Filomat 40:2 (2026), 619–648
https://doi.org/10.2298/FIL2602619T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Equivalence of some factorization properties in topological algebra

Mikhail Tkachenko[a]

[a]*Departamento de Matemáticas Universidad Autónoma Metropolitana Av. San Rafael Atlixco 186, Col. Vicentina, Iztapalapa C.P. 09340, Mexico City, Mexico*

Abstract. We show that the original concept of $\mathbb{R}$-factorizability, as well as some of its modifications, examined in the realms of topological, paratopological, and semitopological groups possess an essential feature of *absoluteness* when transitioning to a broader category. This resolves a certain ambiguity in the research conducted to date and enables us to keep 'old' notation for formally different notions of factorizability.

It is also shown that a paratopological group G is $\mathbb{R}$-factorizable if and only if its T_i-reflection, $T_i(G)$, is $\mathbb{R}$-factorizable for some (equivalently, for each) $i \in \{0, 1, 2, 3\}$, which in turn is equivalent to the regular reflection $Reg(G)$ of G being $\mathbb{R}$-factorizable. When substituting the aforementioned reflections with the quotient group G/N, where N is the closure of the singleton $\{e_G\}$, this result holds true for every topological group G. The latter results indicate a specific form of stability regarding the concept of $\mathbb{R}$-factorizability. Several routes for further investigation are outlined at the end of the article.

Contents

2020 *Mathematics Subject Classification*. Primary 22A05; Secondary 54C30, 54H11.
Keywords. Paratopological group, semitopological group, second-countable, ω-narrow, ω-balanced, continuous homomorphism, $\mathbb{R}$-factorizable group.
Received: 07 March 2025; Revised: 14 July 2025; Accepted: 10 November 2025
Communicated by Ljubiša D. R. Kočinac
The author was supported by grant number CAR-64356 of the Program "Ciencia de Frontera 2019" of the CONAHCyT, Mexico.
Email address: mich@xanum.uam.mx (Mikhail Tkachenko)
ORCID iD: https://orcid.org/0000-0002-1698-381X (Mikhail Tkachenko)

1. Introduction

The concept of $\mathbb{R}$-*factorizability* in topological groups was defined in [23] and studied in detail in [4, Chapter 8]. It has been demonstrated that $\mathbb{R}$-factorizable topological groups exhibit numerous noteworthy properties, in particular in relation to dimensionality and cardinal characteristics. $\mathbb{R}$-factorizable topological groups include precompact groups and Lindelöff groups, and they share several important features with these groups.

Numerous variations of the original concept of $\mathbb{R}$-factorizability have been proposed to date. These include m-factorizability [4, Section 8.5], $\mathcal{M}$-factorizability [13, 32, 33], sm-factorizability [5, 29, 30], $P\mathbb{R}$-factorizability [14] and Ψ_ω-factorizability [34]. Other four variations are introduced in [10]. The list of modifications of the concept continues to grow. Furthermore, several of the aforementioned sources, including those co-authored by the author of this article, apply the concepts of factorizability indistinctively across various categories of topological algebra, such as topological groups, paratopological groups, and semitopological groups. This leads to a number of conceptual and notational issues.

This article has two primary objectives. Initially, in Subsection 1.1, we aim to standardize the notations for various types of factorization in topological algebra. The universality and precision of the proposed notation incurs a cost in terms of length. To mitigate this issue, we suggest two methods for abbreviating the new notation.

The second and main objective is to demonstrate that the concept of $\mathbb{R}$-factorizability, as well as specific variations, remains consistent when the category in which one operates is expanded. Again, this is further clarified with necessary details in Subsection 1.1. Consequently, nearly all results regarding $\mathbb{R}$-factorizability and its variations that have been established in a context of ambiguity are now validated, albeit occasionally requiring additional thorough verification. Furthermore, Theorems 4.13, 4.16, 4.17, 4.22, 4.23, and 4.27 demonstrate that it is possible to retain the traditional notation for various well-established concepts, such as $\mathbb{R}$-factorizability and $\mathcal{M}$-factorizability, without the necessity of designating one of the four categories (topological, paratopological, quasitopological, or semitopological groups). To put it simply, the theorems assert that a topological (paratopological, quasitopological) group G that is factorizable in the broader category $\mathbb{SG}$ of semitopological groups remains factorizable in its own category $\mathbb{TG}$ of topological groups (respectively, in the categories $\mathbb{PG}$ or $\mathbb{QG}$ of paratopological or quasitopological groups).[1)] This is referred to as the *absoluteness* of factorization properties. In a sense, our results expand upon the research conducted in [28], which demonstrates that the notions of $\mathbb{R}_0$-, $\mathbb{R}_1$-, $\mathbb{R}_2$-, and $\mathbb{R}_3$-factorizability in paratopological groups are equivalent to $\mathbb{R}$-factorizability, where no separation restrictions are imposed on groups.

The proofs of the main results rely on the concept of *quasitopological group associated to a semitopological group*, which we introduce and discuss in Section 3. A similar concept for paratopological groups was previously considered by T. Banakh and A. Ravsky in [7]. In categorical terms, a quasitopological group associated to a given semitopological group G, and denoted by G^*, is algebraically the same group that possesses the coarsest quasitopological group topology, which is finer than the topology of G. We also examine the connections between G and G^*, and establish a number of helpful facts that are crucial to Section 4. Both constructions mentioned are partial instances of the concept known as *symmetrization* within the framework of bitopological spaces (refer to [2] for further details).

A considerable amount of our discussion focuses on preserving $\mathbb{R}$-factorizability in (para)topological and semitopological groups when taking *reflections* of the groups, as defined by the author in [26, 27]. The research in this direction started in [15] by L.-X. Peng and Y.-M. Deng. They demonstrated, among other results, that a semitopological group G is $\mathcal{F}\mathbb{R}$-factorizable (resp., $\mathcal{F}m$-factorizable) if and only if the Hausdorff reflection of G, $T_2(G)$, possesses the same property. Also, according to [15, Theorem 3.12], a paratopological group G is $\mathcal{F}\mathbb{R}$-factorizable (resp., $\mathcal{F}m$-factorizable) if and only if so is the regular reflection of the group G, denoted by $Reg(G)$.

We prove in Theorem 4.18 that a paratopological group G is $\mathbb{R}$-factorizable if and only if $T_i(G)$ is $\mathbb{R}$-factorizable for some $i \in \{0, 1, 2, 3\}$ if and only if $Reg(G)$ is $\mathbb{R}$-factorizable, where $T_i(G)$ is the T_i-reflection of

1) It is important to inform the reader that the proofs regarding quasitopological groups in Theorems 4.6 and 4.27 are dependent upon the additional assumption of complete regularity of the groups.

G. An analogous result holds true for a topological group G (see Theorem 4.17). In the latter case, however, the five reflections of G considered in Theorem 4.18 merge into a singular object namely the quotient group G/N. Here, N is the closure in G of the singleton $\{e\}$, where e is the identity of G. Therefore, a topological group G is $\mathbb{R}$-factorizable if and only if so is G/N (see Corollary 2.8).

A significant amount of attention has been dedicated to $\mathcal{M}$-factorizable groups, which are topological groups that admit a continuous factorization of continuous real-valued functions through a continuous homomorphism onto a Hausdorff first-countable (equivalently, metrizable) topological group [13, 32, 33]. There has been a lack of extensive research on a similar concept in the category of paratopological groups. One of the reasons for this is that regular first-countable paratopological groups are not necessarily metrizable or even normal. In Theorem 4.16, we show that the factorizability of continuous real-valued functions on a paratopological group G in first-countable *semitopological* groups is equivalent to the factorizability of these functions on G in first-countable *paratopological* groups. Theorem 4.19 asserts that the continuous real-valued functions on a paratopological group G are factorizable through first-countable paratopological groups if and only if the regular reflection of G, $Reg(G)$, has the same property. In Theorems 4.22 and 4.23, we extend the conclusions of the aforementioned results to $\mathcal{M}$-factorizable (para)topological groups.

Theorem 4.25 shows that the factorizability of the continuous real-valued functions on a completely regular semitopological group G in first-countable (resp., second-countable) semitopological groups is equivalent to their factorizability in *completely regular* first-countable (resp., second-countable) semitopological groups. We do not know, however, whether the regular or Tychonoff reflection of an arbitrary $\mathbb{R}$-factorizable semitopological group is $\mathbb{R}$-factorizable (see Problems 6.1 and 6.2). We conclude by complementing Theorem 4.25 with Theorems 4.26, 4.27, and Corollary 4.28.

In a sense, the validity of our main results on the absoluteness (that is, non-dependence on a category of objects and/or axioms of separation) and stability (non-dependence on taking reflections and/or quotient groups) of the aforementioned factorization properties is due to a strong categorical flavor of the original definition of $\mathbb{R}$-factorizability. However, our proofs also contain a substantial topological component. In particular, we frequently use the celebrated result by T. Banakh and A. Ravsky from [8] stating that a regular paratopological group is completely regular. A more general form of the latter result, for topological monoids with open shifts, is given by the same authors in [8, Corollary 4]. Proposition 4.4 and Lemma 4.24, which connect the continuity of operations in a subgroup of a Cartesian product of semitopological groups or the complete regularity of the subgroup with those of its projections to countable subproducts, are also crucial components of our research.

Additionally, in Remark 4.29 and in a brief Section 5, we suggest a program for further study in the field.

In Section 6, several open problems are presented, aimed at determining which equivalencies established in Theorems 4.13, 4.18 and 4.25 continue to hold for semitopological or quasitopological groups. Also, we recall two open problems from [21] regarding the behavior of $\mathbb{R}$-factorizability in paratopological groups and the associated topological groups.

1.1. Notation, terminology, and preliminary facts

The identity element of a group G is denoted by e_G or, in instances where no ambiguity arises, simply by e. The kernel of a group homomorphism $p\colon G \to H$ is denoted as $\ker p$.

A semitopological group G is *ω-narrow* if for each neighborhood U of the identity in G, there is a countable set $C \subset G$ such that $CU = G = UC$. The aforementioned equalities for a quasitopological group G can be reduced into a single one, $G = UC$ or, alternatively, $G = CU$ (see [4, Section 3.4]).

We adhere to [4] in terms of notation and terminology pertaining to topological algebra, with the exception that we do not require *a priori* that the objects we are considering satisfy the T_1 separation axiom.

For every $i \in \{0, 1, 2, 3, r, 3.5, t\}$ we consider the classes of spaces and groups with topologies satisfying the T_i separation axiom, with 'r' reserved for 'regular' and 't' for 'Tychonoff'. If a space satisfies both the T_3 and T_1 separation axioms, the space is said to be *regular* or a T_r-space. Similarly, completely regular (equivalently, Tychonoff) spaces are exactly T_1-spaces in which continuous real-valued functions separate points and closed sets. In other words, Tychonoff = $T_{3.5}$ & T_1. This is in contrast to [11], in which T_3-spaces are defined as regular T_1-spaces.

When factoring a continuous mapping $f\colon G \to X$ of a given topological (paratopological, quasitopological, semitopological, etc.) group G to a space X from a class $\mathbb{X}$ (say, X is Hausdorff second-countable or metrizable), one chooses a class $\mathbb{O}$ of objects of topological algebra and a class $\mathbb{H}$ of continuous homomorphisms.

Definition 1.1. A group G with a topology is $(\mathbb{X}, \mathbb{O}, \mathbb{H})$-factorizable if for every continuous mapping $f\colon G \to X$ to a space $X \in \mathbb{X}$, there exists a continuous surjective homomorphism $p\colon G \to H$, where $H \in \mathbb{O}$ and $p \in \mathbb{H}$, such that $f = g \circ p$, for some continuous mapping $g\colon H \to X$.

We introduce the following notation:

- $\overline{\mathbb{R}}$ is the class that contains only one space, the real line $\mathbb{R}$;
- $\mathbb{SC}$ is the class of second-countable spaces (no separation restrictions on spaces are imposed);
- $\mathbb{TG}$, $\mathbb{PG}$, $\mathbb{QG}$, and $\mathbb{SG}$ are the classes of topological, paratopological, quasitopological and semitopological groups, respectively;
- $\mathbb{FCTG}$, $\mathbb{FCPG}$, $\mathbb{FCQG}$, and $\mathbb{FCSG}$ are the classes of first-countable topological, paratopological, quasitopological and semitopological groups, respectively;
- $\mathbb{SCTG}$, $\mathbb{SCPG}$, $\mathbb{SCQG}$, and $\mathbb{SCSG}$ are the classes of second-countable topological, paratopological, quasitopological and semitopological groups, respectively;
- $\mathbb{MS}$ is the class of metrizable spaces;
- $\mathbb{MTG}$, $\mathbb{MPG}$, $\mathbb{MQG}$, and $\mathbb{MSG}$ are the classes of metrizable topological, paratopological, quasitopological and semitopological groups, respectively;
- Given $k \in \{0, 1, 2, 3, r, 3.5, t\}$ (with r standing for *regular* and t for *Tychonoff*), and a topological or topological algebra category $\mathbb{O}$, we denote by $\mathbb{O}_k$ the full subcategory of $\mathbb{O}$ consisting of the objects $G \in \mathbb{O}$ that satisfy the T_k separation axiom;
- $\mathbb{CH}$ is the class of continuous homomorphisms (of arbitrary objects of topological algebra where the concept of homomorphism is defined);
- $\mathbb{PH}$ is the class of perfect homomorphisms;
- $\mathbb{OH}$ is the class of open continuous homomorphisms.

By Definition 1.1, we see that the $(\overline{\mathbb{R}}, \mathbb{SCTG}_2, \mathbb{CH})$-factorizability or, by Proposition 2.4, $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{CH})$-factorizability of a topological group abbreviates to $\mathbb{R}$-factorizability. Likewise, the $\mathbb{R}$-factorizability of a paratopological group G abbreviates its $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{CH})$-factorizability. According to [28, Theorem 3.8], the latter property is equivalent to $(\overline{\mathbb{R}}, \mathbb{SCPG}_2, \mathbb{CH})$-factorizability, which in turn is equivalent to $(\overline{\mathbb{R}}, \mathbb{SCPG}_r, \mathbb{CH})$-factorizability of G.

Further, *openly factorizable* topological groups, a concept that implicitly traces back to [6], are exactly $(\mathbb{SC}_r, \mathbb{SCTG}_2, \mathbb{OH})$-factorizable. Also, $\mathcal{M}$-*factorizable* topological groups introduced and studied in [32], [33] and [13] are $(\overline{\mathbb{R}}, \mathbb{MTG}, \mathbb{CH})$-factorizable groups. $\mathcal{F}\mathbb{R}$- and $\mathcal{F}m$-factorizable semitopological groups from [15] are, respectively, $(\overline{\mathbb{R}}, \mathbb{FCSG}, \mathbb{CH})$- and $(\mathbb{MS}, \mathbb{FCSG}, \mathbb{CH})$-factorizable groups. To somewhat finalize this list, we note that $P\mathbb{R}$-factorizable topological groups, as defined in [14], correspond to $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{PH})$-factorizable groups, while Ψ_ω-factorizable groups in [34] are $(\overline{\mathbb{R}}, \Psi_\omega, \mathbb{CH})$-factorizable, where Ψ_ω is the class of topological groups with countable pseudocharacter. It is worth noting that in the last two paragraphs, $\overline{\mathbb{R}}$ can be replaced with $\mathbb{SC}_r$ (see Proposition 2.1 and Corollary 2.2 or [17, Lemma 2.3]).

The new notation seems to be heavily formalized. Nonetheless, it enables us to bypass an issue that requires our attention here.

Every topological group is naturally a paratopological and quasitopological group, and every paratopological group can be considered as a semitopological group. Given a topological group G, one can formally apply four different definitions of $\mathbb{R}$-factorizability to the group:

- $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{CH})$-factorizability,
- $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{CH})$-factorizability,
- $(\overline{\mathbb{R}}, \mathbb{SCQG}, \mathbb{CH})$-factorizability, and
- $(\overline{\mathbb{R}}, \mathbb{SCSG}, \mathbb{CH})$-factorizability.

The new notation reveals the difference between the four notions, while the "generic" reference to $\mathbb{R}$-factorizability of G is ambiguous because it does not mention the category we use to factorize continuous real-valued functions on G. A second-countable codomain, H, of a continuous homomorphism $p\colon G \to H$ may be a topological, paratopological, quasitopological, or semitopological group, contingent upon the selection of a rigorous definition of $\mathbb{R}$-factorizability. A similar, somewhat less critical issue occurs in the case of a paratopological group G, to which the second and fourth notions mentioned are naturally applicable.

To shorten new notation, we apply the following simple rules:

(A) The "old" short notation is used for 'well-established' concepts, like $\mathbb{R}$-*factorizable topological (paratopological, quasitopological) group* or $\mathcal{M}$-*factorizable topological (paratopological, quasitopological) group*.

(B) The third argument, $\mathbb{CH}$, of the parenthetical modifier is omitted if $p\colon G \to H$ is a continuous homomorphism (not a perfect or open one) in the definition of some kind of factorizability, that is, $p \in \mathbb{CH}$. [In particular, $(\overline{\mathbb{R}}, \mathbb{SCSG}, \mathbb{CH})$-factorizability shortens to $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizability.]

Our Theorems 4.6, 4.13, 4.17, 4.22 and 4.23 support the convention in the aforementioned item (A) by stating that the concept remains unaffected by the choice of a broader category for codomains of continuous homomorphisms $p\colon G \to H$ in the cases of $\mathbb{R}$- or $\mathcal{M}$-factorizability.

2. Elementary results

In this section, we compile several results regarding the equivalence of a variety of factorization properties. The proofs of these results are either easily obtainable from previously established facts or are of elementary complexity.

As usual, a class $\mathbb{O}$ of groups with topologies is called *countably productive* if the Cartesian product $\prod_{n\in\omega} G_n$ is in $\mathbb{O}$, provided each factor G_n is in $\mathbb{O}$. Additionally, if every subgroup H of a group $G \in \mathbb{O}$ with the topology inherited from G is also in $\mathbb{O}$, we say that the class $\mathbb{O}$ is *hereditary*.

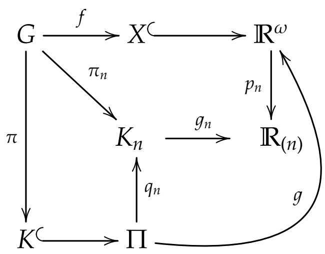
The following proposition has its origin in [4, Lemma 8.1.2]; it elucidates the reasoning behind the substitution of the real line with the class $\mathbb{SC}_r$ of regular second-countable spaces in various modifications of $\mathbb{R}$-factorizability. Also, it presents a considerably more general version of [17, Lemma 2.3]. In fact, our argument in the proof of the proposition is close to the one in [4].

Proposition 2.1. *Let $\mathbb{O}$ be a countably productive and hereditary class of groups with topologies and assume that $\mathbb{H} \in \{\mathbb{CH}, \mathbb{PH}\}$. Then $(\overline{\mathbb{R}}, \mathbb{O}, \mathbb{H})$-factorizability and $(\mathbb{SC}_r, \mathbb{O}, \mathbb{H})$-factorizability coincide.*

Proof. It is clear that $(\mathbb{SC}_r, \mathbb{O}, \mathbb{H})$-factorizability implies $(\overline{\mathbb{R}}, \mathbb{O}, \mathbb{H})$-factorizability. So we must derive the converse implication.

Let G be an $(\overline{\mathbb{R}}, \mathbb{O}, \mathbb{H})$-factorizable group with a topology and $f\colon G \to X$ be a continuous mapping of G to a regular second-countable space X. According to [11, Theorem 2.3.23], we can identify X with a subspace of $\mathbb{R}^\omega$. For every $n \in \omega$, denote by p_n the projection of $\mathbb{R}^\omega$ to the nth factor. Then $p_n \circ f$ is a continuous real-valued function on G, so we can find a continuous homomorphism $\pi_n\colon G \to K_n$ onto a group $K_n \in \mathbb{O}$ and a continuous real-valued function g_n on K_n such that $\pi_n \in \mathbb{H}$ and $p_n \circ f = g_n \circ \pi_n$. Denote by π the diagonal of the family $\{\pi_n : n \in \omega\}$. Then $\pi\colon G \to \prod_{n\in\omega} K_n$ is a continuous homomorphism and the image $K = \pi(G)$ is a subgroup of the group $\Pi = \prod_{n\in\omega} K_n$, where $\Pi \in \mathbb{O}$. Since the class $\mathbb{O}$ is hereditary, we see that $K \in \mathbb{O}$. If $\mathbb{H} = \mathbb{PH}$, then each π_n is a perfect homomorphism, and so is the diagonal π of the family $\{\pi_n : n \in \omega\}$ (see [11, Theorem 3.7.10]). Hence, $\pi \in \mathbb{PH}$ as well. If each π_n is in $\mathbb{CH}$, so is π.

For every $n \in \omega$, let $q_n\colon \Pi \to K_n$ be the projection. Then the equality $\pi_n = q_n \circ \pi$ holds for each $n \in \omega$. Finally, denote by g the Cartesian product of the family $\{g_n : n \in \omega\}$ (equivalently, the diagonal of the family $\{g_n \circ q_n : n \in \omega\}$). Then the mapping $g\colon \Pi \to \mathbb{R}^\omega$ is continuous. In addition, the diagram below commutes.



Indeed, it suffices to verify that $f = g \circ \pi$ or, equivalently, that $p_n \circ f = p_n \circ g \circ \pi$ for each $n \in \omega$. The latter equality follows from our definition of g and π:

$$p_n \circ g \circ \pi = g_n \circ q_n \circ \pi = g_n \circ \pi_n = p_n \circ f.$$

Therefore, the homomorphism $\pi\colon G \to K$ and the mapping $h = g{\restriction}K$ satisfy the equality $f = h \circ \pi$. This proves that G is $(\mathbb{SC}_r, \mathbb{O}, \mathbb{H})$-factorizable. □

The subsequent assertion is derived from Proposition 2.1, as the classes of Hausdorff second-countable or metrizable topological (paratopological, quasitopological, semitopological) groups are countably productive and hereditary.

Corollary 2.2. *The notions of factorizability in each of items* (1)–(8) *below coincide, where* $\mathbb{H} \in \{\mathbb{CH}, \mathbb{PH}\}$:

(1) $(\overline{\mathbb{R}}, \mathbb{SCSG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{SCSG}, \mathbb{H})$*-factorizability in the category of semitopological groups.*
(2) $(\overline{\mathbb{R}}, \mathbb{SCQG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{SCQG}, \mathbb{H})$*-factorizability in the category of quasitopological groups.*
(3) $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{SCPG}, \mathbb{H})$*-factorizability in the category of paratopological groups.*
(4) $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{SCTG}, \mathbb{H})$*-factorizability in the category of topological groups.*
(5) $(\overline{\mathbb{R}}, \mathbb{MTG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{MTG}, \mathbb{H})$*-factorizability in the category of topological groups.*
(6) $(\overline{\mathbb{R}}, \mathbb{MPG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{MPG}, \mathbb{H})$*-factorizability in the category of paratopological groups.*
(7) $(\overline{\mathbb{R}}, \mathbb{MQG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{MQG}, \mathbb{H})$*-factorizability in the category of quasitopological groups.*
(8) $(\overline{\mathbb{R}}, \mathbb{MSG}, \mathbb{H})$*-factorizability and* $(\mathbb{SC}_r, \mathbb{MSG}, \mathbb{H})$*-factorizability in the category of semitopological groups.*

In reference to the categories of topological, paratopological, etc., groups mentioned in items (1) through (8), we indicate that the group G in Definition 1.1 comes from one of these categories.

Since the classes $\mathbb{FCSG}$, $\mathbb{FCQG}$, $\mathbb{FCPG}$ and $\mathbb{FCTG}$ of first-countable groups are also countably productive and hereditary, one can add four new items pertaining to these categories to Corollary 2.2. In particular, for $\mathbb{H} \in \{\mathbb{CH}, \mathbb{PH}\}$, the notions of $(\overline{\mathbb{R}}, \mathbb{FCPG}, \mathbb{H})$- and $(\mathbb{SC}_r, \mathbb{FCPG}, \mathbb{H})$-factorizability coincide in the class of paratopological groups.

It is natural to ask whether items (1)–(4) of Corollary 2.2 are valid for the class $\mathbb{H} = \mathbb{OH}$ of continuous open homomorphisms. The difficulty is that, even in the category of Hausdorff topological groups, the diagonal of a family of continuous open homomorphisms can fail to be open. We answer this question affirmatively in Proposition 4.1.

When G is a topological group, the following lemma is well-known. We skip the lemma's proof because its extension to quasitopological groups is nearly automatic.

Lemma 2.3. *Let G be a quasitopological group, N be the closure of the singleton $\{e\}$ in G, where e is the identity of G, and π the canonical mapping of G onto G/N. Then the following hold:*

(a) *N is a closed invariant subgroup of G, π is an open continuous homomorphism, and the quotient quasitopological group G/N is a T_1-space.*
(b) *Every open set U in G satisfies $U = \pi^{-1}\pi(U)$.*
(c) *For every continuous mapping $f\colon G \to X$ to a T_0-space X, there exists a continuous mapping $h\colon G/N \to X$ satisfying $f = h \circ \pi$. If in addition X is a semitopological group and f is a continuous homomorphism, then h is also a continuous homomorphism.*

Lemma 2.3 can be regarded as a very special case of a considerably more general construction of the *T_k-reflection*, for $k \in \{0, 1, 2, 3, r, 3.5, t\}$, of a semitopological group, as defined in [26]. We employ 'r' and 't' to abbreviate 'regular' and 'Tychonoff', respectively, as in [26]. According to Proposition 2.5 of [26], for every semitopological group G, there exists a continuous *canonical homomorphism* $\varphi_{G,k}\colon G \to T_k(G)$ onto a semitopological group $T_k(G)$ with the following properties:

(i) The group $T_k(G)$ satisfies the T_k separation axiom.
(ii) For every continuous mapping $f\colon G \to X$ to a T_k-space X, there exists a continuous mapping $h\colon T_k(G) \to X$ satisfying $f = h \circ \varphi_{G,k}$. [Hence, if X is a semitopological group and f is a continuous homomorphism, then h is also a continuous homomorphism.]

The group $T_k(G)$, also referred to as the *T_k-reflection* of G, is unique up to a topological isomorphism, which allows a natural triangle diagram to commute.

It is now clear that the quotient group G/N in Lemma 2.3 is both the T_0-reflection and T_1-reflection of the quasitopological group G. Furthermore, if G is a topological group, then the group G/N is also the T_k-reflection of G for each $k \in \{0, 1, 2, r, t\}$. Hence, $G/N = T_t(G) = Tych(G)$, the Tychonoff reflection of G.

If G is a paratopological group, then each T_k-reflection of G is also a paratopological group [27, pp. 201–202]. An easy verification shows that the similar statement is valid for quasitopological groups as well. It is important to note that each topological group is a T_3-space.

Throughout the article, we will frequently employ the T_2- and T_r-reflections of semitopological and paratopological groups. Let us demonstrate how Hausdorff reflections work.

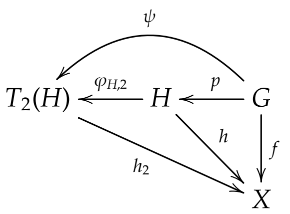
For a given class $\mathbb{O}$ of objects (semitopological groups, paratopological groups, etc.), we denote by $\mathbb{SCO}$ (resp., $\mathbb{FCO}$, $\mathbb{MO}$) the class of second countable (resp., first-countable or metrizable) objects from the class $\mathbb{O}$. This convention will be applied in Propositions 2.4, 2.5, and 2.6.

In the special case of a semitopological or paratopological group G and $\mathbb{X} = \overline{\mathbb{R}}$, item (b) of the next result can be found in [15, Lemma 3.3].

Proposition 2.4. *Let $\mathbb{X}$ be a class of Hausdorff spaces, $\mathbb{O} \in \{\mathbb{SG}, \mathbb{QG}, \mathbb{PG}, \mathbb{TG}\}$, and assume that $G \in \mathbb{O}$. Then the properties of G in each of the items below are equivalent:*

(a) $(\mathbb{X}, \mathbb{SCO})$*-factorizability and* $(\mathbb{X}, \mathbb{SCO}_2)$*-factorizability;*
(b) $(\mathbb{X}, \mathbb{FCO})$*-factorizability and* $(\mathbb{X}, \mathbb{FCO}_2)$*-factorizability.*

Proof. We start with item (a) and assume that the group G is $(\mathbb{X}, \mathbb{SCO})$-factorizable. Let $f\colon G \to X$ be a continuous mapping to a space $X \in \mathbb{X}$. By the assumption, we can find a continuous homomorphism $p\colon G \to H$ onto a second-countable group $H \in \mathbb{O}$ and a continuous mapping $h\colon H \to X$ such that $f = h \circ p$. Denote by $\varphi_{H,2}$ the canonical homomorphism of H onto the Hausdorff reflection $T_2(H)$ of H. According to [26, Proposition 2.5], the homomorphism $\varphi_{H,2}$ is open, so $T_2(H)$ is a quotient group of H. Hence, the Hausdorff group $T_2(H)$ is in the same class $\mathbb{O}$. In addition, the quotient group $T_2(H)$ is second-countable. Hence, $T_2(H)$ is a Hausdorff second-countable group. Applying property (ii) of the Hausdorff reflection $T_2(H)$, we infer that there exists a continuous mapping $h_2\colon T_2(H) \to X$ satisfying $h = h_2 \circ \varphi_{H,2}$.



Hence, the continuous homomorphism $\psi = \varphi_{H,2} \circ p$ of G onto $T_2(H)$ and a continuous mapping h_2 of $T_2(H)$ to X satisfy the equality $f = h_2 \circ \psi$. This implies that G is $(\mathbb{X}, \mathbb{SCO}_2)$-factorizable. The converse implication is evident.

A similar argument applies to item (b), as continuous open homomorphisms preserve first-countability. □

As of now, we have examined various factorization properties for the same group. Conversely, a particular factorization property may be examined within various interrelated objects of topological algebra. We start this in the following proposition that will be applied in the proof of Proposition 2.6 and in Section 4. In the proposition, the case where $\mathbb{O} = \mathbb{TG}$ is partially addressed by Lemma 2.3 (see also Theorem 4.23).

Proposition 2.5. *Let* $\mathbb{O} \in \{\mathbb{SG}, \mathbb{QG}, \mathbb{PG}, \mathbb{TG}\}$ *and* $k \in \{0, 1, 2, 3, r, 3.5, t\}$. *The following are equivalent for a group* $G \in \mathbb{O}$*:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCO}_k)$-factorizable);*

(b) *$T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCO}_k)$-factorizable).*

Furthermore, the group G is $(\overline{\mathbb{R}}, \mathbb{MO})$-factorizable if and only if so is $T_k(G)$.

Proof. Since the argument in the parenthetical case of $(\overline{\mathbb{R}}, \mathbb{SCO}_k)$-factorizability in items (a) and (b) is almost the same, we only take into consideration the case of $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizability.

Let $\varphi_k\colon G \to T_k(G)$ be the canonical homomorphism. The implication (b) ⇒ (a) is almost evident. Indeed, let f be a continuous real-valued function on G. Assume that the group $T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizable. Since the real line $\mathbb{R}$ is a T_k-space, the definition of $T_k(G)$ implies that there exists a continuous real-valued function h on $T_k(G)$ satisfying $f = h \circ \varphi_k$. By the above assumption, we can find a continuous homomorphism $p\colon T_k(G) \to H$ onto a first-countable group $H \in \mathbb{O}$ satisfying the T_k separation axiom and a continuous function h^* on H such that $h = h^* \circ p$. Then the continuous homomorphism $\psi = p \circ \varphi_k$ of G onto H satisfies $f = h^* \circ \psi$. It follows that G is $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizable.

Conversely, assume that G is $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizable and let h be a continuous real-valued function on $T_k(G)$. By our assumption, the continuous function $f = h \circ \varphi_k$ on G can be represented as the composition $f = h^* \circ p$, where p is a continuous homomorphism of G onto a first-countable semitopological group $H \in \mathbb{O}$ satisfying the T_k separation axiom, and h^* a continuous function on H. Then, by the definition of the reflection $T_k(G)$, there exists a continuous homomorphism $\psi\colon T_k(G) \to H$ satisfying $p = \psi \circ \varphi_k$. We have therefore that $h = h^* \circ \psi$, which implies the $(\overline{\mathbb{R}}, \mathbb{FCO}_k)$-factorizability of the group $T_k(G)$.

Metrizable spaces satisfy the T_k separation axiom for each $k \in \{0, 1, 2, 3, r, 3.5, t\}$. Therefore, repeating the above argument in the case of $(\overline{\mathbb{R}}, \mathbb{MO})$-factorizability, with a metrizable group $H \in \mathbb{O}$, we obtain the last statement of the proposition. □

The list of equivalences in Proposition 2.4 can be expanded further by comparing factorization properties of the groups G and $T_2(G)$. Item (b) of the following proposition, in the special case of a semitopological or paratopological group G, follows from [15, Lemma 3.8].

Proposition 2.6. *Let $\mathbb{O} \in \{\mathbb{SG}, \mathbb{QG}, \mathbb{PG}, \mathbb{TG}\}$ and assume that $G \in \mathbb{O}$. Then the following are valid:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{SCO})$-factorizable iff $T_2(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCO}_2)$-factorizable;*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{FCO})$-factorizable iff $T_2(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCO}_2)$-factorizable;*
(c) *G is $(\overline{\mathbb{R}}, \mathbb{MO})$-factorizable iff $T_2(G)$ is $(\overline{\mathbb{R}}, \mathbb{MO})$-factorizable iff the regular reflection of G, $Reg(G)$, is $(\overline{\mathbb{R}}, \mathbb{MO})$-factorizable.*

Proof. It suffices to combine Proposition 2.4 with $\mathbb{X} = \overline{\mathbb{R}}$ and Proposition 2.5, taking $k = 2$ and $k = r$ in the latter one. □

Proposition 2.6 asserts that the study of well-established variants of the concept of $\mathbb{R}$-factorizability is entirely reducible to the Hausdorff context. Given that $\mathbb{R}$-factorizability fundamentally involves continuous real-valued functions, it is reasonable to presume that this reduction can be extended to the Tychonoff scenario. We will demonstrate in Theorems 4.13, 4.22, and 4.19 that this is the case for the category of paratopological groups, whereas Theorems 4.25 and 4.26 partially validate the hypothesis for completely regular semitopological groups.

By adhering to the notation of Lemma 2.3, we can observe that $T_0(G) = T_1(G) = G/N$ for any quasitopological group G. The equalities $T_2(G) = Tych(G) = G/N$ are also valid if G is a topological group. Therefore, the subsequent two results follow from the combination of Lemma 2.3 and Proposition 2.6.

Proposition 2.7. *Let G be a quasitopological group with identity e and N be the closure of the singleton $\{e\}$ in G. Then the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{SCQG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{FCQG})$-factorizable);*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{SCQG}_2)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{FCQG}_2)$-factorizable);*
(c) *G/N is $(\overline{\mathbb{R}}, \mathbb{SCQG}_2)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{FCQG}_2)$-factorizable).*

In presenting the subsequent result, immediate from (c) of Proposition 2.6 and Proposition 2.7, we follow the convention outlined in (A) on page 623. Theorem 4.17 will furnish a complete justification for undertaking this action.

Corollary 2.8. *Let G be a topological group with identity e and N be the closure of the singleton $\{e\}$ in G. Then the group G is $\mathbb{R}$-factorizable if and only if the Hausdorff topological group G/N is $\mathbb{R}$-factorizable. Similarly, G is $\mathcal{M}$-factorizable if and only if so is G/N.*

3. Quasitopological group associated to a semitopological group

All $\mathbb{R}$-factorizable topological groups are ω-narrow, by virtue of [4, Proposition 8.1.3]. Nonetheless, it is assumed that all spaces and topological groups in [4] satisfy the T_1 separation axiom. For this reason, a more general result is necessary, where semitopological and paratopological groups are not subject to any separation restrictions. This requires the notion of *quasitopological group associated to a semitopological group*.

Given a semitopological group G with topology τ, we consider the semitopological group topology $\tau^{-1} = \{U^{-1} : U \in \tau\}$ on G. Clearly, the groups (G, τ) and (G, τ^{-1}) are homeomorphic. Consider the family

$$\mathcal{B} = \{U \cap U^{-1} : e \in U \in \tau\},$$

where e is the identity of G. It is easy to verify that $\mathcal{B}$ is a local base at the identity of G for a quasitopological group topology λ on G which is finer than both τ and τ^{-1}. The group (G, λ) is referred to as the *quasitopological group associated to* G and is denoted by G^*. The identity mapping of G^* onto G is a continuous homomorphism. If G is a paratopological group, then G^* is a topological group (see [9] or [25, Section 4.1]).

In some special cases, the following auxiliary lemma is well-known (see Corollary 3.2).

Lemma 3.1. *Let $p\colon G \to H$ be a continuous homomorphism of a quasitopological group G to a semitopological group H. Then p remains continuous as a homomorphism of G to the associated quasitopological group H^*.*

Proof. Since both G and H are semitopological groups, it suffices to verify the continuity of the homomorphism $p\colon G \to H^*$ at the identity element e of G. Take an open neighborhood U of the identity e_H in H^*. By the definition of H^*, there exists an open neighborhood V of e_H in H such that $V \cap V^{-1} \subset U$. As $p\colon G \to H$ is continuous, there exists an open symmetric neighborhood O of e in G such that $p(O) \subset V$. Then the image $p(O)$ is a symmetric subset of H, so $p(O) = (p(O))^{-1} \subset V^{-1}$. Hence, we have the inclusions $p(O) \subset V \cap V^{-1} \subset U$. This proves the continuity of the homomorphism $p\colon G \to H^*$. □

Corollary 3.2. *If $p\colon G \to H$ is a continuous homomorphism of a topological group G to a paratopological group H, then p remains continuous as a homomorphism of G to the associated topological group H^*.*

The subsequent definition originates from [1] concerning the specific instance of property $\mathcal{P}$ as countable compactness within the category of paratopological groups.

Definition 3.3. Let $\mathcal{P}$ be a topological (or an algebraic-topological) property. A semitopological group G is said to be *totally* $\mathcal{P}$ if the associated quasitopological group G^* has $\mathcal{P}$.

A set of properties $\mathcal{P}$ is conveyed from a paratopological group G to the associated topological group G^*. Metrizability, first-countability, countable weight, countable network weight, and σ-compactness (with the requirement of the T_1 separation property of G in the latter case) are all found among these properties, as referenced in [25, Corollary 4.3]. A similar result, excluding σ-compactness, is valid for semitopological groups.

Proposition 3.4. *Let G be a semitopological group and G^* the quasitopological group associated to G. If G possesses any of the following properties, then G^* does as well:*

(a) *metrizability;*
(b) *first-countability;*
(c) *countable weight;*
(d) *countable network;*
(e) *the T_i separation property, for $i \in \{0, 1, 2, 3, r, 3.5, t\}$.*

Proof. Let τ be the topology of the group G. We denote the semitopological group (G, τ^{-1}) be G'. Consider the diagonal in the product $G \times G'$, $\Delta = \{(x, x) : x \in G\}$, and let p_1 and p_2 be the projections of $G \times G'$ to the first and second factor, respectively. Let also $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in G. It is clear that the family

$$\{U \times U^{-1} : U \in \mathcal{N}(e)\}$$

constitutes a neighborhood base at the identity (e, e) of the product group $G \times G'$. Also, Δ is a subgroup of $G \times G'$. Notice that the equality

$$(U \times U^{-1}) \cap \Delta = \{(x, x) : x \in U \cap U^{-1}\}$$

holds for every $U \in \mathcal{N}(e)$. Therefore, it follows from the definition of the group G^* that the correspondence $(x, x) \mapsto x$, for $x \in G$, is a topological isomorphism of Δ onto G^*. Therefore, we can identify G^* with the subgroup Δ of $G \times G'$.

Each of the properties in (a)–(e) of the proposition is finitely productive and hereditary, while the spaces G and G' are homeomorphic. Hence, if G possesses one of these properties, so does the diagonal Δ as a subspace of $G \times G'$. □

To examine the preservation of σ-compactness in the transition from G to G^*, we recall that a bitopological space (X, τ, σ) is called 2-*Hausdorff* if for every pair of distinct points $x, y \in X$, there exist elements $U \in \tau$ and $V \in \sigma$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Lemma 3.5. *The following are equivalent for a semitopological group G with topology τ and identity element e:*

(a) *the diagonal Δ is closed in $G \times G'$, where $G' = (G, \tau^{-1})$;*
(b) $\{e\} = \bigcap\{U^2 : e \in U \in \tau\}$;
(c) *the bitopological space (G, τ, τ^{-1}) is 2-Hausdorff.*

Proof. (a) ⇒ (b). Take an arbitrary point $(x, y) \in G \times G'$ with $x \neq y$. Since Δ is closed in $G \times G'$, we can find an open neighborhood U of e in G such that the open neighborhood $xU \times yU^{-1}$ of the point (x, y) in $G \times G'$ is disjoint from Δ. The latter implies that the sets xU and yU^{-1} are disjoint, so $x^{-1}y \notin U^2$. Hence, (b) follows.

(b) ⇒ (c). If $x, y \in G$ and $x \neq y$, choose an open neighborhood U of e in G such that $x^{-1}y \notin U^2$. Then $y \notin xU^2$, so the sets xU and yU^{-1} are disjoint. We see that the space (G, τ, τ^{-1}) is 2-Hausdorff.

(c) ⇒ (a). This implication is clear after the above arguments. □

Proposition 3.6. *Let G be a σ-compact semitopological group such that for every $x \in G$ distinct from the identity element e, there exists a neighborhood U of e in G such that $x \notin U^2$. Then the quasitopological group G^* associated to G is also σ-compact.*

Proof. According to the assumptions of the proposition, G satisfies (b) of Lemma 3.5, so the diagonal Δ is closed in $G \times G'$. Since the spaces G and G' are homeomorphic, we see that the product $G \times G'$ and its closed subspace Δ are σ-compact. It remains to note that Δ is topologically isomorphic with the group G^*. □

The proofs of the two subsequent results are evident, hence omitted.

Lemma 3.7. *Let H be a subgroup of a semitopological group G. Then H^* is naturally topologically isomorphic to a subgroup of G^*.*

Proposition 3.8. *Let $\Pi = \prod_{i \in I} S_i$ be the Cartesian product of a family of semitopological groups. Then the quasitopological group Π^* associated to Π is topologically isomorphic to the Cartesian product $\prod_{i \in I} S_i^*$.*

Another important property of semitopological groups is total ω-narrowness. The following fact will be used in Section 4.

Proposition 3.9. *Every subgroup of the Cartesian product $\Pi = \prod_{i \in I} S_i$ of a family of semitopological groups with countable networks is totally ω-narrow.*

Proof. Let H be an arbitrary subgroup of the Cartesian product Π. Applying Lemmas 3.7 and 3.8, we can identify H^*, algebraically and topologically, with a subgroup of $\Pi^* \cong \prod_{i \in I} S_i^*$. So it suffices to show that this subgroup is ω-narrow.

For a nonempty subset J of the index set I, let p_J be the projection of Π^* onto $\Pi_J^* = \prod_{i \in J} S_i^*$. Take an open neighborhood U of the identity e in H^*. There exists a canonical open set V in Π^* such that $e \in H^* \cap V \subset U$. Then $V = p_J^{-1} p_J(V)$, for a finite subset J of I. By (d) of Proposition 3.4, the semitopological group $\Pi_J^* \cong \prod_{i \in J} S_i^*$ and its subgroup $K = p_J(H^*)$ have countable networks, so the group K is ω-narrow. Since $O = p_J(V)$ is an open neighborhood of the identity element in Π_J^*, there exists a countable set $C \subset K$ such that $K \subset CO \cap OC$. Choose a countable set $F \subset H^*$ such that $p_J(F) = C$. Let us show that $H^* \subset FV \cap VF$.

Take an arbitrary element $h \in H^*$. Then $p_J(h) \in yO$, for some $y \in C$. Choose $x \in F$ with $p_J(x) = y$. Then $p_J(h) \in p_J(xV)$ or, equivalently, $p_J(x^{-1}h) \in p_J(V)$. Since $V = p_J^{-1} p_J(V)$, we conclude that $x^{-1}h \in V$, whence $h \in xV$. This implies that $H^* \subset FV$. A similar argument shows that $H^* \subset VF$. Finally, since $F \subset H^*$ and H^* is a subgroup of Π^*, the latter inclusions imply that $H^* = F(H^* \cap V)$ and $H^* = (H^* \cap V)F$. This proves that $FU = H^* = UF$, so the group H^* is ω-narrow. Therefore, the group H is totally ω-narrow. □

Remark 3.10. The condition on the factors S_i in Proposition 3.9 is close to being optimal. Indeed, the second diagonal in the square of the Sorgenfrey line is a typical example of how the product of two hereditarily separable, hereditarily Lindelöf *paratopological* groups can contain a closed discrete subgroup of cardinality 2^ω.

4. Absoluteness of various concepts of factorizability

To begin, we extend Corollary 2.2 to the class $\mathbb{OH}$ of open continuous homomorphisms, restricted to (para)topological groups. As we noted following Corollary 2.2, the challenge is that, even when viewed as a mapping onto its image, the diagonal of a family of open continuous homomorphisms need not be open.

Proposition 4.1. *The notions of factorizability in each of items* (1), (2) *and* (3) *below coincide for each topological group G:*

(1) $(\mathbb{SC}_r, \mathbb{FCTG}_2, \mathbb{OH})$*-factorizability and* $(\overline{\mathbb{R}}, \mathbb{FCTG}, \mathbb{OH})$*-factorizability;*
(2) $(\mathbb{SC}_r, \mathbb{SCTG}_2, \mathbb{OH})$*-factorizability and* $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{OH})$*-factorizability;*
(3) $(\mathbb{SC}_r, \mathbb{MTG}, \mathbb{OH})$*-factorizability and* $(\overline{\mathbb{R}}, \mathbb{MTG}, \mathbb{OH})$*-factorizability.*

Proof. We only prove the equivalences in items (1) and (2), the reader is left to perform a similar verification in (3). Clearly, it suffices to verify that in topological groups, $(\overline{\mathbb{R}}, \mathbb{FCTG}, \mathbb{OH})$-factorizability implies $(\mathbb{SC}_r, \mathbb{FCTG}_2, \mathbb{OH})$-factorizability and, similarly, $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{OH})$-factorizability implies $(\mathbb{SC}_r, \mathbb{SCTG}_2, \mathbb{OH})$-factorizability. So we assume that the group G is $(\overline{\mathbb{R}}, \mathbb{FCTG}, \mathbb{OH})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCTG}, \mathbb{OH})$-factorizable). Let us prove that G has the following important property:

Claim. *For every closed subgroup P of type G_δ in G, the quotient space G/P is first-countable (second-countable).*

Let P be a closed subgroup of type G_δ in G. Take a family $\{U_n : n \in \omega\}$ of open neighborhoods of the identity e in G such that $P = \bigcap_{n\in\omega} U_n$. By induction we define a sequence $\{V_n : n \in \omega\}$ of open symmetric neighborhoods of the e in G such that $V_0 \subset U_0$ and $V_{n+1}^2 \subset V_n \cap U_n$ for each $n \in \omega$. According to [4, Lemma 3.3.10], there exists a continuous prenorm N on the group G satisfying

$$\{x \in G : N(x) < 1/2^n\} \subset V_n \subset \{x \in G : N(x) \le 2/2^n\},$$

for each $n \in \omega$. By our choice of N, the closed subgroup $K = \{x \in G : N(x) = 0\}$ of G satisfies $K = \bigcap_{n\in\omega} V_n \subset P$. Considering N as a continuous real-valued function on G and making use of our assumption about G, we find an open continuous homomorphism $\pi\colon G \to H$ onto a first-countable (resp., second-countable) topological group H and a continuous real-valued function h on H such that $N(x) = h(\pi(x))$, for each $x \in G$. The latter equality implies that $\ker \pi \subset K$, because $h(e_H) = N(e) = 0$. Also, since $K \subset P$, there exists a mapping $\varphi\colon H \to G/P$ satisfying $p = \varphi \circ \pi$, where $p\colon G \to G/P$ is the canonical quotient mapping. As the mappings π and p are continuous, open and surjective, so is φ. Therefore, the space G/P is first-countable (resp., second-countable) as an open continuous image of the first-countable (resp., second-countable) space H. This proves the claim.

Let $f\colon G \to X$ be a continuous mapping to a regular second-countable space X. We can consider X as a subspace of $\mathbb{R}^\omega$. By (4) of Corollary 2.2 (with $\mathbb{H} = \mathbb{CH}$) and (a) of Proposition 2.4 (with $\mathbb{X} = \mathbb{SC}_r$ and $\mathbb{O} = \mathbb{TG}$), the group G is $(\mathbb{SC}_r, \mathbb{FCTG}_2, \mathbb{CH})$-factorizable (resp., $(\mathbb{SC}_r, \mathbb{SCTG}_2, \mathbb{CH})$-factorizable). Hence, there exists a continuous homomorphism π of G onto a Hausdorff first-countable (resp., second-countable) topological group H such that $f = k \circ \pi$, for some continuous real-valued function k on H.

Let P be the kernel of the homomorphism π. Clearly, P is closed subgroup of type G_δ in G. Denote by p the quotient homomorphism of G onto G/P. By the above Claim, the Hausdorff topological group G/P is first-countable (resp., second-countable). There exists a continuous bijection j of G/P onto H such that $j \circ p = \pi$. Then $f = k \circ \pi = k \circ j \circ p$, where $k \circ j$ is a continuous mapping of G/P to X. We conclude, therefore, that the group G is $(\mathbb{SC}_r, \mathbb{FCTG}_2, \mathbb{OH})$-factorizable (resp., $(\mathbb{SC}_r, \mathbb{SCTG}_2, \mathbb{OH})$-factorizable). □

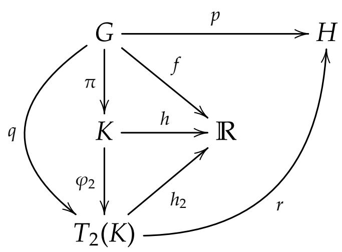
We extend Proposition 4.1 to paratopological groups, but at the expense of additional technical details. This requires the subsequent lemma.

Lemma 4.2. *Let p be a continuous homomorphism of an* $(\overline{\mathbb{R}}, \mathbb{FCPG}, \mathbb{OH})$*-factorizable (resp.,* $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{OH})$*-factorizable) paratopological group G to a semitopological group H with a countable family γ of open neighborhoods of the identity e_H such that $\{e_H\} = \bigcap_{U\in\gamma} \overline{U}$. Then there exist continuous homomorphisms $q\colon G \to L$ and $r\colon L \to H$ satisfying $p = r \circ q$, where q is open and L is a Hausdorff first-countable (resp., second-countable) paratopological group.*

Proof. It follows from the assumptions of the lemma and the homogeneity of H that the space H is Hausdorff. Let $\gamma = \{U_n : n \in \omega\}$. For every $n \in \omega$, put $V_n = p^{-1}(U_n)$. Then each V_n is an open neighborhood of the identity e_G in G. It follows from the continuity of p that $p(\overline{V_n}) \subset \overline{U_n}$ for each $n \in \omega$. Therefore, $\ker p = p^{-1}(e_H) = \bigcap_{n\in\omega} \overline{V_n}$.

According to [8, Corollary 3], for every $n \in \omega$, there exists a continuous real-valued function f_n on G with values in $[0,1]$ such that $f_n(e_G) = 0$ and $f_n^{-1}([0,1)) \subset \overline{V_n}$. Then the real-valued function $f = \sum_{n\in\omega} 2^{-n} f_n$ on G is continuous and satisfies $f^{-1}(0) \subset \ker p$.

By the lemma's assumptions, we can find an open continuous homomorphism π of G onto a first-countable (second-countable) paratopological group K and a continuous real-valued function h on K satisfying $f = h \circ \pi$. Let φ_2 be the canonical homomorphism of K onto the Hausdorff reflection $T_2(K)$ of the group K. By [26, Proposition 2.5], the homomorphism φ_2 is open and onto. Hence, the group $T_2(K)$ is also first-countable (second-countable). It follows from the definition of $T_2(K)$ that this group is Hausdorff and that there exists a continuous real-valued function h_2 on $T_2(K)$ such that $h = h_2 \circ \varphi_2$. Clearly, $q = \varphi_2 \circ \pi$ is an open continuous homomorphism of G onto $T_2(K)$ satisfying $f = h_2 \circ q$. Notice that $h_2(e_2) = f(e_G) = 0$, where e_2 is the identity of $T_2(G)$.



It is easy to see that $\ker q \subset \ker p$. This follows from the inclusions

$$\ker q = q^{-1}(e_2) \subset q^{-1}(h_2^{-1}(0)) = f^{-1}(0) \subset \ker p.$$

Therefore, there exists a homomorphism r of $L = T_2(K)$ to H satisfying $p = r \circ q$. Since p is continuous and q is open, we conclude that the homomorphism r is continuous. This completes the proof. □

Proposition 4.3. *The notions of factorizability in each of items* (1)–(3) *below coincide for every paratopological group G:*

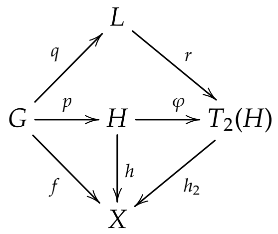
(1) $(\overline{\mathbb{R}}, \mathbb{FCPG}, \mathbb{OH})$-*factorizability,* $(\mathbb{SC}_r, \mathbb{FCPG}, \mathbb{OH})$-*factorizability, and* $(\mathbb{SC}_r, \mathbb{FCPG}_2, \mathbb{OH})$-*factorizability;*

(2) $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{OH})$-*factorizability,* $(\mathbb{SC}_r, \mathbb{SCPG}, \mathbb{OH})$-*factorizability, and* $(\mathbb{SC}_r, \mathbb{SCPG}_2, \mathbb{OH})$-*factorizability;*

(3) $(\overline{\mathbb{R}}, \mathbb{MPG}, \mathbb{OH})$-*factorizability and* $(\mathbb{SC}_r, \mathbb{MPG}, \mathbb{OH})$-*factorizability.*

Proof. First, we simultaneously establish the equivalence of the three notions in items (1) and (2). It suffices to show that $(\overline{\mathbb{R}}, \mathbb{FCPG}, \mathbb{OH})$-factorizability of a given paratopological group G implies its $(\mathbb{SC}_r, \mathbb{FCPG}_2, \mathbb{OH})$-factorizability and, similarly, $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{OH})$-factorizability of G implies its $(\mathbb{SC}_r, \mathbb{SCPG}_2, \mathbb{OH})$-factorizability.

Let $f\colon G \to X$ be a continuous mapping of an $(\overline{\mathbb{R}}, \mathbb{FCPG}, \mathbb{OH})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCPG}, \mathbb{OH})$-factorizable) paratopological group G to a regular second-countable space X. By (3) of Corollary 2.2, the group G is $(\mathbb{SC}_r, \mathbb{FCPG}, \mathbb{CH})$-factorizable (resp., $(\mathbb{SC}_r, \mathbb{SCPG}, \mathbb{CH})$-factorizable). Therefore, we can find a continuous homomorphism $p\colon G \to H$ onto a first-countable (resp., second-countable) paratopological group H and a continuous mapping $h\colon H \to X$ such that $f = h \circ p$. Let $\varphi\colon H \to T_2(H)$ be the canonical open homomorphism of H onto the Hausdorff reflection $T_2(H)$ of H. Since the space X is regular (hence, Hausdorff), there exists a continuous mapping $h_2\colon T_2(H) \to X$ satisfying $h = h_2 \circ \varphi$. Notice that the Hausdorff group $T_2(H)$ is first-countable (resp., second-countable) as an open continuous image of H. In either case, $T_2(H)$ is first-countable, so there exists a countable family γ of open neighborhoods of the identity e in $T_2(H)$ such that the intersection of the closures of the elements of γ contains only the identity e. Applying Lemma 4.2, we find an open continuous homomorphism $q\colon G \to L$ onto a Hausdorff first-countable (resp., second-countable) paratopological group L and a continuous homomorphism $r\colon L \to T_2(H)$ satisfying $\varphi \circ p = r \circ q$.



Therefore, the continuous mapping $g = h_2 \circ r$ of L to X satisfies the equality $f = g \circ q$, where q is an open continuous homomorphism. This proves that the group G is $(\mathbb{SC}_r, \mathbb{FCPG}_2, \mathbb{OH})$-factorizable (resp., $(\mathbb{SC}_r, \mathbb{SCPG}_2, \mathbb{OH})$-factorizable).

The argument is a little bit shorter for item (3). Given a continuous mapping $f\colon G \to X$ of an $(\overline{\mathbb{R}}, \mathbb{MPG}, \mathbb{OH})$-factorizable paratopological group G to a regular second-countable space X, we use item (6) of Corollary 2.2 to find a continuous (not necessarily open) homomorphism $p\colon G \to H$ onto a metrizable paratopological group H and a continuous mapping $h\colon H \to X$ such that $f = h \circ p$. Since H is metrizable, we can find a continuous real-valued function g on H such that $g^{-1}(0) = \{e_H\}$. Then the continuous function $g_* = g \circ p$ on G satisfies $g_*^{-1}(0) = \ker p$. By the assumption about G, there exists an open continuous homomorphism $q\colon G \to L$ onto a metrizable paratopological group L such that $g_* = j \circ q$, for some continuous real-valued function j on L. It is clear that $j(e_L) = 0$. Therefore, $\ker q = q^{-1}(e_L) \subset g_*^{-1}(0) = \ker p$. The latter inclusion implies that there exists a homomorphism $\varphi\colon L \to H$ satisfying $p = \varphi \circ q$. Since q is an open continuous homomorphism, we conclude that the homomorphism φ is continuous.

The continuous mapping $h_* = h \circ \varphi$ of the metrizable paratopological group L to X satisfies the equality $f = h_* \circ q$. This proves that the group G is $(\mathbb{SC}_r, \mathbb{MPG}, \mathbb{OH})$-factorizable. □

The following two results, each with its own value, are necessary to unify diverse notions of factorizability of a (para)topological group in multiple topological-algebraic categories.

Proposition 4.4. *Let a topological (paratopological, quasitopological) group G be embedded, algebraically and topologically, as a subgroup to a product $\Pi = \prod_{i\in I} S_i$ of first-countable semitopological groups. Then for every countable subset C of the index set I, there exists a countable set J with $C \subset J \subset I$ such that the subgroup $p_J(G)$ of $\Pi_J = \prod_{i\in J} S_i$ is a topological (paratopological, quasitopological) group, where $p_J\colon \Pi \to \Pi_J$ is the projection.*

Proof. We exclusively examine the case of a topological group G. Several simplifications are evident in our argument when considering a paratopological or quasitopological group G. To shorten the argument, we assume that for some $i_0 \in I$, the factor S_{i_0} is a singleton.

We let $J_0 = C \cup \{i_0\}$ and $\mathcal{B}_0 = \{\Pi\}$. Suppose that for some $n \in \omega$, we have defined countable subsets $J_0 \subset \cdots \subset J_n$ of I and countable families $\mathcal{B}_0 \subset \cdots \subset \mathcal{B}_n$ of canonical open neighborhoods of the identity e in Π. For every $V \in \mathcal{B}_n$, we choose a finite set $F_V \subset I$ such that $V = p_{F_V}^{-1} p_{F_V}(V)$. Then

$$J_{n+1} = J_n \cup \bigcup \Big\{F_V : V \in \mathcal{B}_n\Big\}$$

is a countable subset of I and $J_n \subset J_{n+1}$. Let $\mathcal{N}_{n+1}$ be a countable local base at the identity e_{n+1} of the semitopological group $\Pi_{J_{n+1}}$. Since G is a topological group and the projection $p_{J_{n+1}}$ is continuous, we can choose a countable family $\mathcal{B}_{n+1}$ of canonical open neighborhoods of e in Π with $\mathcal{B}_n \subset \mathcal{B}_{n+1}$ such that for each element $U \in \mathcal{N}_{n+1}$, there exists $V \in \mathcal{B}_{n+1}$ satisfying

$$V \cup (G \cap V)^{-1} \cup (G \cap V)^2 \subset p_{J_{n+1}}^{-1}(U). \qquad (1)$$

This completes our construction of the sets J_n and countable families $\mathcal{B}_n$ of canonical open neighborhoods of e in Π, where $n \in \omega$.

Let $J = \bigcup_{n\in\omega} J_n$ and $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$. Clearly, $C \subset J_0 \subset J$ and both J and $\mathcal{B}$ are countable. Notice that the equality $V = p_J^{-1} p_J(V)$ holds for each $V \in \mathcal{B}$. We claim that the subgroup $p_J(G)$ of Π_J is a paratopological group when endowed with the subspace topology.

Let e_J be the identity element of Π_J. Since $K = p_J(G)$ is a semitopological group, as a subgroup of Π_J, it suffices to verify the continuity of multiplication at the identity e_J of the group K. Take an arbitrary open neighborhood O of e_J in the group Π_J. We can assume without loss of generality that O is a canonical open set in Π_J. Hence, O depends on a finite set of coordinates, say, $F \subset J$. Then $F \subset J_k$, for some integer $k \geq 1$. Since $\mathcal{N}_k$ is a local base at the identity of Π_{J_k}, there exists an element $U \in \mathcal{N}_k$ such that $U \subset p^J_{J_k}(O)$, where $p^J_{J_k}$ is the projection of Π_J onto Π_{J_k}. It follows from the above inclusion and $F \subset J_k$ that $p_{J_k}^{-1}(U) \subset p_J^{-1}(O)$. By the definition of $\mathcal{B}_k$ (see (1)), we can find an element $V \in \mathcal{B}_k$ satisfying $V \subset p_{J_k}^{-1}(U)$ and $(G \cap V)^2 \subset p_{J_k}^{-1}(U)$. Take arbitrary elements $x, y \in K \cap p_J(V)$. There exist elements $a, b \in G$ such that $p_J(a) = x$ and $p_J(b) = y$. It follows from the equality $V = p_J^{-1} p_J(V)$ that $a, b \in V$. We see that $a, b \in G \cap V$, so our choice of the set V implies that $ab \in (G \cap V)^2 \subset p_{J_k}^{-1}(U) \subset p_J^{-1}(O)$. Therefore, we have that $xy = p_J(ab) \in p_J p_J^{-1}(O) = O$. This proves that $(K \cap p_J(V))^2 \subset O$. So, multiplication in $K = p_J(G)$ is jointly continuous at e_J. We conclude that the subgroup K of Π_J is a paratopological group.

Using the set V that we selected in (1), a similar argument shows that the inclusion $\left(K \cap p_J(V)\right)^{-1} \subset O$ is valid. The latter inclusion implies that inversion in K is continuous at the identity element. Hence, inversion in K is continuous since K is a semitopological group. As a result, $p_J(G) = K$ is a topological group. □

Lemma 4.5. *Let $\mathbb{O}$ be a class of semitopological (quasitopological, paratopological) groups. Let also G be a group with topology and assume that G is completely regular and $(\overline{\mathbb{R}}, \mathbb{O})$-factorizable. Then G is topologically isomorphic to a subgroup of the Cartesian product of a subfamily of $\mathbb{O}$.*

Proof. Denote by $I = C(G)$ the family of continuous real-valued functions on G. Since G is $(\overline{\mathbb{R}}, \mathbb{O})$-factorizable, for every $f \in I$, we can find a group $H_f \in \mathbb{O}$, a continuous surjective homomorphism $p_f \colon G \to H_f$ and a continuous real-valued function h_f on H_f satisfying $f = h_f \circ p_f$. It is easy to see that the diagonal p of the family $\{p_f : f \in I\}$ is a topological monomorphism of G to the Cartesian product $\prod_{f\in I} H_f$. □

When an $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable quasitopological group G is $(\overline{\mathbb{R}}, \mathbb{FCQG})$-factorizable and when the $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizability of G implies its $(\overline{\mathbb{R}}, \mathbb{SCQG})$-factorizability are questions addressed by the following theorem. A stronger form of this result will be presented in Theorem 4.27.

Theorem 4.6. *For a completely regular quasitopological group G, the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable);*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{FCQG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCQG})$-factorizable);*
(c) *G is $(\overline{\mathbb{R}}, \mathbb{FCQG}_2)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCQG}_2)$-factorizable).*

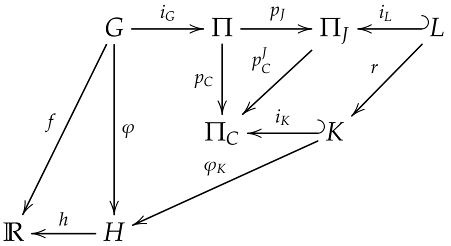
Proof. We focus on $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizability, thus leaving an analogous verification of the parenthetical part of the theorem to the reader.

Clearly, (b) implies (a), while (b) and (c) of the theorem are equivalent by item (b) of Proposition 2.4, where $\mathbb{X} = \overline{\mathbb{R}}$ and $\mathbb{O} = \mathbb{QG}$. Therefore, it suffices to show that (a) implies (b). So we assume that the group G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable.

Since G is completely regular and $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable, Lemma 4.5 implies that G is topologically isomorphic to a subgroup of the topological product of a family of first-countable semitopological groups, say, $\Pi = \prod_{i\in I} S_i$. Let $i_G \colon G \to \Pi$ be a topological monomorphism. For every nonempty subset J of the index set I, we denote by p_J the projection of Π onto $\Pi_J = \prod_{i\in J} S_i$.

Let f be a continuous real-valued function on the quasitopological group G. According to the theorem's assumption regarding G, it is possible to find a first-countable semitopological group H and a continuous homomorphism φ from G onto H such that $f = h \circ \varphi$, where h is a continuous real-valued function on H. Since the group H is first-countable, we can apply [4, Lemma 8.5.4] to find a countable subset C of I and a continuous homomorphism φ_K of $K = p_C(i_G(G))$ to H satisfying $\varphi = \varphi_K \circ p_C \circ i_G$.

By Proposition 4.4, there exists a countable set $J \subset I$ with $C \subset J$ such that the subgroup $L = p_J(i_G(G))$ of Π_J is a quasitopological group. Clearly, the groups Π_J and L are first-countable. Denote by p^J_C the natural projection of Π_J to Π_C satisfying $p_C = p^J_C \circ p_J$. Note that $K = p^J_C(L)$. Then the continuous homomorphism $r = p^J_C \restriction L$ satisfies $\varphi = \varphi_K \circ r \circ p_J \circ i_G$. Let i_K and i_L be the identity embeddings of K to Π_C and L to Π_J, respectively.



Then the continuous real-valued function $h_L = h \circ \varphi_K \circ r$ on L satisfies $f = h_L \circ (p_J \circ i_G)$. Since L is a first-countable quasitopological group, the latter equality implies that G is $(\overline{\mathbb{R}}, \mathbb{FCQG})$-factorizable. Hence, (b) follows from (a). □

It is not clear whether one can weaken the complete regularity of G in Theorem 4.6 to regularity or even the Hausdorff separation property (see Problems 6.6 and 6.7).

We provide the reader with two helpful results regarding Tychonoff reflections of semitopological groups before going deeper into various types of factorizability in (para)topological and semitopological groups. Although not stated directly in [26], the following facts are simple consequences of the information presented there.

Proposition 4.7. *Let $\varphi_t \colon G \to Tych(G)$ be the canonical homomorphism of a semitopological group G onto its Tychonoff reflection $Tych(G)$. Then for each cozero set O in G, the equality $O = \varphi_t^{-1}\varphi_t(O)$ is valid and $\varphi_t(O)$ is cozero set in $Tych(G)$. Hence, φ_t sends zero-sets in G to zero-sets in $Tych(G)$. In particular, the mapping φ_t is z-closed.*

Proof. Let O be a nonempty cozero set in G. There exists a continuous real-valued function f on G with values in $[0, 1]$ such that $O = f^{-1}(J)$, where $J = (0, 1] \subset \mathbb{R}$. By the definition of $Tych(G)$, we can find a continuous real-valued function g on $Tych(G)$ satisfying $f = g \circ \varphi_t$. The latter equality implies that the cozero set $U = g^{-1}(J)$ in $Tych(G)$ satisfies $O = f^{-1}(J) = \varphi_t^{-1}\big(g^{-1}(J)\big) = \varphi_t^{-1}(U)$. Hence, $O = \varphi_t^{-1}\varphi_t(O)$, where $\varphi_t(O) = U = g^{-1}(U)$ is a cozero set in $Tych(G)$.

The remaining statements of the proposition are evident since the complement of a cozero set is a zero-set. □

Let $K = Tych(G)$ be the Tychonoff reflection of a semitopological group G and $\varphi_t \colon G \to K$ be the canonical homomorphism. Denote by $C(G)$ and $C(K)$ the families of continuous real-valued functions on G and K, respectively. We define a mapping $\Phi \colon C(K) \to C(G)$ by letting $\Phi(h) = h \circ \varphi_t$, for each $h \in C(K)$. Since $K = \varphi_t(G)$, the mapping Φ is injective.

Further, according to the definition of $K = Tych(G)$, for every $f \in C(G)$, there exists a continuous real-valued function h_f on K that satisfies $f = h_f \circ \varphi_t$. Such a function h_f is obviously unique. By allowing $\Lambda(f) = h_f$, we obtain a mapping $\Lambda \colon C(G) \to C(K)$. The following proposition explains the properties of Φ and Λ.

Proposition 4.8. *The mapping Φ is a bijection of $C(K)$ onto $C(G)$, while Λ is bijection of $C(G)$ onto $C(K)$. Also, Λ is the inverse of Φ, so $\Phi \circ \Lambda$ is the identity mapping of $C(G)$ onto itself.*

Proof. It follows from the definitions of Φ and Λ that $\Phi(\Lambda(f)) = \Phi(h_f) = h_f \circ \varphi_t = f$, for each $f \in C(G)$. This implies the last statement of the proposition. Also, it follows from the unicity of h_f that the equalities $\Lambda(\Phi(h)) = \Lambda(h \circ \varphi_t) = h$ are valid for each $h \in C(K)$. Hence, Φ and Λ are bijections and Λ is the inverse of Φ. □

Remark 4.9. Propositions 4.7 and 4.8 are evidently valid if the Tychonoff reflection $Tych(G)$ is replaced with the regular reflection, $Reg(G)$, of a semitopological group G. This follows from the existence of a continuous homomorphism $\psi\colon Reg(G) \to Tych(G)$ satisfying $\varphi_t = \psi \circ \varphi_r$, where $\varphi_r\colon G \to Reg(G)$ and $\varphi_t\colon G \to Tych(G)$ are canonical homomorphisms (see [26, Proposition 3.5]).

The equality $Reg(G) = Tych(G)$ does not necessarily apply to a semitopological or quasitopological group G. According to [18], there exists a regular quasitopological group G with $|G| = 2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\omega}$, such that every continuous real-valued function on G is constant. Therefore, $G = Reg(G)$, while $Tych(G)$ is the trivial one-element group. Nevertheless, the equality $Tych(G) = Reg(G)$ holds for every *paratopological* group G, because the regular paratopological group $Reg(G)$ is Tychonoff according to a Banakh–Ravsky theorem in [8].

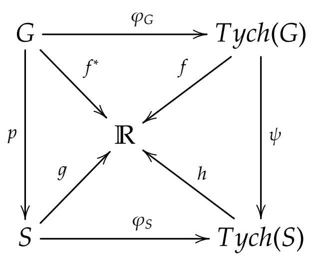
The subsequent lemma shows that, notwithstanding the phenomenon outlined in Remark 4.12, $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable semitopological (hence, paratopological) groups possess a property inherently linked to ω-narrowness.

Lemma 4.10. *If G is an $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable semitopological group, then the Tychonoff reflection of G, $Tych(G)$, is totally ω-narrow. Furthermore, every $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable topological group G is ω-narrow.*

Proof. Let $\varphi_G\colon G \to Tych(G)$ be the canonical continuous surjective homomorphism and f be an arbitrary continuous real-valued function on $Tych(G)$. Then $f^* = f \circ \varphi_G$ is a continuous function on G. Since G is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable, we can find a second-countable semitopological group S and a continuous surjective homomorphism $p\colon G \to S$ such that $f^* = g \circ p$, for some continuous real-valued function g on S. Denote by φ_S the canonical continuous homomorphism of S onto $Tych(S)$. By the definition of $Tych(S)$, there exists a continuous real-valued function h on $Tych(S)$ satisfying $g = h \circ \varphi_S$. Then $\varphi^* = \varphi_S \circ p$ is a continuous homomorphism of G onto the completely regular semitopological group $H_f = Tych(S)$. The group H_f has a countable network as a continuous image of the second-countable group S. Proposition 3.5 from [26] implies that there exists a continuous homomorphism $\psi\colon Tych(G) \to Tych(S)$ satisfying $\varphi^* = \psi \circ \varphi_G$. Hence, we have the equalities

$$f \circ \varphi_G = f^* = g \circ p = h \circ \varphi_S \circ p = h \circ \varphi^* = h \circ \psi \circ \varphi_G,$$

so the diagram below commutes.



We see, therefore, that $f = h \circ \psi$. It follows that every continuous real-valued function f on $Tych(G)$ factorizes through a continuous homomorphism of $Tych(G)$ onto a semitopological group H_f with a countable network. Since the semitopological group $Tych(G)$ is completely regular, we conclude that $Tych(G)$ is topologically isomorphic to a subgroup of the Cartesian product $\prod_{f \in I} H_f$ of semitopological groups H_f with countable networks, where I is the family of all continuous real-valued functions on $Tych(G)$. Applying Proposition 3.9, we conclude that the semitopological group $Tych(G)$ is totally ω-narrow, thus implying the first statement of the lemma.

Assume that G is an $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable topological group and let N be the closure of the singleton $\{e\}$ in G. The quotient group G/N is Hausdorff and regular, where e is the identity of G. Hence, it is easy to show that the group G/N is the Tychonoff reflection of G, $Tych(G) = G/N$. Consequently, the topological group G/N is ω-narrow, as we have just shown. Denote by π the quotient homomorphism of G onto G/N.

Let U be an open neighborhood of the identity e in G. Take an open neighborhood V of e such that $V^2 \subset U$. Then $O = \pi(V)$ is an open neighborhood of the identity in G/N, so there exists a countable set $C \subset G/N$ such that $CO = G/N$. There exists a countable set $F \subset G$ such that $\pi(F) = C$. Let us show that $FU = G$. Indeed, we have the equalities $G/N = CO = \pi(F)\pi(V) = \pi(FV)$. Since N is the kernel of π, it follows from $G/N = \pi(FV)$ that $G = FVN$. Our definition of N as the closure of the singleton $\{e\}$ implies that $N \subset V$. Hence, $G = FVN \subset FVV \subset FU$, and we conclude that $G = FU$. This proves that the group G is ω-narrow. □

The next result follows immediately from Lemma 4.10 since the equality $Tych(G) = Reg(G)$ holds for each paratopological group G (see Remark 4.9).

Corollary 4.11. *If G is an $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable paratopological group, then the regular reflection $Reg(G)$ of G is totally ω-narrow.*

The conclusion of Corollary 4.11 will be improved in Theorem 4.13, where it is demonstrated that if G is an $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable paratopological group, then both G and its regular reflection, $Reg(G)$, are $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable.

Remark 4.12. An $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable paratopological group is not necessarily ω-narrow, so the conclusion of Corollary 4.11 is not valid for the group G in place of $Reg(G)$. Indeed, consider the additive group of a linearly ordered field F endowed with a topology τ whose base is $\{[x, \infty) : x \in F\}$ (see [12]). Then $F^* = (F, \tau)$ is a first-countable paratopological group satisfying the T_0 separation axiom. It is clear that if U and V are open subsets of F^*, then either $U \subset V$ or $V \subset U$. Therefore, every continuous real-valued function on F^* is constant and F^* is both $(\overline{\mathbb{R}}, \mathbb{SCPG}_r)$- and $(\overline{\mathbb{R}}, \mathbb{SCTG}_2)$-factorizable as an additive paratopological group. Nevertheless, if the cofinal character of the linear order on F is uncountable, then F^* is not ω-narrow. A more complicated example of a *Hausdorff* $(\overline{\mathbb{R}}, \mathbb{SCTG}_2)$-factorizable paratopological group G that fails to be ω-narrow can be found in [22, Theorem 1]. To see that the group $G = (\Sigma, \sigma)$ in [22] is $(\overline{\mathbb{R}}, \mathbb{SCTG}_2)$-factorizable, it suffices to verify that the semiregularization of the group G coincides with the countably compact dense subgroup Σ of the compact topological group $\mathbb{T}^{\omega_1}$, where $\mathbb{T}$ is the torus group. Here Σ denotes the Σ-product of ω_1 copies of the group $\mathbb{T}$ considered as a subgroup of $\mathbb{T}^{\omega_1}$. We note that all subgroups of compact topological groups are precompact, hence $\mathbb{R}$-factorizable [4, Corollary 8.1.17].

We can now show that the two '$\mathbb{R}$-factorizabilities' of an arbitrary paratopological group in the categories of second countable paratopological and semitopological groups are equivalent. In other words, none of the two previously listed categories need to be specified when discussing the $\mathbb{R}$-factorizability of a paratopological group.

By [19, Corollary 4], a regular paratopological group G is topologically isomorphic to a subgroup of a Cartesian product of regular second-countable paratopological groups if and only if G is totally ω-narrow. Equivalently, for every neighborhood U of the identity element in the group G, there exists a continuous homomorphism $p\colon G \to H$ onto a regular second-countable paratopological group H such that $p^{-1}(V) \subset U$, for some neighborhood V of the identity in H (see [28, Lemma 2.6]). In other words, every totally ω-narrow paratopological group is *projectively second-countable.* This equivalence is used in the proof of the subsequent theorem.

A real-valued function f on a semitopological group G is called *left* (resp., *right*) *ω-quasi-uniformly continuous* if for every $\varepsilon > 0$, one can find a countable family γ of open neighborhoods of the identity element

e in G with the property that for every $x \in G$, there exists $U \in \gamma$ such that the inequality $|f(ux) - f(x)| < \varepsilon$ (resp., $|f(xu) - f(x)| < \varepsilon$) holds for each $u \in U$. Clearly, every left (resp., right) ω-quasi-uniformly continuous function on G is continuous. If a function is both left and right ω-quasi-uniformly continuous, it is said to be ω-quasi-uniformly continuous. A semitopological group G has *property ω-QU* if every continuous real-valued function on G is ω-quasi-uniformly continuous [28]. Notice that every first-countable semitopological group has property ω-QU.

We recall that a continuous mapping $f\colon X \to Y$ is called *d-open* if for every open subset U of X, the image $f(U)$ is a dense subset of an open set V in Y. Equivalently, $f(U)$ is a subset of the interior of its closure $\overline{f(U)}$ in Y.

Theorem 4.13. *For a paratopological group G, the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable;*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable;*
(c) *G is $(\overline{\mathbb{R}}, \mathbb{SCPG}_r)$-factorizable, that is, $(\overline{\mathbb{R}}, \mathbb{SCPG}_t)$-factorizable;*
(d) *the regular reflection of G, $Reg(G)$, is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable.*

Proof. The implications (c) ⇒ (b) ⇒ (a) are evident. Items (b) and (c) in the theorem are equivalent due to [28, Theorem 3.8]. Also, it follows from Proposition 2.5, with $\mathbb{O} = \mathbb{PG}$ and $k = r$, that (d) implies (b). It only remains to show that (a) implies (d). Suppose that the group G is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable.

Claim. *The group G has property ω-QU.*

Our proof of the claim is very close to the one of [28, Lemma 3.7]. Let f be a continuous real-valued function on the group G. Then we can find a continuous homomorphism $\psi\colon G \to S$ onto a second-countable semitopological group S and a continuous function g on S such that $f = g \circ \psi$. Let γ be a countable local base at the identity of S. Put $\lambda = \{\psi^{-1}(U) : U \in \gamma\}$. One can easily verify that λ is a countable family of open neighborhoods of the identity in G having the property that for every point $x \in G$ and every $\varepsilon > 0$, there exists an element $U \in \lambda$ such that $|f(x) - f(ux)| < \varepsilon$ and $|f(x) - f(xu)| < \varepsilon$ for each $u \in U$. Hence, f is ω-quasi-uniformly continuous. So G has property ω-QU. This proves our Claim.

The canonical surjective homomorphism $\varphi\colon G \to Reg(G)$ is d-open, by [26, Proposition 3.1]. Applying [31, Proposition 2.4], we conclude that the group $Reg(G)$ also has property ω-QU. Also, by Corollary 4.11, the group $Reg(G)$ is totally ω-narrow and, hence, projectively second-countable. According to [16, Lemma 3.7], where $\mathcal{P}$ is the class of second-countable paratopological groups, every projectively second-countable paratopological group with property ω-QU is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable. Hence, the group $Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable. This proves that (a) implies (d) and completes the proof of the theorem. □

It turns out that the conclusion of Theorem 4.13 remains valid if second-countability is substituted with first-countability. To establish this fact we need three lemmas.

Let us recall that a semitopological group G is *ω-balanced* if for every neighborhood U of the identity in G, one can find a countable family γ of open neighborhoods of the identity in G such that for every $x \in G$, there exists $V \in \gamma$ satisfying $x^{-1}Vx \subset U$. The family γ as above is said to be *subordinated* to U. It is clear that every first-countable semitopological group is ω-balanced and that every subgroup of an ω-balanced group is ω-balanced. Also, it is easy to verify that the Cartesian product of a family of ω-balanced groups is ω-balanced (see [3, Sections 8 and 9] or [4, Section 3.4]).

The following auxiliary fact can be obtained by combining Theorem 3.12 and Lemma 3.16, both from [15]. We present a direct argument here.

Lemma 4.14. *Let G be an $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable paratopological group. Then the group $Reg(G)$ is ω-balanced.*

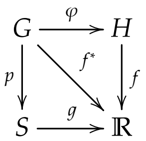
Proof. By a theorem in [8], the regular paratopological group $H = Reg(G)$ is completely regular. Denote by φ the canonical homomorphism of G onto H.

Let U be an open neighborhood of the identity element e_H in H. Take an open neighborhood U_* of e_H in H such that $\overline{U_*} \subset U$. There exists a continuous real-valued function f on H with values in $[0, 1]$ such that

$e_H \in O = f^{-1}(J) \subset U_*$, where $J = (0, 1]$. In particular, $f(e_H) \neq 0$ and f equals zero at the points of $H \setminus U_*$. Then $f^* = f \circ \varphi$ is a continuous real-valued function on G and $f^*(e_G) = f(e_H) \neq 0$. By the lemma's assumptions, we can find a continuous homomorphism $p\colon G \to S$ onto a first-countable semitopological group S and a continuous real-valued function g on S such that $f^* = g \circ p$. Notice that $g(e_S) \neq 0$, so $e_S \in W = g^{-1}(J)$.

Let $\mathcal{N} = \{V_n : n \in \omega\}$ be a local base at the identity of the group S. For every $n \in \omega$, $U_n = p^{-1}(V_n)$ is an open neighborhood of e in G. It is clear from our choice of the family $\mathcal{N}$ that it is subordinated to the open set W. Since p is a continuous homomorphism of G onto S, the family $\{U_n : n \in \omega\}$ is subordinated to the open neighborhood $p^{-1}(W)$ of e_G in G.

It follows from [26, Proposition 3.1] that the homomorphism φ is d-open. Hence, for every $n \in \omega$, there exists an open set O_n in H containing the set $\varphi(U_n)$ as a dense subset. Clearly, $e_H \in \varphi(U_n) \subset O_n$.



We claim that the family $\{O_n : n \in \omega\}$ is subordinated to the open neighborhood U of the identity in G. Indeed, our definition of the sets $O \subset H$ and $W \subset S$ and the equalities $g \circ p = f^* = f \circ \varphi$ imply that $\varphi^{-1}(O) = (f^*)^{-1}(J) = p^{-1}(W)$. Take an arbitrary element $y \in H$ and choose $x \in G$ such that $y = \varphi(x)$. Since the family $\{U_n : n \in \omega\}$ is subordinated to $p^{-1}(W)$, there exists $n \in \omega$ such that $x^{-1}U_n x \subset p^{-1}(W)$. As φ is a homomorphism, we have the inclusions

$$y^{-1}\varphi(U_n)\, y \subset \varphi\big(p^{-1}(W)\big) = \varphi\big(\varphi^{-1}(O)\big) = O.$$

Making use of the continuity of φ and the density of $\varphi(U_n)$ in O_n, we conclude that $y^{-1}O_n\, y \subset \overline{O} \subset \overline{U_*} \subset U$. Hence, the group $H = Reg(G)$ is ω-balanced. □

According to [20, Theorem 2.17], a T_0 paratopological group G is topologically isomorphic to a subgroup of a Cartesian product of first-countable T_0 paratopological groups if and only if G is ω-balanced. In an equivalent formulation, for every neighborhood U of the identity element in the ω-balanced group G, there exists a continuous homomorphism $p\colon G \to H$ onto a first-countable paratopological group H satisfying the T_0 separation axiom such that $p^{-1}(V) \subset U$, for some open neighborhood V of the identity in H. By excluding 'T_0' from this result and replicating the reasoning presented in the proof found in [20], we derive the subsequent assertion, which is applied in the proof of Theorem 4.16.

Lemma 4.15. *Let U be a neighborhood of the identity in an ω-balanced paratopological group G. Then there exists a continuous homomorphism $p\colon G \to H$ onto a first-countable paratopological group H such that $p^{-1}(V) \subset U$, for some open neighborhood V of the identity in H. Therefore, every ω-balanced paratopological group is projectively first-countable.*

The general scheme of reasoning that was employed to prove Theorem 4.13 is also applicable to the proof of the subsequent result. Certain aspects of the argument must be clarified, as we will be employing Lemmas 4.14 and 4.15.

Theorem 4.16. *For a paratopological group G, the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable;*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable;*
(c) *G is $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$-factorizable.*

Proof. Since the class of first-countable paratopological groups is countably productive and hereditary, items (b) and (c) of the theorem are equivalent, by [15, Theorem 3.12]. Therefore, it suffices to show that (a) implies (b). So we assume that the group G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable.

Since every first-countable semitopological group has property ω-QU, it is clear that G also has property ω-QU. The canonical homomorphism $\varphi\colon G \to Reg(G)$ is d-open, by [26, Proposition 3.1]. Applying [31, Proposition 2.4], we see that the group $Reg(G)$ has property ω-QU as well. Also, it follows from Lemma 4.14 that the paratopological group $K = Reg(G)$ is ω-balanced. Hence, Lemma 4.15 implies that K is projectively first-countable. According to [16, Lemma 3.7], every projectively first-countable paratopological group with property ω-QU is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable. It follows that K is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable. Applying [15, Theorem 3.12] once again, we deduce that K is $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$-factorizable. To conclude that G is $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$-factorizable, we use Proposition 2.5 with $\mathbb{O} = \mathbb{PG}$ and $k = r$. This proves the implication (a) $\Rightarrow$ (b) and the theorem. □

Clearly, the subindex "r" in (c) of Theorem 4.16 can be replaced with "t" because every regular paratopological group is completely regular.

By employing Theorem 4.13, we arrive at our next conclusion.

Theorem 4.17. *For a topological group G, the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable;*

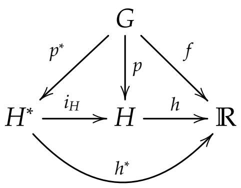
(b) *G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable;*

(c) *G is $(\overline{\mathbb{R}}, \mathbb{SCTG})$-factorizable;*

(d) *G is $(\overline{\mathbb{R}}, \mathbb{SCTG}_2)$-factorizable, equivalently, $\mathbb{R}$-factorizable.*

Proof. The equivalence of items (a) and (b) follows directly from Theorem 4.13. The equivalence of (c) and (d) is a consequence of item (a) from Proposition 2.4 (with $\mathbb{X} = \overline{\mathbb{R}}$). Clearly, (c) implies (b), so it remains to show that (b) implies (c).

Let f be a continuous real-valued function on the group G. By (b), we can find a continuous homomorphism $p\colon G \to H$ onto a second-countable paratopological group H and a continuous real-valued function h on H such that $f = h \circ p$. The group H^* associated to H is a topological group. According to (c) of Proposition 3.4, H^* is second-countable. Denote by p^* the homomorphism of G to H^* coinciding with p pointwise. By Lemma 3.1, p^* is continuous. Denote by h^* the function on H^* that coincides with h pointwise. Evidently, h^* is continuous. The following diagram commutes, where $i_H\colon H^* \to H$ is the identity mapping.



The equality $f = h^* \circ p^*$ implies that the group G is $(\overline{\mathbb{R}}, \mathbb{SCTG})$-factorizable. □

Let us examine the relationship between various kinds of factorizability in topological and paratopological groups as well as their Hausdorff and regular reflections. The research in this direction was started by L.-X. Peng and Y.-M. Deng in [15, Section 3]. Rephrasing their results using the current terminology, they demonstrate that the $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizability of a semitopological group G is equivalent to the $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizability of the group $T_2(G)$. Also, according to [15, Theorem 3.12], a paratopological group G is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable if and only if any (each) of the groups $T_2(G)$ and $Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable. We extend the latter result to $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizability and add T_0-, T_1-, and T_3-reflection in item (b) of the next theorem. In other words, an analogue of Corollary 2.8 is valid for paratopological groups. Additionally, we complement the list of equivalencies in Theorem 4.13.

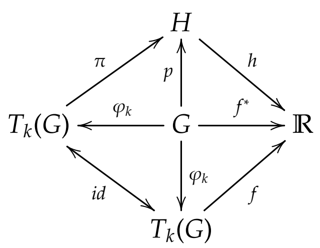
Theorem 4.18. *The following are equivalent for a paratopological group G:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable;*

(b) *$T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable for some $k \in \{0, 1, 2, 3\}$;*
(c) *$Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable;*
(d) *$Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG}_r)$-factorizable.*

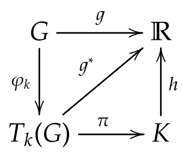
Proof. The equivalence of items (a) and (c) follows from Theorem 4.13, while [28, Theorem 3.8] implies that (c) and (d) are also equivalent. Let us show that (a) and (b) are equivalent as well.

First, we show that (a) implies (b). Assume that (a) is valid, so the group G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable. Take any $k \in \{0, 1, 2, 3\}$ and let $\varphi_k \colon G \to T_k(G)$ be the canonical continuous homomorphism. Also, let f be a continuous real-valued function on the group $T_k(G)$. Then $f^* = f \circ \varphi_k$ is a continuous function on G. Since G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable, we can apply the equivalence of (b) and (c) in Theorem 4.13 (or [28, Theorem 3.8]) to find a continuous homomorphism $p \colon G \to H$ onto a regular second-countable paratopological group H and a continuous real-valued function h on H such that $f^* = h \circ p$. The definition of $T_k(G)$ implies that there is a continuous homomorphism $\pi \colon T_k(G) \to H$ satisfying $p = \pi \circ \varphi_k$, because the regular paratopological group H satisfies the T_k separation axiom.



In the diagram, *id* stands for the identity mapping of $T_k(G)$ onto itself. We see, therefore, that $f = h \circ \pi$, so our choice of H shows that the group $T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG}_r)$-factorizable. Hence, $T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable, which shows that (a) implies (b).

It remains to establish that (b) implies (a). Again, take any $k \in \{0, 1, 2, 3\}$ and assume that the group $T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable. Consider an arbitrary continuous real-valued function g on G. By the definition of $T_k(G)$, there exists a continuous function g^* on $T_k(G)$ such that $g = g^* \circ \varphi_k$. Since $T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable, one can find a continuous homomorphism $\pi \colon T_k(G) \to K$ onto a second-countable paratopological group K and a continuous function h on K such that $g^* = h \circ \pi$. Then $g = h \circ (\pi \circ \varphi_k)$, where $\pi \circ \varphi_k$ is a continuous homomorphism of G onto the second-countable paratopological group K.



This shows that G is $(\overline{\mathbb{R}}, \mathbb{SCPG})$-factorizable. Therefore, (b) implies (a). It follows that (a)–(d) of the theorem are equivalent. □

The next result is an analogue of Theorem 4.18 for $(\overline{\mathbb{R}}, \mathbb{FCSG})$- and $\mathbb{FCPG}$-factorizability. Combining Theorem 3.12 from [15] and our Theorem 4.16 suffices to deduce it. To avoid ambiguity, we present a brief proof of the theorem.

Theorem 4.19. *The following are equivalent for a paratopological group G:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable;*
(b) *$T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable for some $k \in \{0, 1, 2, 3\}$;*
(c) *$Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable;*

(d) $Reg(G)$ *is* $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$*-factorizable.*

Proof. Evidently, (d) implies (c). Since every regular space satisfies the T_k separation axiom for each $k \in \{0, 1, 2, 3\}$, the definition of the reflections $Reg(G)$ and $T_k(G)$ implies that there exists a continuous homomorphism $\psi_k\colon Reg(G) \to T_k(G)$ satisfying $\varphi_k = \psi_k \circ \varphi_r$, where $\varphi_r\colon G \to Reg(G)$ and $\varphi_k\colon G \to T_k(G)$ are canonical homomorphisms. One can easily demonstrate that (c) implies (b) and (b) implies (a) via the fact that the homomorphism ψ_k is surjective.

Therefore, all we have to do is to show that (a) implies (d). Let G be an $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable paratopological group. By [15, Theorem 3.12], the group $Reg(G)$ is also $(\overline{\mathbb{R}}, \mathbb{FCPG})$-factorizable. Applying the equivalence of (b) and (c) in Theorem 4.16 we conclude that the group $Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$-factorizable. Thus, (d) follows from (a), as claimed. □

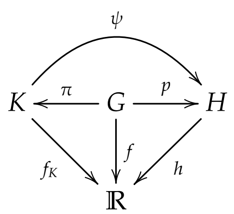
A version of Theorem 4.16 for $(\overline{\mathbb{R}}, \mathbb{MPG})$-factorizability is what we intend to present next. This requires a lemma that admits multiple forms for various categories of topological algebra, from which we choose the one below.

Lemma 4.20. *Suppose that* $p\colon G \to H$ *is a continuous homomorphism of an* $(\overline{\mathbb{R}}, \mathbb{FCSG})$*-factorizable paratopological group* G *onto a first-countable semitopological group* H*. If either* G *is regular or* H *is completely regular, then one can find a regular first-countable paratopological group* K *and continuous surjective homomorphisms* $\pi\colon G \to K$ *and* $\psi\colon K \to H$ *satisfying* $p = \psi \circ \pi$*.*

Proof. First we assume that the semitopological group H is completely regular. Let $\{V_n : n \in \omega\}$ be a countable local base at the identity e_H of H. We can assume that $\overline{V_{n+1}} \subset V_n$ and that the closed sets $\overline{V_{n+1}}$ and $H \setminus V_n$ are functionally separated, for every $n \in \omega$. Take a continuous real-valued function h_n on H with values in $[0, 1]$ such that $h_n(\overline{V_{n+1}}) = \{1\}$ and $h_n(H \setminus V_n) = \{0\}$. Then $h = \sum_{n\in\omega} 2^{-n-1}h_n$ is also a continuous function on H with values in $[0, 1]$ satisfying $h(e_H) = 1$ and $h^{-1}(P_n) \subset V_n$ for each $n \in \omega$, where $P_n = (1 - 2^{-n}, 1]$.

Let $f = h \circ p$. By the lemma's assumption, the group G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable. It follows from Theorem 4.16 that G is $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$-factorizable. Therefore, we can find a continuous homomorphism $\pi\colon G \to K$ onto a regular first-countable paratopological group K and a continuous real-valued function f_K on K such that $f = f_K \circ \pi$. Notice that $f_K(e_K) = f(e_G) = h(e_H) = 1$.

We claim that there exists a continuous homomorphism $\psi\colon K \to H$ satisfying $p = \psi \circ \pi$. According to [4, Proposition 1.5.10] it suffices to verify that for every neighborhood V of the identity in H, there exists an open neighborhood O of the identity in K such that $\pi^{-1}(O) \subset p^{-1}(V)$. For a given neighborhood V of the identity in H, take $n \in \omega$ such that $V_n \subset V$. It follows from our choice of the function h on H that $h^{-1}(P_n) \subset V_n$. Since $f = h \circ p$, we have that $f^{-1}(P_n) = p^{-1}(h^{-1}(P_n)) \subset p^{-1}(V_n) \subset p^{-1}(V)$. Clearly, $O = f_K^{-1}(P_n)$ is an open neighborhood of the identity e_K in K, because $f_K(e_K) = 1$. Applying the equality $f = f_K \circ \pi$, we conclude that the set O satisfies $\pi^{-1}(O) \subset p^{-1}(V)$.



This proves our claim and the lemma in the case of a completely regular group H.

Assume that the paratopological group G is regular. Then G is completely regular, by a Banakh–Ravsky theorem in [8]. Take a local base $\{V_n : n \in \omega\}$ at the identity of H and for each $n \in \omega$, let $U_n = p^{-1}(V_n)$. Since G is completely regular, we can choose a sequence $\{W_n : n \in \omega\}$ of open neighborhoods of the identity in G such that $W_n \subset U_n$, $\overline{W_{n+1}} \subset W_n$, and the sets $\overline{W_{n+1}}$ and $G \setminus W_n$ are functionally separated for each

$n \in \omega$. Similarly to the first part of the proof, we define a continuous real-valued function f on G with values in $[0,1]$, such that $f(e_G) = 1$ and $f^{-1}(P_n) \subset W_n$ for every $n \in \omega$, where $P_n = (1 - 2^{-n}, 1]$. Since G is $(\overline{\mathbb{R}}, \mathbb{FCPG}_r)$-factorizable, there exists a continuous homomorphism $\pi\colon G \to K$ onto a regular first-countable paratopological group K such that $f = f_K \circ \pi$, for some continuous real-valued function f_K on K. Our choice of f, π and f_K implies that for every neighborhood V of the identity in H, there exists $n \in \omega$ such that $\pi^{-1}\big(f_K^{-1}(P_n)\big) \subset p^{-1}(V)$. Therefore, applying [4, Proposition 1.5.10] once again, we find a continuous homomorphism $\psi\colon K \to H$ satisfying $p = \psi \circ \pi$. This completes the proof of the lemma. □

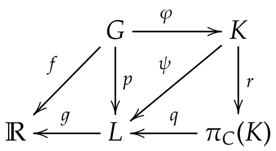
The following proposition is a crucial step in proving Theorem 4.22.

Proposition 4.21. *Every $(\overline{\mathbb{R}}, \mathbb{MSG})$-factorizable paratopological group is $(\overline{\mathbb{R}}, \mathbb{MPG})$-factorizable.*

Proof. Assume that G is an $(\overline{\mathbb{R}}, \mathbb{MSG})$-factorizable paratopological group. By (c) Proposition 2.6, with $\mathbb{O} = \mathbb{SG}$, the group $Reg(G)$ is also $(\overline{\mathbb{R}}, \mathbb{MSG})$-factorizable. Since the regular paratopological group $K = Reg(G)$ is completely regular, it follows from Lemma 4.5 that K is topologically isomorphic to a subgroup of the product of a family of metrizable semitopological groups, say, $M = \prod_{i\in I} M_i$. For every nonempty subset J of I, denote by π_J the projection of M onto the subproduct $M_J = \prod_{i\in J} M_i$ of M.

Let f be a continuous real-valued function on G. By the assumption, there exists a continuous homomorphism $p\colon G \to L$ onto a metrizable semitopological group L such that $f = g \circ p$, for some continuous real-valued function g on L. It is clear that the space L is normal, hence regular. Hence, there exists a continuous homomorphism $\psi\colon K \to L$ satisfying $p = \psi \circ \varphi$, where $\varphi\colon G \to K$ is the canonical homomorphism.

Since the group L is first-countable, we can apply [4, Lemma 8.5.4] to find a nonempty countable subset C of I and a continuous homomorphism $q\colon \pi_C(K) \to L$ such that $\psi = q \circ \pi_C \restriction K$. Let r be the restriction of the projection π_C to K. Then $\psi = q \circ r$.



By Proposition 4.4, there exists a countable set $J \subset I$ such that $C \subset J$ and $M^* = \pi_J(K)$ is a paratopological group when considered as a subgroup of the semitopological group M_J. Since J is countable, the groups M_J and M^* are metrizable. Let π^J_C be the projection of M_J onto M_C. Clearly, π^J_C is a continuous homomorphism satisfying $r = \pi_C \restriction K = \pi^J_C \circ \pi_J \restriction K$. We denote the restriction $\pi_J \restriction K$ by s, thus implying that $r = \pi^J_C \circ s$. Hence, $\lambda = s \circ \varphi$ is a continuous homomorphism of G onto the metrizable paratopological group M^* satisfying the equality $f = (g \circ q \circ \pi^J_C) \circ \lambda$, where $g \circ q \circ \pi^J_C \restriction M^*$ is a continuous real-valued function on M^*. We conclude that the group G is $(\overline{\mathbb{R}}, \mathbb{MPG})$-factorizable. □

Theorem 4.22. *For a paratopological group G, the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{MSG})$-factorizable;*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{MPG})$-factorizable;*
(c) *$T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{MPG})$-factorizable for some (every) $k \in \{0, 1, 2, 3, r\}$.*

Proof. The equivalence of items (a) and (b) of the theorem follows from Proposition 4.21.

The implication (b) ⇒ (c) is quite simple. Indeed, let $\varphi_k\colon G \to T_k(G)$ be the canonical homomorphism, where $k \in \{0, 1, 2, 3, r\}$. Then one uses the fact that for every continuous homomorphism $p\colon G \to M$ to a metrizable paratopological group M, there exists a continuous homomorphism $\psi\colon T_k(G) \to M$ satisfying $p = \psi \circ \varphi_k$. This is a direct consequence of the regularity of M and the definition of $T_k(G)$.

Finally, to deduce the implication (c) ⇒ (b), it suffices to notice that for every continuous real-valued function f on G, there exists a continuous real-valued function h on $T_k(G)$ satisfying $f = h \circ \varphi_k$. Actually, the proof of the implication (c) ⇒ (b) adheres to the pattern established in the proof of Theorem 4.18. □

There exists a counterpart to Theorem 4.22 applicable to topological groups. The proof is omitted, as it is now evident following the aforementioned theorem and Corollary 2.8.

Theorem 4.23. *Let G be a topological group with identity e and N be the closure of the singleton $\{e\}$ in G. Then the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{MSG})$-factorizable;*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{MPG})$-factorizable;*
(c) *G is $(\overline{\mathbb{R}}, \mathbb{MTG})$-factorizable or, equivalently, $\mathcal{M}$-factorizable;*
(d) *the quotient group G/N is $(\overline{\mathbb{R}}, \mathbb{MTG})$-factorizable.*

It is plausible to conjecture that every continuous real-valued function on a Tychonoff $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable semitopological group can be factorized via a continuous homomorphism onto a Tychonoff first-countable semitopological group. This is proved in Theorem 4.25 below, which complements [15, Lemma 3.3] and item (b) of Proposition 2.4.

First, we present a key lemma that, conceptually, parallels Proposition 4.4.

Lemma 4.24. *Let S be a subgroup of a product $\Pi = \prod_{i\in I} G_i$ of first-countable semitopological groups. If S is completely regular and $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable, then for every countable set $C \subset I$, there exists a countable set J with $C \subset J \subset I$ such that the subgroup $\pi_J(S)$ of $\Pi_J = \prod_{i\in J} G_i$ is also completely regular, where $\pi_J \colon \Pi \to \Pi_J$ is the projection.*

Proof. Since C is countable and the factors G_i are first-countable, the group Π_C is first-countable as well. We put $J_0 = C$. Let e be the identity element of S.

Assume that for some $n \in \omega$, we have defined countable subsets $J_0 \subset \cdots \subset J_n$ of the index set I. Take a countable local base γ_n at the identity element of the group Π_{J_n}. For every $U \in \gamma_n$, there exists a continuous real-valued function f_U on S such that $f_U(e) = 1$ and $f_U(S \setminus \pi_{J_n}^{-1}(U)) = \{0\}$. Since S is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable, we can find a continuous homomorphism φ_U of S onto a first-countable semitopological group K_U and a continuous real-valued function h_U on K_U such that $f_U = h_U \circ \varphi_U$. According to [4, Lemma 8.5.4], there exist a countable set $A_U \subset I$ and a continuous homomorphism $r_U \colon \pi_{A_U}(S) \to K_U$ satisfying $\varphi_U = r_U \circ \pi_{A_U} \restriction S$. Hence, we have the equality $f_U = h_U \circ r_U \circ \pi_{A_U} \restriction S$. Then $J_{n+1} = J_n \cup \bigcup\{A_U : U \in \gamma_n\}$ is a countable subset of the index set I with $J_n \subset J_{n+1}$.

Clearly, $J = \bigcup_{n\in\omega} J_n$ is a countable subset of I and $C = J_0 \subset J$. We claim that the subgroup $S^* = \pi_J(S)$ of Π_J is completely regular. By the homogeneity argument, it suffices to verify that for every open neighborhood V of the identity e^* in Π_J, there exists a continuous real-valued function g on S^* such that $g(e^*) = 1$ and $g(S^* \setminus V) = \{0\}$.

Take a canonical open neighborhood O of e^* in Π_J such that $O \subset V$. Then there exists a finite set $F \subset J$ such that $O = p_F^{-1} p_F(O)$, where $p_F \colon \Pi_J \to \Pi_F$ is the projection. It follows from the definition of J that there exists $n \in \omega$ such that $F \subset J_n$. Since γ_n is a local base at the identity of the group Π_{J_n}, there exists $U \in \gamma_n$ satisfying $p_F^{J_n}(U) \subset p_F(O)$, where $p_F^{J_n} \colon \Pi_{J_n} \to \Pi_F$ is the projection. Then $p_{J_n}^{-1}(U) \subset p_F^{-1} p_F(O) = O \subset V$. At the step n of the above inductive construction, we have chosen a continuous real-valued function f_U on S satisfying $f_U(e) = 1$ and $f_U(S \setminus \pi_{J_n}^{-1}(U)) = \{0\}$. Since $f_U = h_U \circ r_U \circ \pi_{A_U} \restriction S$ and $A_U \subset J_{n+1} \subset J$, we can define a real-valued function g on S^* by letting $g(\pi_J(x)) = f_U(x)$, for each $x \in S$. It follows from our definition of g that the equality $g = h_U \circ r_U \circ p_{A_U} \restriction S^*$ holds, where $p_{A_U} \colon \Pi_J \to \Pi_{A_U}$ is the projection. So, the definition of g is correct and g is continuous. It also clear that $g(e^*) = f_U(e) = 1$, while the inclusion $p_{J_n}^{-1}(U) \subset V$ and the choice of f_U imply that $g(S^* \setminus V) = \{0\}$. Hence, the function g is as required and the group $S^* = \pi_J(S)$ is completely regular. □

Theorem 4.25. *The following hold for a completely regular semitopological group G:*

(a) *The $(\overline{\mathbb{R}}, \mathbb{FCSG})$- and $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizability of G are equivalent.*
(b) *The $(\overline{\mathbb{R}}, \mathbb{SCSG})$- and $(\overline{\mathbb{R}}, \mathbb{SCSG}_t)$-factorizability of G are equivalent.*

Proof. Clearly, it suffices to verify that the $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizability of G implies its $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizability and, similarly, the $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizability of G implies its $(\overline{\mathbb{R}}, \mathbb{SCSG}_t)$-factorizability. We prove both implications simultaneously, noting in parentheses the required modifications to the argument for item (b).

Since G is completely regular and $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable), it follows from Lemma 4.5 that G is a subgroup of the Cartesian product of a family $\{G_i : i \in I\}$ of first-countable (resp., second-countable) semitopological groups. Let $\Pi = \prod_{i\in I} G_i$. If f is a continuous real-valued function on G, we can find a continuous homomorphism $\varphi\colon G \to H$ onto a first-countable (resp., second-countable) semitopological group H and a continuous real-valued function h on H satisfying $f = h \circ \varphi$. By [4, Lemma 8.5.4], the homomorphism φ can be represented as the composition $\varphi = \psi \circ \pi_C \restriction S$, where C is a countable subset of I, $\pi_C\colon \Pi \to \Pi_C$ is the projection and $\psi\colon \pi_C(G) \to H$ is a continuous homomorphism. It follows that $f = h \circ \psi \circ \pi_C \restriction S$.

By Lemma 4.24, there exists a countable subset J of I with $C \subset J$ such that the subgroup $G^* = \pi_J(G)$ of Π_J is completely regular. The inclusion $C \subset J$ implies that the continuous real-valued function $g = h \circ \psi \circ \pi^J_C \restriction G^*$ on G^* satisfies the equality $f = g \circ \pi_J \restriction S$, where $\pi^J_C\colon \Pi_J \to \Pi_C$ is the projection. Since J is countable, the group Π_J and its subgroup G^* are first-countable (resp., second-countable). We have thus shown that G is $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG}_t)$-factorizable). □

Theorem 4.25 is complemented by the following result, in which the semitopological group G is not assumed to satisfy any separation axiom.

Theorem 4.26. *The following are equivalent for a semitopological group G:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG}_t)$-factorizable);*
(b) *$Tych(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable);*
(c) *$Tych(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG}_t)$-factorizable).*

Proof. It suffices to consider the case of $(\overline{\mathbb{R}}, \mathbb{FCSG})$- and $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizability. Clearly, (c) implies (b), while (a) and (c) are equivalent by Proposition 2.5, where $\mathbb{O}$ is the class of semitopological groups and $k = t$. Finally, (b) implies (c) according to Theorem 4.25. □

A version of Theorem 4.25 for quasitopological groups is given below. It refines Theorem 4.6.

Theorem 4.27. *The following are equivalent for every completely regular quasitopological group G:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable);*
(b) *G is $(\overline{\mathbb{R}}, \mathbb{FCQG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCQG})$-factorizable);*
(c) *G is $(\overline{\mathbb{R}}, \mathbb{FCQG}_t)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCQG}_t)$-factorizable).*

Proof. The proof of the parenthetical part of the theorem is left to the reader. Since, by Theorem 4.6, the properties in items (a) and (b) of the theorem are equivalent for every completely regular quasitopological group, it suffices to verify that (b) implies (c).

It follows from the assumption in (b) and Lemma 4.5 that G is a subgroup of the product of a family $\{G_i : i \in I\}$ of first-countable quasitopological groups. Let $\Pi = \prod_{i\in I} G_i$. Assume that f is a continuous real-valued function on the group G. Arguing as in the proof of Theorem 4.25 and keeping notation introduced there, we find a countable subset J of the index set I and a continuous function g on the subgroup $G^* = \pi_J(G)$ of Π_J such that the group G^* is completely regular and the equality $f = g \circ \pi_J \restriction G^*$ holds. Since the subgroup G^* of the group Π_J is first-countable, we conclude that G is $(\overline{\mathbb{R}}, \mathbb{FCQG}_t)$-factorizable. □

We recall that a nonempty open subset U of a semitopological group G is *ω-good* [15, 24] if there exists a countable family γ of open neighborhoods of the identity in G such that for each $x \in U$, one can find $V \in \gamma$ with $xV \subset U$. A semitopological group G is *locally ω-good* if it has a base of ω-good sets. Clearly, every first-countable semitopological group is locally ω-good. Every paratopological group is locally ω-good [24, Lemma 2.5], yet this assertion does not hold for semitopological groups.

The following result appears as Lemma 3.16 in [15]. We offer a shorter alternative proof of it based on Theorem 4.25. The definition of the *index of regularity* of a regular semitopological group G, denoted by $Ir(G)$, can be found in [24].

Corollary 4.28. *Every completely regular* $(\overline{\mathbb{R}}, \mathbb{FCSG})$*-factorizable semitopological group G is ω-balanced, locally ω-good, and satisfies $Ir(G) \le \omega$. Also, every completely regular* $(\overline{\mathbb{R}}, \mathbb{SCSG})$*-factorizable semitopological group G is totally ω-narrow.*

Proof. By (a) of Theorem 4.25, the group G is $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$-factorizable. Hence, by Lemma 4.5, G is topologically isomorphic to a subgroup of the Cartesian product Π of a family λ of completely regular first-countable semitopological groups. Each element $H \in \lambda$ is an ω-balanced locally ω-good group satisfying $Ir(H) \le \omega$. Since each group $H \in \lambda$ is (completely) regular, and each of the three properties considered in the corollary is productive and hereditary with respect to taking arbitrary subgroups, we can apply [24, Corollary 3.4] and [15, Lemma 3.15] to conclude that every subgroup of Π has the required properties. This proves the first statement of the corollary.

If a completely regular semitopological group G is $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable, then $G = Tych(G)$, so the group G is is totally ω-narrow according to Lemma 4.10. □

Remark 4.29. Several results presented in Section 4 can be extended to other factorization properties, including $\mathcal{F}m$-, $P\mathbb{R}$-, and Pm-factorizability, contingent upon the appropriate reformulation of these properties in accordance with the new framework outlined in Subsection 1.1. This also pertains to factorization properties that are expressed in terms of the classes $\mathbb{OH}$ and $\mathbb{PH}$ of open and perfect homomorphisms. We delegate this task to readers who are interested due to space constraints.

5. Further generalizations, a discussion

The properties of groups within the classes $\mathbb{SG}$, $\mathbb{QG}$, and $\mathbb{PG}$ show significant sensitivity to separation axioms, even with the presence of good factorization properties (see e.g. Corollary 4.11 and Remark 4.12). This is why we pay much attention to separation axioms in Sections 2–4. Nonetheless, there exist several important subclasses within the aforementioned categories that have demonstrated a close relationship with certain classes of "factorizable" groups. It is sufficient to refer to *precompact* topological groups or to arbitrary subgroups of regular σ-compact paratopological groups. The groups from both subclasses are $\mathbb{R}$-factorizable, as established in [4, Corollary 8.1.17] and [21, Corollary 3.14], respectively. In other terms, various weak compactness properties imply $\mathbb{R}$-factorizability or similar characteristics within the classes $\mathbb{TG}$ and $\mathbb{PG}$, particularly when these properties are combined with suitable separation axioms.

Therefore, it would be natural and quite helpful to reformulate (and generalize) the results of this article in terms of *subvarieties* of $\mathbb{SG}, \mathbb{QG}, \mathbb{PG}$ and $\mathbb{TG}$ and *reflections* of groups in those subvarieties. Let us present a very succinct description of the suggested program.

Let $\mathcal{C}$ be a *PS-variety* of groups, that is, a subclass of $\mathbb{SG}$ closed under arbitrary Cartesian products and taking arbitrary subgroups. Given a semitopological group G, one defines a $\mathcal{C}$*-reflection* of G as a pair $\left(H, \varphi_G^{\mathcal{C}}\right)$, where $H \in \mathcal{C}$ and $\varphi_G^{\mathcal{C}} \colon G \to H$ is a continuous onto homomorphism such that for every continuous homomorphism $p \colon G \to C$ with $C \in \mathcal{C}$, one can find a continuous homomorphism $q \colon H \to C$ satisfying $p = q \circ \varphi_G^{\mathcal{C}}$. An argument similar to the one in the proof of [26, Theorem 2.3] shows that a $\mathcal{C}$-reflection of G exists and is unique up to a topological isomorphism. We denote the group H as above by $\mathcal{C}(G)$ and abusing of terminology also call it the $\mathcal{C}$-reflection of G.

For example, we can introduce the subvarieties $\mathbb{TN}(\omega) \cap \mathbb{PG}$ and $\mathbb{TN}(\omega) \cap \mathbb{TG}$ of the respective varieties $\mathbb{PG}$ and $\mathbb{TG}$, where $\mathbb{TN}(\omega)$ stands for "totally ω-narrow". Also, one can replace total ω-narrowness with the property of being ω-balanced, locally ω-good, etc. We can thus deal with totally ω-narrow reflections and ω-balanced reflections of the groups from the varieties $\mathbb{PG}$ and $\mathbb{TG}$ in the case of total ω-narrowness and the four varieties $\mathbb{SG}, \mathbb{QG}, \mathbb{PG}$ and $\mathbb{TG}$ in the case of ω-balancedness or local ω-goodness. Noteworthy, the corresponding reflection $\mathcal{C}(G)$ of G is in the same basic variety the group G belongs to. Incorporating algebraic restrictions into groups or merging algebraic and topological constraints provides an additional method for defining noteworthy subvarieties of the four basic varieties deserving of examination.

Let us call a category of topologized groups a *PSQ-category* if it is a *PS*-category closed with respect to taking quotient groups. With the concept of $\mathcal{C}$-reflection in hand, we can generalize Proposition 2.4 as follows:

Proposition 5.1. *Let $\mathcal{C}$ be a PSQ-subvariety of $\mathbb{SG}$ and G be semitopological group. Then the following hold:*

(a) *The group G is $(\overline{\mathbb{R}}, \mathbb{SC}\mathcal{C})$-factorizable iff G is $(\overline{\mathbb{R}}, \mathbb{SC}\mathcal{C}_2)$-factorizable iff $T_2(G)$ is $(\overline{\mathbb{R}}, \mathbb{SC}\mathcal{C}_2)$-factorizable.*

(b) *The group G is $(\overline{\mathbb{R}}, \mathbb{FC}\mathcal{C})$-factorizable iff G is $(\overline{\mathbb{R}}, \mathbb{FC}\mathcal{C}_2)$-factorizable iff $T_2(G)$ is $(\overline{\mathbb{R}}, \mathbb{FC}\mathcal{C}_2)$-factorizable.*

It would be interesting to find out if the results from Sections 2 and 4 can be generalized to *PS*- or *PSQ*-subvarieties of the four basic varieties, or to provide particular examples of subvarieties where such generalization is possible. Proposition 5.1 is an example of how this strategy works.

6. Open problems and comments

In addition to potential generalizations of the results discussed in Sections 2–4 and proposed in Section 5, there are several specific questions that are closely connected to the subject matter of this article.

We do not know whether all the items in Theorems 4.13 and 4.19 remain equivalent in the category of semitopological groups:

Problem 6.1. *Let G be an $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable semitopological group. Is the group $Reg(G)$ or $Tych(G)$ $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable?*

We pose a similar problem for $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizability:

Problem 6.2. *Let G be an $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable semitopological group. Is the group $Reg(G)$ or $Tych(G)$ $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable?*

Since the canonical homomorphism $\varphi_{G,k}\colon G \to T_k(G)$ is open for every semitopological group G and every $k = 0, 1, 2$, Proposition 2.4 and items (a) and (b) of Proposition 2.6 (see also Lemmas 3.3 and 3.8 from [15]) can be complemented as follows:

Proposition 6.3. *For every semitopological group G, the following are equivalent:*

(a) *G is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable);*

(b) *G is $(\overline{\mathbb{R}}, \mathbb{FCSG}_2)$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG}_2)$-factorizable);*

(c) *$T_k(G)$ is $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizable (resp., $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable) for each $k \in \{0, 1, 2\}$.*

Consequently, we inquire in Problems 6.1 and 6.2 if it is possible to add the new item, containing the regular reflection $Reg(G)$ or the Tychonoff reflection $Tych(G)$ of G, to item (c) of Proposition 6.3.

The following open problem arises from the equivalence of (a) and (b) in Proposition 6.3. The question examines the possibility of dropping the assumption of complete regularity of the group G in Theorem 4.27 without loosing the equivalence of items (a) and (b) of the theorem.

Problem 6.4. *Let G be an $(\overline{\mathbb{R}}, \mathbb{SCSG})$-factorizable semitopological group. Is G $(\overline{\mathbb{R}}, \mathbb{SCSG}_r)$-factorizable?*

It is easy to see that the affirmative response to Problem 6.4 implies that the answer to Problem 6.1 is also affirmative. Indeed, assume that a semitopological group G is $(\overline{\mathbb{R}}, \mathbb{SCSG}_r)$-factorizable. Denote by φ the canonical homomorphism of G onto $Reg(G)$. Let g be a continuous real-valued function on the group $Reg(G)$. Then $f = g \circ \varphi$ is a continuous real-valued function on G, so we can find a continuous homomorphism p of G onto a regular second-countable semitopological group H and a continuous real-valued function h on

H satisfying $f = h \circ p$. Since H is regular, there exists a continuous homomorphism $\psi\colon Reg(G) \to H$ such that $p = \psi \circ \varphi$. Therefore, the equality $g = h \circ \psi$ is valid, implying that the group $Reg(G)$ is $(\overline{\mathbb{R}}, \mathbb{SCSG}_r)$-factorizable. The same argument works for the group $Tych(G)$ because the regular second-countable group H in the above argument is normal, hence completely regular.

The following is a version of Problem 6.4 for $(\overline{\mathbb{R}}, \mathbb{FCSG})$-factorizability:

Problem 6.5. *Does the* $(\overline{\mathbb{R}}, \mathbb{FCSG})$*-factorizability of a semitopological group G imply its* $(\overline{\mathbb{R}}, \mathbb{FCSG}_r)$*-factorizability or even* $(\overline{\mathbb{R}}, \mathbb{FCSG}_t)$*-factorizability?*

Again, the affirmative response to Problem 6.5 implies that the answer to Problem 6.2 is affirmative in the case of the group $Reg(G)$.

The following two problems offer a way of generalizing Theorem 4.6.

Problem 6.6. *Does the* $(\overline{\mathbb{R}}, \mathbb{FCSG})$*-factorizability of a quasitopological group G imply its* $(\overline{\mathbb{R}}, \mathbb{FCQG})$*-factorizability? What if G is regular?*

Problem 6.7. *Does the* $(\overline{\mathbb{R}}, \mathbb{SCSG})$*-factorizability of a quasitopological group G imply its* $(\overline{\mathbb{R}}, \mathbb{SCQG})$*-factorizability? What if G is regular?*

The first step in addressing Problems 6.1 and 6.4 may involve resolving the subsequent three problems:

Problem 6.8. *Let G be a second-countable semitopological (quasitopological) group. Is the regular reflection* $Reg(G)$ *or the Tychonoff reflection* $Tych(G)$ *of G second-countable?*

Problem 6.9. *Let G be a first-countable semitopological (quasitopological) group. Is the regular reflection* $Reg(G)$ *or the Tychonoff reflection* $Tych(G)$ *of G first-countable?*

Since the group $Reg(G)$ in Problem 6.8 is a continuous homomorphic image of G, the former group has a countable network. We also note that for $T_2(G)$ in place of $Reg(G)$, the answer to Problems 6.8 and 6.9 is affirmative because the Hausdorff reflection $T_2(G)$ of G is an open continuous homomorphic image of G [26, Proposition 2.5].

The *index of regularity* of regular semitopological and paratopological groups is defined in [24, Section 3].

Problem 6.10. *Is it true that every regular semitopological group G with a countable network has countable index of regularity?*

According to [24, Proposition 3.5], every regular Lindelöf paratopological group has countable index of regularity.

Problem 6.11. *Is every Hausdorff (or regular) σ-compact semitopological group* $\mathbb{R}$*-factorizable?*

For a regular paratopological group G, the answer to Problem 6.11 is affirmative, see [21, Corollary 3.14]. Actually, the regularity of G can be omitted, in accordance with Theorem 4.13.

We also recall two open problems that were formulated as a single Problem 5.1 in [21]. We believe they have a close connection to the subject matter of this article.

Problem 6.12. *Let G be a regular* $\mathbb{R}$*-factorizable paratopological group. Is the associated topological group* G^* $\mathbb{R}$*-factorizable?*

For Hausdorff $\mathbb{R}$-factorizable paratopological groups, the response to Problem 6.12 is negative. Indeed, as it is mentioned in Remark 4.12, there exists a Hausdorff paratopological group G such that the semiregularization (hence, the regular reflection) of G is a countably compact topological group, which is $\mathbb{R}$-factorizable, but G is not ω-narrow. Since the topology of the associated topological group G^* is finer than the topology of G, the group G^* is not ω-narrow either. We conclude that G^* cannot be $\mathbb{R}$-factorizable since every $\mathbb{R}$-factorizable topological group is ω-narrow, as stated in [4, Proposition 8.1.3]. It should be noted that the the group $Reg(G)$ is $\mathbb{R}$-factorizable if and only if so is G, by Theorem 4.18. Thus, while G^* is not $\mathbb{R}$-factorizable, the group G is.

Problem 6.13. *Let G be a paratopological group such that the associated topological group G^* is $\mathbb{R}$-factorizable. Must the group G be $\mathbb{R}$-factorizable? What if G is regular?*

References

[1] O. Alas, M. Sanchis, *Countably compact paratopological groups*, Semigroup Forum **74** (2007), 423–438.

[2] S. Andima, R. Kopperman, P. Nickolas, S. Popvassilev, *A family of asymmetric Ellis-type theorems*, Houston J. Math. **36** (2010), 181–198.

[3] A. V. Arhangel'skii, *Classes of topological groups*, Russian Math. Surveys **36** (1981), 151–174. Russian original in: Uspekhy Mat. Nauk **36** no. 3 (1981), 127–146.

[4] A. V. Arhangel'skii, M. G. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics, vol. 1, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, New York, 2008 (xiv+781 pp.) ISBN: 978-90-78677-06-2.

[5] A. V. Arhangel'skii, M. G. Tkachenko, *C-extensions of topological groups*, Topol. Appl. **235** (2018), 54–72.

[6] T. Banakh, S. Dimitrova, *Openly factorizable spaces and compact extensions of topological semigroups*, Comment. Math. Univ. Carolinae **51** (2010), 113–131.

[7] T. Banakh, A. Ravsky, *On subgroups of saturated or totally bounded paratopological groups*, Algebra Discrete Math. no. 4 (2003), 1–20.

[8] T. Banakh, A. Ravsky, *Each regular paratopological group is completely regular*, Proc. Amer. Math. Soc. **145** (2017), 1373–1382.

[9] T. Banakh, A. Ravsky, *The regularity of quotient paratopological groups*, Mat. Studii **49** no. 2 (2018), 144–149.

[10] M. Bao, X. Xu, *On some kinds of factorizable topological groups*, arXiv:2205.01511v2 [math.GN], May 14, 2022.

[11] R. Engelking, *General Topology*, Heldermann Verlag, Berlin 1989.

[12] P. Hafner, G. Mazzola, *The cofinal character of uniform spaces and ordered fields*, Z. Math. Logik Grundl. **17** (1971), 377–384.

[13] W. He, D. Peng, M. Tkachenko, H. Zhang, *$\mathcal{M}$-factorizable feathered topological groups*, Topol. Appl. **289** (2021), 107481.

[14] L.-X. Peng, Y. Liu, *On Lindelöf feathered topological groups*, Topol. Appl. **285** (2020), 107405.

[15] L.-X. Peng,Y.-M. Deng, *On MM-ω-balancedness and $\mathcal{F}\mathbb{R}(\mathcal{F}m)$-factorizable semi(para)topological groups*, Topol. Appl. **360** (2025), 109183.

[16] L.-X. Peng, C.-J. Ma, Y.-M. Deng, *On bM-ω-balancedness and $\mathcal{M}$-factorizability of para(semi)topological groups*, Topol. Appl. **337** (2023), 108639.

[17] L.-X. Peng, Y. Wang, *On some properties of $\mathcal{P}\mathbb{R}$ ($\mathcal{P}m$)-factorizable semitopological groups*, Filomat **39** (2025), 4079–4093.

[18] E. Reznichenko, M. Tkachenko, *Regular rigid Korovin orbits*, *RACSAM* **120** no. 6 (2026).

[19] I. Sánchez, *Cardinal invariants of paratopological groups*, Topol. Algebra Appl. **1** (2013), 37–45.

[20] I. Sánchez, *Subgroups of products of paratopological groups*, Topol. Appl. **163** (2014), 160–173.

[21] M. Sanchis, M. G. Tkachenko, *$\mathbb{R}$-factorizable paratopological groups*, Topol. Appl. **157** (2010), 800–808.

[22] M. Sanchis, M. G. Tkachenko, *Feebly compact paratopological groups and real-valued functions*, Monatshefte Math. **168** (2012), 579–597.

[23] M. G. Tkachenko, *Factorization theorems for topological groups and their applications*, Topol. Appl. **38** (1991), 21–37.

[24] M. G. Tkachenko, *Embedding paratopological groups into topological products*, Topol. Appl. **156** (2009), 1298–1305.

[25] M. G. Tkachenko, *Paratopological and Semitopological Groups Versus Topological Groups*, Ch. 20 in *Recent Progress in General Topology III* (K.P. Hart, J. van Mill, P. Simon, Eds.), Atlantis Press, 2014; pp. 825–882.

[26] M. G. Tkachenko, *Axioms of separation in semitopological groups and related functors*, Topol. Appl. **161** (2014), 364–376.

[27] M. G. Tkachenko, *Axioms of separation in paratopological groups and reflection functors*, Topol. Appl. **179** (2015), 200–214.

[28] L.-H. Xie, S. Lin, M. Tkachenko, *Factorization properties of paratopological groups*, Topol. Appl. **160** (2013), 1902–1917.

[29] L.-H. Xie, M. G. Tkachenko, *Simply sm-factorizable (para)topological groups and their completions*, Monatshefte für Mathematik **193:2** (2020), 507–529.

[30] L.-H. Xie, M. G. Tkachenko, *Simply sm-factorizable (para)topological groups and their quotients*, Topol. Appl. **301** (2021), 107537.

[31] L.-H. Xie, P.-F. Yan, The continuous d-open homomorphism images and subgroups of $\mathbb{R}$-factorizable paratopological groups, Topol. Appl. **300** (2021), 107627.

[32] H. Zhang, D. Peng, W. He, *On M-factorizable topological groups*, Topol. Appl. **274** (2020), 107126.

[33] H. Zhang, W. He, D. Peng, M. G. Tkachenko, *On $\mathcal{M}$-factorizable P-groups*, Topol. Appl. **290** (2021), 107578.

[34] H. Zhang, W. Xi, Y. Wu, H. Li, *On Ψ_ω-factorizable groups*, Topol. Appl. **358** (2024), 109129.