Filomat 40:2 (2026), 661–675
https://doi.org/10.2298/FIL2602661L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Comparison theorems for nonnegative splittings of tensors

Yan-Cun Li[a,*], Shu-Xin Miao[a]

[a]*College of Mathematics and Statistics, Northwest Normal University, Lanzhou, 730070, P.R. China*

Abstract. In this paper, by analogy with the definition of the nonnegative splitting of a matrix, we introduce the definition of the nonnegative splitting of a tensor. Considering the case that nonnegative splitting of a strong $\mathcal{M}$-tensor is not necessarily convergent, we establish a new convergence theorem. Since comparison theorems involving the spectral radius of iterative tensors are useful tools in the analysis of convergence rate of tensor splitting iterative (TSI) methods, we derive several comparison theorems for nonnegative splittings of tensors in this paper. These results generalize the previous ones.

1. Introduction

In recent years, multi-linear systems have found increasing applications in engineering and scientific computing, drawing significant attention from researchers. Examples include higher-order Markov chains, graph analysis, chemometrics, diffusion tensor imaging, image processing, and the multilinear PageRank problem, see, e.g. [1, 3, 5, 8, 18, 21] and the references therein.

Consider a multi-linear system:

$$\mathcal{A}\mathbf{x}^{m-1} = \mathbf{b}, \tag{1}$$

where $\mathcal{A}$ is an order m dimension n tensor, $\mathbf{b}$ is an n-dimension vector, $\mathcal{A}\mathbf{x}^{m-1}$ is an n-dimension vector, and the i-th component of $\mathcal{A}\mathbf{x}^{m-1}$ is defined as [16]:

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m}, \quad i = 1, 2, \cdots, n, \tag{2}$$

where x_i denotes the i-th component of $\mathbf{x}$.

In order to solve the multi-linear system with tensor splitting iterative (TSI) method [10, 11], the coefficient tensor $\mathcal{A}$ is split into

$$\mathcal{A} = \mathcal{E} - \mathcal{F}, \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 15A69; Secondary 65F10.
Keywords. Nonnegative splitting, comparison theorem, multi-linear system
Received: 24 May 2024; Revised: 17 July 2025; Accepted: 09 November 2025
Communicated by Marko Petković
* Corresponding author: Yan-Cun Li
Email addresses: yancun0326@163.com (Yan-Cun Li), miaosx@nwnu.edu.cn (Shu-Xin Miao)
ORCID iDs: https://orcid.org/0009-0001-7738-6840 (Yan-Cun Li), https://orcid.org/0000-0002-0905-7770 (Shu-Xin Miao)

where $\mathcal{E}$ is left-nonsingular [12], then an iterative scheme for solving the system (1) can be described as follows

$$\mathbf{x}_k = \left[M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}_{k-1}^{m-1} + M(\mathcal{E})^{-1}\mathbf{b}\right]^{\left[\frac{1}{m-1}\right]}, \quad k = 1, 2, \cdots, \tag{4}$$

where $\mathbf{x}_0$ is an arbitrarily chosen initial vector and tensor $M(\mathcal{E})^{-1}\mathcal{F}$ is called the iterative tensor. The choice of different splitting tensors $\mathcal{E}$ and $\mathcal{F}$ corresponds to different iterative methods for solving (1). Therefore, selecting optimal $\mathcal{E}$ and $\mathcal{F}$ is crucial for enhancing the convergence rate of the TSI method. In [10], the authors analyzed the convergence rate of the TSI method and demonstrated that the spectral radius $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$ can serve as an approximate convergence rate of the iteration (4). Numerical examples in [10] further verified that $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$ effectively characterizes the proposed approximate convergence rate of the iteration (4). These results imply that a smaller $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$ corresponds to faster convergence of the iterative scheme (4).

In matrix analysis, the comparison theorems for splittings of matrices serve as a useful tool for analyzing the convergence rates of iterative methods [20]. In 2018, by analogy with the comparison theorems for (weak) regular splittings of matrices, Liu, Li and Vong established analogous comparison theorems for (weak) regular splittings of tensors [11]. It is well-known that nonnegative splitting of a matrix has a larger range and fewer constraints than (weak) regular splitting of a matrix [6, 20, 22]. However, prior works have not extended the concept of nonnegative splitting to the tensor setting. In this paper, we investigate the nonnegative splitting of a tensor and analyze the comparison of the proposed approximate convergence rate for tensor splitting iterative methods.

The contributions of this paper are twofold. First, we define the nonnegative splitting of a tensor, and establish a novel convergence theorem for nonnegative splitting of a tensor. Second, we propose comparison theorems for nonnegative splittings of tensors, which are more computationally tractable and require fewer constraints.

The structure of this paper is organized as follows. In Section 2, we introduce fundamental definitions, notations and preliminary lemmas that are used in subsequent sections. Section 3 presents the concept of nonnegative splitting of a tensor and establishes a novel convergence theorem. Based on the theory of nonnegative tensors, we develop comparison theorems for nonnegative splittings of tensors in Section 4. Numerical experiments in Section 5 demonstrate the validity of the proposed comparison theorems. Finally, concluding remarks are provided in Section 6.

2. Preliminaries

For convenience we shall now briefly explain some of the terminology used in the next section.

Let n be a positive integer, by $\langle n\rangle$ we denote the set $\{1, \cdots, n\}$. A tensor $\mathcal{A}$ consists of $n_1 \times \cdots \times n_m$ entries in the real field $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1\cdots i_m}), a_{i_1\cdots i_m} \in \mathbb{R}, i_j \in \left\langle n_j\right\rangle, \quad j = 1, \cdots, m.$$

If $n_1 = \cdots = n_m = n$, $\mathcal{A}$ is called an order m dimension n tensor. We denote the set of all order m dimension n tensors by $\mathbb{R}^{[m,n]}$. When $m = 2$, $\mathbb{R}^{[2,n]}$ denotes the set of all $n \times n$ real matrices. When $m = 1$, $\mathbb{R}^{[1,n]}$ is simplified as $\mathbb{R}^n$, which is the set of all n-dimension real vectors. Similarly, the above notions can be generalized to the complex number field $\mathbb{C}$.

Let $\mathbf{0}$, O, $\mathcal{O}$ denote a zero vector, a zero matrix and a zero tensor, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be two tensors with the same sizes. The order $\mathcal{A} \geq \mathcal{B}$ ($\mathcal{A} > \mathcal{B}$) means that each entry of $\mathcal{A}$ is no less than (larger than) corresponding one of $\mathcal{B}$, calling $\mathcal{A}$ nonnegative if $\mathcal{A} \geq \mathcal{O}$. These definitions can be applied immediately to matrices by identifying them with order 2 dimension n tensors.

Next we introduce the definition of several special tensors.

Definition 2.1. *Let $\mathcal{I} = (\delta_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$. $\mathcal{I}$ is called the unit tensor if $\delta_{i_1\cdots i_m}$ satisfies*

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & i_1 = \cdots = i_m, \\ 0, & else. \end{cases}$$

Definition 2.2. *([19]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called row diagonal if $a_{ii_2\cdots i_m}$ can take nonzero value only when $i_2 = ... = i_m$.*

Definition 2.3. *([15]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. The majorization matrix associated to the tensor $\mathcal{A}$ is the matrix of coefficients of the non-mixed terms and is denoted $M(\mathcal{A})$, i.e., $M(\mathcal{A})_{ij} = a_{ii_2...i_m}$ where $i_2 = \cdots = i_m = j$.*

Definition 2.4. *([19]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. Then $\mathcal{A}$ is row diagonal if and only if $\mathcal{A} = M(\mathcal{A})\mathcal{I}$.*

Definition 2.5. *([12]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. If $M(\mathcal{A})$ is a nonsingular matrix and $\mathcal{A} = M(\mathcal{A})\mathcal{I}$, then $\mathcal{A}$ is called left-invertible or left-nonsingular, $M(\mathcal{A})^{-1}$ is called the order 2 left-inverse of $\mathcal{A}$.*

We now define the product between a matrix and a tensor, which is a special case of the tensor product introduced in [2, 18].

Definition 2.6. *([10]) Let $A = (a_{ij})$ be an n-dimensional square matrix and $\mathcal{B} = (b_{i_1\cdots i_m})$ is an order m dimension n tensor, then tensor $\mathcal{C} = A\mathcal{B}$ is an order m dimension n tensor, and its entries are gives as follows:*

$$c_{ji_2\cdots i_m} = \sum_{j_2=1}^{n} a_{jj_2} b_{j_2 i_2\cdots i_m}, \qquad 1 \le j, i_r \le n, r = 2, ..., m. \tag{5}$$

The formula (5) can be expressed as follows [8]

$$\mathcal{C}_{(1)} = (A\mathcal{B})_{(1)} = A\mathcal{B}_{(1)}. \tag{6}$$

Where $\mathcal{C}_{(1)}$ and $\mathcal{B}_{(1)}$ are the matrix obtained from $\mathcal{C}$ and $\mathcal{B}$ flattened along the first index [8, 9]. For example, if $\mathcal{B}_{ijk}$ is an order 3 dimension n tensor, then

$$\mathcal{B}_{(1)} = \begin{bmatrix} b_{111} & \cdots & b_{1n1} & b_{112} & \cdots & b_{1n2} & \cdots & b_{11n} & \cdots & b_{1nn} \\ b_{211} & \cdots & b_{2n1} & b_{212} & \cdots & b_{2n2} & \cdots & b_{21n} & \cdots & b_{2nn} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n11} & \cdots & b_{nn1} & b_{n12} & \cdots & b_{nn2} & \cdots & b_{n1n} & \cdots & b_{nnn} \end{bmatrix}. \tag{7}$$

The definitions of tensor eigenvalues and eigenvectors are introduced as follows.

Definition 2.7. *([13, 16]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]},$$

where $\mathbf{x}^{[m-1]} = \left(x_1^{m-1}, \ldots, x_n^{m-1}\right)^T$. We call $(\lambda, \mathbf{x})$ an H-eigenpair if both λ and $\mathbf{x}$ are real.

Let $\sigma(\mathcal{A})$ be the set of all eigenvalues of $\mathcal{A}$, the spectral radius of $\mathcal{A}$ is defined by $\rho(\mathcal{A}) = \max\{|\lambda| \mid \lambda \in \sigma(\mathcal{A})\}$.

In [7], Ding, Qi and Wei gave the definitions of the $\mathcal{Z}$-tensor, $\mathcal{M}$-tensor and strong $\mathcal{M}$-tensor.

Definition 2.8. *([7]) Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. $\mathcal{A}$ is called a $\mathcal{Z}$-tensor if its off-diagonal entries are non-positive. $\mathcal{A}$ is called an $\mathcal{M}$-tensor if there exist a nonnegative tensor $\mathcal{B}$ and a positive real number $\eta \ge \rho(\mathcal{B})$ such that*

$$\mathcal{A} = \eta\mathcal{I} - \mathcal{B}.$$

If $\eta > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor.

Based on this definition, Liu, Li and Vong [10] gave the definition of the tensor splitting.

Definition 2.9. *([10]) Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is said to be a splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular; a regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $M(\mathcal{E})^{-1} \geq O$ and $\mathcal{F} \geq \mathcal{O}$; a weak regular splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular with $M(\mathcal{E})^{-1} \geq O$ and $M(\mathcal{E})^{-1}\mathcal{F} \geq \mathcal{O}$; a convergent splitting if $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1$.*

The following are equivalent conditions for a strong $\mathcal{M}$-tensor based on tensor splitting.

Lemma 2.10. *([10, 11]) If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a $\mathcal{Z}$-tensor, then the following conditions are equivalent:*
(1) *$\mathcal{A}$ is a strong $\mathcal{M}$-tensor.*
(2) *There exist an inverse-positive $\mathcal{Z}$-matrix B and a semi-positive $\mathcal{Z}$-tensor $\mathcal{C}$ with $\mathcal{A} = B\mathcal{C}$.*
(3) *$\mathcal{A}$ has a convergent (weak) regular splitting.*
(4) *All (weak) regular splittings of $\mathcal{A}$ are convergent.*

To facilitate subsequent proofs, we introduce the following lemmas regarding tensor properties.

Lemma 2.11. *([4, 23]) (weak Perron-Frobenius theorem) If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is a nonnegative tensor, then there exist vector $\mathbf{x} \geq \mathbf{0}, \mathbf{x} \neq \mathbf{0}$ such that*

$$\mathcal{A}\mathbf{x}^{m-1} = \rho(\mathcal{A})\mathbf{x}^{[m-1]}.$$

Lemma 2.12. *([23, Lemma 3.3 and Lemma 3.5]) Let $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{[m,n]}$. If $\mathcal{O} \leq \mathcal{A} \leq (<)\mathcal{B}$, then $\rho(\mathcal{A}) \leq \rho(\mathcal{B})$.*

Lemma 2.13. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor, and $\mu, \nu \in \mathbb{R}$:*
(1) *[11] If $\mu\mathbf{x}^{[m-1]} \leq (<)\mathcal{A}\mathbf{x}^{m-1}$, $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ such that $\mu \leq (<)\rho(\mathcal{A})$.*
(2) *[17] If $\upsilon\mathbf{x}^{[m-1]} \geq \mathcal{A}\mathbf{x}^{m-1}$, $\mathbf{x} > \mathbf{0}$ such that $\upsilon \geq \rho(\mathcal{A})$.*

3. Nonnegative splitting and convergence theorem

We can define the definition of nonnegative splitting of a tensor.

Definition 3.1. *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is said to be a splitting of $\mathcal{A}$ if $\mathcal{E}$ is left-nonsingular; a nonnegative splitting of $\mathcal{A}$ if $M(\mathcal{E})^{-1}\mathcal{F} \geq \mathcal{O}$.*

The concept of nonnegative splitting of a tensor generalizes the nonnegative splitting of a matrix. When $m = 2$, the tensor $\mathcal{A}$ reduces to a matrix case. Note that for matrices, the left inverse of a matrix is equal to the right inverse. However, this property does not hold for the general tensors [18].

Clearly, we can draw the following corollary from the Definition 2.9 and Definition 3.1.

Corollary 3.2. *Any regular splitting of $\mathcal{A}$ is a weak regular splitting of $\mathcal{A}$, and any weak regular splitting of $\mathcal{A}$ is a nonnegative splitting of $\mathcal{A}$. In general, the converses are not true.*

The following example will show that the nonnegative splitting is not necessarily a weak regular splitting.

Example 3.3. *If $\mathcal{A} \in \mathbb{R}^{[3,2]}$. Let*

$$\mathcal{A}_{(1)} = \begin{pmatrix} -2 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & -2 \end{pmatrix},$$

be splitted as $\mathcal{A} = \mathcal{E} - \mathcal{F}$, where

$$\mathcal{E}_{(1)} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & -2 \end{pmatrix} \text{ and } \mathcal{F}_{(1)} = \begin{pmatrix} 0 & -0.25 & -0.25 & -0.25 \\ 0 & -0.25 & -0.25 & 0 \end{pmatrix}.$$

Some calculation gives

$$M(\mathcal{E})^{-1} = \begin{pmatrix} -0.5 & 0 \\ -0.0625 & -0.5 \end{pmatrix} \text{ and } (M(\mathcal{E})^{-1}\mathcal{F})_{(1)} = \begin{pmatrix} 0 & 0.125 & 0.125 & 0.125 \\ 0 & 0.1406 & 0.1406 & 0.0156 \end{pmatrix}.$$

It is easy to check that $M(\mathcal{E})^{-1} \leq O$, $M(\mathcal{E})^{-1}\mathcal{F} \geq \mathcal{O}$. Then the splitting is nonnegative, and is not weak regular. So the nonnegative splitting is not necessarily weak regular splitting.

Recall that the regular (weak regular) splitting of a strong $\mathcal{M}$-tensor is a convergent splitting by Lemma 2.10 (see also [10, Theorem 3.18]). However, unlike the regular (weak regular) splitting, this property does not hold for nonnegative splitting of a tensor in general. The following example demonstrates that the nonnegative splitting of a strong $\mathcal{M}$-tensor may not be convergent.

Example 3.4. *If $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$ with $a_{111} = 2$, $a_{222} = 2$ and all other entries $a_{ijk} = -0.25$. Clearly, $\mathcal{A}$ is a strong $\mathcal{M}$-tensor [11, Remark 3.5]. Let*

$$\mathcal{E}_{(1)} = \begin{pmatrix} 0 & -0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0 \end{pmatrix} \text{ and } \mathcal{F}_{(1)} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

We get

$$M(\mathcal{E})^{-1} = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \text{ and } (M(\mathcal{E})^{-1}\mathcal{F})_{(1)} = \begin{pmatrix} 0 & 0 & 0 & 8 \\ 8 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $M(\mathcal{E}_1)^{-1}\mathcal{F} \geq \mathcal{O}$. Then the splitting $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is nonnegative, and by using the power method [14] that we get $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) = 8 > 1$. In this case, the nonnegative splitting of a strong $\mathcal{M}$-tensor is not convergent splitting.

For nonnegative splitting of a tensor, we have the following convergence theorem.

Theorem 3.5. *Let $\mathcal{A}, \mathcal{E}, \mathcal{F} \in \mathbb{R}^{[m,n]}$. Suppose that $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is a nonnegative splitting. Let $\mathbf{x}$ be the Perron vector of $M(\mathcal{E})^{-1}\mathcal{F}$. Then the following statements are equivalent:*
(1) $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1$.
(2) *If* $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$, *then* $\mathcal{F}\mathbf{x}^{m-1} > \mathbf{0}$.
(3) $M(\mathcal{E})^{-1}\mathcal{A}\mathbf{x}^{m-1} \leq \mathbf{x}^{[m-1]}$.
(4) $\rho\left(M(\mathcal{E})^{-1}\mathcal{A}\right) > 0$.

Proof. As $\mathcal{A} = \mathcal{E} - \mathcal{F}$ is a nonnegative splitting, then $M(\mathcal{E})^{-1}\mathcal{F} \geq \mathcal{O}$. By the weak Perron-Frobenius theorem (see Lemma 2.11), there exists a vector $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ such that

$$M(\mathcal{E})^{-1}\mathcal{F}\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)\mathbf{x}^{[m-1]}.$$

It is easy to see that

$$M(\mathcal{E})^{-1}\mathcal{A} = M(\mathcal{E})^{-1}(\mathcal{E} - \mathcal{F}) = \mathcal{I} - M(\mathcal{E})^{-1}\mathcal{F}.$$

By post-multiplying by $\mathbf{x}^{m-1}$ we get

$$M(\mathcal{E})^{-1}\mathcal{A}\mathbf{x}^{m-1} = (\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{F})\mathbf{x}^{m-1} = (1 - \rho(M(\mathcal{E})^{-1}\mathcal{F}))\mathbf{x}^{[m-1]}. \tag{8}$$

Based on (8), the following proof are dived into four parts.

First, we prove "(1) ⇒ (2)". It is obvious that

$$\mathcal{A}\mathbf{x}^{m-1} = (1 - \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right))\mathcal{E}\mathbf{x}^{m-1}.$$

By $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1$ and $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$, we have

$$\mathcal{A}\mathbf{x}^{m-1} < \mathcal{E}\mathbf{x}^{m-1},$$

Hence,

$$\mathcal{F}\mathbf{x}^{m-1} > \mathbf{0}.$$

Second, we prove "(2) ⇒ (1)". Since $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$ and $\mathcal{F}\mathbf{x}^{m-1} > \mathbf{0}$, we have

$$M(\mathcal{E})\mathbf{x}^{[m-1]} > 0.$$

Moreover, since

$$\mathbf{0} \leq \mathcal{A}\mathbf{x}^{m-1} = M(\mathcal{E})(\mathcal{I} - M(\mathcal{E})^{-1}\mathcal{F})\mathbf{x}^{m-1} = (1 - \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right))M(\mathcal{E})\mathbf{x}^{[m-1]}$$

and $M(\mathcal{E})\mathbf{x}^{[m-1]} > \mathbf{0}$, we get $\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1$.

Third, we prove (1) ⇔ (3). By relation (8) and the fact that $\mathbf{x} \geq \mathbf{0}$ with $\mathbf{x} \neq \mathbf{0}$, we have

$$\begin{aligned} M(\mathcal{E})^{-1}\mathcal{A}\mathbf{x}^{m-1} \leq \mathbf{x}^{[m-1]} &\Leftrightarrow (1 - \rho(M(\mathcal{E})^{-1}\mathcal{F}))\mathbf{x}^{[m-1]} \leq \mathbf{x}^{[m-1]} \\ &\Leftrightarrow 0 < 1 - \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1 \\ &\Leftrightarrow \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1. \end{aligned}$$

Last, we prove (1) ⇔ (4). From (8), we get that $1 - \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$ is an eigenvalues of $M(\mathcal{E})^{-1}\mathcal{A}$. Then

$$\rho\left(M(\mathcal{E})^{-1}\mathcal{A}\right) \geq 1 - \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right).$$

Hence,

$$\rho\left(M(\mathcal{E})^{-1}\mathcal{A}\right) > 0 \Leftrightarrow 0 < 1 - \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1 \Leftrightarrow \rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right) < 1.$$

From this we can obtain the required results. □

4. The comparison theorem

In this section, we investigate comparison results for different splittings of a given tensor $\mathcal{A}$. To this end, consider the following two splittings

$$\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2, \tag{9}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are left-nonsingular.

When one splitting is a nonnegative splitting and the other is a weak regular splitting in (9), we obtain the following comparison results.

Theorem 4.1. *Let the splittings given in (9) be convergent, where the first splitting is a nonnegative splitting and the second is a weak regular splitting. Suppose there exists a constant $\alpha \in (0, 1]$ such that*

$$\mathcal{F}_1 \leq \alpha\mathcal{F}_2.$$

Then, the inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ holds when $\alpha = 1$, and the strict inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ holds when $0 < \alpha < 1$.

Proof. Noting that the splittings given in (9) are convergent, it follows that $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < 1$ and $\rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$. Therefore, it suffices to prove the inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq (<)\ \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$.

Because $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1$ is a nonnegative splitting, then $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$. By the weak Perron-Frobenius theorem (see Lemma 2.11), there exists a vector $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ such that

$$M(\mathcal{E}_1)^{-1}\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]},$$

which implies:

$$\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)M(\mathcal{E}_1)\mathbf{x}^{[m-1]} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathcal{E}_1\mathbf{x}^{m-1}. \tag{10}$$

By $\mathcal{E}_1 = \mathcal{A} + \mathcal{F}_1$ and (10), we get

$$\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathcal{A}\mathbf{x}^{m-1} = (1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathcal{F}_1\mathbf{x}^{m-1}.$$

By $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < 1$ and $\mathcal{F}_1 \leq \alpha\mathcal{F}_2$, we get

$$\alpha(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathcal{F}_2\mathbf{x}^{m-1} \geq (1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathcal{A}\mathbf{x}^{m-1}. \tag{11}$$

Noting that $\mathcal{A} = \mathcal{E}_2 - \mathcal{F}_2$. Then by (11) we get

$$(\alpha - \alpha\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) + \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathcal{F}_2\mathbf{x}^{m-1} \geq \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)M(\mathcal{E}_2)\mathbf{x}^{[m-1]}.$$

By $M(\mathcal{E}_2)^{-1} \geq \mathcal{O}$, we have

$$(\alpha - \alpha\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) + \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))M(\mathcal{E}_2)^{-1}\mathcal{F}_2\mathbf{x}^{m-1} \geq \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]},$$

i.e.,

$$M(\mathcal{E}_2)^{-1}\mathcal{F}_2\mathbf{x}^{m-1} \geq \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)/(\alpha - \alpha\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) + \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}. \tag{12}$$

Which, by Lemma 2.13 and (12), implies

$$\rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) \geq \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)/(\alpha - \alpha\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) + \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)).$$

From this we can obtain the required results, $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ whenever $\alpha = 1$ and $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ whenever $0 < \alpha < 1$. □

For the case when $\alpha = 1$ and $\mathcal{F}_1 \leq \mathcal{F}_2$ the equality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ can be proved by [11, Theorem 3.4 (1)].

Theorem 4.2. *Let the splittings given in (9) be convergent, where the first splitting is a nonnegative splitting and the second is a weak regular splitting. If there exists* α *with* $0 < \alpha \leq 1$ *such that*

$$M(\mathcal{E}_1) \leq \alpha M(\mathcal{E}_2).$$

Then, the inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ *holds when* $\alpha = 1$*, and the strict inequality* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ *holds when* $0 < \alpha < 1$.

Proof. In what follows, it suffices to prove that $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$ or $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$, since both splittings are assumed to be convergent.

Because $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1$ is a nonnegative splitting, then $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$. By Lemma 2.11, there exists a vector $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ such that

$$M(\mathcal{E}_1)^{-1}\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]}.$$

By $\mathcal{F}_1 = \mathcal{E}_1 - \mathcal{A}$, we get

$$M(\mathcal{E}_1)^{-1}(\mathcal{E}_1 - \mathcal{A})\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]},$$

which implies:

$$M(\mathcal{E}_1)^{-1}\mathcal{E}_1\mathbf{x}^{m-1} - M(\mathcal{E}_1)^{-1}\mathcal{A}\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]}. \tag{13}$$

By $\mathcal{E}_1 = M(\mathcal{E}_1)\mathcal{I}$ and (13), we get

$$M(\mathcal{E}_1)^{-1}\mathcal{A}\mathbf{x}^{m-1} = (1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]}.$$

Multiply both sides of $M(\mathcal{E}_1) \le \alpha M(\mathcal{E}_2)$ by $(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]}$ gives us

$$\mathcal{A}\mathbf{x}^{m-1} = (1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))M(\mathcal{E}_1)\mathbf{x}^{[m-1]} \le \alpha(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))M(\mathcal{E}_2)\mathbf{x}^{[m-1]}.$$

By $M(\mathcal{E}_2)^{-1} \ge O$, we have

$$M(\mathcal{E}_2)^{-1}(\mathcal{E}_2 - \mathcal{F}_2)\mathbf{x}^{m-1} \le \alpha(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}.$$

We have

$$M(\mathcal{E}_2)^{-1}\mathcal{F}_2\mathbf{x}^{m-1} \ge (1-\alpha(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}. \tag{14}$$

Which, by Lemma 2.13 and (14), implies

$$\rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) \ge 1-\alpha+\alpha\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right).$$

From this we can obtain the required results, $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \le \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ whenever $\alpha = 1$ and $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ whenever $0 < \alpha < 1$. □

Now, according to Theorem 4.2, the Corollary 4.3 is given.

Corollary 4.3. *Let the splittings given in (9) be convergent, where the first splitting is a nonnegative splitting and the second is a weak regular splitting. If* $M(\mathcal{E}_1) < M(\mathcal{E}_2)$*, then there exists* α *with* $0 < \alpha < 1$ *such that* $M(\mathcal{E}_1) \le \alpha M(\mathcal{E}_2)$ *and the strict inequality* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ *is valid.*

Proof. Denoting

$$M(\mathcal{E}_1) = \left(m_{ij}^{(1)}\right)_{n\times n}, \quad M(\mathcal{E}_2) = \left(m_{ij}^{(2)}\right)_{n\times n}.$$

From $M(\mathcal{E}_1) < M(\mathcal{E}_2)$ it gets

$$m_{ij}^{(1)} < m_{ij}^{(2)}, \qquad i,j = 1,2,...,n.$$

If there exists $m_{ij}^{(1)} > 0$ then let

$$\alpha = \max_{1\le i,j\le n}\left\{\frac{m_{ij}^{(1)}}{m_{ij}^{(2)}} \mid m_{ij}^{(1)} > 0\right\}.$$

Otherwise $m_{ij}^{(1)} \le 0$, i.e., $M(\mathcal{E}_1) \le O$, then $0 < \alpha < 1$ is arbitrary.

We get $0 < \alpha < 1$ and

$$m_{ij}^{(1)} \le \alpha m_{ij}^{(2)}, \qquad i,j = 1,2,...,n,$$

i.e.,

$$M(\mathcal{E}_1) \le \alpha M(\mathcal{E}_2). \tag{15}$$

By the Theorem 4.2 and (15) we can prove that strict inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ is true. □

We now discuss why some hypotheses in the results of Theorem 4.1, Theorem 4.2 and Corollary 4.3 cannot be weakened. For instance, if the given splittings are replaced by both nonnegative splittings, Theorem 4.1 may no longer hold. The following example demonstrates that even when the splittings in (9) are nonnegative, the comparison results stated in Theorem 4.1 are not valid.

Example 4.4. *If* $\mathcal{A} \in \mathbb{R}^{[3,2]}$. *Let*

$$\mathcal{A}_{(1)} = \begin{pmatrix} -2 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & -2 \end{pmatrix},$$

and let

$$(\mathcal{E}_1)_{(1)} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, (\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & -0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 0 \end{pmatrix},$$

$$(\mathcal{E}_2)_{(1)} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & -2 \end{pmatrix}, (\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & -0.25 & -0.25 & -0.25 \\ 0 & -0.25 & -0.25 & 0 \end{pmatrix}.$$

We get

$$M(\mathcal{E}_1)^{-1} = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}, M(\mathcal{E}_2)^{-1} = \begin{pmatrix} -0.5 & 0 \\ -0.0625 & -0.5 \end{pmatrix},$$

$$(M(\mathcal{E}_1)^{-1}\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & 0.125 & 0.125 & 0.125 \\ 0.125 & 0.125 & 0.125 & 0 \end{pmatrix}, (M(\mathcal{E}_2)^{-1}\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & 0.125 & 0.125 & 0.125 \\ 0 & 0.1406 & 0.1406 & 0.0156 \end{pmatrix}.$$

It is easy to check that $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ *are two nonnegative splittings with* $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$ *and* $M(\mathcal{E}_2)^{-1}\mathcal{F}_2 \geq \mathcal{O}$. *Moreover, since* $\mathcal{F}_1 \leq \mathcal{F}_2$, *it follows from Theorem 4.1 that* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$. *But by calculating that we get* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) = 0.3750 > \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) = 0.3273 < 1$. *In this case, theorem 4.1 is not true.*

Similarly, if the above splittings is replaced by both nonnegative splittings then theorem 4.2 and Corollary 4.3 may not hold. we see from example 4.4 that $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) = 0.3750 > \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) = 0.3273 < 1$, *when* $M(\mathcal{E}_1) \leq M(\mathcal{E}_2)$.

When both splittings in (9) are nonnegative splittings, the following comparison results hold.

Theorem 4.5. *Let the splittings given in* (9) *be convergent and nonnegative. Suppose there exists a constant* $\alpha \in (0, 1]$ *such that*

$$\alpha M(\mathcal{E}_1)^{-1} \geq M(\mathcal{E}_2)^{-1}.$$

If the Perron vector $\mathbf{x}$ *of* $M(\mathcal{E}_1)^{-1}\mathcal{F}_1$ *satisfies* $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$, *then the inequality* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ *holds when* $\alpha = 1$ *and the strict inequality* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ *holds when* $0 < \alpha < 1$.

Proof. Because $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1$ is a nonnegative splitting, then $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$. By Lemma 2.11, $M(\mathcal{E}_1)^{-1}\mathcal{F}_1$ has a Perron vector $\mathbf{x}$ such that

$$M(\mathcal{E}_1)^{-1}\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]}.$$

By $\mathcal{F}_1 = \mathcal{E}_1 - \mathcal{A}$, we get

$$M(\mathcal{E}_1)^{-1}(\mathcal{E}_1 - \mathcal{A})\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]},$$

which implies:

$$M(\mathcal{E}_1)^{-1}\mathcal{E}_1\mathbf{x}^{m-1} - M(\mathcal{E}_1)^{-1}\mathcal{A}\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]}. \tag{16}$$

By $\mathcal{E}_1 = M(\mathcal{E}_1)\mathcal{I}$ and (16), we get

$$M(\mathcal{E}_1)^{-1}\mathcal{A}\mathbf{x}^{m-1} = (1 - \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}.$$

Because the Perron vector $\mathbf{x}$ of $M(\mathcal{E}_1)^{-1}\mathcal{F}_1$ satisfies $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$. Multiply both sides of $\alpha M(\mathcal{E}_1)^{-1} \geq M(\mathcal{E}_2)^{-1}$ by $\mathcal{A}\mathbf{x}^{m-1}$ gives us

$$\alpha M(\mathcal{E}_1)^{-1}\mathcal{A}\mathbf{x}^{m-1} \geq M(\mathcal{E}_2)^{-1}\mathcal{A}\mathbf{x}^{m-1},$$

i.e.,

$$\alpha(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]} \geq M(\mathcal{E}_2)^{-1}\mathcal{A}\mathbf{x}^{m-1}. \tag{17}$$

By $\mathcal{A} = \mathcal{E}_2 - \mathcal{F}_2$ and (17), we have

$$M(\mathcal{E}_2)^{-1}\mathcal{F}_2\mathbf{x}^{m-1} \geq (1-\alpha(1-\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}. \tag{18}$$

Which, by Lemma 2.13 and (18), implies

$$\rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) \geq 1-\alpha+\alpha\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right).$$

From this we can obtain the required results, $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ whenever $\alpha = 1$ and $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ whenever $0 < \alpha < 1$. □

For the case when $\alpha = 1$ and $M(\mathcal{E}_1)^{-1} \geq M(\mathcal{E}_2)^{-1}$ the equality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ can be proved by [11, Lemma 5.3].

Corollary 4.6. *Let the splittings given in (9) be convergent and nonnegative. If the Perron vector* $\mathbf{x}$ *of* $M(\mathcal{E}_1)^{-1}\mathcal{F}_1$ *satisfies* $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$ *and* $M(\mathcal{E}_1)^{-1} > M(\mathcal{E}_2)^{-1}$, *then there exists* α *with* $0 < \alpha < 1$ *such that* $\alpha M(\mathcal{E}_1)^{-1} \geq M(\mathcal{E}_2)^{-1}$ *and the strict inequality* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ *is valid.*

Proof. Denoting

$$M(\mathcal{E}_1)^{-1} = \left(m_{ij}^{(1)}\right)_{n\times n}, \quad M(\mathcal{E}_2)^{-1} = \left(m_{ij}^{(2)}\right)_{n\times n}.$$

From $M(\mathcal{E}_1)^{-1} > M(\mathcal{E}_2)^{-1}$ it gets

$$m_{ij}^{(1)} > m_{ij}^{(2)}, \qquad i,j = 1,2,\ldots,n.$$

If there exists $m_{ij}^{(1)} > 0$ then let

$$\alpha = \max_{1\leq i,j\leq n}\left\{\frac{m_{ij}^{(1)}}{m_{ij}^{(2)}} \mid m_{ij}^{(1)} > 0\right\}.$$

Otherwise $m_{ij}^{(1)} \leq 0$, i.e., $M(\mathcal{E}_1)^{-1} \leq O$, then $0 < \alpha < 1$ is arbitrary.

We get $0 < \alpha < 1$ and

$$\alpha m_{ij}^{(1)} \geq m_{ij}^{(2)}, \qquad i,j = 1,2,\ldots,n,$$

i.e.,

$$\alpha M(\mathcal{E}_1)^{-1} \geq M(\mathcal{E}_2)^{-1}. \tag{19}$$

By the Theorem 4.5 and (19) we can prove that strict inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ is true. □

Theorem 4.7. *Let the splittings given in (9) be convergent and nonnegative.*
(1) *If either* $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) \geq I$ *or* $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) \leq I$, *then*

$$\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1. \tag{20}$$

(2) *If there exists* α *with* $0 < \alpha < 1$ *such that either* $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) \geq 1/\alpha I$ *or* $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) \leq \alpha I$, *then*

$$\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1. \tag{21}$$

Proof. Because $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1$ is a nonnegative splitting, then $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq O$. By Lemma 2.11, $M(\mathcal{E}_1)^{-1}\mathcal{F}_1$ has a Perron vector $\mathbf{x}$ such that

$$M(\mathcal{E}_1)^{-1}\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)\mathbf{x}^{[m-1]},$$

which implies:

$$\mathcal{F}_1\mathbf{x}^{m-1} = \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right)M(\mathcal{E}_1)\mathbf{x}^{[m-1]}. \tag{22}$$

By $\mathcal{F}_1 = \mathcal{E}_1 - \mathcal{A}$ and (22), we get

$$\mathcal{A}\mathbf{x}^{m-1} = (1 - \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))M(\mathcal{E}_1)\mathbf{x}^{[m-1]},$$

i.e.,

$$M(\mathcal{E}_2)^{-1}\mathcal{A}\mathbf{x}^{m-1} = (1 - \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1)\mathbf{x}^{[m-1]}. \tag{23}$$

Let $\beta = 1$ or α. If $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) \leq \beta I$. Because splittings given in (9) be convergent, then $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < 1$ and $\rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$. By $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < 1$, $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) \leq \beta I$ and (23), we get

$$(1 - \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1)\mathbf{x}^{[m-1]} \leq \beta(1 - \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))I\mathbf{x}^{[m-1]},$$

i.e.,

$$M(\mathcal{E}_2)^{-1}\mathcal{A}\mathbf{x}^{m-1} \leq \beta(1 - \rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}. \tag{24}$$

By $\mathcal{A} = \mathcal{E}_2 - \mathcal{F}_2$ and (24), we get

$$M(\mathcal{E}_2)^{-1}\mathcal{F}_2\mathbf{x}^{m-1} \geq (1 - \beta + \beta\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right))\mathbf{x}^{[m-1]}. \tag{25}$$

Which, by Lemma 2.13 and (25), implies

$$\rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) \geq 1 - \beta + \beta\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right).$$

Then (20) is valid whenever $\beta = 1$ and the strict inequality (21) is valid whenever $\beta = \alpha$.

For the case when $\beta = 1$ or α, If $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) \geq 1/\beta I$, the proof is similar. □

Corollary 4.8. *Let the splittings given in (9) be convergent and nonnegative. If either $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) > I$ or $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) < I$, then the strict inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ is valid.*

Proof. If $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) > I$ then we denote $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) = \left(\tilde{m}_{ij}\right)_{n\times n}$ and we define α by

$$\alpha = \max_{1\leq i\leq n}\left\{\frac{1}{\tilde{m}_{ii}}\right\}.$$

If $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) < I$ then we denote $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) = \left(\hat{m}_{ij}\right)_{n\times n}$ and define α by

$$\alpha = \max_{1\leq i\leq n}\{\hat{m}_{ii} \mid \hat{m}_{ii} > 0\},$$

whenever there exists at least a diagonal element $\hat{m}_{ii} > 0$. Otherwise, $0 < \alpha < 1$ is arbitrary.

It is easy to verify that $0 < \alpha < 1$ and

$$M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) \geq 1/\alpha I,$$

whenever $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) > I$ or

$$M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) \leq \alpha I,$$

whenever $M(\mathcal{E}_2)^{-1}M(\mathcal{E}_1) < I$. By the Theorem 4.7 (2) we can prove that strict inequality $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$ is true. □

Theorem 4.9. *Let the splittings given in (9) be convergent and nonnegative. If* $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \leq (<) M(\mathcal{E}_2)^{-1}\mathcal{F}_2$*, then* $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) < 1$.

Proof. Because $\mathcal{A} = \mathcal{E}_1 - \mathcal{F}_1 = \mathcal{E}_2 - \mathcal{F}_2$ are nonnegative splittings, then $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$ and $M(\mathcal{E}_2)^{-1}\mathcal{F}_2 \geq \mathcal{O}$. By Lemma 2.12 and $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \leq (<) M(\mathcal{E}_2)^{-1}\mathcal{F}_2$, we have $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) \leq \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$. Noting the splittings given in (9) be convergent, from this we can obtain the required results. □

5. Numerical examples

In this section, we will demonstrate the validity of the comparison theorems through several elementary multi-linear systems. All tests will be done in MATLAB R2016b with the configuration: Intel(R) Core(TM)i7-8750H CPU 2.20 GHz and 2.21 GHz.

All numerical experiments were initialized with an appropriate starting value $\mathbf{x}_0$, and the iterative process was terminated when either of the following criteria was met: (1) $\mathbf{x}_k$ satisfies

$$\text{RES} = \|\mathbf{b} - \mathcal{A}\mathbf{x}_k^{m-1}\|_2 \leq 10^{-7}$$

or (2) the number of the prescribed maximum iteration steps 1000 is exceeded. In the following examples, two aspects are given to check the efficiency of the proposed comparison theorems: the number of iteration steps (denoted by IT), the CPU time in seconds (denoted by CPU(s)).

Example 5.1. *If* $\mathcal{A} \in \mathbb{R}^{[3,2]}$. *Let*

$$\mathcal{A}_{(1)} = \begin{pmatrix} 2 & -0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 2 \end{pmatrix},$$

and let

$$(\mathcal{E}_1)_{(1)} = \begin{pmatrix} 2 & 0 & 0 & -0.5 \\ -0.5 & 0 & 0 & 2 \end{pmatrix}, (\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 & 0 \end{pmatrix},$$

$$(\mathcal{E}_2)_{(1)} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, (\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 & 0 \end{pmatrix}.$$

We get

$$M(\mathcal{E}_2)^{-1} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix},$$

$$(M(\mathcal{E}_1)^{-1}\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & 0.3333 & 0.3333 & 0 \\ 0 & 0.3333 & 0.3333 & 0 \end{pmatrix}, (M(\mathcal{E}_2)^{-1}\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0 \end{pmatrix}.$$

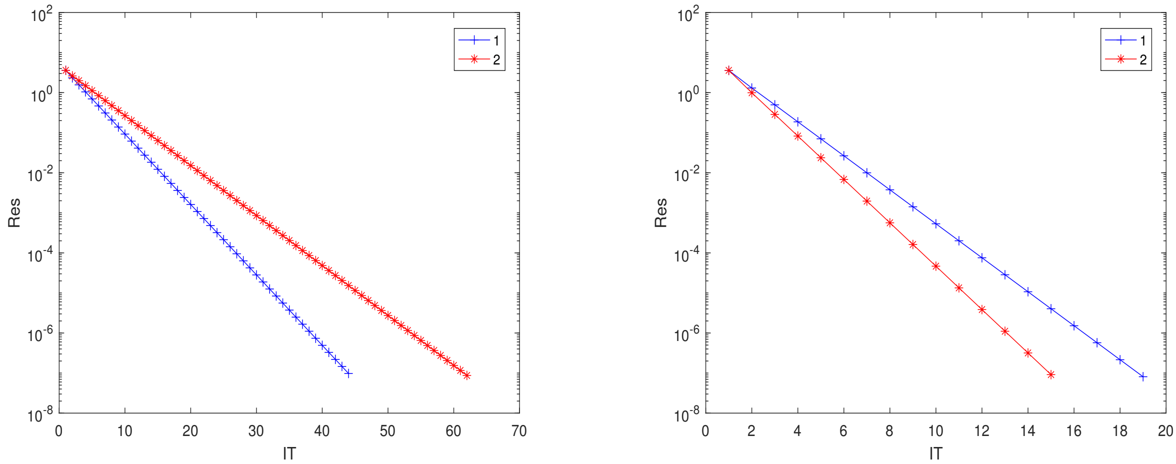
It is easy to check that $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$, $M(\mathcal{E}_2)^{-1} \geq O$ and $M(\mathcal{E}_2)^{-1}\mathcal{F}_2 \geq \mathcal{O}$. So the splittings are nonnegative and weak regular, respectively, and $\mathcal{F}_1 \leq \mathcal{F}_2$. By Theorem 4.1, we get $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$. This result is further confirmed numerically using the power method, which yields $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) = 0.6666 < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) = 0.7500$.

We set $\mathbf{b} = [2, 3]^T$ and initial vector $\mathbf{x}_0 = [0.1, 0.1]^T$. By the power method and the TSI method, we get Table 1. As seen in Table 1, the spectral radius of iteration tensor, the number of iteration steps and CPU time of choice $\mathcal{E}_1, \mathcal{F}_1$ outperform choice $\mathcal{E}_2, \mathcal{F}_2$.

To further demonstrate the efficiency of Theorem 4.1, in Figure 1 (a), we plot the values of residuals (RES) of $\mathcal{E}_1$ and $\mathcal{E}_2$ with respect to the iteration steps (IT). From Figure 1 (a), comparing $\mathcal{E}_1$ and $\mathcal{E}_2$, we find that $\mathcal{E}_1$ is more effective and practical than $\mathcal{E}_2$.

Table 1 Comparison results of the Example 5.1.

$\mathcal{E}$ and $\mathcal{F}$	IT	CPU (s)	$\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$
$\mathcal{E}_1, \mathcal{F}_1$	44	0.0015	0.6666
$\mathcal{E}_2, \mathcal{F}_2$	62	0.0021	0.7500



(a) Example 5.1 (b) Example 5.2

Figure 1: *RES versus IT for (a) Example 5.1 and (b) Example 5.2.*

Example 5.2. *If* $\mathcal{A} \in \mathbb{R}^{[3,2]}$. *Let*

$$\mathcal{A}_{(1)} = \begin{pmatrix} 2 & -0.25 & -0.25 & -0.25 \\ -0.25 & -0.25 & -0.25 & 2 \end{pmatrix},$$

and let

$$(\mathcal{E}_1)_{(1)} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, (\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0 \end{pmatrix},$$

$$(\mathcal{E}_2)_{(1)} = \begin{pmatrix} 2 & 0 & 0 & -0.25 \\ -0.25 & 0 & 0 & 2 \end{pmatrix}, (\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & 0.25 & 0.25 & 0 \\ 0 & 0.25 & 0.25 & 0 \end{pmatrix}.$$

We get

$$M(\mathcal{E}_1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, M(\mathcal{E}_2) = \begin{pmatrix} 2 & -0.25 \\ -0.25 & 2 \end{pmatrix},$$

$$(M(\mathcal{E}_1)^{-1}\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & 0.125 & 0.125 & 0.125 \\ 0.125 & 0.125 & 0.125 & 0 \end{pmatrix}, (M(\mathcal{E}_2)^{-1}\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & 0.1429 & 0.1429 & 0 \\ 0 & 0.1429 & 0.1429 & 0 \end{pmatrix}.$$

It is easy to check that $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$, $M(\mathcal{E}_2)^{-1} > \mathcal{O}$ and $M(\mathcal{E}_2)^{-1}\mathcal{F}_2 \geq \mathcal{O}$. So the first splitting is a nonnegative and the second is a weak regular, and $M(\mathcal{E}_1) \geq M(\mathcal{E}_2)$. It follows from Theorem 4.2 that $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) > \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$. In fact, we have $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) = 0.3750 > \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) = 0,2858$.

In this example, we choose $\mathbf{b} = [2,3]^T$ and initial vector $\mathbf{x}_0 = [0.1, 0.1]^T$. From Table 2 and Figure 1 (b), we can validate the conclusions of Theorem 4.2.

Table 2. Comparison results of the Example 5.2.

$\mathcal{E}$ and $\mathcal{F}$	IT	CPU (s)	$\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$
$\mathcal{E}_1, \mathcal{F}_1$	19	0.0010	0.3750
$\mathcal{E}_2, \mathcal{F}_2$	15	6.1725e-04	0.2858

Example 5.3. *If* $\mathcal{A} \in \mathbb{R}^{[3,2]}$. *Let*

$$\mathcal{A}_{(1)} = \begin{pmatrix} -3 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & -3 \end{pmatrix},$$

and let

$$(\mathcal{E}_1)_{(1)} = \begin{pmatrix} -3 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & -3 \end{pmatrix}, (\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0 & -0.3 & -0.3 & 0 \\ -0.3 & -0.3 & -0.3 & 0 \end{pmatrix},$$

$$(\mathcal{E}_2)_{(1)} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, (\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & -0.3 & -0.3 & -0.3 \\ -0.3 & -0.3 & -0.3 & 0 \end{pmatrix}.$$

We get

$$M(\mathcal{E}_1)^{-1} = \begin{pmatrix} -0.3333 & -0.0333 \\ 0 & -0.3333 \end{pmatrix}, M(\mathcal{E}_2) = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix},$$

$$(M(\mathcal{E}_1)^{-1}\mathcal{F}_1)_{(1)} = \begin{pmatrix} 0.01 & 0.11 & 0.11 & 0 \\ 0.1 & 0.1 & 0.1 & 0 \end{pmatrix}, (M(\mathcal{E}_2)^{-1}\mathcal{F}_2)_{(1)} = \begin{pmatrix} 0 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0 \end{pmatrix}.$$

It is easy to check that $M(\mathcal{E}_1)^{-1}\mathcal{F}_1 \geq \mathcal{O}$ and $M(\mathcal{E}_2)^{-1}\mathcal{F}_2 \geq \mathcal{O}$. So the splittings are nonnegative. The condition $M(\mathcal{E}_1)^{-1}M(\mathcal{E}_2) \geq I$ of Theorem 4.7 is satisfied, hence, $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right)$. In fact, we have $\rho\left(M(\mathcal{E}_1)^{-1}\mathcal{F}_1\right) = 0.2572 < \rho\left(M(\mathcal{E}_2)^{-1}\mathcal{F}_2\right) = 0.3000$.

In Example 5.3, we take $\mathbf{b} = [2,3]^T$ and initial vector $\mathbf{x}_0 = [0.1, 0.1]^T$. From Table 3, we observe that the spectral radius of iteration tensor, number of iteration steps and CPU time of choice $\mathcal{E}_1$, $\mathcal{F}_1$ outperform choice $\mathcal{E}_2, \mathcal{F}_2$. From Figure 2, comparing $\mathcal{E}_1$ and $\mathcal{E}_2$, we find that $\mathcal{E}_1$ is more effective and practical than $\mathcal{E}_2$. These results confirm the validity of Theorem 4.7.

Table 3 Comparison results of the Example 5.3.

$\mathcal{E}$ and $\mathcal{F}$	IT	CPU (s)	$\rho\left(M(\mathcal{E})^{-1}\mathcal{F}\right)$
$\mathcal{E}_1, \mathcal{F}_1$	14	4.9451e-04	0.2572
$\mathcal{E}_2, \mathcal{F}_2$	16	6.4237e-04	0.3000

6. Concluding remark

In this paper, we present the definition of nonnegative splitting of a tensor. We establish a new convergence theorem that addresses the case where nonnegative splitting of strong $\mathcal{M}$-tensor is not necessarily convergent. Theoretically, we prove comparison theorems for nonnegative splittings of tensors. Numerical examples demonstrate that the comparison theorems for nonnegative splittings of tensors are effective for solving multi-linear systems using the TSI method. The obtained results improve and/or generalize the previous results. Applications of these comparison results to evaluate the efficiency of preconditioners for multi-linear systems warrant further study.

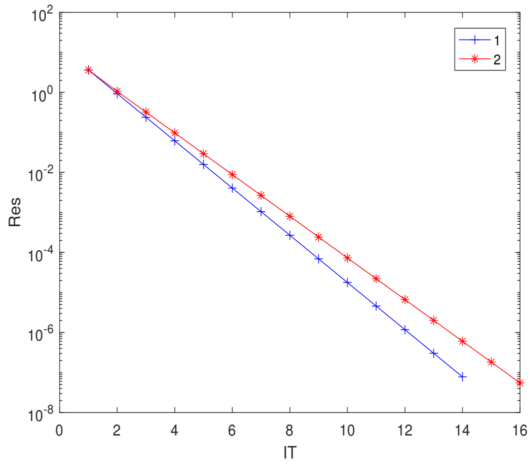


Figure 2: RES versus IT for Example 5.3

Acknowledgments

The authors are very thankful to the referees for their constructive comments and valuable suggestions, which greatly improved the original manuscript of this paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

[1] M. Boubekraoui, *Extrapolated splitting methods for multilinear PageRank computations*, Appl. Numer. Math. **208** (2025), 92–103.
[2] C. Bu, X. Zhang, J. Zhou, et al., *The inverse, rank and product of tensors*, Linear Algebra Appl. **446** (2014), 269–280.
[3] M. Brazell, N. Li, C. Navasca and C. Tamon, *Solving multilinear systems via tensor inversion*, SIAM J. Matrix Anal. Appl. **34**(2) (2013), 542–570.
[4] K. C. Chang, K. Pearson and T. Zhang, *Perron–Frobenius theorem for nonnegative tensors*, Comm. Math. Sci. **6** (2008), 507–520.
[5] J. Cooper and A. Dutle, *Spectra of uniform hypergraphs*, Linear Algebra Appl. **436**(9) (2012), 3268–3292.
[6] G. Csordas and R. S. Varga, *Comparisons of regular splittings of matrices*, Numer. Math. **44** (1984), 23–35.
[7] W. Ding, L. Qi and Y. Wei, *$\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors*, Linear Algebra Appl. **439**(10) (2013), 3264–3278.
[8] T. G. Kolda and B. W. Bader, *Tensor decompositions and applications*, SIAM Rev. **51**(3) (2009), 455–500.
[9] T. G. Kolda, *Multilinear operators for higher-order decompositions*, Tech. Report SAND2006-2081, Sandia National Laboratories, Albuquerque, NM and Livermore, CA, 2006.
[10] W. Li, D. D. Liu and S. W. Vong, *The tensor splitting with application to solve multi-linear systems*, J. Comput. Appl. Math. **330** (2018), 75–94.
[11] W. Li, D. D. Liu and S. W. Vong, *Comparison results for splitting iterations for solving multi-linear systems*, Appl. Numer. Math. **134** (2018), 105–121.
[12] W. Liu and W. Li, *On the inverse of a tensor*, Linear Algebra Appl. **495** (2016), 199–205.
[13] L. H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, In: Proc. IEEE Int. Workshop on Computational Advances in Multi-Sensor Adaptive Processing, Vol. 1, CAMSAP 05, IEEE Computer Society Press, Piscataway, NJ, 2005, 129–132.
[14] M. Ng, L. Qi and G. Zhou, *Finding the largest eigenvalue of a nonnegative tensor*, SIAM J. Matrix Anal. Appl. **31** (2009), 1090–1099.
[15] K. Pearson, *Essentially positive tensors*, Int. J. Algebra **4** (2010), 421–427.
[16] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput. **40** (2005), 1302–1324.
[17] L. Qi and Z. Luo, *Tensor Analysis: Spectral Theory and Special Tensors*, SIAM, Philadelphia, 2017.
[18] J. Y. Shao, *A general product of tensors with applications*, Linear Algebra Appl. **439**(8) (2013), 2350–2366.
[19] J. Y. Shao and L. You, *On some properties of three different types of triangular blocked tensors*, Linear Algebra Appl. **511** (2016), 110–140.
[20] Y. Song, *Comparison theorems for splittings of matrices*, Numer. Math. **92** (2002), 563–591.
[21] A. Smilde, R. Bro and P. Geladi, *Multi-way analysis: applications in the chemical sciences*, Wiley, West Sussex, England, 2004.
[22] Z. Woźnicki, *Nonnegative splitting theory*, Japan J. Indust. Appl. Math. **11** (1994), 289–342.
[23] Y. Yang and Q. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl. **31** (2010), 2517–2530.