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Pseudo-conformal structure preserving explicit numerical methods for the stochastic linearly damped Hamiltonian systems

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Abstract. In this paper, the linearly damped stochastic differential equations with invariants are examined. These invariants follow a linear differential equation with coefficients that are either linear constants or time-dependent. To preserve the essential characteristics of these linearly damped stochastic differential equations, a stochastic exponential integrator is utilized. Moreover, the stochastic pseudo-conformal symplectic methods are constructed and their pseudo-conformal symplectic orders for the stochastic damped Hamiltonian systems with additive noises are analyzed. All of these methods are explicit so that the implementations become more easier than implicit methods. Particularly, these methods have desired properties in accuracy and approximately preserved symplectic structure of the systems through some numerical experiments, especially including Schrödinger equation.

1. Introduction

Stochastic differential equations (SDEs) play a vital role in modeling real-life systems influenced by noise. Despite their practical significance, the number of exact solutions known is still limited, necessitating the use of numerical integration methods to solve these equations. However, it is challenging to construct numerical schemes that can efficiently approximate accurate solutions for the underlying dynamics and how to preserve its structure is remain concern. Moreover, when applying numerical methods, the decision to maintain certain geometric attributes of SDEs is crucial, especially during long-term integration. It is just as significant as in deterministic scenarios [6, 7, 21]. Forturately, numerous notable studies have been conducted in this domain [10, 11, 15–17].

In the deterministic realm, Aubry and Chartier [1] introduced pseudo-symplectic methods tailored for Hamiltonian systems. Later on, in the stochastic setting, the relationship between preserving quadratic invariants and the symplectic structure was explored in [8]. Under certain conditions, an SRK (stochastic Runge-Kutta) method preserving quadratic invariants is established, which is symplectic in nature. In a subsequent study [19], three distinct pseudo-symplectic methodologies were developed, and their orders of

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pseudo-symplectic were confirmed. Recently, [4] pinpointed similar conditions for SRK methods regarding the near-preservation of quadratic invariants and pseudo-symplectic traits. Alongside, they proposed a methodology to formulate explicit SRK pseudo-symplectic schemes, leveraging colored trees and B-series to address order conditions.

A key consideration in modeling classical mechanics and quantum systems is incorporating dissipation. However, traditional structure preserving methods are unable to maintain the dissipation of the system. Consequently, accurately simulating the dissipation in these systems remains an urgent and unresolved issue. Addressing this challenge would lead to significant advancements in simulating and understanding complex systems subject to stochastic influences and dissipation.

Several structure-preserving algorithms for damped systems have been developed in the literature. For instance, [12] introduced the concept of a conformal symplectic structure for Hamiltonian ordinary differential equations with linear damping, which is referred as the conformal Hamiltonian system. This idea was expanded to multisymplectic PDEs by Moore in [18]. For stochastic systems, [14] offers a thorough examination of the stochastic Hamiltonian system and its corresponding algorithms. In another work, [2] presented exponential Runge–Kutta (RK) and partitioned exponential RK methods to efficiently solve linearly damped ordinary differential equations with either constant or time-dependent coefficients. Furthermore, in [22] it not only discusses the development of structure-preserving stochastic exponential integrators tailored for conservative stochastic differential equations (SDEs) with potential linear, time-dependent damping terms but also demonstrates their application in solving damped stochastic differential equations using the conformal exponential integrator. Moreover, [5] constructs the exponential integrators that incorporate both the linear drift and diffusion terms by employing the entire class of stochastic SRK schemes and a stochastic extension of Lawson-type schemes for both Stratonovich and Itô integrals. It is well recognized that these stochastic conformal symplectic methods are inherently implicit when applied to general stochastic damped Hamiltonian systems, inevitably leading to increased computational complexity.

This article addresses a set of stochastic pseudo-conformal symplectic methods for stochastic damped Hamiltonian systems, concentrating on the research of the following system:

$$\begin{cases} dP = f\left(P,Q\right)dt - \alpha\left(t\right)P\left(t\right)dt + \sum_{r=1}^{m}\sigma_r\left(t\right)dW_r\left(t\right), \\ dQ = g\left(P,Q\right)dt - \beta\left(t\right)Q\left(t\right)dt + \sum_{r=1}^{m}\gamma_r\left(t\right)dW_r\left(t\right), \\ f\left(P,Q\right) = -\frac{\partial H(P,Q)}{\partial Q},\ g\left(P,Q\right) = \frac{\partial H(P,Q)}{\partial P}, \end{cases} \tag{1}$$

where $P, Q, f, g, \sigma_r, \gamma_r$ are $n-$dimensional column-vectors, H is a Hamiltonian, and damped parts $\alpha(t)$ and $\beta(t)$ are depend on time, W_r are independent standard Wiener process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $r = 1, \cdots, m$. The Hamiltonians are assumed to belong to the set $\mathscr{C}_b^K$ for certain $K \in \mathbb{N}$, where the function space consists of functions that are $K-$times continuously differentiable with bounded derivatives up to order K.

The stochastic pseudo-conformal symplectic method and stochastic pseudo-conformal symplectic order for numerical method is defined as follows. The time interval $[t_0, t_0 + T]$ is divided into N equal segments, producing a partition $t_0 < t_1 < \cdots < t_N = t_0 + T$. Let $h = T/N$ be the step size, denote $t_n = t_0 + nh$ and $t_{n+\frac{1}{2}} = \frac{t_n + t_{n+1}}{2}$. We denote the approximations of the solution to the equation at time t_n by P_n and Q_n, respectively, and represent $y_n = (P_n, Q_n)$. Then use the form $y_{n+1} = \phi_h(y_n)$ to represent one-step approximation.

Definition 1.1. *If a numerical method based on the one-step approximation $y_{n+1} = \phi_h\left(y_n\right)$ with mean-square order M for the stochastic damped Hamiltonian systems* (1) *satisfy*

$$\left\|\left(\frac{\partial \phi_h}{\partial y_n}\right)^{\top} J \left(\frac{\partial \phi_h}{\partial y_n}\right) - Je^{-\int_{t_n}^{t_{n+1}} A(s)ds}\right\|_{L^2(\Omega)} := \left(E\left\|\left(\frac{\partial \phi_h}{\partial y_n}\right)^{\top} J \left(\frac{\partial \phi_h}{\partial y_n}\right) - Je^{-\int_{t_n}^{t_{n+1}} A(s)ds}\right\|^2\right)^{1/2} = O(h^{L+1}),$$

with $L > M$, $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, I denotes the identity matrix and $A(t) = \alpha(t) + \beta(t)$, then this method is called a pseudo-conformal symplectic method of mean-square order (M, L), and L is called the pseudo-conformal symplectic *order.*

The reminder of this paper is structured as follows. In Sect. 2, a series of pseudo-conformal symplectic methods for (1) are constructed and present their pseudo-conformal symplectic orders. In Sect. 3, numerical experiments for three pseudo-conformal symplectic methods are conducted and compared with the Euler-Maruyama method to demonstrate their favorable properties in long-time simulations . Furthermore, the pseudo-conformal symplectic midpoint method is applied to damped-driven stochastic Schrödinger equation [20], showing that it also preserves relevant properties, such as charge conservation, when applied to partial differential equations. Finally, in Sect. 4, a summary of completed work is provided.

2. Stochastic pseudo-conformal symplectic method for stochastic damped Hamiltonian systems

2.1. Pseudo-conformal symplectic methods based on midpoint and trapezoidal methods

In this section, we will construct a class of pseudo-conformal symplectic methods based on commonly used Euler, midpoint and trapezoidal techniques.

Denote $\delta_n W_r := W_r(t_n + h) - W_r(t_n)$, $r = 1, \cdots, m$, $n = 0, \cdots, N$, $p = P_n$, $q = Q_n$ and define

$$\begin{cases}\tilde{p} = e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\left(p + f(p,q)h + \sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r\right),\\ \tilde{q} = e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\left(q + g(p,q)h + \sum_{r=1}^{m}\gamma_r(t_n)\delta_n W_r\right).\end{cases} \tag{2}$$

For $a \in [0,1]$, define

$$\begin{cases}\bar{P} = e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\left(p + af(p,q)h\right) + (1-a)f(\tilde{p},\tilde{q})h + e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r,\\ \bar{Q} = e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\left(q + ag(p,q)h\right) + (1-a)g(\tilde{p},\tilde{q})h + e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\sum_{r=1}^{m}\gamma_r(t_n)\delta_n W_r,\end{cases} \tag{3}$$

and

$$\begin{cases}\hat{P} = e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p + e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds}\Big(f(ae^{\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds}p + (1-a)e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds}\tilde{p},\\ \quad ae^{\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\beta(s)ds}q + (1-a)e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\beta(s)ds}\tilde{q})h + \sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r\Big),\\ \hat{Q} = e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q + e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\beta(s)ds}\Big(g(ae^{\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds}p + (1-a)e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds}\tilde{p},\\ \quad ae^{\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\beta(s)ds}q + (1-a)e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\beta(s)ds}\tilde{q})h + \sum_{r=1}^{m}\gamma_r(t_n)\delta_n W_r\Big).\end{cases} \tag{4}$$

We called (2) + (3) and (2) + (4) as pseudo-conformal symplectic trapezoidal (PCST) method and pseudo-conformal symplectic midpoint (PCSM) method respectly.

Theorem 2.1. *Assume that $H \in \mathscr{C}_b^2$. Then there exists a positive constant $C_1 = C_1(H,a)$ such that the methods* (2) + (3) *and* (2) + (4) *for the system satisfy*

$$\left\|\left(\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)}\right)^{\top} J\left(\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)}\right) - Je^{-\int_{t_n}^{t_{n+1}}A(s)ds}\right\|_{L^2(\Omega)} = C_1h^2 + O\left(h^3\right). \tag{5}$$

Moreover, if $H \in \mathscr{C}_b^4$, then there exists a positive constant $C_2 = C_2(H)$ such that

$$\left\|\left(\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)}\right)^{\top} J\left(\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)}\right) - Je^{-\int_{t_n}^{t_{n+1}}A(s)ds}\right\|_{L^2(\Omega)} = |2a-1|C_2h^2 + O\left(h^3\right). \tag{6}$$

Proof. We prove the result for the method (2) + (3). Rewrite (2) + (3) as

$$\tilde{F} := \tilde{p} - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\left(p + f(p,q)h + \sum_{r=1}^{m}\sigma_r(t_n)\,\delta_n W_r\right) = 0,$$
$$\tilde{G} := \tilde{q} - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\left(p + g(p,q)h + \sum_{r=1}^{m}\gamma_r(t_n)\,\delta_n W_r\right) = 0,$$

and

$$\bar{F} := \bar{P} - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}ahf(p,q) - (1-a)hf(\tilde{p},\tilde{q}) - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\sum_{r=1}^{m}\sigma_r(t_n)\,\delta_n W_r = 0,$$
$$\bar{G} := \bar{Q} - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}ahg(p,q) - (1-a)hg(\tilde{p},\tilde{q}) - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\sum_{r=1}^{m}\gamma_r(t_n)\,\delta_n W_r = 0.$$

According to the hypothesis of the $H \in \mathscr{C}_b^2$, we have

$$\frac{\partial(\bar{F},\bar{G})}{\partial(\bar{P},\bar{Q})}\cdot\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)} + \frac{\partial(\bar{F},\bar{G})}{\partial(\tilde{p},\tilde{q})}\cdot\frac{\partial(\tilde{p},\tilde{q})}{\partial(p,q)} + \frac{\partial(\bar{F},\bar{G})}{\partial(p,q)} = 0,$$

then

$$\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)} = -\frac{\partial(\bar{F},\bar{G})}{\partial(\tilde{p},\tilde{q})}\cdot\frac{\partial(\tilde{p},\tilde{q})}{\partial(p,q)} - \frac{\partial(\bar{F},\bar{G})}{\partial(p,q)} =: \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Denote

$$\tilde{H}_{pp} = \frac{\partial^2 H(\tilde{p},\tilde{q})}{\partial p^2},\ \tilde{H}_{pq} = \frac{\partial^2 H(\tilde{p},\tilde{q})}{\partial p\partial q},\ \tilde{H}_{qq} = \frac{\partial^2 H(\tilde{p},\tilde{q})}{\partial q^2},$$
$$H_{pp} = \frac{\partial^2 H(p,q)}{\partial p^2},\ H_{pq} = \frac{\partial^2 H(p,q)}{\partial p\partial q},\ H_{qq} = \frac{\partial^2 H(p,q)}{\partial q^2}.$$

Simple calculations yield

$$\frac{\partial(\bar{F},\bar{G})}{\partial(\bar{P},\bar{Q})} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$
$$\frac{\partial(\bar{F},\bar{G})}{\partial(\tilde{p},\tilde{q})} = \begin{pmatrix} (1-a)\tilde{H}_{pq}h & (1-a)\tilde{H}_{qq}h \\ -(1-a)\tilde{H}_{pp}h & -(1-a)\tilde{H}_{pq}h \end{pmatrix},$$
$$\frac{\partial(\tilde{p},\tilde{q})}{\partial(p,q)} = \begin{pmatrix} e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\left(I - H_{pq}h\right) & -e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}H_{qq}h \\ e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}H_{pp}h & e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}(I + H_{pq}h) \end{pmatrix},$$
$$\frac{\partial(\bar{F},\bar{G})}{\partial(p,q)} = \begin{pmatrix} -e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}(I - aH_{pq}h) & e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}aH_{qq}h \\ -e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}aH_{pp}h & -e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}(I + aH_{pq}h) \end{pmatrix}.$$

For convenience, denote that

$$m = e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds},\ n = e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}.$$

By introducing these notations, we can express Σ as

$$\Sigma_{11} = mI - m\left(aH_{pq} + (1-a)\tilde{H}_{pq}\right)h + (1-a)\left(m\tilde{H}_{pq}H_{pq} - n\tilde{H}_{qq}H_{pp}\right)h^2,$$
$$\Sigma_{12} = -\left(maH_{qq} + n(1-a)\tilde{H}_{qq}\right)h + (1-a)\left(m\tilde{H}_{pq}H_{qq} - n\tilde{H}_{qq}H_{pq}\right)h^2,$$
$$\Sigma_{21} = \left(naH_{pp} + m(1-a)\tilde{H}_{pp}\right)h - (1-a)\left(m\tilde{H}_{pp}H_{pq} - n\tilde{H}_{pq}H_{pp}\right)h^2,$$
$$\Sigma_{22} = nI + n\left(aH_{pq} + (1-a)\tilde{H}_{pq}\right)h - (1-a)\left(m\tilde{H}_{pp}H_{qq} - n\tilde{H}_{pq}H_{pq}\right)h^2,$$

thus

$$\left(\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)}\right)^{\top} J\left(\frac{\partial(\bar{P},\bar{Q})}{\partial(p,q)}\right)=\begin{pmatrix} \Sigma_{11}^{\top}\Sigma_{21}-\Sigma_{21}^{\top}\Sigma_{11} & \Sigma_{11}^{\top}\Sigma_{22}-\Sigma_{21}^{\top}\Sigma_{12} \\ \Sigma_{12}^{\top}\Sigma_{21}-\Sigma_{22}^{\top}\Sigma_{11} & \Sigma_{12}^{\top}\Sigma_{22}-\Sigma_{22}^{\top}\Sigma_{12} \end{pmatrix}.$$

In what follow we only estimate $\Sigma_{11}^{\top}\Sigma_{21}-\Sigma_{21}^{\top}\Sigma_{11}$, while an analogous idea can be applied to estimate the term $\Sigma_{11}^{\top}\Sigma_{22}-\Sigma_{21}^{\top}\Sigma_{12}$, $\Sigma_{12}^{\top}\Sigma_{21}-\Sigma_{22}^{\top}\Sigma_{11}$ and $\Sigma_{12}^{\top}\Sigma_{22}-\Sigma_{22}^{\top}\Sigma_{12}$. Direct calculations yield that

$$\begin{aligned} &\left\|\Sigma_{11}^{\top}\Sigma_{21}-\Sigma_{21}^{\top}\Sigma_{11}\right\|_{L^2(\Omega)} \\ =&\Big\|mna^2\left(H_{pp}H_{pq}-H_{pq}H_{pp}\right)+m^2(1-a)^2\left(\tilde{H}_{pp}\tilde{H}_{pq}-\tilde{H}_{pq}\tilde{H}_{pp}\right) \\ &-(1-a)m^2\left(\tilde{H}_{pp}H_{pq}-H_{pq}\tilde{H}_{pp}\right)-(1-a)mn\left(H_{pp}\tilde{H}_{pq}-\tilde{H}_{pq}H_{pp}\right) \\ &+mna(1-a)\left(H_{pp}\tilde{H}_{pq}-\tilde{H}_{pq}H_{pp}\right)+m^2a(1-a)\left(\tilde{H}_{pp}H_{pq}-\tilde{H}_{pq}H_{pp}\right)\Big\|_{L^2(\Omega)}h^2 \\ &+O\left(h^3\right). \end{aligned} \tag{7}$$

Thus (5) holds for certain C_1.

We prove the result for the method (2) + (4). For convenience, denote that

$$j=e^{-\int_{t_{n+\frac12}}^{t_{n+1}}\alpha(s)ds},\ v=e^{\int_{t_{n+\frac12}}^{t_{n+1}}\alpha(s)ds},\ k=e^{-\int_{t_{n+\frac12}}^{t_{n+1}}\beta(s)ds},\ w=e^{\int_{t_{n+\frac12}}^{t_{n+1}}\beta(s)ds}.$$

Rewrite (4) as

$$\begin{aligned} \hat{F}&:=\hat{P}-mp-j\left(f\left(avp+(1-a)\,j\tilde{p},awq+(1-a)\,k\tilde{q}\right)h+\sum_{r=1}^{m}\sigma_r\left(t_n\right)\delta_n W_r\right)=0, \\ \hat{G}&:=\hat{Q}-nq-k\left(g\left(avp+(1-a)\,j\tilde{p},awq+(1-a)\,k\tilde{q}\right)h+\sum_{r=1}^{m}\gamma_r\left(t_n\right)\delta_n W_r\right)=0. \end{aligned}$$

Simple calculations yield

$$\begin{aligned} \frac{\partial(\hat{F},\hat{G})}{\partial(\hat{P},\hat{Q})}&=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \\ \frac{\partial(\hat{F},\hat{G})}{\partial\left(\tilde{p},\tilde{q}\right)}&=\begin{pmatrix} j^2(1-a)\tilde{H}_{pq}h & jk(1-a)\tilde{H}_{qq}h \\ -kj(1-a)\tilde{H}_{pp}h & -k^2(1-a)\tilde{H}_{pq}h \end{pmatrix}, \\ \frac{\partial(\tilde{p},\tilde{q})}{\partial(p,q)}&=\begin{pmatrix} m\left(I-H_{pq}h\right) & -mH_{qq}h \\ nH_{pp}h & n(I+H_{pq}h) \end{pmatrix}, \\ \frac{\partial(\hat{F},\hat{G})}{\partial(p,q)}&=\begin{pmatrix} -(mI-jvaH_{pq}h) & jwaH_{qq}h \\ -kvaH_{pp}h & -(nI+kwaH_{pq}h) \end{pmatrix}. \end{aligned}$$

Then

$$\frac{\partial(\hat{P},\hat{Q})}{\partial(p,q)}=-\frac{\partial(\hat{F},\hat{G})}{\partial(\tilde{p},\tilde{q})}\cdot\frac{\partial(\tilde{p},\tilde{q})}{\partial(p,q)}-\frac{\partial(\hat{F},\hat{G})}{\partial(p,q)}=:\begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}.$$

By introducing these notations, we can express η as

$$\begin{aligned} \eta_{11}=&mI-\left(jvaH_{pq}+j^2m(1-a)\tilde{H}_{pq}\right)h+(1-a)\left(j^2m\tilde{H}_{pq}H_{pq}-jkn\tilde{H}_{qq}H_{pp}\right)h^2, \\ \eta_{12}=&-j\left(waH_{qq}+kn(1-a)\tilde{H}_{qq}\right)h-j(1-a)\left(jm\tilde{H}_{pq}H_{qq}-kn\tilde{H}_{qq}H_{pq}\right)h^2, \\ \eta_{21}=&k\left(vaH_{pp}+jm(1-a)\tilde{H}_{pp}\right)h-(1-a)k\left(jm\tilde{H}_{pp}H_{pq}-kn\tilde{H}_{pq}H_{pp}\right)h^2, \\ \eta_{22}=&nI+k\left(waH_{pq}+nk(1-a)\tilde{H}_{pq}\right)h+k(1-a)\left(nk\tilde{H}_{pq}H_{pq}-mj\tilde{H}_{pp}H_{qq}\right)h^2, \end{aligned}$$

thus

$$\left(\frac{\partial(\hat{P},\hat{Q})}{\partial(p,q)}\right)^{\top} J\left(\frac{\partial(\hat{P},\hat{Q})}{\partial(p,q)}\right) = \begin{pmatrix} \eta_{11}^{\top}\eta_{21} - \eta_{21}^{\top}\eta_{11} & \eta_{11}^{\top}\eta_{22} - \eta_{21}^{\top}\eta_{12} \\ \eta_{12}^{\top}\eta_{21} - \eta_{22}^{\top}\eta_{11} & \eta_{12}^{\top}\eta_{22} - \eta_{22}^{\top}\eta_{12} \end{pmatrix}.$$

In what follows we only estimate $\eta_{11}^{\top}\eta_{21} - \eta_{21}^{\top}\eta_{11}$, while an analogous idea can be applied to estimate the term $\eta_{11}^{\top}\eta_{22} - \eta_{21}^{\top}\eta_{12}$, $\eta_{12}^{\top}\eta_{21} - \eta_{22}^{\top}\eta_{11}$ and $\eta_{12}^{\top}\eta_{22} - \eta_{22}^{\top}\eta_{12}$. Direct calculations yield that

$$\begin{aligned}
&\left\|\eta_{11}^{\top}\eta_{21} - \eta_{21}^{\top}\eta_{11}\right\|_{L^2(\Omega)} \\
=&\Big\|kjv^2a^2\left(H_{pp}H_{pq} - H_{pq}H_{pp}\right) + kj^3m^2(1-a)^2\left(\tilde{H}_{pp}\tilde{H}_{pq} - \tilde{H}_{pq}\tilde{H}_{pp}\right) \\
&- kjm^2(1-a)\left(\tilde{H}_{pp}H_{pq} - H_{pq}\tilde{H}_{pp}\right) - k^2mn(1-a)\left(H_{pp}\tilde{H}_{pq} - \tilde{H}_{pq}H_{pp}\right) \\
&+vkj^2ma(1-a)\left(H_{pp}\tilde{H}_{pq} - H_{pq}\tilde{H}_{pp}\right) + kj^2vma(1-a)\left(\tilde{H}_{pp}H_{pq} - \tilde{H}_{pq}H_{pp}\right)\Big\|_{L^2(\Omega)} h^2 \\
&+ O\left(h^3\right).
\end{aligned} \tag{8}$$

Thus (5) holds for certain C_1. In PCST and PCSM methods, the constant denoted by C_1 differs, but for the sake of convenience, we will uniformly refer to this constant as C_1.

Now we assume that $H \in \mathscr{C}_b^4$.

For $\tilde{p} = e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\left(p + f(p,q)h + \sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r\right)$, perform a Taylor expansion of $e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}$ to get

$$\begin{aligned}
&\tilde{p} - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p = e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\left(f(p,q)h + \sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r\right), \\
&\tilde{p} - (1-\alpha(t_n)h + O(h^2))p = (1-\alpha(t_n)h + O(h^2))\left(f(p,q)h + \sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r\right),
\end{aligned}$$

thus we have

$$\tilde{p} - p = \sum_{r=1}^{m}\sigma_r(t_n)\delta_n W_r + O(h),$$

which means terms $\tilde{p} - p$ and $\tilde{q} - q$ are both $O(h^{\frac{1}{2}})$ in the sense of $L^2(\Omega)$ norm.

Expanding $\tilde{H}_{pp}$ and $\tilde{H}_{pq}$ at (p,q), we have

$$\begin{aligned}
\tilde{H}_{pp} &= \frac{\partial^2 H(\tilde{p},\tilde{q})}{\partial p^2} \\
&= \frac{\partial^2 H(p,q)}{\partial p^2} + \frac{\partial^3 H(p,q)}{\partial p^3}\otimes(\tilde{p}-p) + \frac{\partial^3 H(p,q)}{\partial p^2\partial q}\otimes(\tilde{q}-q) + C_3h \\
&= H_{pp} + H_{ppp}\otimes(\tilde{p}-p) + H_{ppq}\otimes(\tilde{q}-q) + C_3h, \\
\tilde{H}_{pq} &= \frac{\partial^2 H(\tilde{p},\tilde{q})}{\partial p\partial q} \\
&= \frac{\partial^2 H(p,q)}{\partial p\partial q} + \frac{\partial^3 H(p,q)}{\partial p^2\partial q}\otimes(\tilde{p}-q) + \frac{\partial^3 H(p,q)}{\partial p\partial q^2}\otimes(\tilde{q}-q) + C_3h \\
&= H_{pq} + H_{ppq}\otimes(\tilde{p}-p) + H_{pqq}\otimes(\tilde{q}-q) + C_4h,
\end{aligned}$$

here $\otimes$ means tensor product.

Substituting the expansion of $\tilde{H}_{pp}$ and $\tilde{H}_{pq}$ to (7) and by the fact that $\tilde{p}-p$ and $\tilde{q}-q$ are both $O(h^{\frac{1}{2}})$, we have the coefficient of h^2 term in the estimation of $\Sigma_{11}^{\top}\Sigma_{21}-\Sigma_{21}^{\top}\Sigma_{11}$ is $|mna^2+m^2(1-a)^2-m^2(1-a)-mn(1-a)+mna(1-a)+m^2a(1-a)| := \bar{B}$. Furthermore using the Taylor expansion

$$\begin{aligned} m &= e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds} = 1-\alpha(t_n)h+O(h^2),\\ n &= e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds} = 1-\beta(t_n)h+O(h^2), \end{aligned}$$

we can easily check that the coefficient of h^2 term is $|a^2+(1-a)^2-(1-a)-(1-a)+a(1-a)+a(1-a)|$ which vanishes if and only if $a=\frac{1}{2}$. Thus, there exists a positive constant C_2 such that

$$\begin{aligned} \left\|\Sigma_{11}^{\top}\Sigma_{21}-\Sigma_{21}^{\top}\Sigma_{11}\right\|_{L^2(\Omega)} &= |a^2+(1-a)^2-(1-a)-(1-a)+a(1-a)+a(1-a)|C_2h^2+O(h^3)\\ &= |2a-1|C_2h^2+O(h^3). \end{aligned}$$

Substituting the expansion of $\tilde{H}_{pp}$ and $\tilde{H}_{pq}$ to (8) and by the fact that $\tilde{p}-p$ and $\tilde{q}-q$ are both $O(h^{\frac{1}{2}})$, we have the coefficient of h^2 term in the estimation of $\eta_{11}^{\top}\eta_{21}-\eta_{21}^{\top}\eta_{11}$ is $|kjv^2a^2+kj^3m^2(1-a)^2-kjm^2(1-a)-k^2mn(1-a)+vkj^2ma(1-a)+kj^2vma(1-a)| := \hat{B}$. Furthermore using the Taylor expansion

$$\begin{aligned} j &= e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds} = 1-\frac{1}{2}\alpha(t_n)h+O(h^2),\\ v &= e^{\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\alpha(s)ds} = 1+\frac{1}{2}\alpha(t_n)h+O(h^2),\\ k &= e^{-\int_{t_{n+\frac{1}{2}}}^{t_{n+1}}\beta(s)ds} = 1-\frac{1}{2}\beta(t_n)h+O(h^2), \end{aligned}$$

we can easily check that the coefficient of h^2 term is $|a^2+(1-a)^2-(1-a)-(1-a)+a(1-a)+a(1-a)|$ which vanishes if and only if $a=\frac{1}{2}$. Thus, there exists a positive constant C_2 such that

$$\begin{aligned} \left\|\eta_{11}^{\top}\eta_{21}-\eta_{21}^{\top}\eta_{11}\right\|_{L^2(\Omega)} &= |a^2+(1-a)^2-(1-a)-(1-a)+a(1-a)+a(1-a)|C_2h^2+O(h^3)\\ &= |2a-1|C_2h^2+O(h^3). \end{aligned}$$

In PCST method and PCSM method, the constant denoted by C_2 differs, but for the sake of convenience, we will uniformly refer to this constant as C_2. □

According to Theorem2.1, the result expressed in relation (6) demonstrates that the PCST method of (2) + (3) and PCSM method of (2) + (4) possesses a pseudo-conformal symplectic order of $(1,2)$ if and only if $a=\frac{1}{2}$.

2.2. *Stochastic pseudo-conformal symplectic Runge-Kutta methods*

Among the numerical methods for stochastic damped Hamiltonian systems, Runge-Kutta method belongs to an important class of methods. However, they may bring more complexity to the calculations because they can be implicit. According to Lawson schemes [5] and to fit the content of the article, we construct a class of s-stage pseudo-conformal symplectic Runge-Kutta (PCSRK) methods in this subsection.

We propose the following scheme

$$
\begin{aligned}
\hat{\mathscr{P}}_1 &= \hat{P}_n + e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\varphi_1,\\
\hat{\mathscr{Q}}_1 &= \hat{Q}_n + e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\psi_1,\\
\hat{\mathscr{P}}_i &= \hat{P}_n + h\sum_{j=1}^{i-1} e^{-\int_{t_n}^{t_{n+c_jh}}\alpha(s)ds}a_{ij}f\left(e^{\int_{t_n}^{t_{n+c_jh}}\alpha(s)ds}\hat{\mathscr{P}}_j, e^{\int_{t_n}^{t_{n+c_jh}}\beta(s)ds}\hat{\mathscr{Q}}_j\right) + e^{-\int_{t_n}^{t_{n+c_{i-1}h}}\alpha(s)ds}\varphi_i,\\
\hat{\mathscr{Q}}_i &= \hat{Q}_n + h\sum_{j=1}^{i-1} e^{-\int_{t_n}^{t_{n+c_jh}}\beta(s)ds}a_{ij}g\left(e^{\int_{t_n}^{t_{n+c_jh}}\alpha(s)ds}\hat{\mathscr{P}}_j, e^{\int_{t_n}^{t_{n+c_jh}}\beta(s)ds}\hat{\mathscr{Q}}_j\right) + e^{-\int_{t_n}^{t_{n+c_{i-1}h}}\beta(s)ds}\psi_i,\\
\hat{P}_{n+1} &= \hat{P}_n + h\sum_{i=1}^{s} e^{-\int_{t_n}^{t_{n+c_ih}}\alpha(s)ds}b_if\left(e^{\int_{t_n}^{t_{n+c_ih}}\alpha(s)ds}\hat{\mathscr{P}}_i, e^{\int_{t_n}^{t_{n+c_ih}}\beta(s)ds}\hat{\mathscr{Q}}_i\right) + e^{-\int_{t_n}^{t_{n+c_ih}}\alpha(s)ds}\eta,\\
\hat{Q}_{n+1} &= \hat{Q}_n + h\sum_{i=1}^{s} e^{-\int_{t_n}^{t_{n+c_ih}}\beta(s)ds}b_ig\left(e^{\int_{t_n}^{t_{n+c_ih}}\alpha(s)ds}\hat{\mathscr{P}}_i, e^{\int_{t_n}^{t_{n+c_ih}}\beta(s)ds}\hat{\mathscr{Q}}_i\right) + e^{-\int_{t_n}^{t_{n+c_ih}}\beta(s)ds}\zeta,\\
P_{n+1} &= e^{\int_{t_n}^{t_{n+1}}\alpha(s)ds}\hat{P}_{n+1},\\
Q_{n+1} &= e^{\int_{t_n}^{t_{n+1}}\beta(s)ds}\hat{Q}_{n+1},
\end{aligned}
\tag{9}
$$

random terms φ_i, ψ_i, η, ζ, $i = 1, \cdots s$ need to be fixed, ensuring their independence not only from p and q but also from the parameters a_{ij}, b_i.

We first construct a 2-stage pseudo-conformal symplectic Runge-Kutta method,

$$
\begin{aligned}
\mathscr{P}_1 =& e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p + e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\varphi_1,\\
\mathscr{Q}_1 =& e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q + e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\psi_1,\\
\mathscr{P}_2 =& e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p + ha_{21}e^{-\int_{t_n}^{t_{n+c_1h}}\alpha(s)ds}f\left(e^{\int_{t_n}^{t_{n+c_1h}}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}\mathscr{Q}_1\right) + e^{-\int_{t_n}^{t_{n+c_1h}}\alpha(s)ds}\varphi_2,\\
\mathscr{Q}_2 =& e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q + ha_{21}e^{-\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}g\left(e^{\int_{t_n}^{t_{n+c_1h}}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}\mathscr{Q}_1\right) + e^{-\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}\psi_2,\\
P =& e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p + hb_1e^{-\int_{t_n}^{t_{n+c_1h}}\alpha(s)ds}f\left(e^{\int_{t_n}^{t_{n+c_1h}}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}\mathscr{Q}_1\right)\\
&+ hb_2e^{-\int_{t_n}^{t_{n+c_2h}}\alpha(s)ds}f\left(e^{\int_{t_n}^{t_{n+c_2h}}\alpha(s)ds}\mathscr{P}_2, e^{\int_{t_n}^{t_{n+c_2h}}\beta(s)ds}\mathscr{Q}_2\right) + e^{-\int_{t_n}^{t_{n+c_2h}}\alpha(s)ds}\eta,\\
Q =& e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q + hb_1e^{-\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}g\left(e^{\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_{n+c_1h}}\beta(s)ds}\mathscr{Q}_1\right)\\
&+ hb_2e^{-\int_{t_n}^{t_{n+c_2h}}\beta(s)ds}g\left(e^{\int_{t_n}^{t_{n+c_2h}}\alpha(s)ds}\mathscr{P}_2, e^{\int_{t_n}^{t_{n+c_2h}}\beta(s)ds}\mathscr{Q}_2\right) + e^{-\int_{t_n}^{t_{n+c_2h}}\beta(s)ds}\zeta.
\end{aligned}
\tag{10}
$$

Let

$$
\begin{aligned}
&\varphi_1 = \sum_{r=1}^{m}\sigma_r(t_n)\left(\lambda_1 J_{r0} + \mu_1\delta_n W_r\right),\ \psi_1 = \sum_{r=1}^{m}\gamma_r(t_n)\left(\lambda_1 J_{r0} + \mu_1\delta_n W_r\right),\\
&\varphi_2 = \sum_{r=1}^{m}\sigma_r(t_n)\left(\lambda_2 J_{r0} + \mu_2\delta_n W_r\right),\ \psi_2 = \sum_{r=1}^{m}\gamma_r(t_n)\left(\lambda_2 J_{r0} + \mu_2\delta_n W_r\right),\\
&\eta = \sum_{r=1}^{m}\sigma_r\delta_n W_r + \sum_{r=1}^{m}\sigma_r' I_{0r},\ \zeta = \sum_{r=1}^{m}\gamma_r\delta_n W_r + \sum_{r=1}^{m}\gamma_r' I_{0r},
\end{aligned}
\tag{11}
$$

where

$$\delta_n W_r := h^{1/2}\xi_{rn},\ J_{r0} := h^{1/2}\left(\frac{\xi_{rn}}{2} + \frac{\eta_{rn}}{\sqrt{12}}\right),\ I_{0r} := h^{3/2}\left(\frac{\xi_{rn}}{2} - \frac{\eta_{rn}}{\sqrt{12}}\right), \tag{12}$$

with ξ_{rn} and η_{rn}, $n = 0, \cdots, N-1$, being a sequence of independent $N(0,1)-$distributed random variables and where the parameters satisfy

$$\begin{aligned} &b_1 + b_2 = 1,\ a_{21}b_2 = 1/2,\ \sum_{i=1}^{2} b_i\lambda_i = 1,\ \sum_{i=1}^{2} b_i\mu_i = 0, \\ &\sum_{i=1}^{2} b_i\left(\lambda_i J_{r0} + \mu_i\delta_n W_r\right)\left(\lambda_i J_{l0} + \mu_i\delta_n W_l\right) = \frac{h}{2}\delta_{rl}. \end{aligned} \tag{13}$$

It has been determined that the method (10) − (12), satisfying (13), converges in the mean-square sense with an order of $\frac{3}{2}$ [14]. In this work, we refrain from considering any higher-order schemes due to the requirement of simulating multiple Wiener integrals [23] for schemes with mean-square order higher than $\frac{3}{2}$. The following theorem establishes that (10) is pseudo-conformal symplectic with pseudo-conformal symplectic order 2, provided appropriate assumptions on the Hamiltonian H are met.

Theorem 2.2. *Assume that $H \in \mathscr{C}_b^4$. Then the* 2−*stage pseudo-conformal symplectic Runge-Kutta method* (10)-(12) *is pseudo-conformal symplectic of order* $(\frac{3}{2}, 2)$ *if the conditions* (13) *and*

$$\begin{aligned} e^{-v}b_1^2 + e^{-d}b_2^2 - \left(e^{-d} + e^{-v}\right)a_{21}b_2 + \left(e^{-d} + e^{-v}\right)b_1b_2 = 0, \\ e^{-d}b_2^2 + e^{-v}b_1b_2 - e^{-v}a_{21}b_2 = 0, \end{aligned} \tag{14}$$

hold, where $v = \int_{t_n}^{t_n+c_1h}\beta(s)ds - \int_{t_n}^{t_n+c_1h}\alpha(s)ds$, $d = \int_{t_n}^{t_n+c_2h}\beta(s)ds - \int_{t_n}^{t_n+c_2h}\alpha(s)ds$.

Proof. Review (10), we can get

$$\begin{aligned} F_1 &:= \mathscr{P}_1 - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}\varphi_1 = 0, \\ G_1 &:= \mathscr{Q}_1 - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}\psi_1 = 0, \\ F_2 &:= \mathscr{P}_2 - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p - ha_{21}e^{-\int_{t_n}^{t_n+c_1h}\alpha(s)ds}f\left(e^{\int_{t_n}^{t_n+c_1h}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_n+c_1h}\beta(s)ds}\mathscr{Q}_1\right) \\ &\quad - e^{-\int_{t_n}^{t_n+c_1h}\alpha(s)ds}\varphi_2 = 0, \\ G_2 &:= \mathscr{Q}_2 - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q - ha_{21}e^{-\int_{t_n}^{t_n+c_1h}\beta(s)ds}g\left(e^{\int_{t_n}^{t_n+c_1h}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_n+c_1h}\beta(s)ds}\mathscr{Q}_1\right) \\ &\quad - e^{-\int_{t_n}^{t_n+c_1h}\beta(s)ds}\psi_2 = 0, \\ F &:= P - e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}p - hb_1e^{-\int_{t_n}^{t_n+c_1h}\alpha(s)ds}f\left(e^{\int_{t_n}^{t_n+c_1h}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_n+c_1h}\beta(s)ds}\mathscr{Q}_1\right) \\ &\quad - hb_2e^{-\int_{t_n}^{t_n+c_2h}\alpha(s)ds}f\left(e^{\int_{t_n}^{t_n+c_2h}\alpha(s)ds}\mathscr{P}_2, e^{\int_{t_n}^{t_n+c_2h}\beta(s)ds}\mathscr{Q}_2\right) - e^{-\int_{t_n}^{t_n+c_2h}\alpha(s)ds}\eta = 0, \\ G &:= Q - e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}q - hb_1e^{-\int_{t_n}^{t_n+c_1h}\beta(s)ds}g\left(e^{\int_{t_n}^{t_n+c_1h}\alpha(s)ds}\mathscr{P}_1, e^{\int_{t_n}^{t_n+c_1h}\beta(s)ds}\mathscr{Q}_1\right) \\ &\quad - hb_2e^{-\int_{t_n}^{t_n+c_2h}\beta(s)ds}g\left(e^{\int_{t_n}^{t_n+c_2h}\alpha(s)ds}\mathscr{P}_2, e^{\int_{t_n}^{t_n+c_2h}\beta(s)ds}\mathscr{Q}_2\right) - e^{-\int_{t_n}^{t_n+c_2h}\beta(s)ds}\zeta = 0. \end{aligned}$$

The formulations above readily lead to

$$\frac{\partial(F,G)}{\partial(P,Q)}\cdot\frac{\partial(P,Q)}{\partial(p,q)} + \frac{\partial(F,G)}{\partial(\mathscr{P}_2,\mathscr{Q}_2)}\cdot\frac{\partial(\mathscr{P}_2,\mathscr{Q}_2)}{\partial(p,q)} + \frac{\partial(F,G)}{\partial(\mathscr{P}_1,\mathscr{Q}_1)}\cdot\frac{\partial(\mathscr{P}_1,\mathscr{Q}_1)}{\partial(p,q)} + \frac{\partial(F,G)}{\partial(p,q)} = 0,$$

then

$$\frac{\partial(P,Q)}{\partial(p,q)}=-\frac{\partial(F,G)}{\partial(\mathscr{P}_2,\mathscr{Q}_2)}\cdot\frac{\partial(\mathscr{P}_2,\mathscr{Q}_2)}{\partial(p,q)}-\frac{\partial(F,G)}{\partial(\mathscr{P}_1,\mathscr{Q}_1)}\cdot\frac{\partial(\mathscr{P}_1,\mathscr{Q}_1)}{\partial(p,q)}-\frac{\partial(F,G)}{\partial(p,q)}=:\begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix}.$$

Denote

$$H_{pp}^{(1)}=\frac{\partial^2 H(\mathscr{P}_1,\mathscr{Q}_1)}{\partial p^2},\ H_{pq}^{(1)}=\frac{\partial^2 H(\mathscr{P}_1,\mathscr{Q}_1)}{\partial p\partial q},\ H_{qq}^{(1)}=\frac{\partial^2 H(\mathscr{P}_1,\mathscr{Q}_1)}{\partial q^2},$$
$$H_{pp}^{(2)}=\frac{\partial^2 H(\mathscr{P}_2,\mathscr{Q}_2)}{\partial p^2},\ H_{pq}^{(2)}=\frac{\partial^2 H(\mathscr{P}_2,\mathscr{Q}_2)}{\partial p\partial q},\ H_{qq}^{(2)}=\frac{\partial^2 H(\mathscr{P}_2,\mathscr{Q}_2)}{\partial q^2}.$$

Subsequently, we can obtain

$$\frac{\partial(\mathscr{P}_1,\mathscr{Q}_1)}{\partial(p,q)}=\begin{pmatrix}e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}I & 0\\ 0 & e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}I\end{pmatrix},\ \frac{\partial(F,G)}{\partial(p,q)}=\begin{pmatrix}-e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}I & 0\\ 0 & -e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}I\end{pmatrix},$$
$$\frac{\partial(\mathscr{P}_2,\mathscr{Q}_2)}{\partial(p,q)}=\begin{pmatrix}e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}(I-a_{21}H_{pq}^{(1)}h) & -e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}e^{\int_{t_n}^{t_{n+c_1h}}(\beta(s)-\alpha(s))ds}a_{21}H_{qq}^{(1)}h\\ e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds}e^{\int_{t_n}^{t_{n+c_1h}}(\alpha(s)-\beta(s))ds}a_{21}H_{pp}^{(1)}h & e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}(I+a_{21}H_{pq}^{(1)}h)\end{pmatrix},$$
$$\frac{\partial(F,G)}{\partial(\mathscr{P}_2,\mathscr{Q}_2)}=\begin{pmatrix}b_2H_{pq}^{(2)}h & e^{\int_{t_n}^{t_{n+c_2h}}(\beta(s)-\alpha(s))ds}b_2H_{qq}^{(2)}h\\ -e^{\int_{t_n}^{t_{n+c_2h}}(\alpha(s)-\beta(s))ds}b_2H_{pp}^{(2)}h & -b_2H_{pq}^{(2)}h\end{pmatrix},$$
$$\frac{\partial(F,G)}{\partial(\mathscr{P}_1,\mathscr{Q}_1)}=\begin{pmatrix}b_1H_{pq}^{(1)}h & e^{\int_{t_n}^{t_{n+c_1h}}(\beta(s)-\alpha(s))ds}b_1H_{qq}^{(1)}h\\ -e^{\int_{t_n}^{t_{n+c_1h}}(\alpha(s)-\beta(s))ds}b_1H_{pp}^{(1)}h & -b_1H_{pq}^{(1)}h\end{pmatrix},$$

with

$$\begin{aligned}
\theta_{11}=&kI-k\left(b_1H_{pq}^{(1)}+b_2H_{pq}^{(2)}\right)h+a_{21}b_2k\left(H_{pq}^{(2)}H_{pq}^{(1)}-e^{d-v}H_{qq}^{(2)}H_{qq}^{(1)}\right)h^2,\\
\theta_{12}=&-u\left(e^{v}b_1H_{qq}^{(1)}+e^{d}b_2H_{qq}^{(2)}\right)h+a_{21}b_2u\left(e^{v}H_{pq}^{(2)}H_{qq}^{(1)}-e^{d}H_{qq}^{(2)}H_{pq}^{(1)}\right)h^2,\\
\theta_{21}=&k\left(e^{-v}b_1H_{pp}^{(1)}+e^{-d}b_2H_{pp}^{(2)}\right)h-a_{21}b_2k\left(e^{-d}H_{pp}^{(2)}H_{pq}^{(1)}-e^{-v}H_{pq}^{(2)}H_{pp}^{(1)}\right)h^2,\\
\theta_{22}=&uI+u\left(b_1H_{pq}^{(1)}+b_2H_{pq}^{(2)}\right)h-ua_{21}b_2\left(H_{pp}^{(2)}H_{qq}^{(1)}-H_{pq}^{(2)}H_{pq}^{(1)}\right)h^2.
\end{aligned}$$

In a more concise notation, denote that

$$k=e^{-\int_{t_n}^{t_{n+1}}\alpha(s)ds},\ u=e^{-\int_{t_n}^{t_{n+1}}\beta(s)ds}.$$

Simultaneously, we have

$$\left(\frac{\partial(P,Q)}{\partial(p,q)}\right)^{\top}J\left(\frac{\partial(P,Q)}{\partial(p,q)}\right)=\begin{pmatrix}\theta_{11}^{\top}\theta_{21}-\theta_{21}^{\top}\theta_{11} & \theta_{11}^{\top}\theta_{22}-\theta_{21}^{\top}\theta_{12}\\ \theta_{12}^{\top}\theta_{21}-\theta_{22}^{\top}\theta_{11} & \theta_{12}^{\top}\theta_{22}-\theta_{22}^{\top}\theta_{12}\end{pmatrix}. \tag{15}$$

Moving forward, our attention turns to the estimation of $\theta_{11}^{\top}\theta_{21}-\theta_{21}^{\top}\theta_{11}$. A comparable methodology can be utilized to estimate the other elements of the aforementioned matrix.

Straightforward calculations show that

$$\begin{aligned}
&\left\|\theta_{11}^{\top}\theta_{21}-\theta_{21}^{\top}\theta_{11}\right\|_{L^2(\Omega)}\\
=&\Big\|-b_1^2\left(H_{pq}^{(1)}H_{pp}^{(1)}-H_{pp}^{(1)}H_{pq}^{(1)}\right)k^2e^{-v}-b_2^2\left(H_{pq}^{(2)}H_{pp}^{(2)}-H_{pp}^{(2)}H_{pq}^{(2)}\right)k^2e^{-d}\\
&+a_{21}b_2\left(H_{pq}^{(2)}H_{pp}^{(1)}-H_{pp}^{(1)}H_{pq}^{(2)}\right)k^2e^{-v}+a_{21}b_2\left(H_{pq}^{(1)}H_{pp}^{(2)}-H_{pp}^{(2)}H_{pq}^{(1)}\right)k^2e^{-d}\\
&-b_1b_2\left(H_{pq}^{(2)}H_{pp}^{(1)}-H_{pp}^{(1)}H_{pq}^{(2)}\right)k^2e^{-v}-b_1b_2\left(H_{pq}^{(1)}H_{pp}^{(2)}-H_{pp}^{(2)}H_{pq}^{(1)}\right)k^2e^{-d}\Big\|_{L^2(\Omega)}h^2+O\left(h^{5/2}\right).
\end{aligned}$$

Expanding $H_{pp}^{(2)}$ and $H_{pq}^{(2)}$ at $(\mathscr{P}_1, \mathscr{Q}_1)$, we have

$$\begin{aligned}
H_{pp}^{(2)} &= \frac{\partial^2 H(\mathscr{P}_2, \mathscr{Q}_2)}{\partial p^2} \\
&= \frac{\partial^2 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p^2} + \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p^3} \otimes (\mathscr{P}_2 - \mathscr{P}_1) + \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p^2 \partial q} \otimes (\mathscr{Q}_2 - \mathscr{Q}_1) + C_3 h \\
&:= H_{pp}^{(1)} + H_{ppp}^{(1)} \otimes (\mathscr{P}_2 - \mathscr{P}_1) + H_{ppq}^{(1)} \otimes (\mathscr{Q}_2 - \mathscr{Q}_1) + O(h), \\
H_{pq}^{(2)} &= \frac{\partial^2 H(\mathscr{P}_2, \mathscr{Q}_2)}{\partial p \partial q} \\
&= \frac{\partial^2 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p \partial q} + \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p^2 \partial q} \otimes (\mathscr{P}_2 - \mathscr{P}_1) + \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p \partial q^2} \otimes (\mathscr{Q}_2 - \mathscr{Q}_1) + C_3 h \\
&:= H_{pp}^{(1)} + H_{ppq}^{(1)} \otimes (\mathscr{P}_2 - \mathscr{P}_1) + H_{pqq}^{(1)} \otimes (\mathscr{Q}_2 - \mathscr{Q}_1) + O(h),
\end{aligned}$$

where $\otimes$ means tensor product, $H_{ppp}^{(1)} = \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p^3}, H_{ppq}^{(1)} = \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p^2 \partial q}, H_{pqq}^{(1)} = \frac{\partial^3 H(\mathscr{P}_1, \mathscr{Q}_1)}{\partial p \partial q^2}$.

In what follows we only estimate $\Sigma_{11}^\top \theta_{21} - \theta_{21}^\top \theta_{11}$, while an analogous idea can be applied to estimate the terms $\theta_{11}^\top \theta_{22} - \theta_{21}^\top \theta_{12}$, $\theta_{12}^\top \theta_{21} - \theta_{22}^\top \theta_{11}$ and $\theta_{12}^\top \theta_{22} - \theta_{22}^\top \theta_{12}$. Direct calculations yield that

$$\begin{aligned}
&\left\| \theta_{11}^\top \theta_{21} - \theta_{21}^\top \theta_{11} \right\|_{L^2(\Omega)} \\
=& k^2 |e^{-v} b_1^2 + e^{-d} b_2^2 - \left(e^{-d} + e^{-v}\right) a_{21} b_2 + \left(e^{-d} + e^{-v}\right) b_1 b_2| \cdot \left\| H_{pq}^{(1)} H_{pp}^{(1)} - H_{pp}^{(1)} H_{pq}^{(1)} \right\|_{L^2(\Omega)} h^2 \\
&+ k^2 e^{-d} |b_2^2 - a_{21} b_2 + b_1 b_2| \cdot \left\| \mathbb{M} - \mathbb{M}^\top \right\|_{L^2(\Omega)} h^{5/2} \\
&+ k^2 |e^{-d} b_2^2 + e^{-v} b_1 b_2 - e^{-v} a_{21} b_2| \cdot \left\| \mathbb{M}' - \mathbb{M}'^\top \right\|_{L^2(\Omega)} h^{5/2} + O(h^3),
\end{aligned}$$

with matrices

$$\begin{aligned}
\mathbb{M} &= H_{pp}^{(1)} H_{ppp}^{(1)} \otimes (\mathscr{P}_2 - \mathscr{P}_1) + H_{pq}^{(1)} H_{ppq}^{(1)} \otimes (\mathscr{Q}_2 - \mathscr{Q}_1), \\
\mathbb{M}' &= H_{ppq}^{(1)} \otimes (\mathscr{P}_2 - \mathscr{P}_1) H_{pp}^{(1)} + H_{pqq}^{(1)} \otimes (\mathscr{Q}_2 - \mathscr{Q}_1) H_{pp}^{(1)},
\end{aligned}$$

which are typically not symmetric. Then the use of Taylor expansion gets

$$\begin{aligned}
e^{-v} &= e^{\int_{t_n}^{t_n + c_1 h} \alpha(s) ds - \int_{t_n}^{t_n + c_1 h} \beta(s) ds} \\
&= 1 - c_1 \left(\alpha(t_n) - \beta(t_n)\right) h + O(h^2), \\
e^{-d} &= e^{\int_{t_n}^{t_n + c_2 h} \alpha(s) ds - \int_{t_n}^{t_n + c_2 h} \beta(s) ds} \\
&= 1 - c_2 \left(\alpha(t_n) - \beta(t_n)\right) h + O(h^2).
\end{aligned}$$

Considering that our analysis can only attain a precision up to $h^{\frac{5}{2}}$ order, we will retain solely the first term in the Taylor expansion. Previous analysis has shown that the two-stage pseudo-conformal symplectic Runge-Kutta method (10) achieves a pseudo-conformal symplectic order of $(\frac{3}{2}, 2)$ if condition (14) is met. Moreover, the fulfillment of (14) is achieved when $b_2 = \frac{1}{2}$, $b_1 = \frac{1}{2}$, and $a_{21} = 1$. □

For general s-stage pseudo-conformal symplectic Runge-Kutta method (9) with

$$\begin{aligned}
\varphi_i &= \sum_{r=1}^{m} \sigma_r \left(\lambda_i J_{r0} + \mu_i \delta_n W_r\right), \\
\psi_i &= \sum_{r=1}^{m} \gamma_r \left(\lambda_i J_{r0} + \mu_i \delta_n W_r\right), \; i = 2, \cdots, s,
\end{aligned} \tag{16}$$

it is convergent of mean-square order $\frac{3}{2}$ when the parameters satisfy [14]

$$\begin{aligned} &\sum_{i=1}^{s} b_i = 1, \quad \sum_{i=1}^{s}\sum_{j=1}^{i-1} b_i a_{ij} = \frac{1}{2}, \\ &\sum_{i=1}^{s} b_i \lambda_i = 1, \quad \sum_{i=1}^{s} b_i \mu_i = 0, \\ &\sum_{i=1}^{s} b_i \left(\lambda_i J_{r0} + \mu_i \delta_n W_r\right)\left(\lambda_i J_{r0} + \mu_i \delta_n W_l\right) = \frac{h}{2}\delta_{rl}. \end{aligned} \tag{17}$$

It can be proven analogously that if an s-stage method complies with conditions (17) and (18) along with certain A_j, B_j, C_j coefficients related to damping terms,

$$\sum_{j=1}^{k-1} e^{A_j} b_k a_{kj} + \sum_{j\geq 1}^{s} e^{B_j} b_k a_{jk} - b_k \left(\sum_{i=1}^{s} e^{C_j} b_i\right) = 0,\ k = 2, ..., s, \tag{18}$$

then such a method exhibits pseudo-conformal symplectic order $(\frac{3}{2}, 2)$.

Importantly, for $s = 2$, the method achieves order $(\frac{3}{2}, 2)$ pseudo-conformal symplectic only when the stipulations within Theorem2.2 are satisfied; i.e. the condition $b_2 = \frac{1}{2}$, $b_1 = \frac{1}{2}$, and $a_{21} = 1$ is fulfilled.

For $s > 2$, there are many admissible parameters that enable this method to achieve order $(\frac{3}{2}, 2)$. By making appropriate choices of parameters, one can construct higher-order explicit pseudo-conformal symplectic methods based on the explicit exponential Runge-Kutta method (9).

3. Numerical experiments

In this section, we present numerical experiments to simulate the stochastic damped Hamiltonian system (1). We demonstrate the effectiveness of our approach by considering both linear oscillator equation and a stochastic nonlinear Schrödinger equation.

3.1. *Damped oscillator with additive noise*

Consider the 2−dimensional damped oscillator with additive noise as follow

$$d\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} dt - \begin{pmatrix} 2\gamma p \\ 0 \end{pmatrix} dt + \begin{pmatrix} -\sigma \\ \sigma \end{pmatrix} dW(t), \tag{19}$$

with initial condition $p(0) = p_0$, $q(0) = q_0$.

As we know, a differentiable and non-constant scalar function $\mathcal{I}(p, q) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is called a (stochastic) comformal invariant of (19) if

$$\mathcal{I}(p(t), q(t)) e^{\int_0^t (\alpha(s)+\beta(s))ds} = \mathcal{I}(p_0, q_0), \text{ a.s.},$$

where $\alpha(t), \beta(t) : \mathbb{R} \to \mathbb{R}$ [22].

For the quadratic function

$$\mathcal{I}(p, q) = \frac{1}{2}(p^2 + q^2) + \gamma pq,$$

we can check that when $\gamma = 1$,

$$
\begin{aligned}
d(\mathcal{I}e^{2t}) =& d((\frac{1}{2}p^2 + \frac{1}{2}q^2 + pq)e^{2t}) \\
=& (de^{2t})(\frac{1}{2}p^2 + \frac{1}{2}q^2 + pq) + d(\frac{1}{2}p^2 + \frac{1}{2}q^2 + pq)e^{2t} \\
=& 2e^{2t}(\frac{1}{2}p^2 + \frac{1}{2}q^2 + pq)dt + (pdp + qdq + pdq + qdp)e^{2t} \\
=& e^{2t}(\frac{1}{2}p^2 + \frac{1}{2}q^2 + pq)dt + (-pqdt - 2p^2dt - p\sigma dW(t))e^{2t} \\
&+ (pqdt + q\sigma dW(t))e^{2t} + (p^2dt + p\sigma dW(t))e^{2t} \\
&+ (-q^2dt - 2pqdt - q\sigma dW(t))e^{2t} \\
=& 0,
\end{aligned}
$$

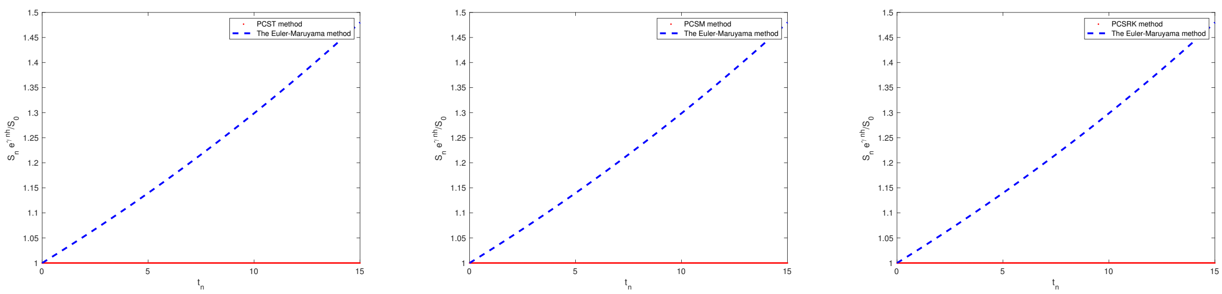
i.e.

$$\mathcal{I}(p, q) = e^{-2t}\mathcal{I}(p_0, q_0),$$

which means that $\mathcal{I}(p, q) = \frac{1}{2}(p^2 + q^2) + pq$ is a comformal quadratic invariant of (19).

To validate that Methods (3), (4) and (10) can effectively preserve the conformal symplectic structure, we choose a step size with $h = 2^{-5}$, set $\sigma = 0.5$, and $\gamma = 0.3$, with initial conditions $p_0 = 0.5$, $q_0 = 0$. We observe the variable $S_n e^{2\gamma t_n}/S$ which results in Fig.1, observes the value $S_n e^{2\gamma t_n}/S$ for three integrator methods in this article and the Euler–Maruyama method, where S_n represents the area of the triangle at time t_n. Conformal symplecticity dictates that this value should remain at 1 along the exact flow. Simulations are conducted over the time interval $[0, 15]$, revealing that these integrator methods in this artical successfully preserves conformal symplecticity, while the Euler–Maruyama method does not. Fig.2 illustrates the trend in triangle area under damping.

Subsequently, to examine whether these methods maintain the conformal quadratic invariant, we set $\gamma = 1$ while keeping the remaining parameters unchanged. In Fig.3, we present the errors $\eta_n = |\mathcal{I}_n e^{2t_n} - \mathcal{I}_0|$ in the conformal quadratic invariant $\mathcal{I}$, which are generated by both the integrator we constructed within this article and the Euler–Maruyama method [13] over the interval $[0, 15]$.



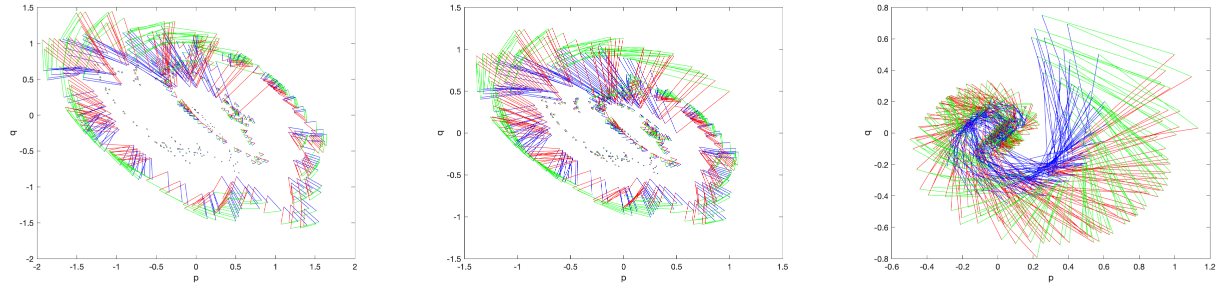
(a) PCST (b) PCSM (c) PCSRK

Figure 1: The value $S_n e^{2\gamma t_n}/S$ of the stochastic exponential integrator and the Euler-Maruyama method for solving (19) with $\gamma = 0.3$

3.2. *Damped stochastic nonlinear Schrödinger equation*

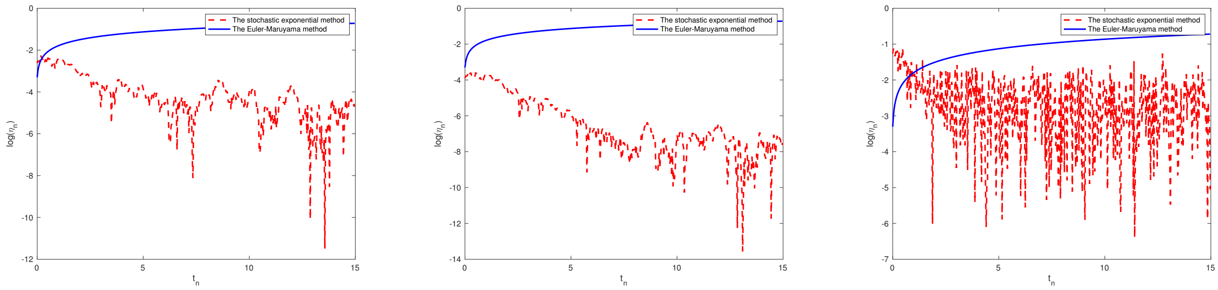
In this part, we consider the following damped stochastic nonlinear Schrödinger equation with an additive noise

$$du + (\gamma u - iu_{xx} - 2i|u|^2u)dt = \epsilon Q dW, \; x \in [-30, 30], \; t \geq 0, \tag{20}$$



(a) PCST (b) PCSM (c) PCSRK

Figure 2: The numbercial triangles produced by the stochastic exponential integrator for solving (19) with $\gamma = 0.3$



(a) PCST (b) PCSM (c) PCSRK

Figure 3: Error of the conformal quadratic invariant $\mathcal{I}$ of (19) with $\gamma = 1$

with initial condition $u(x, 0) = u_0(x)$ and appropriate boundary conditions, where $u(x,t) = p(x,t) + iq(x,t)$ is a complex-valued function and W is a complex-valued Wiener process defined on a filtered probability $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We consider the equivalent form of equation (20), as derived according to [3]. Let $\{e_k : k \in N_+\}$ be an orthonormal basis of $L^2([-30, 30]; \mathbb{R})$. Then there exist a sequence of independent $\mathcal{F}_t$-Brownian motions $\{\beta_k : k \in N_+\}$ such that

$$W(t, x, \omega) = \sum_{k=1}^{\infty} \phi e_k(x)\beta_k(t, \omega),\ \omega \in \Omega, \tag{21}$$

where $\phi \in \mathcal{L}_2(\mathbb{L}^2, \mathbb{H}^2)$ is called the Hilbert-Schmidt operator. Let $\{\eta_k : k \in N_+\}$ denote the eigenvalues of the operator Q on the orthonormal basis . Then $\sum_{k\in N_+} \eta_k < \infty$ and the equivalent form of (21) is

$$W(t, x, \omega) = \sum_{k=1}^{\infty} \sqrt{\eta_k} e_k(x)\beta_k(t, \omega). \tag{22}$$

Notice that h is the uniform spatial step and $u_j := u_j(t)$ denotes $u(x_j, t)$ with $x_j = jh$, $j = 0, 1, \cdots, J + 1$. Inserting (22) into (20), one can obtain the equivalent form of (20),

$$du + (\gamma u - iu_{xx} - 2i|u|^2 u)dt = \epsilon \sum_{k=1}^{\infty} \sqrt{\eta_k} e_k(x) d\beta_k(t, \omega). \tag{23}$$

We truncate the noise with the first P terms [9], and utilize central finite difference scheme to perform space semi-discretization on (23), i.e.$u_{xx} = \frac{u_{j+1}-2u_j+u_{j-1}}{h^2}$ can get

$$du_j + (\gamma u_j - i\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} - 2i|u_j|^2 u_j)dt = \epsilon \sum_{k=1}^{P} \sqrt{\eta_k} e_k(x_j) d\beta_k(t). \tag{24}$$

We rewrite some parts of (24) get

$$dU + (\gamma U - i\frac{A}{h^2}U - 2i|U|^2 U)dt = \epsilon\sigma\Lambda d\beta, \tag{25}$$

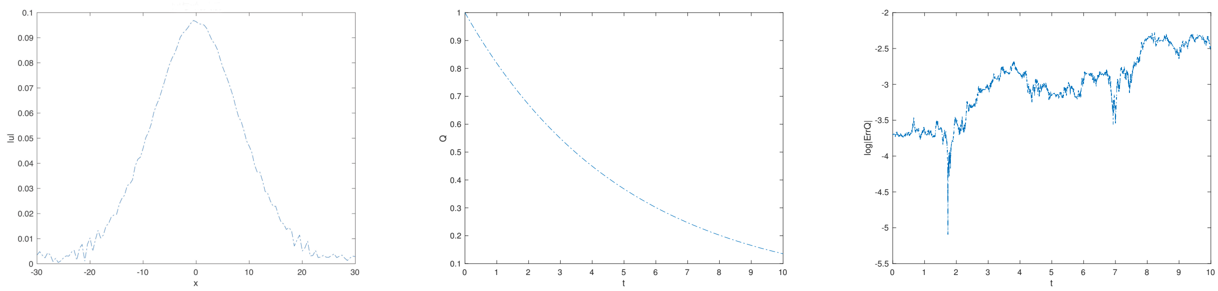
where $U = (u_1, \cdots, u_J)^\top \in \mathbb{C}^J$, $|U|^2 = diag\{|u_1|^2, \cdots, |u_J|^2\}$, $\beta = (\beta_1, \cdots, \beta_P)^\top \in \mathbb{C}^P$, $\Lambda = diag\{\sqrt{\eta_1}, \cdots, \sqrt{\eta_P}\}$,

$$A = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{bmatrix} \quad and \quad \sigma = \begin{bmatrix} e_1(x_1) & \cdots & e_P(x_1) \\ \vdots & & \vdots \\ e_1(x_J) & \cdots & e_P(x_J) \end{bmatrix}.$$

The PCSM method is used in the time domain to derive a fully discrete scheme.

Equation (20) exhibits a global charge conservation property and symmetry. In the forthcoming numerical experiments, we will demonstrate that the fully discretized system also possesses discrete global charge conservation and retains its symmetry.

In the sequel, taking $\gamma = 0.1$, $\epsilon = 0.1$, $e_k(x) = \sin(k\pi x)$, $h = 0.5$, $\tau = 0.001$ and to verify the correctness of the numerical method, in practical applications truncating the infinite series of Wiener process till $P = 100$[9]. Fig.4(a) shows the relationship of x and $|u(x,t)|$ when $t = 10$, i.e.the waveforms of method and the symmetry of the solution with respect to space.



(a) (b) (c)

Figure 4: (a) shows the relationship of x and $|u|$, (b) shows the relationship of t and Q, (c) shows error of the global charge of (20).

Meanwhile, Q means global charge of (20), where $Q(p,q) = \int_{-30}^{30}(p^2+q^2)dx$ [20]. And (20) exhibits charge conservation law, and the demonstration is provided below

$$Q(p(t), q(t)) = e^{-2\theta(t)} Q(p_0, q_0), \theta(t) = \int_0^t \gamma(s)ds.$$

Fig.4(b) shows the relationship of t and Q, which illustrates the charge evolution for our numerical methods. Means that the Q is decaying.

In Fig.4(c), the relationship between $log|ErrQ|$ and time t is depicted, with $ErrQ$ being defined as $ErrQ = |e^{2\theta(t)}Q(p(t), q(t)) - Q(p_0, q_0)|$, and $t \in [0, 10]$, indicating that the method maintains the discrete charge conservation of (20) over long time intervals.

4. Conclusion

This paper presents a class of stochastic pseudo-conformal symplectic methods suitable for stochastic damped Hamiltonian systems with additive multi-dimensional Wiener processes and linear damping. Compared to existing methods, these proposed stochastic pseudo-conformal symplectic methods offer a significant advantage of reduced computational cost while maintaining the symplectic structure of the system with a certain level of accuracy over relatively long time periods. The pseudo-conformal symplectic orders of the methods are analyzed. The theoretical findings are validated through applications to linear oscillators and spatially discretized stochastic nonlinear Schrödinger equations. For future research, a more general class of stochastic pseudo-conformal symplectic schemes with higher orders, applicable to partial differential equations, will be constructed.

References

[1] A. Aubry and P. Chartier, *Pseudo-symplectic Runge–Kutta methods*, BIT **38** (1998), 439–461.
[2] A. Bhatt and B. E. Moore, *Structure-preserving exponential Runge–Kutta methods*, SIAM J. Sci. Comput. **39**(2) (2017), A593–A612.
[3] C. Chen, J. Hong and L. Ji, *Stochastic conformal multi-symplectic method for damped stochastic nonlinear Schrödinger equation*, arXiv preprint arXiv:1803.10885 (2018).
[4] C. A. Envelope, *Explicit pseudo-symplectic Runge–Kutta methods for stochastic Hamiltonian systems*, Appl. Numer. Math. **185** (2023), 18–37.
[5] K. Debrabant, A. Kværnø and N. C. Mattsson, *Runge–Kutta Lawson schemes for stochastic differential equations*, BIT **61** (2021), 381–409.
[6] K. Feng and M. Qin, *Symplectic geometric algorithms for Hamiltonian systems*, Springer, Berlin, 2010.
[7] E. Hairer, C. Lubich and G. Wanner, *Geometric numerical integration*, Oberwolfach Rep. **3**(1) (2006), 805–882.
[8] J. Hong, D. Xu and P. Wang, *Preservation of quadratic invariants of stochastic differential equations via Runge–Kutta methods*, Appl. Numer. Math. **87** (2015), 38–52.
[9] J. Hong, L. Ji and X. Wang, *Convergence in probability of an ergodic and conformal multi-symplectic numerical scheme for a damped stochastic NLS equation*, arXiv preprint arXiv:1611.08778 (2016).
[10] X. Li, C. Zhang, Q. Ma and X. Ding, *Discrete gradient methods and linear projection methods for preserving a conserved quantity of stochastic differential equations*, Int. J. Comput. Math. **95**(12) (2018), 2511–2524.
[11] Q. Ma, D. Ding and X. Ding, *Symplectic conditions and stochastic generating functions of stochastic Runge–Kutta methods for stochastic Hamiltonian systems with multiplicative noise*, Appl. Math. Comput. **219**(2) (2012), 635–643.
[12] R. I. McLachlan and M. Perlmutter, *Conformal Hamiltonian systems*, J. Geom. Phys. **39**(4) (2001), 276–300.
[13] G. N. Milstein, *Numerical integration of stochastic differential equations*, Springer Science & Business Media, 2013.
[14] G. N. Milstein and M. V. Tretyakov, *Stochastic numerics for mathematical physics*, Springer, Berlin, 2004.
[15] G. N. Milstein, Y. M. Repin and M. V. Tretyakov, *Symplectic integration of Hamiltonian systems with additive noise*, SIAM J. Numer. Anal. **39**(6) (2002), 2066–2088.
[16] G. N. Milstein, Y. M. Repin and M. V. Tretyakov, *Numerical methods for stochastic systems preserving symplectic structure*, SIAM J. Numer. Anal. **40**(4) (2002), 1583–1604.
[17] T. Misawa, *Energy conservative stochastic difference scheme for stochastic Hamilton dynamical systems*, Japan J. Indust. Appl. Math. **17** (2000), 119–128.
[18] B. E. Moore, *Conformal multi-symplectic integration methods for forced-damped semi-linear wave equations*, Math. Comput. Simulation **80**(1) (2009), 20–28.
[19] X. Niu, J. Cui, J. Hong and Z. Liu, *Explicit pseudo-symplectic methods for stochastic Hamiltonian systems*, BIT **58** (2018), 163–178.
[20] M. Song, S. Song, W. Zhang, J. You and X. Qian, *Conformal structure-preserving schemes for damped-driven stochastic nonlinear Schrödinger equation with multiplicative noise*, Numer. Methods Partial Differential Equations **39**(2) (2023), 1706–1728.
[21] B. Wang, H. Yang and F. Meng, *Sixth-order symplectic and symmetric explicit ERKN schemes for solving multi-frequency oscillatory nonlinear Hamiltonian equation*, Calcolo **54** (2016), 1–24.
[22] G. Yang, Q. Ma and X. Li, *Structure-preserving stochastic conformal exponential integrator for linearly damped stochastic differential equations*, Calcolo **56** (2019), 1–20.
[23] W. Zhou, J. Zhang and J. Hong, *Stochastic symplectic Runge–Kutta methods for the strong approximation of Hamiltonian systems with additive noise*, J. Comput. Appl. Math. **325** (2017), 134–148.