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New approaches to solving initial value problems of mixed differential equations

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Abstract. An initial value problem involving mixed (ordinary and fractional) derivatives for non-autonomous variable-order differential equations is presented. We investigate the existence and uniqueness of solutions, as well as their Ulam-Hyers stability. Finally, we illustrate our results through numerical examples.

1. Introduction

Functional differential equations find extensive applications across diverse disciplines, including biology, physics, and engineering; see, for example, [2, 4, 8]. In recent years, variable-order fractional calculus of variations has emerged as an effective tool in research, control, and optimization, providing robust analytical and numerical methods to model complex phenomena under varying conditions. An advanced extension of variable-order fractional calculus has also been developed; for further details, see [1, 5–7, 10, 11, 18–21]. In this framework, the order function, originally denoted by $\mathfrak{I}(\ell)$, is generalized to a modified form, $\mathfrak{I}(\ell, \hbar(\ell))$, to capture more intricate dynamic behaviors.

In [12], the authors investigated the existence of solutions for a specific class of variable-order fractional differential equations, which are formulated as follows:

$$\begin{cases} \mathbb{D}_{\varsigma^+}^{\mathfrak{I}(\ell,\hbar(\ell))}\hbar(\ell) = \Theta(\ell, \hbar(\ell)), \\ \hbar(h) = \hbar_0, \end{cases}$$

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where the operator $\mathbb{D}_{h^+}^{\Im(\ell,\hbar(\ell))}$ denotes the Riemann-Liouville fractional derivative of variable order $\Im(\ell,\hbar(\ell))$, and Θ is a continuous function defined within the problem framework.

Motivated by these developments, we investigate a related non-autonomous initial value problem (NAVOIVP) involving variable-order fractional derivatives, formulated as follows:

$$\begin{cases} \mathbb{D}_{0^+}^{\Im(\ell,\hbar(\ell))}\hbar(\ell) + \varsigma\hbar'(\ell) = \Theta(\ell,\hbar(\ell)), \ \ \ell \in \tilde{\psi}_\ell := [0,\tilde{\ell}], \\ \hbar(0) = 0. \end{cases} \tag{1}$$

In this context, the interval is defined as $0 < \tilde{\ell} < +\infty$, with ς being a positive parameter. The functions are specified as $\Im : \tilde{\psi}\ell \times \mathbb{R} \to (0,1)$ and $\Theta : \tilde{\psi}\ell \times \mathbb{R} \to \mathbb{R}$, both of which are continuous. The operator $\mathbb{D}_{0^+}^{\Im(\ell,\hbar(\ell))}$ denotes the Riemann-Liouville fractional derivative of variable order $\Im(\ell,\hbar(\ell))$.

The main objective of this study is to establish novel criteria for the existence and uniqueness of solutions to the NAVOIVP (1). Numerical examples are presented at the end to illustrate and validate the theoretical results.

2. Preliminary

This section introduces some of the notations, definitions, and basic concepts used in this work. Note that $\mathbb{E} = C(\tilde{\psi}_\ell,\mathbb{R})$ denotes the Banach space of continuous functions $\hbar$ mapping $\tilde{\psi}_\ell$ into $\mathbb{R}$, equipped with the norm

$$\|\hbar\| = \sup\{|\hbar(\ell)| \mid \ell \in \tilde{\psi}_\ell\}.$$

Definition 2.1. *[13, 16, 17] Let $\Im : \tilde{\psi}_\ell \times \mathbb{R} \to (0,1)$ be a continuous function. The left Riemann-Liouville fractional integral of variable order $\Im(\ell,\hbar(\ell))$ for the function $\hbar(\ell)$ is defined by*

$$\mathbb{I}_0^{\Im(\ell,\hbar(\ell))}\hbar(\ell) = \int_0^\ell \frac{(\ell-\aleph)^{\Im(\aleph,\hbar(\aleph))-1}}{\Gamma(\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\,d\aleph, \quad \ell > 0, \tag{2}$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. *[14, 17, 22] Consider a continuous function $\Im : \tilde{\psi}_\ell \times \mathbb{R} \to (0,1)$. The left Riemann-Liouville fractional derivative of variable order $\Im(\ell,\hbar(\ell))$, applied to the function $\hbar(\ell)$, is defined as*

$$\mathbb{D}_0^{\Im(\ell,\hbar(\ell))}\hbar(\ell) = \Big(\frac{d}{d\ell}\Big)\mathbb{I}_0^{1-\Im(\ell,\hbar(\ell))}\hbar(\ell) = \Big(\frac{d}{d\ell}\Big)\int_0^\ell \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\,d\aleph, \quad \ell > 0. \tag{3}$$

Remark 2.3. *[24–26] When examining general functions such as $\Im(\ell,\hbar(\ell))$ and $v(\ell,\hbar(\ell))$, it becomes evident that the semigroup property does not hold, i.e.,*

$$\mathbb{I}_{a^+}^{\Im(\ell,\hbar(\ell))}\,\mathbb{I}_{a^+}^{v(\ell,\hbar(\ell))}\hbar(\ell) \neq \mathbb{I}_{a^+}^{\Im(\ell,\hbar(\ell))+v(\ell,\hbar(\ell))}\hbar(\ell).$$

Lemma 2.4. *[27] Suppose $\Im : \tilde{\psi}_\ell \times \mathbb{R} \to (0,1)$ is a continuous mapping. Then, for any function y belonging to the space*

$$C_\delta(\tilde{\psi}_\ell,\mathbb{R}) := \{y(\ell) \in C(\tilde{\psi}_\ell,\mathbb{R}) \mid \ell^\delta y(\ell) \in C(\tilde{\psi}_\ell,\mathbb{R}),\ 0 \le \delta < \min \Im(\ell,\hbar(\ell))\},$$

the fractional integral of variable order, denoted by $\mathbb{I}_{0^+}^{\Im(\ell,\hbar(\ell))}y(\ell)$, is well-defined for every ℓ in the domain $\tilde{\psi}_\ell$.

Lemma 2.5. *[27] Let $\Im \in C(\tilde{\psi}_\ell \times \mathbb{R},(0,1])$ be a continuous function. Then, for any $y \in C(\tilde{\psi}_\ell,\mathbb{R})$,*

$$\mathbb{I}_{0^+}^{\Im(\ell,\hbar(\ell))}y(\ell) \in C(\tilde{\psi}_\ell,\mathbb{R}).$$

Theorem 2.6. *[3] Let K be a closed, bounded, and convex subset of a real Banach space $\Im$, and let $\wp_1$ and $\wp_2$ be operators on K satisfying the following conditions:*

1. *$\wp_1(K) + \wp_2(K) \subset K$,*
2. *$\wp_1$ is continuous on K and $\wp_1(K)$ is relatively compact subset of $\Im$,*
3. *$\wp_2$ is a strict contraction on K, i.e, there exists $\tilde{p} \in [0,1)$ such that*

$$\| \wp_2(\hbar) - \wp_2(\kappa) \| \le \tilde{p} \, \| \hbar - \kappa \|,$$

for every $\hbar,\kappa \in K$. Then, there exists $\hbar \in K$ such that $\wp_1(\hbar) + \wp_2(\hbar) = \hbar$.

3. Conditions for Existence

We now present the following hypotheses:

(SY1) There exist positive constants $0 < \sigma < \min \Im(\ell, \hbar(\ell))$ and $\Lambda > 0$ such that the expression $\ell^{\sigma}\Theta$ remains continuous over the domain $\tilde{\psi}_{\ell} \times \mathbb{R}$, and

$$\ell^{\sigma} \left|\Theta(\ell, \hbar(\ell)) - \Theta(\ell, y(\ell))\right| \leq \Lambda \left|\hbar(\ell) - y(\ell)\right|, \quad \forall \hbar, y \in \mathbb{R},\ \ell \in \tilde{\psi}_{\ell}.$$

(SY2) $\Im : \tilde{\psi}_{\ell} \times \mathbb{R} \to (0, \Im^{*}]$ is a continuous function such that

$$0 \leq \Im(\ell, \hbar(\ell)) \leq \Im^{*} < 1.$$

Remark 3.1. *[23]*

1. *The function $\Gamma(2 - \Im(\ell, \hbar(\ell)))$ arises from the composition of two continuous functions and is therefore itself continuous. Consequently, we define*

$$M_{\Gamma} = \max \left|\frac{1}{\Gamma(2 - \Im(\ell, \hbar(\ell)))}\right|.$$

2. *Due to the continuity of the function $\Im(\ell, \hbar(\ell))$, we have*

$$\tilde{\ell}^{1-\Im(\ell,\hbar(\ell))} \leq 1 \quad \text{if } 1 \leq \tilde{\ell} < \infty, \quad \text{and} \quad \tilde{\ell}^{1-\Im(\ell,\hbar(\ell))} \leq \tilde{\ell}^{1-\Im^{*}} \quad \text{if } 0 \leq \tilde{\ell} \leq 1.$$

Therefore, we conclude that

$$\tilde{\ell}^{1-\Im(\ell,\hbar(\ell))} \leq \max\{1, \tilde{\ell}^{1-\Im^{*}}\} = \tilde{\ell}^{*}.$$

Remark 3.2. *[9] Assuming that X and Y are two real numbers, then*

$$| \varsigma X - \beta Y | \leq 2 \max(\varsigma, \beta) \mid X - Y \mid,$$

where ς and β are positive real numbers.

Lemma 3.3. *[23] Let (SY2) hold and let $\hbar_n, \hbar \in C[0, \tilde{\ell}]$. Assume that $\hbar_n(\ell) \to \hbar(\ell)$ for all $\ell \in [0, \tilde{\ell}]$ as $n \to \infty$. Then*

$$\int_0^{\ell} \frac{(\ell - \aleph)^{-\Im(\aleph,\hbar_n(\aleph))}}{\Gamma(1 - \Im(\aleph, \hbar_n(\aleph)))} \hbar_n(\aleph)\, d\aleph \longrightarrow \int_0^{\ell} \frac{(\ell - \aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1 - \Im(\aleph, \hbar(\aleph)))} \hbar(\aleph)\, d\aleph, \quad \ell \in [0, \tilde{\ell}],$$

as $n \to \infty$.

Furthermore, the following lemma will be needed to solve the NAVOIVP (1).

Proposition 3.4. *The function $\hbar \in \mathbb{E}$ forms a solution of the NAVOIVP (1) if and only if $\hbar$ fulfills the integral equation*

$$\hbar(\ell) = \frac{1}{\varsigma}\Big[\int_0^{\ell} \Theta(\aleph, \hbar(\aleph))\ d\aleph - \int_0^{\ell} \frac{(\ell - \aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1 - \Im(\aleph, \hbar(\aleph)))}\hbar(\aleph)\ \ d\aleph\Big]. \tag{4}$$

Proof Utilizing the definition of the variable-order fractional derivative given in (3), the NAVOIVP in (1) can be reformulated as follows:

$$\left(\frac{d}{dt}\right) \int_0^{\ell} \frac{(\ell - \aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1 - \Im(\aleph, \hbar(\aleph)))}\hbar(\aleph)\ \ d\aleph + \varsigma\hbar^{'}(\ell) = \Theta(\ell, \hbar(\ell)).$$

Then,

$$\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph + \varsigma\hbar(\ell) = \int_0^{\ell} \Theta(\aleph,\hbar(\aleph))\ d\aleph + c_1. \tag{5}$$

Now, the evaluation of equation (5) at $\ell = 0$ give us $c_1 = 0$. Thus, we have

$$\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph + \varsigma\hbar(\ell) = \int_0^{\ell} \Theta(\aleph,\hbar(\aleph))\ d\aleph,$$

so

$$\hbar(\ell) = \frac{1}{\varsigma}\Big[\int_0^{\ell} \Theta(\aleph,\hbar(\aleph))\ d\aleph - \int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph\Big].$$

On the other hand, differentiating both sides of the equation (4), we have

$$(\frac{d}{dt})\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph + \varsigma\hbar'(\ell) = \Theta(\ell,\hbar(\ell)),$$

which means the NAVOIVP (1).
Now, we will prove the existence of solutions for the NAVOIVP (1). Theorem 2.6 forms the basis of the first finding.

Theorem 3.5. *Suppose that the assumptions (SY1) and (SY2) are satisfied. If the inequality*

$$\frac{M_{\Gamma},\tilde{\ell}\tilde{\ell}}{1-\Im} + \Lambda,\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} < \varsigma, \tag{6}$$

holds, then the (NAVOIVP) given in (1) admits at least one solution in the space $\mathbb{E}$.

Proof First, we introduce and define the following operators

$$\wp_1,\ \wp_2 : \mathbb{E} \to \mathbb{E},$$

as

$$\wp_1\hbar(\ell) = \frac{1}{\varsigma}\Big[-\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph\Big],\quad \wp_2\hbar(\ell) = \frac{1}{\varsigma}\int_0^{\ell} \Theta(\aleph,\hbar(\aleph))\ d\aleph.$$

Next, consider the set

$$B_R = \{\hbar \in \mathbb{E}, \parallel \hbar \parallel \leq R\},$$

where

$$R \geq \frac{\frac{\Theta^*\tilde{\ell}}{\varsigma}}{1-\frac{1}{\varsigma}\Big[\frac{M_\Gamma\tilde{\ell}^*\tilde{\ell}}{1-\Im^*} + \Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\Big]},$$

and

$$\Theta^* = \sup_{\ell\in\tilde{\psi}_\ell} |\ \Theta(\ell,0)\ |.$$

It is evident that B_R is nonempty, bounded, convex, and closed. Next, we aim to demonstrate that the functions $\wp_1$ and $\wp_2$ satisfy the conditions stated in Theorem 2.6. This will be established through the following four-step approach:
Step 1:$\wp_1(B_R) + \wp_2(B_R) \subseteq B_R$. For $\hbar \in B_R$, we obtain

$$|\ \wp_1\hbar(\ell) + \wp_2\hbar(\ell)\ | \leq \frac{1}{\varsigma}\Big[\ |\int_0^{\ell} \Theta(\aleph,\hbar(\aleph))\ d\aleph\ | + |\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph\ |\ \Big]$$

$$
\begin{aligned}
&\leq \frac{1}{\varsigma}\Big[\int_0^{\ell} \mid \Theta(\aleph,\hbar(\aleph)) - \Theta(\aleph,0) + \Theta(\aleph,0) \mid d\aleph + \int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))} \mid \hbar(\aleph) \mid \quad d\aleph\Big]\\
&\leq \frac{1}{\varsigma}\Big[\int_0^{\ell} \aleph^{-\sigma}\aleph^{\sigma} \mid \Theta(\aleph,\hbar(\aleph)) - \Theta(\aleph,0) \mid d\aleph + \int_0^{\ell} \mid \Theta(\aleph,0) \mid d\aleph\\
&+ M_\Gamma \int_0^{\ell} \tilde{\ell}^{-\Im(\aleph,\hbar(\aleph))}\Big(\frac{\ell-\aleph}{\tilde{\ell}}\Big)^{-\Im(\aleph,\hbar(\aleph))} \mid \hbar(\aleph) \mid d\aleph\Big]\\
&\leq \frac{1}{\varsigma}\Big[\int_0^{\ell} \Lambda \mid \hbar(\aleph) \mid \aleph^{-\sigma} d\aleph + \int_0^{\ell} \Theta^* d\aleph + M_\Gamma \tilde{\ell}^{*} \int_0^{\ell} \Big(\frac{\ell-\aleph}{\tilde{\ell}}\Big)^{-\Im(\aleph,\hbar(\aleph))} \mid \hbar(\aleph) \mid d\aleph\Big]\\
&\leq \frac{1}{\varsigma}\Big[\Lambda \parallel \hbar \parallel \frac{\ell^{-\sigma+1}}{-\sigma+1} + \Theta^*\ell + \frac{M_\Gamma \tilde{\ell}^*}{\tilde{\ell}^{-\Im^*}} \parallel \hbar \parallel \int_0^{\ell} (\ell-\aleph)^{-\Im^*} d\aleph\Big]\\
&\leq \frac{1}{\varsigma}\Big[\Lambda \parallel \hbar \parallel \frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} + \Theta^*\tilde{\ell} + \frac{M_\Gamma \tilde{\ell}^*}{\tilde{\ell}^{-\Im^*}} \frac{\ell^{1-\Im^*}}{(1-\Im^*)} \parallel \hbar \parallel \Big]\\
&\leq \frac{1}{\varsigma}\Big[\Lambda \parallel \hbar \parallel \frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} + \Theta^*\tilde{\ell} + \frac{M_\Gamma \tilde{\ell}^*\tilde{\ell}}{(1-\Im^*)} \parallel \hbar \parallel \Big]\\
&\leq \frac{1}{\varsigma}\Big[\frac{M_\Gamma \tilde{\ell}^*\tilde{\ell}}{1-\Im^*} + \Lambda \frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\Big] \parallel \hbar \parallel + \frac{\Theta^*\tilde{\ell}}{\varsigma}\\
&\leq \frac{1}{\varsigma}\Big[\frac{M_\Gamma \tilde{\ell}^*\tilde{\ell}}{1-\Im^*} + \Lambda \frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\Big] R + \frac{\Theta^*\tilde{\ell}}{\varsigma}\\
&\leq R,
\end{aligned}
$$

which means that $\wp_1(B_R) + \wp_2(B_R) \subseteq B_R$.
Step 02: The function $\wp_1$ is continuous. Consider a sequence $\hbar_n$ in $\mathbb{E}$ such that $\hbar_n \to \hbar$. For a fixed $\ell \in \tilde{\psi}_\ell$, we proceed to derive an estimate:

$$
\mid \wp_1\hbar_n(\ell) - \wp_1\hbar(\ell) \mid \;\leq\; \frac{1}{\varsigma}\Big[\mid \int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar_n(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar_n(\aleph)))}\hbar_n(\aleph)d\aleph - \int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)d\aleph \mid \Big].
$$

By using lemma 3.3, we have

$$
\parallel \wp_1\hbar_n(\ell) - \wp_1\hbar(\ell) \parallel \to 0, \;\; n \to \infty.
$$

The preceding analysis confirms that the operator $\wp_1$ is continuous on the space $\mathbb{E}$.
Step 3: We now demonstrate the compactness of $\wp_1$. To do so, we aim to show that the image $\wp_1(B_R)$ is relatively compact, which directly implies that $\wp_1$ is a compact operator. From Step 1, it follows that $\wp_1(B_R)$ is uniformly bounded. Specifically, we observe:

$$
\wp_1(B_R) = \{\wp_1(\hbar) : \hbar \in B_R\} \subset \wp_1(B_R) + \wp_2(B_R) \subset B_R,
$$

which indicates that for all $\hbar \in B_R$, the norm $\parallel \wp_1(\hbar) \parallel \leq R$ thus confirming uniform boundedness of the image set $\wp_1(B_R)$. Now, consider the function defined as $w(\ell) = a^\ell - b^\ell$ for $\ell \in (-1, 0)$ with constants $0 < a < b < 1$. This function is strictly decreasing. Indeed, given that $\ln a < \ln b < 0$ and $a^\ell > b^\ell > 0$, we compute:

$$
w'(\ell) = a^\ell \ln a - b^\ell \ln b < b^\ell \ln a - b^\ell \ln b = b^\ell(\ln a - \ln b) < 0,
$$

which confirms the monotonicity of $w(\ell)$. Applying this idea, consider the expression $\varkappa(\aleph) = \left(\frac{\ell_1-\aleph}{\tilde{\ell}}\right)^{-\Im(\aleph,\hbar(\aleph))} - \left(\frac{\ell_2-\aleph}{\tilde{\ell}}\right)^{-\Im(\aleph,\hbar(\aleph))}$, where the ratios satisfy $0 < \frac{\ell_1-\aleph}{\tilde{\ell}} < \frac{\ell_2-\aleph}{\tilde{\ell}} < 1$. Noting the structural similarity to $w(\aleph)$, we deduce that $\varkappa(\aleph)$ also decreases as the exponent $-\Im(\aleph,\hbar(\aleph))$ increases. Consequently, for $\ell_1, \ell_2 \in \tilde{\psi}_\ell$ with $\ell_1 < \ell_2$, and for any $\hbar \in B_R$, we find that

$$
\mid \wp_1\hbar(\ell_2) - \wp_1\hbar(\ell_1) \mid \;\leq\; \frac{1}{\varsigma}\Big[\mid \int_0^{\ell_1} \frac{(\ell_1-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph))))}\hbar(\aleph) - \int_0^{\ell_2} \frac{(\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)d\aleph \mid \Big]
$$

$$
\begin{aligned}
&\leq \frac{1}{\varsigma}\Big[\mid \int_0^{\ell_1} \frac{(\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph) - \frac{(\ell_1-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)d\aleph \mid \\
&+ \mid \int_{\ell_1}^{\ell_2} \frac{(\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)d\aleph \mid \Big] \\
&\leq \frac{1}{\varsigma}\Big[\int_0^{\ell_1} \mid \frac{1}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))} \parallel (\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))} - (\ell_1-\aleph)^{-\Im(\aleph,\hbar(\aleph))} \parallel \hbar(\aleph) \mid d\aleph \\
&+ \int_{\ell_1}^{\ell_2} \frac{(\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))} \mid \hbar(\aleph) \mid d\aleph \Big] \\
&\leq \frac{1}{\varsigma}\Big[M_\Gamma \parallel \hbar \parallel \int_0^{\ell_1} (\ell_1-\aleph)^{-\Im(\aleph,\hbar(\aleph))} - (\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))} d\aleph \\
&+ \int_{\ell_1}^{\ell_2} \frac{(\ell_2-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))} \mid \hbar(\aleph) \mid d\aleph \Big] \\
&\leq \frac{1}{\varsigma}\Big[M_\Gamma \parallel \hbar \parallel \int_0^{\ell_1} \tilde{\ell}^{-\Im(\aleph,\hbar(\aleph))}\Big(\Big(\frac{\ell_1-\aleph}{\tilde{\ell}}\Big)^{-\Im(\aleph,\hbar(\aleph))} - \Big(\frac{\ell_2-\aleph}{\tilde{\ell}}\Big)^{-\Im(\aleph,\hbar(\aleph))}\Big) d\aleph \\
&+ M_\Gamma \parallel \hbar \parallel \int_{\ell_1}^{\ell_2} \tilde{\ell}^{-\Im(\aleph,\hbar(\aleph))}\Big(\frac{\ell_2-\aleph}{\tilde{\ell}}\Big)^{-\Im(\aleph,\hbar(\aleph))} d\aleph \Big] \\
&\leq \frac{1}{\varsigma}\Big[M_\Gamma \parallel \hbar \parallel \tilde{\ell}^* \int_0^{\ell_1} \Big(\Big(\frac{\ell_1-\aleph}{\tilde{\ell}}\Big)^{-\Im^*} - \Big(\frac{\ell_2-\aleph}{\tilde{\ell}}\Big)^{-\Im^*}\Big) d\aleph + M_\Gamma \parallel \hbar \parallel \tilde{\ell}^* \int_{\ell_1}^{\ell_2} \Big(\frac{\ell_2-\aleph}{\tilde{\ell}}\Big)^{-\Im^*} d\aleph \Big] \\
&\leq \frac{M_\Gamma \parallel \hbar \parallel \tilde{\ell}^*}{\varsigma[\tilde{\ell}^{-\Im^*}(1-\Im^*)]}\Big[(\ell_1)^{1-\Im^*} - (\ell_2)^{1-\Im^*} + 2(\ell_2-\ell_1)^{1-\Im^*} \Big].
\end{aligned}
$$

Hence, $\mid \wp_1\hbar(\ell_2) - \wp_1\hbar(\ell_1) \mid \to 0$ as $\ell_2 \to \ell_1$. It implies that $\wp_1(B_R)$ is equicontinuous.
Step 4: $\wp_2$ is a strict contraction. For $\hbar, \kappa \in \mathbb{E}$ and $\ell \in \tilde{\psi}_\ell$, we obtain

$$
\begin{aligned}
\mid \wp_2\hbar(\ell) - \wp_2\kappa(\ell) \mid &\leq \frac{1}{\varsigma}\Big[\int_0^{\ell} \aleph^{-\sigma}\aleph^{\sigma} \mid \Theta(\aleph,\hbar(\aleph)) - \Theta(\aleph,\kappa(\aleph)) \mid d\aleph \Big] \\
&\leq \frac{1}{\varsigma}\Big[\Lambda \parallel \hbar-\kappa \parallel \int_0^{\ell} \aleph^{-\sigma} d\aleph \Big] \\
&\leq \frac{1}{\varsigma}\Big[\Lambda \parallel \hbar-\kappa \parallel \frac{\ell^{-\sigma+1}}{-\sigma+1} \Big] \\
&\leq \frac{\Lambda}{\varsigma}\Big[\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} \Big] \parallel \hbar-\kappa \parallel .
\end{aligned}
$$

Consequently by (6), the operator $\wp_2$ is a strict contraction.
Therefore, all conditions of Theorem 2.6 are fulfilled. We infer that the NAVOIVP (1) has at least one solution in $\mathbb{E}$.

4. Results of Uniqueness

In the next result, we shall demonstrate the uniqueness of solutions for the NAVOIVP (1) using the Banach contraction principle.

Theorem 4.1. *Assume that conditions (SY1) and (SY2) hold. If*

$$
\frac{1}{\varsigma}\left[\Lambda \frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma \tilde{\ell}^* \tilde{\ell} \right] < 1, \tag{7}
$$

then the NAVOIVP (1) has a unique solution in $\mathbb{E}$.

Proof Consider the operator

$$\wp : \mathbb{E} \to \mathbb{E},$$

as follows

$$\wp\hbar(\ell) = \wp_1\hbar(\ell) + \wp_2\hbar(\ell), \; for\, \hbar \in \mathbb{E}.$$

For $\hbar, \hbar^* \in \mathbb{E}$, we can write

$$\begin{aligned}
|\, \wp\hbar(\ell) - \wp\hbar^*(\ell)\,| &\le \frac{1}{\varsigma}\Big[\int_0^\ell \aleph^{-\sigma}\aleph^{\sigma}\,|\,\Theta(\aleph,\hbar(\aleph)) - \Theta(\aleph,\hbar^*(\aleph))\,|\,d\aleph \\
&+ \Big|\int_0^\ell \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)d\aleph - \int_0^\ell \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar^*(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar^*(\aleph)))}\hbar^*(\aleph)d\aleph\,\Big|\Big] \\
&\le \frac{1}{\varsigma}\Big[\Lambda \,\|\, \hbar - \hbar^* \,\| \int_0^\ell \aleph^{-\sigma}d\aleph \\
&+ \Big|\int_0^\ell \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)d\aleph - \int_0^\ell \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar^*(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar^*(\aleph)))}\hbar^*(\aleph)d\aleph\,\Big|\Big] \\
&\le \frac{1}{\varsigma}\Big[\Lambda \,\|\, \hbar - \hbar^* \,\|\, \frac{\ell^{-\sigma+1}}{-\sigma+1} \\
&+ 2\int_0^\ell \Big(\sup_{\ell\in\tilde{\psi}_\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))} + \sup_{\ell\in\tilde{\psi}_\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar^*(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar^*(\aleph)))}\Big)\,|\,\hbar - \hbar^*\,|\,d\aleph\Big] \\
&\le \frac{1}{\varsigma}\Big[\Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\,\|\,\hbar-\hbar^*\,\| + 2\,\|\,\hbar-\hbar^*\,\| \int_0^\ell (M_\Gamma\tilde{\ell}^* + M_\Gamma\tilde{\ell}^*)d\aleph\Big] \\
&\le \frac{1}{\varsigma}\Big[\Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\,\|\,\hbar-\hbar^*\,\| + 4M_\Gamma\tilde{\ell}^*\,\|\,\hbar-\hbar^*\,\|\,\ell\Big] \\
&\le \frac{1}{\varsigma}\Big[\Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma\tilde{\ell}^*\tilde{\ell}\Big]\,\|\,\hbar-\hbar^*\,\|\,.
\end{aligned}$$

As a consequence of Equation (7), the operator $\wp$ acts as a contraction. Therefore, by applying the Banach fixed-point theorem, we conclude that $\wp$ has a unique fixed point, which corresponds to the unique solution of the NAVOIVP (1) in the space $\mathbb{E}$.

5. Ulam-Hyers Stability

Definition 5.1. *[15] In connection with the NAVOIVP* (1), *consider the inequality*

$$|D_{0^+}^{\Im(\ell,\vartheta(\ell))}\vartheta(\ell) + \varsigma\vartheta'(\ell) - \Theta(\ell,\vartheta(\ell))| \le \epsilon, \;\; \ell \in \tilde{\psi}_\ell. \tag{8}$$

We say that the NAVOIVP (1) *exhibits Ulam-Hyers stability if there exists a constant $c_\Theta > 0$ such that, for any $\epsilon > 0$ and for every function $\vartheta \in C(\tilde{\psi}_\ell, \mathbb{R})$ satisfying inequality (8), there exists a solution $\hbar \in C(\tilde{\psi}_\ell, \mathbb{R})$ of the original problem* (1) *for which*

$$|\vartheta(\ell) - \hbar(\ell)| \le c_\Theta\epsilon, \quad \ell \in \tilde{\psi}_\ell.$$

Theorem 5.2. *Suppose that assumptions (SY1) and (SY2), along with inequality (7), are satisfied. Then the NAVOIVP* (1) *is Ulam-Hyers stable.*

Proof Let $\epsilon > 0$ be any arbitrary number. Suppose a function $\vartheta(\ell) \in C(\tilde{\psi}_\ell, \mathbb{R})$ satisfies the inequality

$$|D_{0^+}^{\Im(\ell,\vartheta(\ell))}\vartheta(\ell) + \varsigma\vartheta'(\ell) - \Theta(\ell,\vartheta(\ell))| \le \epsilon, \;\; \ell \in \tilde{\psi}_\ell, \tag{9}$$

integrating both sides of the above (9), we have

$$\Big|\vartheta(\ell) + \frac{1}{\varsigma}\Big[\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph - \int_0^{\ell}\Theta(\aleph,\hbar(\aleph))\ d\aleph\Big]\Big| \le \epsilon\tilde{\ell}.$$

Let $\ell \in \tilde{\psi}_\ell$, then

$$\begin{aligned}
\mid \vartheta(\ell)-\hbar(\ell)\mid &= \Big|\vartheta(\ell) + \frac{1}{\varsigma}\Big[\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph - \int_0^{\ell}\Theta(\aleph,\hbar(\aleph))\ d\aleph\Big]\Big| \\
&= \Big|\vartheta(\ell) + \frac{1}{\varsigma}\Big[\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph - \int_0^{\ell}\Theta(\aleph,\hbar(\aleph))\ d\aleph\Big] \\
&+ \frac{1}{\varsigma}\Big[\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\vartheta(\aleph))}}{\Gamma(1-\Im(\aleph,\vartheta(\aleph)))}\vartheta(\aleph)\ d\aleph - \int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\vartheta(\aleph))}}{\Gamma(1-\Im(\aleph,\vartheta(\aleph)))}\vartheta(\aleph)\ d\aleph\Big] \\
&+ \frac{1}{\varsigma}\Big[\int_0^{\ell}\Theta(\aleph,\vartheta(\aleph))\ d\aleph - \int_0^{\ell}\Theta(\aleph,\vartheta(\aleph))\ d\aleph\Big]\Big| \\
&= \Big|\vartheta(\ell) + \frac{1}{\varsigma}\Big[\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\vartheta(\aleph))}}{\Gamma(1-\Im(\aleph,\vartheta(\aleph)))}\vartheta(\aleph)\ d\aleph - \int_0^{\ell}\Theta(\aleph,\vartheta(\aleph))\ d\aleph\Big]\Big| \\
&+ \frac{1}{\varsigma}\Big[\Big|\int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))}\hbar(\aleph)\ d\aleph - \int_0^{\ell} \frac{(\ell-\aleph)^{-\Im(\aleph,\vartheta(\aleph))}}{\Gamma(1-\Im(\aleph,\vartheta(\aleph)))}\vartheta(\aleph)\ d\aleph\Big|\Big] \\
&+ \frac{1}{\varsigma}\Big[\int_0^{\ell}\mid\Theta(\aleph,\vartheta(\aleph))-\Theta(\aleph,\hbar(\aleph))\mid d\aleph\Big] \\
&\le \epsilon\tilde{\ell} + \frac{1}{\varsigma}\Big[2\int_0^{\ell}\Big(\sup_{\ell\in\tilde{\psi}_\ell}\frac{(\ell-\aleph)^{-\Im(\aleph,\hbar(\aleph))}}{\Gamma(1-\Im(\aleph,\hbar(\aleph)))} + \sup_{\ell\in\tilde{\psi}_\ell}\frac{(\ell-\aleph)^{-\Im(\aleph,\vartheta(\aleph))}}{\Gamma(1-\Im(\aleph,\vartheta(\aleph)))}\Big)\mid\hbar-\vartheta\mid d\aleph \\
&+ \int_0^{\ell}\aleph^{-\sigma}\aleph^{\sigma}\mid\Theta(\aleph,\vartheta(\aleph))-\Theta(\aleph,\hbar(\aleph))\mid d\aleph\Big] \\
&\le \epsilon\tilde{\ell} + \frac{1}{\varsigma}\Big[2\parallel\vartheta-\hbar\parallel\int_0^{\ell}(M_\Gamma\tilde{\ell}^* + M_\Gamma\tilde{\ell}^*)d\aleph + \Lambda\parallel\vartheta-\hbar\parallel\int_0^{\ell}\aleph^{-\sigma}d\aleph\Big] \\
&\le \epsilon\tilde{\ell} + \frac{1}{\varsigma}\Big[4M_\Gamma\tilde{\ell}^*\parallel\vartheta-\hbar\parallel\int_0^{\ell}d\aleph + \Lambda\parallel\vartheta-\hbar\parallel\frac{\ell^{-\sigma+1}}{-\sigma+1}\Big] \\
&\le \epsilon\tilde{\ell} + \frac{1}{\varsigma}\Big[4M_\Gamma\tilde{\ell}^*\ell\parallel\vartheta-\hbar\parallel + \Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\parallel\vartheta-\hbar\parallel\Big] \\
&\le \epsilon\tilde{\ell} + \frac{1}{\varsigma}\Big[4M_\Gamma\tilde{\ell}^*\tilde{\ell}\parallel\vartheta-\hbar\parallel + \Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\parallel\vartheta-\hbar\parallel\Big] \\
&\le \epsilon\tilde{\ell} + \frac{1}{\varsigma}\Big[4M_\Gamma\tilde{\ell}^*\tilde{\ell} + \Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\Big]\parallel\vartheta-\hbar\parallel,
\end{aligned}$$

then

$$\parallel\vartheta-\hbar\parallel\Big(1-\frac{1}{\varsigma}\Big[4M_\Gamma\tilde{\ell}^*\tilde{\ell} + \Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\Big]\Big) \le \epsilon\tilde{\ell}.$$

Now, for each $\ell \in \tilde{\psi}_\ell$, we have

$$\mid\vartheta(\ell)-\hbar(\ell)\mid \le \parallel\vartheta-\hbar\parallel \le \frac{\tilde{\ell}}{1-\frac{1}{\varsigma}\Big[4M_\Gamma\tilde{\ell}^*\tilde{\ell} + \Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1}\Big]}\epsilon = c_\Theta\epsilon.$$

Thus, the NAVOIVP (1) is Ulam-Hyers stable.

6. Numerical Examples

Example 6.1. *Consider the following NAVOIVP:*

$$\begin{cases} D^{\frac{\ell}{3}+\frac{1}{2}}\hbar(\ell) + 3\hbar'(\ell) = \frac{\exp(-\ell+4)}{\sqrt{2\pi\ell+\frac{5}{6}}}(\ell^2) + \frac{1}{3}\hbar, \ \ \ell \in [0,1], \\ \hbar(0) = 0. \end{cases} \tag{10}$$

Also, assume $\varsigma = 3$ *and* $\Im(\ell,\hbar(\ell)) = \frac{\ell}{3}+\frac{1}{2}$. *Then,* $\Im$ *is a continuous function with* $0 < \Im(\ell,\hbar(\ell)) < \frac{1}{3}+\frac{1}{2} = \frac{5}{6} = \Im^* < 1$, *and* $\min_{\ell\in\tilde{\psi}_\ell} \mid \Im(\ell,\hbar(\ell)) \mid = \frac{1}{2}$, *and*

$$\begin{aligned} \ell^{\frac{1}{6}} \mid \Theta(\ell,\hbar) - \Theta(\ell,y) \mid &= \ell^{\frac{1}{6}} \mid \frac{\exp(-\ell+4)}{\sqrt{2\pi\ell+\frac{5}{6}}}(\ell^2) + \frac{1}{3}\hbar - \frac{\exp(-\ell+4)}{\sqrt{2\pi\ell+\frac{5}{6}}}(\ell^2) - \frac{1}{3}y \mid \\ &= \ell^{\frac{1}{6}} \mid \frac{1}{3}\hbar - \frac{1}{3}y \mid \\ &\leq \ell^{\frac{1}{6}}\frac{1}{3} \mid \hbar - y \mid \\ &\leq \frac{1}{3} \mid \hbar - y \mid . \end{aligned}$$

Hence, the conditions (SY1) and (SY2) are fulfilled with the parameters $\Lambda = \frac{1}{3}$ *and* $\sigma = \frac{1}{6}$. *Moreover, we compute the following:*

$$\begin{aligned} \frac{1}{\varsigma}\Big[\Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma\tilde{\ell}^*\tilde{\ell}\Big] &= \frac{1}{3}\Big[\frac{1}{3}\frac{1}{\frac{5}{6}} + 4\frac{1}{\sqrt{\pi}}\Big] = \frac{1}{3}\Big[\frac{1}{3}\frac{6}{5} + \frac{4}{\sqrt{\pi}}\Big] = \frac{1}{3}\Big[\frac{2}{5} + \frac{4}{\sqrt{\pi}}\Big] \\ &= \frac{1}{3}\Big[\frac{2\sqrt{\pi}+20}{5\sqrt{\pi}}\Big] = \frac{23.54}{26.58} \approx 0.89 < 1. \end{aligned}$$

Therefore, by invoking Theorem 4.1, the NAVOIVP (10) possesses a unique solution. Furthermore, Theorem 5.2 guarantees that this problem also exhibits Ulam-Hyers stability.

Example 6.2. *Consider the following NAVOIVP:*

$$\begin{cases} D^{\frac{\ell}{2}+\frac{1}{4}}\hbar(\ell) + 6\hbar'(\ell) = \frac{8(\ell+1)}{4\sqrt{2\pi}} + (\exp(\sqrt{\ell+1})) + \frac{\pi}{4}\hbar, \ \ \ell \in [0,1], \\ \hbar(0) = 0, \end{cases} \tag{11}$$

and assume $\varsigma = 6$ *and* $\Im(\ell,\hbar(\ell)) = \frac{\ell}{2}+\frac{1}{4}$. *Then,* $\Im$ *is a continuous function with* $0 < \Im(\ell,\hbar(\ell)) < \frac{1}{2}+\frac{1}{4} = \frac{6}{8} = \Im^* < 1$, *and* $\min_{\ell\in\tilde{\psi}_\ell} \mid \Im(\ell,\hbar(\ell)) \mid = \frac{1}{4}$. *Furthermore, we have*

$$\begin{aligned} \ell^{\frac{1}{16}} \mid \Theta(\ell,\hbar) - \Theta(\ell,y) \mid &= \ell^{\frac{1}{16}} \mid \frac{8(\ell+1)}{4\sqrt{2\pi}} + (\exp(\sqrt{\ell+1})) + \frac{\pi}{4}\hbar - \frac{8(\ell+1)}{4\sqrt{2\pi}} - (\exp(\sqrt{\ell+1})) - \frac{\pi}{4}y \mid \\ &\leq \ell^{\frac{1}{16}}\frac{\pi}{4} \mid \hbar - y \mid \\ &\leq \frac{\pi}{4} \mid \hbar - y \mid . \end{aligned}$$

So (SY 1), (SY 2) satisfied with $\Lambda = \frac{\pi}{4}$ *and* $\sigma = \frac{1}{16}$. *In addition to this, we have*

$$\begin{aligned} \frac{1}{6}\Big[\Lambda\frac{\tilde{\ell}^{-\sigma+1}}{-\sigma+1} + 4M_\Gamma\tilde{\ell}^*\tilde{\ell}\Big] &= \frac{1}{6}\Big[\frac{\pi}{4}\frac{1}{\frac{15}{16}} + 4\times(1.2254)\Big] = \frac{1}{6}\Big[\frac{\pi}{4}\frac{16}{15} + 4.9016\Big] \\ &= \frac{1}{6}\Big[\frac{4\pi}{15} + 4.9016\Big] \approx 0.96 < 1. \end{aligned}$$

According to Theorem 4.1, the NAVOIVP (11) has a unique solution. Moreover, by Theorem 5.2, the NAVOIVP (11) is Ulam-Hyers stable.

7. Conclusion

In this study, we rigorously established the existence, uniqueness, and Ulam-Hyers stability of solutions to the NAVOIVP (1), which modeled non-autonomous differential equations of variable fractional order, encompassing both classical and fractional derivatives with $0 < \mathfrak{I}(\ell, \hbar(\ell)) < 1$. The theoretical analysis leveraged the Ulam-Hyers stability framework (Theorem 5.2) in conjunction with fixed-point techniques, as formalized in Theorems 3.5 and 4.1. The analytical results were further substantiated through representative numerical examples. These findings provided a robust foundation for the study of variable-order fractional differential equations and suggested potential applications across a broad spectrum of scientific and engineering problems.

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