}^b_{\delta_R}(a) = \begin{cases} \delta^*_R(\overline{\mathfrak{U}}(b), \overline{\mathfrak{U}}(a)), & \text{if } a \neq \top_X \\ \top, & \text{if } a = \top_X. \end{cases}$$

$\mathcal{D}^a_{\delta_R}(b) = 0.6 \quad \mathcal{D}^d_{\delta_R}(a) = 1 \quad \mathcal{D}^a_{\delta_R}(c) = 0.6$
$\mathcal{D}^d_{\delta_R}(b) = 1 \quad \mathcal{D}^b_{\delta_R}(c) = 0.4 \quad \mathcal{D}^d_{\delta_R}(c) = 1$

Define, $\delta_{\mathcal{D}_R}(a, b) = \bigvee_{a_\iota \in X} \overline{\mathfrak{U}}(a)(a_\iota) \wedge \mathcal{D}^*_R(b)$.

$\delta_{\mathcal{D}_{\delta_R}}(a, b) = 0.4 \quad \delta_{\mathcal{D}_{\delta_R}}(a, c) = 0.4 \quad \delta_{\mathcal{D}_{\delta_R}}(a, d) = 0$
$\delta_{\mathcal{D}_{\delta_R}}(b, c) = 0.6 \quad \delta_{\mathcal{D}_{\delta_R}}(b, d) = 0 \quad \delta_{\mathcal{D}_{\delta_R}}(c, d) = 0$

6. Conclusion and works ahead

In such consideration, we study the relationships between Čech L-fuzzy rough proximities and Čech L-fuzzy rough closure spaces. We prove that there exists a functor connecting between Čech L-fuzzy rough proximity spaces and Čech L-fuzzy rough closure spaces; in light of that, we discuss the relations between their categories. Additionally, we introduce L-fuzzy rough ideals and discover the connection between L-fuzzy rough ideals and Čech L-fuzzy rough proximity spaces. A direction worthy of future work is to study the theory of topogenous and uniformity via rough sets.

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