Filomat 40:2 (2026), 693–702
https://doi.org/10.2298/FIL2602693C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Double domination in some graph operators

Abel Cabrera-Martínez[a,*], Ismael Rios-Villamar[b], José M. Sigarreta[b]

[a]*Universidad de Córdoba, Departamento de Matemáticas, Campus de Rabanales, 14071, Córdoba, Spain*
[b]*Universidad Autónoma de Guerrero, Facultad de Matemáticas, Carlos E. Adame 54, Col. La Garita 39650, Acapulco, Mexico*

Abstract. Let G be a nontrivial graph. A set $D \subseteq V(G)$ is a double dominating set of G if $|N_G[v] \cap D| \geq 2$ for every vertex $v \in V(G)$, where $N_G[v]$ represents the closed neighborhood of v. The double domination number of G is the minimum cardinality among all double dominating sets of G. In this paper we study this domination parameter in some well-known graph operators defined from a connected graph G.

1. Introduction

In this paper, we consider finite, simple and nontrivial connected graphs G with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$. Given a vertex v_i of G, $N_G(v_i)$ and $N_G[v_i]$ represent the *open neighborhood* and the *closed neighborhood* of v_i, respectively. A set $D \subseteq V(G)$ is a 2-packing of G if $N_G[v_i] \cap N_G[v_j] = \emptyset$ for every different vertices $v_i, v_j \in D$. For any $i \in [n-1]$, let $V_i(G) = \{v_j \in V(G) : |N_G(v_j)| = i\}$. By $\mathcal{S}(G) = \{v_i \in V(G) : N_G(v_i) \cap V_1(G) \neq \emptyset\}$ and $\mathcal{S}_s(G) = \{v_i \in \mathcal{S}(G) : |N_G(v_i) \cap V_1(G)| \geq 2\}$ we denote the sets of *support vertices* and *strong support vertices* of G, respectively. In addition, let us consider the set $V_E(G) = \{v^{i,j} : v_i v_j \in E(G)\}$ (observe that $v^{i,j} = v^{j,i}$), which will play an important role in the definitions of graph operators.

Domination theory in graphs is one of the most active and popular research areas within graph theory. The most studied domination variant in graphs is the total domination, which was introduced in 1980 by Cockayne, Dawes and Hedetniemi [14]. Given a nontrivial connected graph G, a set $D \subseteq V(G)$ is a *total dominating set* of G if $N_G(v_i) \cap D \neq \emptyset$ for every vertex $v_i \in V(G)$. Two well-known variants related to total domination in graphs are the double domination and the total Italian domination, which were introduced in [16] and [6], respectively. Given a nontrivial connected graph G,

- a set $D \subseteq V(G)$ is a *double dominating set* of G if $|N_G[v_i] \cap D| \geq 2$ for every vertex $v_i \in V(G)$. The *double domination number* of G, denoted by $\gamma_{\times 2}(G)$, is the minimum cardinality among all double dominating sets of G.

2020 *Mathematics Subject Classification*. Primary 05C69; Secondary 05C76.
Keywords. Double domination; Total Italian domination; Graph operators.
Received: 10 December 2024; Revised: 16 November 2025; Accepted: 17 November 2025
Communicated by Paola Bonacini
* Corresponding author: Abel Cabrera-Martínez
Email addresses: acmartinez@uco.es (Abel Cabrera-Martínez), 18305783@uagro.mx (Ismael Rios-Villamar), josemariasigarretaalmira@hotmail.com (José M. Sigarreta)
ORCID iDs: https://orcid.org/0000-0003-2806-4842 (Abel Cabrera-Martínez), https://orcid.org/0009-0007-5190-7058 (Ismael Rios-Villamar), https://orcid.org/0000-0003-4352-5109 (José M. Sigarreta)

- a function $f : V(G) \to \{0, 1, 2\}$ is a *total Italian dominating function* (TIDF) on G if $\{v_i \in V(G) : f(v_i) \geq 1\}$ is a total dominating set of G, and every vertex $v_i \in V(G)$ for which $f(v_i) = 0$ satisfies that $\sum_{v_j \in N(v_i)} f(v_j) \geq 2$. The *total Italian domination number* of G, denoted by $\gamma_{tI}(G)$, is the minimum weight $\omega(f) = \sum_{v_i \in V(G)} f(v_i)$ among all TIDF f on G. Observe that the function f generates three sets W_0, W_1 and W_2, where $W_i = \{v_j \in V(G) : f(v_j) = i\}$ for $i \in \{0, 1, 2\}$. In such a sense, we write $f(W_0, W_1, W_2)$ so as to refer to the TIDF f.

In addition, these two domination parameters are related. Observe that a set $D \subseteq V(G)$ is a double dominating set of G if and only if there exists a TIDF $f(W_0, W_1, W_2)$ such that $W_1 = D$ and $W_2 = \emptyset$. By this previous equivalence and by the definitions, it follows that $\gamma_t(G) \leq \gamma_{tI}(G) \leq \gamma_{\times 2}(G)$. Recent studies on the double domination and total Italian domination in graphs are for instance [1, 7, 9–11, 17] and [2, 3, 8, 12, 29], respectively.

In this paper we study the double domination number of the following well-known graph operators defined from a nontrivial connected graph G: the central graph $\mathtt{C}(G)$, the graph operator $\mathtt{R}(G)$, the middle graph $\mathtt{M}(G)$ and the Mycielskian graph $\mu(G)$. In Section 2 we obtain the exact values for the double domination numbers of the graph operators $\mathtt{R}(G)$ and $\mathtt{C}(G)$. In Section 3 we first prove that the double domination number and the total Italian domination number coincide for middle graphs. Subsequently we obtain a closed formula and tight bounds for $\gamma_{\times 2}(\mathtt{M}(G))$ in terms of some invariants of G. Finally, Section 4 deals with the case of Mycielskian graph $\mu(G)$, where, as a main result, we obtain a closed formula for the double domination number of $\mu(G)$ in terms of the total Italian domination number of graph G.

2. Double domination in $\mathtt{R}(G)$ and $\mathtt{C}(G)$

We begin this section by defining the graph operators $\mathtt{R}(G)$ and $\mathtt{C}(G)$.

Definition 2.1. *Given a nontrivial connected graph G,*

- *the operator $\mathtt{R}(G)$ is the graph obtained from G by subdividing each edge exactly once and adding the edges v_iv_j whenever $v_iv_j \in E(G)$. Formally, $V(\mathtt{R}(G)) = V(G) \cup V_E(G)$ and $E(\mathtt{R}(G)) = E(G) \cup \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in V_E(G)\}$,*
- *the central graph $\mathtt{C}(G)$ is the graph obtained from G by subdividing each edge exactly once and adding the edges v_iv_j whenever $v_iv_j \notin E(G)$. Formally, $V(\mathtt{C}(G)) = V(G) \cup V_E(G)$ and $E(\mathtt{C}(G)) = E(\overline{G}) \cup \{v_iv^{i,j}, v_jv^{i,j} : v^{i,j} \in V_E(G)\}$, where $\overline{G}$ is the complement graph of G.*

As you can see, both graph operators have the same set of vertices. However, the main difference is that one keeps the edges of G while the other does not keep them and also adds the edges of $\overline{G}$. In Figure 1 we show a graph G and its corresponding graphs $\mathtt{R}(G)$ and $\mathtt{C}(G)$.

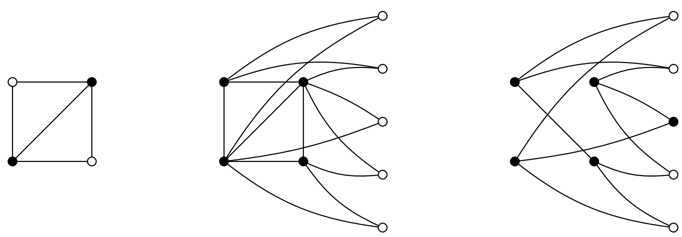

Figure 1: From left to right, a graph G and its corresponding graphs $\mathtt{R}(G)$ and $\mathtt{C}(G)$, respectively. In each case, the set of black vertices describes a double dominating set of minimum cardinality.

Several studies have explored the behavior of domination parameters under these graph operators, including, for example, [4, 5, 18, 26, 28]. Following this line of research, we now aim to determine the exact value of the double domination number for these operators. The next theorem addresses the case of the graph operator $\mathtt{R}(G)$.

Theorem 2.2. *For any nontrivial connected graph G of order n,*

$$\gamma_{\times 2}(\mathtt{R}(G)) = n.$$

Proof. Let D be a $\gamma_{\times 2}(\mathtt{R}(G))$-set such that $|D \cap V(G)|$ is maximum among all $\gamma_{\times 2}(\mathtt{R}(G))$-sets. Suppose that there exists a vertex $v^{i,j} \in D$. Since $|N_{\mathtt{R}(G)}(v^{i,j}) \cap D| \geq 1$, it follows that $D \cap \{v_i, v_j\} \neq \emptyset$. Observe that $D' = (D \setminus \{v^{i,j}\}) \cup \{v_i, v_j\}$ is a $\gamma_{\times 2}(\mathtt{R}(G))$-set such that $|D' \cap V(G)| > |D \cap V(G)|$, which is a contradiction. Hence $D \cap V_E(G) = \emptyset$, which leads to $D \subseteq V(G)$. Now, if there exists a vertex $v_i \in V(G) \setminus D$, then for any vertex $v_j \in N_G(v_i)$ we have that $v^{i,j} \notin D$ and $|N_{\mathtt{R}(G)}(v^{i,j}) \cap D| \leq 1$, a contradiction. Therefore $D = V(G)$, which implies that $\gamma_{\times 2}(\mathtt{R}(G)) = |D| = |V(G)| = n$. □

Next, we obtain the exact value of the double domination number of graph operator $\mathtt{C}(G)$. First, we need to define two families of graphs and state two useful lemmas.

Definition 2.3. *For any integer $k \geq 4$, let $\mathcal{F}_k$ and $\mathcal{G}_k$ be the families of connected graphs G of order k defined as follows.*

- *$G \in \mathcal{F}_k$ whenever $V_{k-1}(G) = \emptyset$.*
- *$G \in \mathcal{G}_k$ whenever $|V_{k-1}(G)| = 1$, $V_1(G) \neq \emptyset$ and $V_{k-2}(G) = \emptyset$.*

Lemma 2.4. *Let G be a connected graph of order $n \geq 4$,*

$$\gamma_{\times 2}(\mathtt{C}(G)) \leq \begin{cases} n & \text{if } G \in \mathcal{F}_n \cup \mathcal{G}_n, \\ n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. If $G \in \mathcal{F}_n$, then it is easy to check that $V(G)$ is a double dominating set of $\mathtt{C}(G)$, which implies that $\gamma_{\times 2}(\mathtt{C}(G)) \leq |V(G)| = n$, as desired. Now, assume that $G \in \mathcal{G}_n$ and let $V_{n-1}(G) = \{v_1\}$ and $v_2 \in V_1(G)$. Observe that $(V(G) \setminus \{v_2\}) \cup \{v^{1,2}\}$ is a double dominating set of $\mathtt{C}(G)$ because $V_{n-2}(G) = \emptyset$. Hence, $\gamma_{\times 2}(\mathtt{C}(G)) \leq n$, as desired. From now on, we assume that $G \notin \mathcal{F}_n \cup \mathcal{G}_n$. Let $W \subseteq V_E(G)$ be a set of minimum cardinality such that $N_{\mathtt{C}(G)}(v_i) \cap W \neq \emptyset$ for every $v_i \in V_{n-1}(G)$. Since the subgraph induced by $V_{n-1}(G)$ is isomorphic to the complete graph of order $|V_{n-1}(G)|$ and by the minimality of $|W|$, it follows that $|W| \leq \lceil |V_{n-1}(G)|/2 \rceil$. Now, it is straightforward that $V(G) \cup W$ is a double dominating set of $\mathtt{C}(G)$. Therefore, $\gamma_{\times 2}(\mathtt{C}(G)) \leq |V(G) \cup W| \leq n + \lceil |V_{n-1}(G)|/2 \rceil$, which completes the proof. □

Lemma 2.5. *Let D be a $\gamma_{\times 2}(\mathtt{C}(G))$-set such that $|D \cap V(G)|$ is maximum among all $\gamma_{\times 2}(\mathtt{C}(G))$-sets. Then the following statements hold.*

(i) *$N_{\mathtt{C}(G)}(v_i) \cap V_E(G) \subseteq D$ and $N_G(v_i) \subseteq D$ for every vertex $v_i \in V(G) \setminus D$.*

(ii) *$V(G) \subseteq D$ or $V(G) \setminus D$ is a 2-packing of G.*

Proof. First, we proceed to prove (i). Let $v_i \in V(G) \setminus D$. Every vertex $v_j \in N_G(v_i)$ satisfies that $N_{\mathtt{C}(G)}(v^{i,j}) = \{v_i, v_j\}$, which implies that $v_j, v^{i,j} \in D$. Hence, $N_{\mathtt{C}(G)}(v_i) \cap V_E(G) \subseteq D$ and $N_G(v_i) \subseteq D$, as desired. Finally, we proceed to prove (ii). If $V(G) \subseteq D$, then we are done. Now, let us consider that $V(G) \setminus D \neq \emptyset$. If $|V(G) \setminus D| = 1$, then $V(G) \setminus D$ is a 2-packing of G, as desired. Now, we assume that $|V(G) \setminus D| \geq 2$. Let v_i and v_j be any two different vertices in $V(G) \setminus D$. If $v_i v_j \in E(G)$, then $|N_{\mathtt{C}(G)}[v^{i,j}] \cap D| \leq 1$, a contradiction. If there exists $v_k \in N_G(v_i) \cap N_G(v_j)$, then by statement (i) we have that $v_k, v^{i,k}, v^{j,k} \in D$, which leads to $D' = (D \setminus \{v^{i,k}\}) \cup \{v_i\}$ is a $\gamma_{\times 2}(\mathtt{C}(G))$-set such that $|D' \cap V(G)| > |D \cap V(G)|$, a contradiction too. As a consequence, it follows that $N_G(v_i) \cap N_G(v_j) = \emptyset$. Therefore, $V(G) \setminus D$ is a 2-packing of G, as desired. □

We conclude this section by providing the exact value for the double domination number of central graph $\mathtt{C}(G)$.

Theorem 2.6. *Let G be a connected graph of order $n \geq 4$,*

$$\gamma_{\times 2}(\mathtt{C}(G)) = \begin{cases} n & \text{if } G \in \mathcal{F}_n \cup \mathcal{G}_n, \\ n + \left\lceil \frac{|V_{n-1}(G)|}{2} \right\rceil & \text{otherwise.} \end{cases}$$

Proof. Let D be a $\gamma_{\times 2}(\mathtt{C}(G))$-set such that $|D \cap V(G)|$ is maximum among all $\gamma_{\times 2}(\mathtt{C}(G))$-sets. Next, let us consider the following three complementary cases.

Case 1: $G \in \mathcal{F}_n$. By Lemma 2.4 we only need to prove that $\gamma_{\times 2}(\mathtt{C}(G)) \geq n$. Suppose that $|V(G) \setminus D| = r > 0$. Without loss of generality, we assume that $V(G) \setminus D = \{v_1, \ldots, v_r\}$ and $v_{i+r} \in N_G(v_i)$ for every $i \in [r]$. By Lemma 2.5 (i) and (ii) we have that $R = \{v_{r+1}, \ldots, v_{2r}\} \subseteq V(G) \cap D$ and $|R| = r$, respectively. Moreover, by Lemma 2.5 (i) we have that $Q = \cup_{i \in [r]} \{v^{i,i+r}\} \subseteq D$. Let $D' = (D \setminus Q) \cup (V(G) \setminus D)$. As $V_{n-1}(G) = \emptyset$ and $V(G) \subseteq D'$, it is easy to check that D' is a $\gamma_{\times 2}(\mathtt{C}(G))$-set such that $|D' \cap V(G)| = n > |D \cap V(G)|$, a contradiction. Thus $V(G) \subseteq D$, which implies that $\gamma_{\times 2}(\mathtt{C}(G)) = |D| \geq |V(G)| = n$, as desired.

Case 2: $G \in \mathcal{G}_n$. By Lemma 2.4 we only need to prove that $\gamma_{\times 2}(\mathtt{C}(G)) \geq n$. If $V(G) \subseteq D$, then it is straightforward that $\gamma_{\times 2}(\mathtt{C}(G)) = |D| \geq |V(G)| = n$, as desired. From now on, assume that $V(G) \setminus D \neq \emptyset$. Let $V_{n-1}(G) = \{v_j\}$. If $v_j \in V(G) \setminus D$, then by Lemma 2.5 (i) we have that $N_{\mathtt{C}(G)}(v_j) \cap V_E(G) \subseteq D$ and $N_G(v_j) \subseteq D$, which implies that $\gamma_{\times 2}(\mathtt{C}(G)) = |D| \geq |N_{\mathtt{C}(G)}(v_j) \cap V_E(G)| + |N_G(v_j)| = 2(n-1) \geq n$, as desired. Finally, if $v_j \in D$ then by Lemma 2.5 (ii) we have that $|V(G) \setminus D| = 1$, and as a consequence, $|V(G) \cap D| = n - 1$. In addition, by Lemma 2.5 (i) we have that $N_{\mathtt{C}(G)}(v) \cap V_E(G) \subseteq D$, where $v \in V(G) \setminus D$. Therefore, $\gamma_{\times 2}(\mathtt{C}(G) = |D| \geq |D \cap V(G)| + |N_{\mathtt{C}(G)}(v) \cap V_E(G)| \geq n$, as desired.

Case 3: $G \notin \mathcal{F}_n \cup \mathcal{G}_n$. By Lemma 2.4 we only need to prove that $\gamma_{\times 2}(\mathtt{C}(G)) \geq n + \lceil |V_{n-1}(G)|/2 \rceil$. First, we assume that $V(G) \subseteq D$. Since every vertex in $D \cap V_E(G)$ has exactly two neighbors in $V(G)$ and $D \cap N_{\mathtt{C}(G)}(v_i) \subseteq D \cap V_E(G)$ for every vertex $v_i \in V_{n-1}(G)$, we deduce that

$$2|D \cap V_E(G)| = \sum_{v_i \in V_{n-1}(G)} |D \cap N_{\mathtt{C}(G)}(v_i)| \geq |V_{n-1}(G)|.$$

Hence, $\gamma_{\times 2}(\mathtt{C}(G)) = |D| = |D \cap V(G)| + |D \cap V_E(G)| \geq n + \lceil |V_{n-1}(G)|/2 \rceil$, as desired. Finally, we assume that $V(G) \setminus D \neq \emptyset$. Observe that $V_{n-1}(G) \neq \emptyset$ because $G \notin \mathcal{F}_n$. Now, let us consider the following two subcases.

Subcase 3.1: there exists a vertex $v_j \in V_{n-1}(G) \setminus D$. By Lemma 2.5 (i) we have that $N_{\mathtt{C}(G)}(v_j) \cap V_E(G) \subseteq D$ and $N_G(v_j) \subseteq D$, which implies that $\gamma_{\times 2}(\mathtt{C}(G)) = |D| \geq |N_{\mathtt{C}(G)}(v_j) \cap V_E(G)| + |N_G(v_j)| = 2(n-1) \geq n + \lceil |V_{n-1}(G)|/2 \rceil$, as desired.

Subcase 3.2: $V_{n-1}(G) \subseteq D$. By Lemma 2.5 (ii) we have that $|V(G) \setminus D| = 1$, which implies that $|V(G) \cap D| = n - 1$. Without loss of generality, we assume that $V(G) \setminus D = \{v_1\}$ and $v_2 \in V_{n-1}(G)$. If there exists a vertex $v_k \in V(G) \setminus \{v_1, v_2\}$ such that $v_1 v_k \in E(G)$, then $D'' = (D \setminus \{v^{1,2}, v^{1,k}\}) \cup \{v_1, v^{2,k}\}$ is a $\gamma_{\times 2}(\mathtt{C}(G))$-set such that $|D'' \cap V(G)| > |D \cap V(G)|$, a contradiction. Hence, $V_{n-1}(G) = \{v_2\}$ and $v_1 \in V_1(G)$. These previous conditions and the fact that $G \notin \mathcal{G}_n$ lead to $V_{n-2}(G) \neq \emptyset$. Let $v_i \in V_{n-2}(G)$. Since $N_{\mathtt{C}(G)}(v_i) \cap V(G) = \{v_1\}$ and $v_1 \notin D$, it follows that $|(N_{\mathtt{C}(G)}(v_1) \cup N_{\mathtt{C}(G)}(v_i)) \cap D \cap V_E(G)| \geq 2$. Hence, $\gamma_{\times 2}(\mathtt{C}(G)) = |D| = |D \cap V(G)| + |D \cap V_E(G)| \geq (n-1) + 2 = n + \lceil |V_{n-1}(G)|/2 \rceil$, as desired. □

3. Double domination in M(G)

The concept of middle graph $\mathtt{M}(G)$ of a nontrivial connected graph G was introduced by Hamada and Yoshimura in [15]. Several studies have explored the behavior of domination parameters under this operator, including, for example, [19–22]. We begin this section by defining this graph operator.

Definition 3.1. *Given a nontrivial connected graph G, the middle graph $\mathtt{M}(G)$ is obtained from G by subdividing each edge exactly once, and joining pairs of these new vertices if and only if their corresponding edges are adjacent in G. Formally, $V(\mathtt{M}(G)) = V(G) \cup V_E(G)$ and $E(\mathtt{M}(G)) = \{v_i v^{i,j}, v_j v^{i,j} : v^{i,j} \in V_E(G)\} \cup E(\mathtt{L}(G))$, where $\mathtt{L}(G)$ is the line graph of G.*

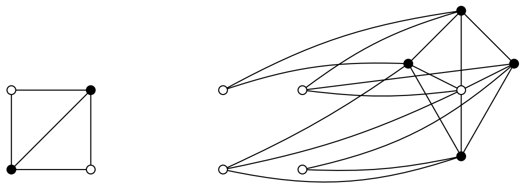

Figure 2: A graph G, and its corresponding graph $\mathtt{M}(G)$. The set of black vertices describes a $\gamma_{\times 2}(\mathtt{M}(G))$-set.

In Figure 2 we show a graph G and its corresponding graph $\mathtt{M}(G)$. Our first result shows that the double domination number and the total Italian domination number coincide for middle graphs.

Theorem 3.2. *For any nontrivial connected graph G,*

$$\gamma_{\times 2}(\mathtt{M}(G)) = \gamma_{tI}(\mathtt{M}(G)).$$

Proof. Let $f(W_0, W_1, W_2)$ be a $\gamma_{tI}(\mathtt{M}(G))$-function such that $|W_2|$ is minimum among all $\gamma_{tI}(\mathtt{M}(G))$-functions. Suppose that $W_2 \neq \emptyset$ and we analyze the next two possible cases.

Case 1: There exists a subscript $i \in [n]$ such that $v_i \in W_2$. Since $N_{\mathtt{M}(G)}(v_i) \subseteq V_E(G)$, there exists $j \in [n] \setminus \{i\}$ such that $v^{i,j} \in W_1 \cup W_2$. As $N_{\mathtt{M}(G)}[v_i] \subseteq N_{\mathtt{M}(G)}[v^{i,j}]$, it follows that the function $f'(W_0', W_1', W_2')$, defined by $W_0' = W_0$, $W_1' = W_1 \cup \{v_i\}$ and $W_2' = W_2 \setminus \{v_1\}$, is a TIDF on $\mathtt{M}(G)$ such that $\omega(f') < \omega(f) = \gamma_{tI}(\mathtt{M}(G))$, a contradiction.

Case 2: There exist two different subscripts $i, j \in [n]$ such that $v^{i,j} \in W_2$. Observe that $N_{\mathtt{M}(G)}(v^{i,j}) \cap (W_1 \cup W_2) \neq \emptyset$. Without loss of generality, we assume that $i \in [n]$ satisfies that $(N_{\mathtt{M}(G)}[v_i] \setminus \{v^{i,j}\}) \cap (W_1 \cup W_2) \neq \emptyset$. As $N_{\mathtt{M}(G)}[v^{i,j}] = N_{\mathtt{M}(G)}[v_i] \cup N_{\mathtt{M}(G)}[v_j]$, it is easy to check that the function $f''(W_0'', W_1'', W_2'')$, defined by $W_0'' = W_0$, $W_1'' = W_1 \cup \{v_j, v^{i,j}\}$ and $W_2'' = W_2 \setminus \{v^{i,j}\}$, is a TIDF on $\mathtt{M}(G)$ such that $\omega(f'') \leq \omega(f)$ and $|W_2''| < |W_2|$, a contradiction.

From the two previous cases, it follows that $W_2 = \emptyset$. As a consequence, W_1 is a double dominating set of $\mathtt{M}(G)$, which implies that $\gamma_{\times 2}(\mathtt{M}(G)) \leq |W_1| = \omega(f) = \gamma_{tI}(\mathtt{M}(G))$. Moreover, by definition it follows that $\gamma_{tI}(\mathtt{M}(G)) \leq \gamma_{\times 2}(\mathtt{M}(G))$. Therefore, $\gamma_{\times 2}(\mathtt{M}(G)) = \gamma_{tI}(\mathtt{M}(G))$, which completes the proof. □

Now, we present a lemma that will be very useful throughout the section.

Lemma 3.3. *If G is a nontrivial connected graph, then there exists a $\gamma_{\times 2}(\mathtt{M}(G))$-set D such that $D \subseteq V_E(G) \cup V_1(G)$.*

Proof. Let D be a $\gamma_{\times 2}(\mathtt{M}(G))$-set such that $|D \cap V_E(G)|$ is maximum among all $\gamma_{\times 2}(\mathtt{M}(G))$-sets. Suppose there exists a subscript $i \in [n]$ such that $v_i \in D \setminus V_1(G)$. Hence, there exist $j, k \in [n] \setminus \{i\}$ such that $v_j, v_k \in N_G(v_i)$ and $D \cap \{v^{i,j}, v^{i,k}\} \neq \emptyset$. If $v^{i,j}, v^{i,k} \in D$, then $D \setminus \{v_i\}$ is a double dominating set of $\mathtt{M}(G)$, a contradiction. So, $|D \cap \{v^{i,j}, v^{i,k}\}| = 1$, which implies that $D' = (D \setminus \{v_i\}) \cup \{v^{i,j}, v^{i,k}\}$ is a $\gamma_{\times 2}(\mathtt{M}(G))$-set such that $|D' \cap V_E(G)| > |D \cap V_E(G)|$, a contradiction too. Therefore, D is a $\gamma_{\times 2}(\mathtt{M}(G))$-set such that $D \subseteq V_E(G) \cup V_1(G)$, which completes the proof. □

The next theorem provides a closed formula for the double domination number of middle graph of every graph G satisfying that $V_1(G) = \emptyset$. For this purpose, let us consider the following parameter, which was recently introduced and studied in [27]. A *total edge cover* of a graph G with minimum degree at least two is a set $F \subseteq E(G)$ such that every vertex of G is incident to at least two edge in F. The *total edge covering number* of G, denoted by $\beta_t'(G)$, is the minimum cardinality among all total edge covers of G.

Theorem 3.4. *For any connected graph G with minimum degree at least two,*

$$\gamma_{\times 2}(\mathtt{M}(G)) = \beta_t'(G).$$

Proof. Let D be a $\gamma_{\times 2}(\mathtt{M}(G))$-set that satisfies the condition given in Lemma 3.3, that is, $D \subseteq V_E(G)$. Let $E = \{v_iv_j \in E(G) : v^{i,j} \in D\}$. Observe that every vertex $v_i \in V(G)$ satisfies that $|N_{\mathtt{M}(G)}(v_i) \cap D| \geq 2$, which implies that v_i is incident to at least two edge in E. Therefore E is a total edge cover of G, and as a consequence, $\gamma_{\times 2}(\mathtt{M}(G)) = |D| = |E| \geq \beta'_t(G)$. On the other hand, let F' be a total edge cover of G such that $|F'| = \beta'_t(G)$ and let $F = \{v^{i,j} \in V_E(G) : v_iv_j \in F'\}$. Observe that $|N_{\mathtt{M}(G)}(v_i) \cap F| \geq 2$ for every vertex $v_i \in V(G)$. In addition, $|N_{\mathtt{M}(G)}(v^{i,j}) \cap F| \geq 2$ for every vertex $v^{i,j} \in V_E(G) \setminus F$ and $|N_{\mathtt{M}(G)}(v^{i,j}) \cap F| \geq 1$ for every vertex $v^{i,j} \in V_E(G) \cap F$. Hence, F is a double dominating set of $\mathtt{M}(G)$, which implies that $\gamma_{\times 2}(\mathtt{M}(G)) \leq |F| = |F'| = \beta'_t(G)$. Therefore, $\gamma_{\times 2}(\mathtt{M}(G)) = \beta'_t(G)$, as desired. □

The following result provides tight lower and upper bounds for the double domination number of middle graph $\mathtt{M}(G)$.

Theorem 3.5. *For any connected graph G of order $n \geq 3$ and size m,*

$$\max\left\{2|V_1(G)|, n + \left\lceil \frac{|V_1(G)|}{2} \right\rceil\right\} \leq \gamma_{\times 2}(\mathtt{M}(G)) \leq m + |V_1(G)|.$$

Furthermore,

(i) $\gamma_{\times 2}(\mathtt{M}(G)) = 2|V_1(G)|$ *if and only if* $V(G) = \mathcal{S}_s(G) \cup V_1(G)$.

(ii) $\gamma_{\times 2}(\mathtt{M}(G)) = m + |V_1(G)|$ *if and only if* $\min\{|N_G(v_i)|, |N_G(v_j)|\} \leq 2$ *for every* $i, j \in [n]$ *such that* $v_iv_j \in E(G)$.

Proof. Let D be a $\gamma_{\times 2}(\mathtt{M}(G))$-set that satisfies Lemma 3.3. Since $D \subseteq V_E(G) \cup V_1(G)$, it follows that $\gamma_{\times 2}(\mathtt{M}(G)) = |D| \leq |V_E(G) \cup V_1(G)| = m + |V_1(G)|$. Now, we proceed to prove the lower bound. Observe that $V_1(\mathtt{M}(G)) = V_1(G)$ and $|V_1(\mathtt{M}(G))| = |\mathcal{S}(\mathtt{M}(G))|$. As $V_1(\mathtt{M}(G)) \cup \mathcal{S}(\mathtt{M}(G)) \subseteq D$ we have that $\gamma_{\times 2}(\mathtt{M}(G)) = |D| \geq |V_1(\mathtt{M}(G)) \cup \mathcal{S}(\mathtt{M}(G))| = 2|V_1(\mathtt{M}(G))| = 2|V_1(G)|$. Moreover, as $D \subseteq V_E(G) \cup V_1(G)$ it follows that $|N_{\mathtt{M}(G)}(v_i) \cap D| \geq 2$ for every $v_i \in V(G) \setminus V_1(G)$ and $|N_{\mathtt{M}(G)}(v_i) \cap D| = 1$ for every $v_i \in V_1(G)$. In addition, we have that $v^{i,j} \in N_{\mathtt{M}(G)}(v_i) \cap N_{\mathtt{M}(G)}(v_j)$ for every $i, j \in [n]$ such that $v_iv_j \in E(G)$. Hence,

$$\begin{aligned} 2|D| &\geq 2|V_1(G)| + \sum_{v_i \in V(G) \setminus V_1(G)} |N_{\mathtt{M}(G)}(v_i) \cap D| + \sum_{v_i \in V_1(G)} |N_{\mathtt{M}(G)}(v_i) \cap D| \\ &\geq 2|V_1(G)| + 2(n - |V_1(G)|) + |V_1(G)| \\ &= 2n + |V_1(G)|. \end{aligned}$$

Therefore $\gamma_{\times 2}(\mathtt{M}(G)) = |D| \geq n + \lceil |V_1(G)|/2 \rceil$, as desired.

Next, we proceed to prove (i). Without loss of generality, we assume that $\mathcal{S}(G) = \{v_1, \ldots, v_k\}$ whenever $V_1(G) \neq \emptyset$. For every $i \in [k]$, let $E_i = \{j \in \{k+1, \ldots, n\} : v_j \in N_G(v_i) \cap V_1(G)\}$. Now, let us consider the set $R = \cup_{i \in [k]}(\cup_{j \in E_i}\{v^{i,j}\})$. Observe that $|R| = |V_1(G)|$ and $R \cup V_1(G) \subseteq D$. If $V(G) = \mathcal{S}_s(G) \cup V_1(G)$, then it is straightforward that $R \cup V_1(G)$ is a double dominating set of $\mathtt{M}(G)$, which implies that $D = R \cup V_1(G)$. Therefore, $\gamma_{\times 2}(\mathtt{M}(G)) = |D| = |R \cup V_1(G)| = 2|V_1(G)|$, as desired. On the other hand, assume that $V(G) \neq \mathcal{S}_s(G) \cup V_1(G)$. So, there exists $j \in [n]$ such that $|N_{\mathtt{M}(G)}(v_j) \cap R| < 2$ and $|N_{\mathtt{M}(G)}(v_j) \cap D| \geq 2$, which implies that $R \cup V_1(G) \subsetneq D$. Hence, $2|V_1(G)| = |R \cup V_1(G)| < |D| = \gamma_{\times 2}(\mathtt{M}(G))$, which completes the proof of (i).

Finally, we proceed to prove (ii). If there exist subscripts $i, j \in [n]$ such that $v_iv_j \in E(G)$ satisfying that $\min\{|N_G(v_i)|, |N_G(v_j)|\} \geq 3$, then it is easy to check that $(V_E(G) \cup V_1(G)) \setminus \{v^{i,j}\}$ is a double dominating set of $\mathtt{M}(G)$. Hence, $\gamma_{\times 2}(\mathtt{M}(G)) \leq |(V_E(G) \cup V_1(G)) \setminus \{v^{i,j}\}| < m + |V_1(G)|$, as desired. Finally, assume that $\gamma_{\times 2}(\mathtt{M}(G)) < m + |V_1(G)|$. This implies that there exist subscripts $i, j \in [n]$ such that $v^{i,j} \notin D$. Hence $v_i, v_j \notin D$, which implies that $\min\{|N_{\mathtt{M}(G)}(v_i) \cap D|, |N_{\mathtt{M}(G)}(v_j) \cap D|\} \geq 2$. As an immediate consequence, it follows that $\min\{|N_G(v_i)|, |N_G(v_j)|\} \geq 3$. Therefore, the proof is complete. □

We conclude this section with a result that provides a lower bound for the double domination number of the middle graph of a tree.

Theorem 3.6. *The following statements hold for any tree T of order $n \geq 3$.*

(i) $\gamma_{\times 2}(\mathtt{M}(T)) \geq n + |V_1(T)| - |\mathcal{S}(T)|$.

(ii) $\gamma_{\times 2}(\mathtt{M}(T)) = n + |V_1(T)| - |\mathcal{S}(T)|$ *if and only if* $V(T) = \mathcal{S}_s(T) \cup V_1(T)$.

Proof. Let D be a $\gamma_{\times 2}(\mathtt{M}(T))$-set that satisfies Lemma 3.3, that is, $D \subseteq V_E(T) \cup V_1(T)$. Without loss of generality, assume that $\mathcal{S}(T) = \{v_1, \ldots, v_k\}$. For every $i \in [k]$, let $E_i = \{j \in \{k+1, \ldots, n\} : v_j \in N_T(v_i) \cap V_1(T)\}$. Now, let us consider the set $R = \cup_{i \in [k]}(\cup_{j \in E_i}\{v^{i,j}\})$. Observe that $|R| = |V_1(T)|$ and $R \cup V_1(T) \subseteq D$. Now, let $A = V(T) \setminus (\mathcal{S}_s(T) \cup V_1(T))$, $A' = \mathcal{S}(T) \setminus \mathcal{S}_s(T)$ and $A'' = V(T) \setminus (\mathcal{S}(T) \cup V_1(T))$. Observe that $A = A' \cup A''$. We next analyze the following three complementary cases.

Case 1: $A = \emptyset$. This implies that $V(T) = \mathcal{S}_s(T) \cup V_1(T)$. By Theorem 3.5 (i) and the fact that $n = |V_1(T)| + |\mathcal{S}(T)|$ we obtain that $\gamma_{\times 2}(\mathtt{M}(T)) = 2|V_1(T)| = n + |V_1(T)| - |\mathcal{S}(T)|$.

Case 2: $A' \neq \emptyset$ and $A'' = \emptyset$. In this case, $V(T) = \mathcal{S}(T) \cup V_1(T)$. Let $D' = D \setminus (R \cup V_1(T))$. For any $v_i \in A' \subseteq V(T) \setminus D$ we have that $|N_{\mathtt{M}(T)}(v_i) \cap D| \geq 2$ and $|N_{\mathtt{M}(T)}(v_i) \cap R| = 1$, which implies that $|N_{\mathtt{M}(T)}(v_i) \cap D'| \geq 1$. As a consequence, $D' \neq \emptyset$ and by the fact that $n = |V_1(T)| + |\mathcal{S}(T)|$ it follows that $\gamma_{\times 2}(\mathtt{M}(T)) = |D| = |R| + |V_1(T)| + |D'| > 2|V_1(T)| = n + |V_1(T)| - |\mathcal{S}(T)|$.

Case 3: $A'' \neq \emptyset$. Let $D' = D \setminus (R \cup V_1(T))$. We claim that $|D'| > |A''|$. For any $v_i \in A'' \subseteq V(T) \setminus D$ we have that $|N_{\mathtt{M}(T)}(v_i) \cap D'| \geq 2$. In particular, and due to the fact that T is a tree, there exists a vertex $v_j \in A''$ such that $|N_T(v_j) \cap A''| \leq 1$. As $|N_{\mathtt{M}(T)}(v_j) \cap D'| \geq 2$, there exists $v_i \in N_T(v_j) \cap \mathcal{S}(T)$ such that $v^{i,j} \in D'$. Hence, $\sum_{v_i \in \mathcal{S}(T)} |N_{\mathtt{M}(T)}(v_i) \cap D'| > 0$. Since $v^{i,j} \in N_{\mathtt{M}(T)}(v_i) \cap N_{\mathtt{M}(T)}(v_j)$ for every $i, j \in [n]$ such that $v_i v_j \in E(T)$, we deduce that

$$2|D'| \geq \sum_{v_i \in A''} |N_{\mathtt{M}(T)}(v_i) \cap D'| + \sum_{v_i \in \mathcal{S}(T)} |N_{\mathtt{M}(T)}(v_i) \cap D'| > 2|A''|.$$

Therefore, $|D'| > |A''|$, as desired. Since $n = |A''| + |V_1(T)| + |\mathcal{S}(T)|$, it follows that $\gamma_{\times 2}(\mathtt{M}(T)) = |D| = |R| + |V_1(T)| + |D'| > 2|V_1(T)| + |A''| = n + |V_1(T)| - |\mathcal{S}(T)|$.

By the three previous cases we obtain that $\gamma_{\times 2}(\mathtt{M}(T)) \geq n + |V_1(T)| - |\mathcal{S}(T)|$ and that $\gamma_{\times 2}(\mathtt{M}(T)) = n + |V_1(T)| - |\mathcal{S}(T)|$ if and only if $V(T) = \mathcal{S}_s(T) \cup V_1(T)$. Therefore, the proof is complete. □

4. Double domination in $\mu(G)$

The Mycielskian graph $\mu(G)$ of a graph G was introduced by J. Mycielski in [25] as a construction that increases the chromatic number while avoiding the creation of triangles. Beyond coloring, several works have examined how domination parameters behave under this operator (see, for instance, [13, 23, 24, 30]). We begin this section by defining this graph operator.

Definition 4.1. *Let G be a nontrivial connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. The Mycielskian graph $\mu(G)$ is obtained from G by adding n new vertices $u_1, \ldots, u_n$, and an additional vertex w and then adding the edges wu_i for every $i \in [n]$. In addition, for each edge $v_i v_j \in E(G)$, we add the edges $u_i v_j$ and $v_i u_j$ to complete the construction of $\mu(G)$.*

In Figure 3 we show an example of a Mycielskian graph. The next lemma provides an upper bound for the double domination number of $\mu(G)$ in terms of $\gamma_{tI}(G)$.

Lemma 4.2. *The following statements hold for any nontrivial connected graph G.*

(i) $\gamma_{\times 2}(\mu(G)) \leq \gamma_{tI}(G) + 2$.

(ii) *If* $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G) + 2$, *then* $\gamma_{\times 2}(G) = \gamma_{tI}(G)$.

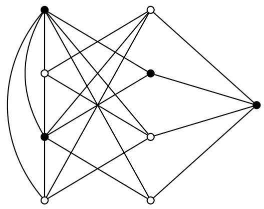

Figure 3: The Mycielskian graph $\mu(G)$, where G is the graph given in Figure 1. The set of black vertices describes a $\gamma_{\times 2}(\mu(G))$-set.

Proof. Let $f(W_0, W_1, W_2)$ be a $\gamma_{tI}(G)$-function. Without loss of generality, suppose $W_2 = \{v_1, \ldots, v_k\}$ whenever $|W_2| = k > 0$. We define a set $D \subseteq V(\mu(G))$ as follows.

$$D = \begin{cases} W_1 \cup \{u_1, w\} & \text{if } |W_2| = 0, \\ W_1 \cup W_2 \cup (\cup_{i\in[k]}\{u_i\}) \cup \{w\} & \text{otherwise.} \end{cases}$$

We claim that D is a double dominating set of $\mu(G)$. For this, let us consider the following two complementary cases.

Case 1: $|W_2| = 0$. By definition, we have that $|N_{\mu(G)}[w] \cap D| = 2$. Moreover, the fact that W_1 is a double dominating set of G and $w \in D$ lead to $|N_{\mu(G)}[v_i] \cap D| \geq 2$ and $|N_{\mu(G)}(u_i) \cap D| \geq 2$ for any $i \in [n]$. Hence, D is a double dominating set of $\mu(G)$, as desired.

Case 2: $|W_2| > 0$. By definition, we have that $|N_{\mu(G)}[w] \cap D| \geq 2$. Moreover, the fact that $W_1 \cup W_2$ is a total dominating set of G and $w \in D$ lead to $|N_{\mu(G)}(u_i) \cap D| \geq 2$ for every $i \in [n]$. In addition, it is straightforward that $|N_{\mu(G)}[v_i] \cap D| \geq 2$ for every $i \in [n]$. Hence, D is a double dominating set of $\mu(G)$, as desired.

From the previous cases we have that $\gamma_{\times 2}(\mu(G)) \leq |D| \leq |W_1| + 2|W_2| + 2 = \gamma_{tI}(G) + 2$, which completes the proof of (i). Now, if $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G) + 2$, then we have equalities in the previous inequality chain. In particular, we have that $|D| = |W_1| + 2|W_2| + 2$, which leads to $W_2 = \emptyset$. Hence, W_1 is a double dominating set of G, and as a consequence, $\gamma_{tI}(G) \leq \gamma_{\times 2}(G) \leq |W_1| = \omega(f) = \gamma_{tI}(G)$. Therefore, $\gamma_{\times 2}(G) = \gamma_{tI}(G)$, which completes the proof. □

The following theorem provides a closed formula for the double domination number of Mycielskian graph $\mu(G)$ in terms of the total Italian domination number of G.

Theorem 4.3. *For any nontrivial connected graph G,*

$$\gamma_{\times 2}(\mu(G)) \in \{\gamma_{tI}(G) + 1, \gamma_{tI}(G) + 2\}.$$

Proof. By Lemma 4.2 (i) we have that $\gamma_{\times 2}(\mu(G)) \leq \gamma_{tI}(G)+2$. We only need to prove that $\gamma_{\times 2}(\mu(G)) \geq \gamma_{tI}(G)+1$. Let D be a $\gamma_{\times 2}(\mu(G))$-set. Let us consider the following two complementary cases.

Case 1: $w \in D$. Let $f(W_0, W_1, W_2)$ be a function defined on G as follows. For each $i \in [n]$,

$$f(v_i) = \begin{cases} 0 & \text{if } |\{v_i, u_i\} \cap D| = 0, \\ 1 & \text{if } |\{v_i, u_i\} \cap D| = 1, \\ 2 & \text{if } |\{v_i, u_i\} \cap D| = 2. \end{cases}$$

Observe that $|N_{\mu(G)}[v_i] \cap (D \setminus \{w\})| \geq 2$ for every vertex $v_i \in V(G)$. As a consequence, $f(N_G(v_i)) \geq 2$ whenever $v_i \in W_0$ and $f(N_G(v_i)) \geq 1$ whenever $v_i \in W_1 \cup W_2$. Hence, f is a TIDF on G, which implies that $\gamma_{\times 2}(\mu(G)) = |D \setminus \{w\}| + 1 = |W_1| + 2|W_2| + 1 = \omega(f) + 1 \geq \gamma_{tI}(G) + 1$, as desired.

Case 2: $w \notin D$. Let $A = \{u_i \in D : v_i \notin D\}$ and $A' = \{v_i \in V(G) : u_i \in A\}$. Observe that $A' \cap D = \emptyset$. Now, it is easy to check that $|N_{\mu(G)}[v_i] \cap (D \cap V(G))| \geq 2$ for every $v_i \in V(G) \setminus A'$. If $A = \emptyset$ then $A' = \emptyset$, which implies that $D \cap V(G)$ is a double dominating set of G. Hence,

$$\gamma_{\times 2}(\mu(G)) = |D| \geq |D| - |D \cap N_{\mu(G)}(w)| + 2 = |V(G) \cap D| + 2 \geq \gamma_{\times 2}(G) + 2 \geq \gamma_{tI}(G) + 2. \quad (1)$$

From now on, assume that $A \neq \emptyset$. Let $A'' = \{v_i \in A' : |N_{\mu(G)}(v_i) \cap (D \cap V(G))| \geq 2\}$. If $A' = A''$, then $D \cap V(G)$ is a double dominating set of G and as a consequence, Inequality chain (1) follows, as desired. Finally, assume that $A' \setminus A'' \neq \emptyset$. Let us fix a vertex $v_i \in A' \setminus A''$. As D is a $\gamma_{\times 2}(\mu(G))$-set, there exists $j \in [n] \setminus \{i\}$ such that $u_j \in D$ and $v_i v_j \in E(G)$. Recall that $v_j \in A'$ if and only if $v_j \notin D$. Now, let us consider the following subset $D_G \subseteq V(G)$.

$$D_G = \begin{cases} (D \cap V(G)) \cup (A' \setminus \{v_j\}) & \text{if } v_j \in A', \\ (D \cap V(G)) \cup A' & \text{otherwise.} \end{cases}$$

Observe that D_G is a double dominating set of G of cardinality $|D_G| = |D \cap V(G)| + |A'| - 1$. Hence, $\gamma_{\times 2}(\mu(G)) = |D| = |D \cap V(G)| + |D \cap \{u_1, \ldots, u_n\}| \geq |D \cap V(G)| + |A'| = |D_G| + 1 \geq \gamma_{\times 2}(G) + 1 \geq \gamma_{tI}(G) + 1$, as desired. □

The following result provides a sufficient condition for $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G)+1$, which follows immediately from Lemma 4.2 (ii) and Theorem 4.3, as well as another sufficient condition that guarantees that $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G) + 2$.

Theorem 4.4. *The following statements hold for any nontrivial connected graph G.*

(i) *If $\gamma_{\times 2}(G) > \gamma_{tI}(G)$, then $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G) + 1$.*

(ii) *If $\gamma_{\times 2}(G) = \gamma_t(G)$, then $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G) + 2$.*

Proof. First, we proceed to prove (i). From Lemma 4.2 (ii), we deduce that if $\gamma_{\times 2}(G) > \gamma_{tI}(G)$, then $\gamma_{\times 2}(\mu(G)) \neq \gamma_{tI}(G) + 2$. Therefore, Theorem 4.3 leads to $\gamma_{\times 2}(\mu(G)) = \gamma_{tI}(G) + 1$, as desired. Finally, we proceed to prove (ii). Assume that $\gamma_{\times 2}(G) = \gamma_t(G)$. By Lemma 4.2 (i) we only need to prove that $\gamma_{\times 2}(\mu(G)) \geq \gamma_{tI}(G) + 2$. Let D be a $\gamma_{\times 2}(\mu(G))$-set. We consider the following two complementary cases.

Case 1: $w \notin D$. For every $i \in [n]$ we have that $|N_{\mu(G)}(u_i) \cap D \cap V(G)| \geq 1$, which implies that $|N_G(v_i) \cap D \cap V(G)| \geq 1$. As a consequence, $D \cap V(G)$ is a total dominating set of G and therefore $|D \cap V(G)| \geq \gamma_t(G)$. Moreover, the fact that $w \notin D$ leads to $|D \setminus V(G)| \geq 2$. Hence, $\gamma_{\times 2}(\mu(G)) = |D| \geq |D \cap V(G)| + |D \setminus V(G)| \geq \gamma_t(G) + 2 = \gamma_{\times 2}(G) + 2 \geq \gamma_{tI}(G) + 2$, as desired.

Case 2: $w \in D$. Let $I = \{i \in [n] : u_i \in D\}$ and let $J = \{i \in I : v_i \in D\}$. Observe that $J \subseteq I$ and $I \neq \emptyset$ because $|N_{\mu(G)}[w] \cap D| \geq 2$. If there exists $j \in J$, then it is easy to check that $D' = (D \cap V(G)) \cup (\cup_{i \in I \setminus \{j\}} \{v_i\})$ is a total dominating set of G satisfying that $|D'| \leq |D| - 2$. Therefore, $\gamma_{\times 2}(\mu(G)) = |D| \geq |D'| + 2 \geq \gamma_t(G) + 2 = \gamma_{\times 2}(G) + 2 \geq \gamma_{tI}(G) + 2$, as desired. From now on, assume that $J = \emptyset$ and we fix a subscript $k \in I$. We claim that $D'' = (D \cap V(G)) \cup (\cup_{i \in I \setminus \{k\}} \{v_i\})$ is a total dominating set of G. If $v_i \in V(G) \setminus N_G(v_k)$, then $|N_G(v_i) \cap D''| = |N_{\mu(G)}(v_i) \cap D| \geq 1$, as desired. Now, suppose that $v_i \in N_G(v_k)$. If $v_i \notin D$, then $|N_{\mu(G)}(v_i) \cap D| \geq 2$, which implies that $|N_G(v_i) \cap D''| \geq 1$, as desired. Moreover, if $v_i \in D$ then $u_i \notin D$. As $v_k \notin D$, then $|N_{\mu(G)}(u_i) \cap D \cap V(G)| \geq 1$, which implies that $|N_G(v_i) \cap D''| \geq 1$, as desired. Hence, D'' is a total dominating set of G satisfying that $|D''| \leq |D| - 2$. Therefore, $\gamma_{\times 2}(\mu(G)) = |D| \geq |D''| + 2 \geq \gamma_t(G) + 2 = \gamma_{\times 2}(G) + 2 \geq \gamma_{tI}(G) + 2$, as desired. □

References

[1] N.A. Abd Aziz, N.J. Rad, H. Kamarulhaili, *A note on the double domination number in maximal outerplanar and planar graphs*, RAIRO Oper. Res. **56(5)** (2022), 3367–3371.

[2] H. Abdollahzadeh Ahangar, M. Chellali, M. Hajjari, S.M. Sheikholeslami, *Further progress on the total Roman* {2}*-domination number of graphs*, Bull. Iranian Math. Soc. **48(3)** (2022), 1111–1119.

[3] H. Abdollahzadeh Ahangar, M. Chellali, S.M. Sheikholeslami, J.C. Valenzuela-Tripodoro, *Total Roman* {2}*-dominating functions in graphs*, Discuss. Math. Graph Theory **42** (2022), 937–958.
[4] L.A. Basilio, J. Leanos, O. Rosario, J.M. Sigarreta, *The differential on graph operator* $R(G)$, RAIRO-Oper. Res. **58** (2024), 5467–5479.
[5] S. Bermudo, *Total domination on tree operators*, Mediterr. J. Math. **20** (2023).
[6] S. Cabrera García, A. Cabrera Martínez, F.A. Hernández Mira, I.G. Yero, *Total Roman* {2}*-domination in graphs*, Quaest. Math. **44(3)** (2022), 411–434.
[7] A. Cabrera Martínez, *New bounds on the double domination number of trees*, Discrete Appl. Math. **315** (2022), 97–103.
[8] A. Cabrera-Martínez, A. Conchado Peiró, J.M. Rueda-Vázquez, *Further results on the total Italian domination number of trees*, Aims Math. **8(5)** (2023), 10654–10664.
[9] A. Cabrera-Martínez, A. Estrada-Moreno, *Double domination in rooted product graphs*, Discrete Appl. Math. **339** (2023), 127–135.
[10] A. Cabrera-Martínez, S. Cabrera García, J.A. Rodríguez-Velázquez, *Double domination in lexicographic product graphs*, Discrete Appl. Math. **284** (2020), 290–300.
[11] A. Cabrera-Martínez, J.A. Rodríguez-Velázquez, *A note on double domination in graphs*, Discrete Appl. Math. **300** (2021), 107–111.
[12] P. Chakradhar, P.V. Subba Reddy, *Algorithmic aspects of total Roman* {2}*-domination in graphs*, Commun. Comb. Optim. **7(2)** (2022), 183–192.
[13] X. Chen, H. Xing, *Domination parameters in Mycielski graphs*, Util. Math. **71** (2006), 235–244.
[14] E.J. Cockayne, R.M. Dawes, S.T. Hedetniemi, *Total domination in graphs*, Networks **10** (1980), 211–219.
[15] T. Hamada, I. Yoshimura, *Traversability and connectivity of the middle graph of a graph*, Discrete Math. **14** (1976), 247–255.
[16] F. Harary, T.W. Haynes, *Double domination in graphs*, Ars Comb. **55** (2000), 201–213.
[17] M.A. Henning, P. Maniya, D. Pradhan, *Paired versus double domination in forbidden graph classes*, Comput. Appl. Math. **44** (2025), 71.
[18] F. Kazemnejad, S. Moradi, *Total domination number of central graphs*, Bull. Korean Math. Soc. **56** (2019), 1059–1075.
[19] F. Kazemnejad, B. Pahlavsay, E. Palezzato, M. Torielli, *Domination number of middle graphs*, Trans. Comb., **12(2)** (2023), 79–91.
[20] F. Kazemnejad, B. Pahlavsay, E. Palezzato, M. Torielli, *Total domination number of middle graphs*, Electron. J. Graph Theory Appl., **10(1)** (2022), 275–288.
[21] K. Kim, *Domination-related parameters in middle graphs*, Discrete Math. Algorithm. Appl. **16(5)** (2024).
[22] K. Kim, *Italian,* 2*-rainbow and Roman domination numbers in middle graphs* RAIRO Oper. Res., **58** (2024), 2045–2053.
[23] Y.S. Kwon, J. Lee, M.Y. Sohn *Domination parameters in Mycielski graphs*, Bull. Korean Math. Soc., **58(4)** (2021), 829–836.
[24] D.A. Mojdeh, N.J. Rad, *On domination and its forcing in Mycielski's graphs*, Sci. Iran. **15** (2008), 218–222.
[25] J. Mycielski, *Sur le coloriage des graphs*, Colloq. Math. **3** (1955), 161–162.
[26] H.P. Patil, R. Pandiya Raj, *On the total graph of Mycielski graphs, central graphs and their covering numbers*, Discuss. Math. Graph Theory, **33** (2013), 361–371.
[27] I. Rios-Villamar, A. Cabrera-Martínez, G. Reyna, J. M. Sigarreta, *Double total domination number of some graph operators*, Submitted.
[28] J.M. Sigarreta, *Total domination on some graph operators*, Mathematics **9(3)** (2021).
[29] S.M. Sheikholeslami, L. Volkmann, *Nordhaus-Gaddum type inequalities on the total Italian domination number in graphs*, RAIRO Oper. Res. **56(4)** (2022), 2235–2243.
[30] J. Varghese, S. Aparna Lakshmanan, *Italian Domination on Mycielskian and Sierpinski Graphs*, Discrete Math. Algorithm. Appl., **13(4)** (2021), 2150037.