Filomat 40:2 (2026), 703–719
https://doi.org/10.2298/FIL2602703X

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized m-Schröder paths and the Chung-Feller property

Hua Xin[a,*], Huan Xiong[a]

[a]*Institute for Advanced Study in Mathematics, Harbin Institute of Techonology, Harbin, 150000, Heilongjiang, P.R. China*

Abstract. In this paper, we introduce a generalization of m-Schröder paths. For a fixed positive integer m, the generalized m-Schröder paths are lattice paths that start at $(0,0)$, use the steps $U=(1,1)$, $H=(1,0)$, $V_1=(0,-1)$, and $V_2=(0,-2)$ which are weighted respectively by 1, h, a and b, remain weakly above the line $y=\frac{m-1}{m}x$, and end on this line. We use generating functions and Riordan arrays to discuss the enumeration of the partial generalized m-Schröder paths and the free generalized m-Schröder paths, and obtain a Chung-Feller property. In particular, when $h=a=b=1$, we find that the number of generalized m-Schröder paths of order n equals the number of hybrid $(m+1)$-ary trees with n internal nodes.

1. Introduction

The Schröder numbers R_n, for $n \geq 0$, occur in many enumeration problems. We list four of them:

(i) R_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(0,1)$, and $(1,1)$, that never go below the line $y=x$, see [2, 5, 30, 33];

(ii) R_n is the number of lattice paths from $(0,0)$ to $(2n,0)$ with steps $(1,1)$, $(2,0)$, and $(1,-1)$, that never go below the line $y=0$, see [30–32];

(iii) R_n is the number of lattice paths from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,0)$, and $(0,-1)$, that never go below the line $y=0$, see [17, 18];

(iv) R_n is the number of di-sk trees with n internal nodes, where a di-sk tree is a complete binary tree in which each internal node is labeled with either 1 or 2, but with the restriction that no internal node has the same label with its left child, see [8–10, 13, 36].

For a positive integer $m \geq 1$, the m-Schröder numbers $R_n^{(m)}$, for $n \geq 0$, also appear in various enumeration problems. We mention four of them:

(i) $R_n^{(m)}$ is the number of lattice paths from $(0,0)$ to (mn,n) with steps $(1,0)$, $(0,1)$, and $(1,1)$, that never go below the line $y=\frac{1}{m}x$, see [11, 28];

(ii) $R_n^{(m)}$ is the number of lattice paths from $(0,0)$ to $((m+1)n,(m-1)n)$ with steps $(1,1)$, $(2,0)$, and $(1,-1)$, that never go below the line $y=\frac{m-1}{m+1}x$, see [37];

2020 *Mathematics Subject Classification*. Primary 11B83; Secondary 05A15, 05A19, 15B36.
Keywords. generalized m-Schröder path; generalized m-Schröder numbers; generating function; Riordan array; Chung-Feller property; hybrid m-ary tree.
Received: 15 April 2025; Accepted: 18 November 2025
Communicated by Paola Bonacini
* Corresponding author: Hua Xin
Email addresses: xinh1028@gmail.com (Hua Xin), huan.xiong.math@gmail.com (Huan Xiong)
ORCID iDs: https://orcid.org/0009-0009-3689-6455 (Hua Xin), https://orcid.org/0000-0001-5401-7834 (Huan Xiong)

(iii) $R_n^{(m)}$ is the number of lattice paths from $(0,0)$ to $((m+1)n, 0)$ with steps $(1,1)$, $(1,-m)$, and $(2,1-m)$, that never go below the line $y=0$, see [24];

(iv) $R_n^{(m)}$ is the number of di-sk $(m+1)$-ary trees with n internal nodes, where a di-sk $(m+1)$-ary tree is a complete $(m+1)$-ary tree in which each internal node is labeled with either 1 or 2, but with the restriction that no internal node has the same label with its leftmost child, see [16, 36].

If $m=1$, then $R_n^{(1)} = R_n$, i.e., the 1-Schröder numbers are the classical Schröder numbers. In this paper, we propose a generalization of m-Schröder numbers as follows.

Definition 1.1. *Let $m \geq 1$. A* generalized m-Schröder path *of order n (or length mn) is a lattice path from $(0,0)$ to $(mn, (m-1)n)$ that never goes below the line $y = \frac{m-1}{m}x$, with steps $U=(1,1)$, $H=(1,0)$, $V_1=(0,-1)$, and $V_2=(0,-2)$, weighted respectively by 1, h, a and b.*

Let α be a path. We define the weight $w(\alpha)$, or $|\alpha|$, to be the product of the weights of all its steps. The weight of a set $\mathcal{A}$ of paths, denoted by $w(\mathcal{A})$ or $|\mathcal{A}|$, is the sum of the total weights of all paths in $\mathcal{A}$.

Definition 1.2. *Denote by $\mathcal{R}_n^{(m)}(h,a,b)$ the set of all generalized m-Schröder paths of order n with respect to this weight assignment. We define $R_n^{(m)}(h,a,b) = |\mathcal{R}_n^{(m)}(h,a,b)|$ the (h,a,b)-generalized m-Schröder numbers, or simply the generalized m-Schröder numbers.*

When $h=a=1$ and $b=0$, the generalized 1-Schröder paths are the Schröder paths considered by [18]. When $h=a=b=1$, there exists a natural bijection between $\mathcal{R}_n^{(1)}(1,1,1)$ and the generalized Schröder paths of order n considered in [35], which are paths from $(0,0)$ to (n,n) with steps $(1,0)$, $(1,1)$, $(0,1)$, and $(0,2)$, and staying weakly below the line $y=x$. In [36], we established a bijection between the set of all hybrid binary trees with n internal nodes and the set of generalized Schröder paths from $(0,0)$ to (n,n). Hence there is a bijection from $\mathcal{R}_n^{(1)}(1,1,1)$ to the set of all hybrid trees with n internal nodes [23, 36].

The main aims of this paper is to give a recurrence relation for the generating function of the generalized m-Schröder numbers $R_n^{(m)}(h,a,b)$, as well as the explicit formula. Based on the generating function $R_m(t,h,a,b) = \sum_{n=0}^{\infty} R_n^{(m)}(h,a,b)t^n$, we obtain the Riordan array expressions for the three combinatorial matrices involving the generalized m-Schröder paths. Moreover, a Chung-Feller property for the generalized m-Schröder paths is derived.

This paper is organized as follows. In the next section, we review the main properties of the Riordan arrays which will be useful in this paper. In Section 3, we discuss the enumeration of generalized m-Schröder paths. Then we count the set of all partial generalized m-Schröder paths from $(0,0)$ to $(mn, (m-1)n+k)$ and staying weakly above the line $y=\frac{m-1}{m}x$. In Section 4, we study the set of all lattice paths from $(0,0)$ to $(mn,(m-1)n+k)$ with no restriction. In Section 5, we prove the Chung-Feller property for the generalized m-Schröder paths.

2. Riordan arrays

In the study of counting lattice paths, the use of Riordan arrays serves as an important tool. In the following, we will recall the Riordan array and the (m,r,s)-half of Riordan array. The concept of Riordan array was introduced in [27, 29] as a generalization of the Pascal matrix. Recently, Riordan arrays have been used widely in the enumeration of lattice paths [4, 14, 21, 25, 29]. Here we briefly recall the notion of Riordan arrays. An infinite lower triangular matrix $G=(g_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $g(t)f(t)^k$, where $g(t)$ and $f(t)$ are formal power series with $g(0)=1$, $f(0)=0$ and $f'(0)\neq 0$. The matrix corresponding to the pair $g(t), f(t)$ is denoted by $(g(t), f(t))$. The set of all Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1,t)$, called the Riordan group. If $(s_n)_{n\in\mathbb{N}}$ is any sequence having $s(t)=\sum_{n=0}^{\infty} s_n t^n$ as its generating function, then for every Riordan array $(g(t), f(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} g_{n,k}s_k = [t^n]g(t)s(f(t)). \tag{1}$$

This is called the fundamental theorem of Riordan arrays and it can be rewritten as

$$(g(t), f(t))\, s(t) = g(t)s(f(t)). \tag{2}$$

For an infinite lower triangular matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$, the vertical half of G is defined as the infinite lower triangular matrix $(g_{2n-k,n})_{n,k\in\mathbb{N}}$. It is known that if $G = (p(t), tq(t))$ is a Riordan array, then its vertical half is also a Riordan array [1, 15, 21, 22, 35, 36]. The following (m, r, s)-half of Riordan arrays are introduced in [37, 38].

Definition 2.1. *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array, and let m, r be positive integers and s a positive fractional number such that ms is integral number. We define the (m, r, s)-half $G^{(m,r,s)}$ of G as the lower triangular infinite matrix whose (n, k) entry is $g_{(m+1)n+(ms-m-1)k+r,mn+(ms-m)k+r}$, for $n \geq k \geq 0$, and it is zero if $k > n$.*

Lemma 2.2. *Let $G = (p(t), tq(t)) = (g_{n,k})_{n,k\geq 0}$ be a Riordan array and let $f(t)$ be the generating function defined by the functional equation $f(t) = q\,(tf(t)^m)$. Then the (m, r, s)-half Riodran array of G is given by*

$$G^{(m,r,s)} \quad = \quad ((tf(t)^m)'p(tf(t)^m)f(t)^{r-m}, tf(t)^{ms})\,.$$

In particular,

$$G^{(m,0,1)} \quad = \quad ((tf(t)^m)'p(tf(t)^m)f(t)^{-m}, tf(t)^m)\,, \tag{3}$$

$$G^{(m,0,\frac{m+1}{m})} \quad = \quad \left((tf(t)^m)'p(tf(t)^m)f(t)^{-m}, tf(t)^{m+1}\right). \tag{4}$$

For example, suppose that we want to count the lattice paths using the steps $U = (1, 1)$ and $H = (1, 0)$. If we assign each element $p_{n,k}$ to the number of such paths from $(0, 0)$ to (n, k), then we get the Pascal matrix $P = (p_{n,k})_{n,k\in\mathbb{N}}$ with $p_{n,k} = \binom{n}{k}$. It can be expressed as the Riordan array $P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$.

On the other hand, if we assign each element $g_{n,k}$ to the number of such paths from $(0, 0)$ to $(k, 2n-k)$, then we get the matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ with $g_{n,k} = \binom{k}{n-k}$. It can be expressed as the Riordan array $G = (1, t(1 + t))$. It is not hard to check that $P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ is the $(1, 0, 1)$-half of $G = (1, t(1 + t))$.

The Lagrange inversion formula will be used in the future. Several forms of the Lagrange inversion formula exist (see [7, 12, 30]). Here we need the following form.

Lemma 2.3. *Let $w = w(t)$ be the solution of the functional equation $w = t\phi(w)$, where $\phi(t)$ is a formal power series such that $\phi(0) \neq 0$, and let $F(t)$ be any formal power series. Then we have*

$$[t^n]F(w(t)) \quad = \quad \frac{1}{n}[t^{n-1}]F'(t)\phi(t)^n. \tag{5}$$

3. The generalized m-Schröder paths and generalized m-Schröder matrix

Recall that a *generalized m-Schröder path* of order n (or length mn) is a lattice path from $(0, 0)$ to $(mn, (m-1)n)$ which never goes below the line $y = \frac{m-1}{m}x$, consists of steps $U = (1, 1)$, $H = (1, 0)$, $V_1 = (0, -1)$, and $V_2 = (0, -2)$, and these steps are weighted by 1, h, a and b, respectively.

Let $\mathcal{R}_n^{(m)}(h, a, b)$ denote the set of all generalized m-Schröder paths of order n with respect to this weight assignment, and let $R_n^{(m)}(h, a, b) = |\mathcal{R}_n^{(m)}(h, a, b)|$.

Theorem 3.1. *The generating function $R_m(t) = R_m(t, h, a, b) = \sum_{n=0}^{\infty} R_n^{(m)}(h, a, b)t^n$ satisfies the following equation*

$$R_m(t) \quad = \quad 1 + ht\,R_m(t)^m + at\,R_m(t)^{m+1} + bt^2\,R_m(t)^{2m+1}, \tag{6}$$

and $R_n^{(m)}(h, a, b)$ is given by

$$R_n^{(m)}(h, a, b) \quad = \quad \frac{1}{mn+1} \sum_{i=0}^{mn+1} \sum_{j=0}^{n-i-j} \binom{mn+1}{i}\binom{n-i-j}{j} h^i a^{n-i-2j} b^j. \tag{7}$$

Proof. Any nonempty path $\alpha \in \mathcal{R}^{(m)}(h,a,b) = \bigcup_{n=0}^{\infty} \mathcal{R}_n^{(m)}(h,a,b)$ has a unique first return decomposition [7] of one of the following forms:

- $\alpha = U_1\beta_1U_2\beta_2\cdots U_{m-1}\beta_{m-1}H\beta_m$,
- $\alpha = U_1\beta_1U_2\beta_2\cdots U_m\beta_m V_1\beta_{m+1}$,
- $\alpha = U_1\beta_1U_2\beta_2\cdots U_{2m-1}\beta_{2m-1}U_{2m}\beta_{2m}V_2\beta_{2m+1}$,

where each β_i denotes arbitrary path in $\mathcal{R}^{(m)}(h,a,b)$, each U_j denotes step $U=(1,1)$, and $H=(1,0)$, $V_1=(0,-1)$, $V_2=(0,-1)$ are steps. See Figure 1 for an illustration of this decomposition for $m=2$.

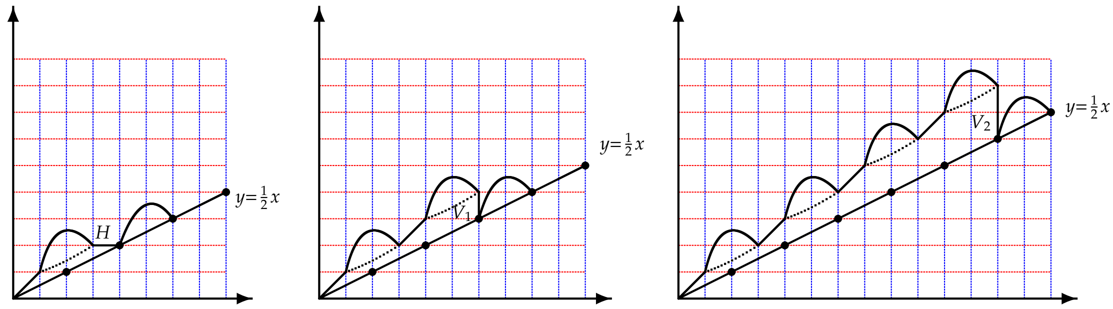

Figure 1: Decompositions of the generalized 2-Schröder paths.

From this decomposition and symbolic method of Flajolet [7] we can see that the generating function $R_m(t)=\sum_{n=0}^{\infty} R_n^{(m)}(h,a,b)t^n$ satisfies the equation

$$R_m(t) = 1 + ht\,R_m(t)^m + at\,R_m(t)^{m+1} + bt^2\,R_m(t)^{2m+1}.$$

By this equation, we know that

$$R_m(t) - at\,R_m(t)^{m+1} - bt^2\,R_m(t)^{2m+1} = 1 + ht\,R_m(t)^m.$$

It follows that

$$R_m(t) = \frac{1 + ht\,R_m(t)^m}{1 - at\,R_m(t)^m - bt^2\,R_m(t)^{2m}}.$$

Then, we can get

$$tR_m(t)^m = t\left(\frac{1 + ht\,R_m(t)^m}{1 - at\,R_m(t)^m - bt^2\,R_m(t)^{2m}}\right)^m.$$

Let $\phi(t) = \frac{1+ht}{1-at-bt^2}$, $w(t) = t\,R_m(t)^m$. Then, $w(t) = t\phi(w(t))^m$ and $R_m(t) = \phi(w(t))$.

Using the Lemma 2.3, we get that

$$\begin{aligned}
R_n^{(m)}(h,a,b) &= [t^n]R_m(t) \\
&= [t^n]\phi(w(t)) \\
&= \frac{1}{n}[t^{n-1}]\phi'(t)\phi(t)^{mn} \\
&= \frac{1}{n}[t^{n-1}]\left(\frac{1}{mn+1}\phi(t)^{mn+1}\right)' \\
&= [t^n]\frac{1}{mn+1}\phi(t)^{mn+1} \\
&= [t^n]\frac{1}{mn+1}\left(\frac{1+ht}{1-at-bt^2}\right)^{mn+1} \\
&= \frac{1}{mn+1}[t^n](1+ht)^{mn+1}\left(\frac{1}{1-at-bt^2}\right)^{mn+1} \\
&= \frac{1}{mn+1}[t^n]\sum_{i=0}^{mn+1}\binom{mn+1}{i}h^it^i\left(\frac{1}{1-at-bt^2}\right)^{mn+1} \\
&= \frac{1}{mn+1}[t^n]\sum_{i=0}^{mn+1}\binom{mn+1}{i}h^it^i\sum_{l=0}^{\infty}(at+bt^2)^l \\
&= \frac{1}{mn+1}[t^n]\sum_{i=0}^{mn+1}\binom{mn+1}{i}h^it^i\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{l}{j}(at)^{l-j}(bt^2)^j \\
&= \frac{1}{mn+1}[t^n]\sum_{i=0}^{mn+1}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{mn+1}{i}\binom{l}{j}h^ia^{l-j}b^jt^{i+j+l} \\
&= \frac{1}{mn+1}\sum_{i=0}^{mn+1}\sum_{j=0}^{n-i-j}\binom{mn+1}{i}\binom{n-i-j}{j}h^ia^{n-i-2j}b^j.
\end{aligned}$$

□

Example 3.2. *Let $h = a = b = 1$. The generating function $R_m(t) = \sum_{n=0}^{\infty} R_n^{(m)}(1,1,1)t^n$ satisfies $R_m(t) = 1 + t\,R_m(t)^m + t\,R_m(t)^{m+1} + t^2\,R_m(t)^{2m+1}$. For $m \geq 1$, we find that $R_m(t,1,1,1)$ is the generating function of the number of hybrid $(m+1)$-ary trees [16, 23] with n internal nodes. The following table displays the beginnings of the sequences $\left(R_n^{(m)}(1,1,1)\right)_{n\geq 0}$, for $1 \leq m \leq 9$.*

$R_n^{(m)}(1,1,1)$	0	1	2	3	4	5	6	7	*Annnnnn*
$R_n^{(1)}(1,1,1)$	1	2	7	31	154	820	4575	26398	*A*007863
$R_n^{(2)}(1,1,1)$	1	2	11	81	684	6257	60325	603641	*A*215654
$R_n^{(3)}(1,1,1)$	1	2	15	155	1854	24124	331575	4736345	*A*239107
$R_n^{(4)}(1,1,1)$	1	2	19	253	3920	66221	1183077	21981764	*A*239108
$R_n^{(5)}(1,1,1)$	1	2	23	375	7138	148348	3262975	74673216	*A*239109
$R_n^{(6)}(1,1,1)$	1	2	27	521	11764	290305	7585749	206294771	*A*245050
$R_n^{(7)}(1,1,1)$	1	2	31	691	18054	515892	15615159	492007235	*A*245051
$R_n^{(8)}(1,1,1)$	1	2	35	885	26264	852909	29347189	1051325430	*A*245052
$R_n^{(9)}(1,1,1)$	1	2	39	1103	36650	1333156	51392991	2062946770	*A*245053

Table 1. $R_n^{(m)}(1,1,1)$ for $m = 1,2,3,4,5,6,7,8,9$

Example 3.3. *Let $h = a = 1$, and $b = 0$. For $m \geq 1$, $R_n^{(m)}(1,1,0)$ is the m-Schröder numbers, and the generating function $R_m(t) = \sum_{n=0}^{\infty} R_n^{(m)}(1,1,0)t^n$ satisfies $R_m(t) = 1 + t\,R_m(t)^m + t\,R_m(t)^{m+1}$. The following table displays the beginnings of the sequences $\left(R_n^{(m)}(1,1,0)\right)_{n\geq 0}$, for $1 \leq m \leq 6$.*

$R_n^{(m)}(1,1,0)$	0	1	2	3	4	5	6	7	*Annnnnn*
$R_n^{(1)}(1,1,0)$	1	2	6	22	90	394	1806	8558	*A*006318
$R_n^{(2)}(1,1,0)$	1	2	10	66	498	4066	34970	312066	*A*027307
$R_n^{(3)}(1,1,0)$	1	2	14	134	1482	17818	226214	2984206	*A*144097
$R_n^{(4)}(1,1,0)$	1	2	18	226	3298	52450	881970	15422018	*A*260332
$R_n^{(5)}(1,1,0)$	1	2	22	342	6202	122762	2571326	56031470	*A*363006
$R_n^{(6)}(1,1,0)$	1	2	26	482	10450	247554	6208970	162064322	*A*371700

Table 2. $R_n^{(m)}(1,1,0)$ for $m = 1,2,3,4,5,6$

Example 3.4. *Let $h = b = 1$, and $a = 0$. For $m \geq 1$, $R_n^{(m)}(1,0,1)$ is the m-Fuss-Catalan numbers [4, 5, 7], and the generating function $R_m(t) = \sum_{n=0}^{\infty} R_n^{(m)}(1,0,1)t^n$ satisfies $R_m(t) = 1 + t\,R_m(t)^m + t^2\,R_m(t)^{2m+1}$. The following table displays the beginnings of the sequences $\left(R_n^{(m)}(1,0,1)\right)_{n\geq 0}$, for $1 \leq m \leq 7$.*

$R_n^{(m)}(1,0,1)$	0	1	2	3	4	5	6	7	*Annnnnn*
$R_n^{(1)}(1,0,1)$	1	1	2	5	14	42	132	429	*A*000108
$R_n^{(2)}(1,0,1)$	1	1	3	12	55	237	1428	7752	*A*001764
$R_n^{(3)}(1,0,1)$	1	1	4	22	140	969	7084	53820	*A*002293
$R_n^{(4)}(1,0,1)$	1	1	5	35	285	2530	23751	231880	*A*002294
$R_n^{(5)}(1,0,1)$	1	1	6	51	506	5481	62832	749398	*A*002295
$R_n^{(6)}(1,0,1)$	1	1	7	70	819	10472	141778	1997688	*A*002296
$R_n^{(7)}(1,0,1)$	1	1	8	92	1240	18278	285384	4638348	*A*007556

Table 3. $R_n^{(m)}(1,0,1)$ for $m = 1,2,3,4,5,6,7$

Example 3.5. *When $m = 1$, we obtain some other interesting sequences, which are listed in the following table.*

$R_n^{(1)}(h,a,b)$	0	1	2	3	4	5	6	7	8	*Annnnnn*
$R_n^{(1)}(0,1,1)$	1	1	3	10	38	154	654	2871	12925	*A*001002
$R_n^{(1)}(1,1,2)$	1	2	8	40	224	1344	8448	54912	366080	*A*151374
$R_n^{(1)}(2,1,2)$	1	3	14	83	554	3966	29756	230915	1838162	*A*215661
$R_n^{(1)}(1,2,2)$	1	3	17	121	965	8247	73841	683713	6493145	*A*216314
$R_n^{(1)}(2,1,0)$	1	3	12	57	300	1686	9912	60213	374988	*A*047891
$R_n^{(1)}(1,2,0)$	1	3	15	93	654	4791	37275	299865	2474025	*A*103210
$R_n^{(1)}(0,1,2)$	1	1	4	15	68	322	1608	8283	43780	*A*250886
$R_n^{(1)}(1,0,2)$	1	1	3	9	33	125	503	2081	8849	*A*049171
$R_n^{(1)}(0,2,1)$	1	2	9	50	311	2072	14460	104346	772255	*A*192945

Table 4. $R_n^{(1)}(h,a,b)$ for some specific h, a, b.

The *partial generalized m-Schröder paths* are the prefixes of the generalized m-Schröder paths. In this part, we enumerate partial generalized m-Schröder paths ending at $(mn,(m-1)n+k)$ with respect to the length. Let $\mathcal{R}_{n,k}^{(m)}(h,a,b)$ be the set of all lattice paths from $(0,0)$ to $(mn,(m-1)n+k)$ and staying weakly above the line $y = \frac{m-1}{m}x$, let $R^{(m)}_{n,k}(h,a,b) = |\mathcal{R}^{(m)}_{n,k}(h,a,b)|$. We call the matrix $R^{(m)}(h,a,b) = \left(R^{(m)}_{n,k}(h,a,b)\right)_{n,k\in\mathbb{N}}$ the *generalized m-Schröder matrix*. We also write $R^{(m)} = \left(R^{(m)}_{n,k}\right)_{n,k\in\mathbb{N}}$ for short.

Let $f_k(t)$ be the generating function for the partial generalized m-Schröder paths ending at $(mn,(m-1)n+k)$, i.e., $f_k(t) = \sum_{n=k}^{\infty} R^{(m)}_{n,k}(h,a,b)t^n$. According to the above result, for $k=0$, we obviously have $f_0(t) = R_m(t)$, here $R_m(t)$ is the generating function of the generalized m-Schröder numbers. For $k>0$, a partial generalized m-Schröder path ending at $(mn,(m-1)n+k)$ is in the form of $\alpha(U\beta)^{mk}$, where $\alpha,\beta\in\mathcal{R}^{(m)}(h,a,b) = \bigcup_{n=0}^{\infty}\mathcal{R}^{(m)}_n(h,a,b)$ and each $U=(1,1)$ is an up step, see the paths in Figure 2 for examples. Then, we obtain $f_k(t) = R_m(t)\left(tR_m(t)^m\right)^k$. As a consequence, we deduce the following result.

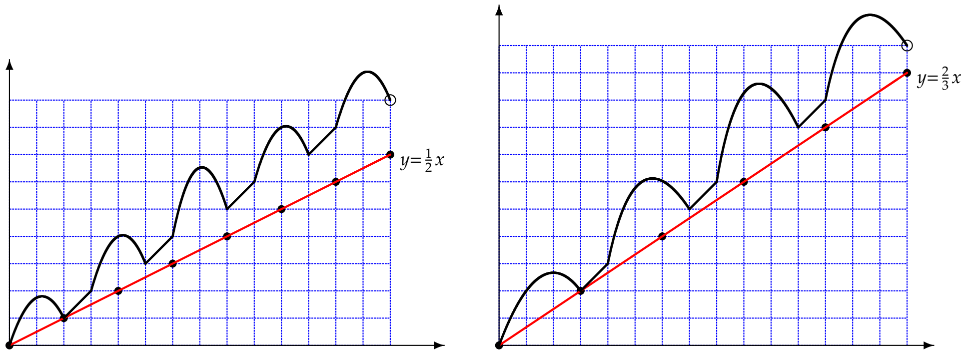

Figure 2: The decompositions of a path $\alpha\in\mathcal{R}^{(2)}_{7,2}$ and $\beta\in\mathcal{R}^{(3)}_{5,1}$.

Theorem 3.6. *Let $\mathcal{R}^{(m)}_{n,k}(h,a,b)$ be the set of all lattice paths from $(0,0)$ to $(mn,(m-1)n+k)$ and staying weakly above the line $y=\frac{m-1}{m}x$, and let $R^{(m)}_{n,k}(h,a,b) = |\mathcal{R}^{(m)}_{n,k}(h,a,b)|$. Then*

$$R^{(m)} = \left(R^{(m)}_{n,k}\right)_{n,k\in\mathbb{N}} = \left(R_m(t), tR_m(t)^m\right), \tag{8}$$

where $R_m(t) = R_m(t,h,a,b) = \sum_{n=0}^{\infty} R^{(m)}_n(h,a,b)t^n$, as defined in (6).

4. The free generalized *m*-Schröder paths

Let $m\geq 1$. A *free generalized m-Schröder path* is a lattice path starting from $(0,0)$, using the steps $\{U=(1,1), H=(1,0), V_1=(0,-1), V_2=(0,-2)\}$ which are weighted respectively by 1, h, a and b. A *grand generalized m-Schröder* path of order n (or length mn) is a free generalized m-Schröder path from $(0,0)$ to $(mn,(m-1)n)$.

Let $G(n,k)$ be the set of all free paths ending at the point $(k,2k-n)$, and let $g_{n,k} = |G(n,k)|$. Then we get the array $G = G(h,a,b) = (g_{n,k})_{n,k\in\mathbb{N}}$. In Figure 3, we give a schematic illustration of dependence of $g_{n+1,k+1}$ on the other elements in the array. Thus we deduce the recurrence:

$$g_{n+1,0} = ag_{n,0} + bg_{n-1,0}, \tag{9}$$
$$g_{n+1,k+1} = g_{n,k} + hg_{n-1,k} + ag_{n,k+1} + bg_{n-1,k+1}, \tag{10}$$

with the initial condition $g_{0,0}=1$.
For $k\geq 0$, let $g_k(t) = \sum_{n=k}^{\infty} g_{n,k}t^n$. Then, from (9) and (10), we obtain that

$$g_0(t) = 1 + atg_0(t) + bt^2g_0(t),$$
$$g_{k+1}(t) = tg_k(t) + ht^2g_k(t) + atg_{k+1}(t) + bt^2g_{k+1}(t).$$

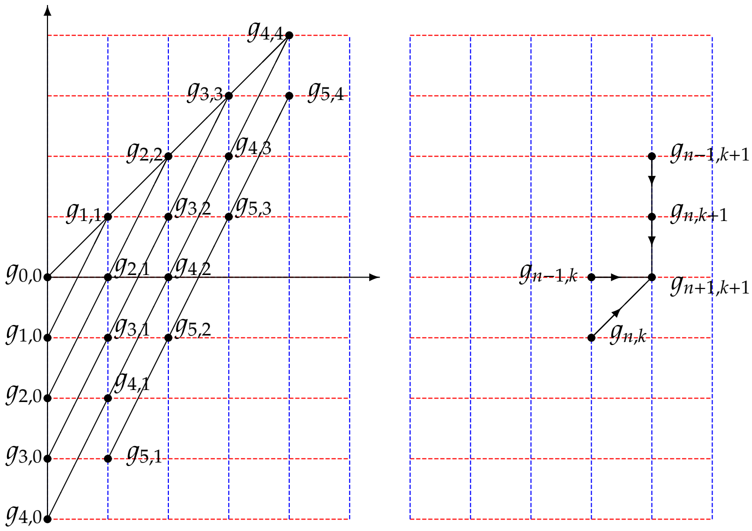

Figure 3: The matrix $(g_{n,k})_{n,k\in\mathbb{N}}$ and recurrence of the entries

Hence, we have

$$g_{k+1}(t) = \frac{t+ht^2}{1-at-bt^2} g_k(t), \quad g_0(t) = \frac{1}{1-at-bt^2}.$$

Consequently,

$$g_k(t) = \frac{1}{1-at-bt^2} \left(\frac{t+ht^2}{1-at-bt^2} \right)^k.$$

Therefore, we proved the following theorem.

Theorem 4.1. *The matrix $G(h,a,b) = (g_{n,k})_{n,k\in\mathbb{N}}$ can be represented by a Riordan array as*

$$G(h,a,b) = \left(\frac{1}{1-at-bt^2}, \frac{t+ht^2}{1-at-bt^2} \right).$$

Theorem 4.2. *The general term of the array $G(h,a,b)$ is*

$$g_{n,k}(h,a,b) \quad = \quad \sum_{j=0}^{n} \sum_{i=0}^{k} \binom{k}{i} \binom{n-i-j}{k} \binom{n-k-i-j}{j} h^i a^{n-k-i-2j} b^j.$$

Proof. From the definition of the Riordan array, we have

$$\begin{aligned} g_{n,k} &= [t^n]\frac{1}{1-at-bt^2}\left(\frac{t+ht^2}{1-at-bt^2}\right)^k \\ &= [t^{n-k}](1+ht)^k\left(\frac{1}{1-at-bt^2}\right)^{k+1} \\ &= [t^{n-k}]\sum_{i=0}^{k}\binom{k}{i}h^it^i\left(\frac{1}{1-at-bt^2}\right)^{k+1} \\ &= [t^{n-k}]\sum_{i=0}^{k}\sum_{l=0}^{\infty}\binom{k}{i}\binom{k+l}{l}h^it^i(at+bt^2)^l \\ &= [t^{n-k}]\sum_{i=0}^{k}\sum_{l=0}^{\infty}\sum_{j=0}^{l}\binom{k}{i}\binom{k+l}{l}\binom{l}{j}h^ia^{l-j}b^jt^{i+j+l} \\ &= \sum_{j=0}^{n}\sum_{i=0}^{k}\binom{k}{i}\binom{n-i-j}{k}\binom{n-k-i-j}{j}h^ia^{n-k-i-2j}b^j. \end{aligned}$$

□

For example, in the case $h = a = b = 1$, the first few terms of the array $G(1,1,1)$ are

$$\left(\frac{1}{1-t-t^2}, \frac{t(1+t)}{1-t-t^2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 7 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 5 & 15 & 16 & 7 & 1 & 0 & 0 & \cdots \\ 8 & 30 & 43 & 29 & 9 & 1 & 0 & \cdots \\ 13 & 58 & 104 & 95 & 46 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{11}$$

The first column is the Fibonacci numbers (A000045 [26]).

Let $\mathcal{U}^{(m)}_{n,k}(h,a,b)$ be the set of all free generalized m-Schröder paths ending at $(mn,(m-1)n+k)$ with no other restriction, and let $U^{(m)}_{n,k} = |\mathcal{U}^{(m)}_{n,k}(h,a,b)|$. Then the array $U^{(m)}(h,a,b) = \left(U^{(m)}_{n,k}\right)_{n,k\in\mathbb{N}}$ is the $(m,0,1)$-half of the matrix $G(h,a,b) = (G_{n,k})_{n,k\in\mathbb{N}}$, i.e, $U^{(m)}_{n,k} = G_{(m+1)n-k,mn}$. We call $U^{(m)}(h,a,b) = \left(U^{(m)}_{n,k}\right)_{n,k\in\mathbb{N}}$ the *grand generalized m-Schröder matrix*. In particular, the element $U^{(m)}_{n,0}$ is the total weight of grand generalized m-Schröder paths of order n.

Theorem 4.3. *The matrix $U^{(m)}(h,a,b) = \left(U^{(m)}_{n,k}\right)_{n,k\in\mathbb{N}}$ is a Riordan array given by*

$$U^{(m)}(h,a,b) = \left(\frac{(tR_m(t)^m)'R_m(t)^{-m+1}}{1+htR_m(t)^m}, tR_m(t)^m\right). \tag{12}$$

where $R_m(t)$ is determined by the equation

$$R_m(t) = 1 + htR_m(t)^m + atR_m(t)^{m+1} + bt^2R_m(t)^{2m+1}.$$

Proof. By the definition of $U^{(m)}(h,a,b)$, we know that $U^{(m)}(h,a,b)$ is the $(m,0,1)$-half of

$$G(h,a,b)=\left(\frac{1}{1-at-bt^2},\frac{t+ht^2}{1-at-bt^2}\right)=(p(t),tq(t)).$$

From Lemma 2.2,

$$U^{(m)}(h,a,b)=\left((tf(t)^m)'\,p\,(tf(t)^m)\,f(t)^{-m},tf(t)^m\right),$$

where $f(t)$ be the generating function defined by the functional equation

$$f(t)=q(tf(t)^m)=\frac{1+htf(t)^m}{1-atf(t)^m-b(tf(t)^m)^2}.$$

Thus, we have

$$f(t) \quad = \quad 1+htf(t)^m+atf(t)^{m+1}+bt^2f(t)^{2m+1}.$$

It follows from (6) that $f(t)=R_m(t)$, the generating function for the generalized m-Schröder paths. Hence,

$$\begin{aligned}
(tf(t)^m)'p(tf(t)^m)f(t)^{-m} &= (tf(t)^m)'\frac{1}{1-atf(t)^m-b(tf(t)^m)^2}f(t)^{-m}\\
&= \frac{(tf(t)^m)'f(t)^{-m+1}}{f(t)(1-atf(t)^m-b(tf(t)^m)^2)}\\
&= \frac{(tf(t)^m)'f(t)^{-m+1}}{1+htf(t)^m}\\
&= \frac{(tR_m(t)^m)'R_m(t)^{-m+1}}{1+htR_m(t)^m}.
\end{aligned}$$

This completes the proof. □

Theorem 4.4. *For $n\geq k\geq 0$, we have*

$$U^{(m)}_{n,k} \quad = \quad \sum_{j=0}^{(m+1)n-k}\sum_{i=0}^{mn}\binom{mn}{i}\binom{(m+1)n-k-i-j}{mn}\binom{n-k-i-j}{j}h^ia^{n-k-i-2j}b^j.$$

Proof. From Theorem 4.2, we get the formula for $U^{(m)}_{n,k}$. □

5. The Chung-Feller property

Denoting by $\mathcal{D}^{(k)}_n$ the sets of lattice paths from $(0,0)$ to $(2n,0)$ using up steps $U=(1,1)$ and down steps $D=(1,-1)$ with exactly k up steps below the line $y=0$. The well-known Chung-Feller theorem asserts that the sets $\mathcal{D}^{(0)}_n,\mathcal{D}^{(1)}_n,\cdots,\mathcal{D}^{(n)}_n$ all have the same cardinality $\frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, see [3, 6, 19, 20, 34]. The Chung-Feller property for other type paths was investigated in [6, 19, 20, 34, 37]. In [35], the authors proved a Chung-Feller property for the generalized Schröder paths.

In this section, we will present a Chung-Feller property for the generalized m-Schröder paths.

Recall that $U^{(m)}_{n,k}=|\mathcal{U}^{(m)}_{n,k}(h,a,b)|$, where $\mathcal{U}^{(m)}_{n,k}(h,a,b)$ is the set of all free generalized m-Schröder paths from $(0,0)$ to $(mn,(m-1)n+k)$, and $R^{(m)}_{n,k}=|\mathcal{R}^{(m)}_{n,k}(h,a,b)|$, where $\mathcal{R}^{(m)}_{n,k}(h,a,b)$ is the set of all generalized m-Schröder paths from $(0,0)$ to $(mn,(m-1)n+k)$ staying on or above the line $y=\frac{m-1}{m}x$. In the following theorem, we give a connection between the first two columns of $U^{(m)}(h,a,b)=\left(U^{(m)}_{n,k}(h,a,b)\right)_{n,k\in\mathbb{N}}$ and the first column of $R^{(m)}(h,a,b)=\left(R^{(m)}_{n,k}(h,a,b)\right)_{n,k\in\mathbb{N}}$.

Theorem 5.1. *For $n \geq 0$, we have*

$$(mn+1)R^{(m)}_{n,0} = U^{(m)}_{n,0} + hU^{(m)}_{n,1}. \tag{13}$$

Proof. From Theorem 4.3,

$$U^{(m)}(h,a,b) = G^{(m,0,1)} = \left(\frac{(tR_m(t)^m)'\, R_m(t)^{-m+1}}{1+htR_m(t)^m}, tR_m(t)^m\right).$$

Applying the fundamental theorem of Riordan arrays (2), we get that

$$\begin{aligned} G^{(m,0,1)} \cdot (1+ht) &= \left(\frac{(tR_m(t)^m)'\, R_m(t)^{-m+1}}{1+htR_m(t)^m}, tR_m(t)^m\right) \cdot (1+ht) \\ &= \frac{(tR_m(t)^m)'\, R_m(t)^{-m+1}}{1+htR_m(t)^m} \cdot (1+htR_m(t)^m) \\ &= (tR_m(t)^m)'\, R_m(t)^{-m+1}. \end{aligned}$$

By extracting the coefficient $[t^n]$ from the generating functions on both sides of the above equation, we arrive at

$$\begin{aligned} U^{(m)}_{n,0} + hU^{(m)}_{n,1} &= [t^n]G^{(m,0,1)} \cdot (1+ht) \\ &= [t^n]\,(tR_m(t)^m)'\, R_m(t)^{-m+1} \\ &= [t^n]\left(R_m(t)^m + mtR_m(t)^{m-1}R_m(t)'\right) R(t)^{1-m} \\ &= [t^n]R_m(t) + [t^n]mtR_m(t)' \\ &= [t^n]R_m(t) + m[t^{n-1}]R_m(t)' \\ &= [t^n]R_m(t) + mn[t^n]R_m(t) \\ &= (mn+1)[t^n]R_m(t) \\ &= (mn+1)R^{(m)}_{n,0}, \end{aligned}$$

hence the result follows. □

When $m = 1$, we can obtain the Theorem 4.1 in [35].

Corollary 5.2. *For $n \geq 0$, we have*

$$(n+1)V^{(1)}_{n,0} = U^{(1)}_{n,0} + hU^{(1)}_{n,1}.$$

In the following, we will give a bijective proof of the Theorem 5.1. We can easily obtain the following two lemmas.

Lemma 5.3. *For a free generalized m-Schröder path α from $(0,0)$ to $(mn+1, mn-n+1)$, the number of up or horizontal steps in α is $mn+1$.*

Lemma 5.4. *Let $\overline{U}^{(m)}(mn+1, mn-n+1)$ be the set of all free generalized m-Schröder paths from $(0,0)$ to $(mn+1, mn-n+1)$ ending with an up or horizontal step. Then*

$$|\overline{U}^{(m)}(mn+1, mn-n+1)| = U^{(m)}_{n,0} + hU^{(m)}_{n,1}.$$

To give a bijection proof of Theorem 5.1, we should introduce a special point for a lattice path. We view a free generalized m-Schröder path $\alpha \in \overline{U}^{(m)}(mn+1, mn-n+1)$ as a sequence of points

$$\alpha = (x_0, y_0)(x_1, y_1)(x_2, y_2)\cdots(x_{n-1}, y_{n-1})(x_l, y_l),$$

where $(x_0, y_0) = (0, 0)$ and $(x_l, y_l) = (mn + 1, mn - n + 1)$, such that $(x_j - x_{j-1}, y_j - y_{j-1}) = s_j \in S = \{(1, 1), (1, 0), (0, -1), (0, -2)\}$ for each $1 \le j \le l$. A *lowest point* (x_i, y_i) is a point in the path α such that $y_i - \frac{m-1}{m}x_i \le y_j - \frac{m-1}{m}x_j$ for all $j = 0, 1, 2, \cdots, l$. A *rightmost lowest point* (x_i, y_i) is a lowest point such that $i > j$ if (x_j, y_j) is also a lowest point and $j \neq i$. For example, the rightmost lowest point of the lattice path in the left of Figure 3 is $(6, 3)$, and the rightmost lowest point of the lattice path in the right of Figure 3 is $(5, 2)$. In Figure 4, the rightmost lowest point of the lattice paths are marked with small circles.

For $j = 1, 2, \ldots, mn, mn+1$, let $\overline{U}^{(m,j)}(mn+1, mn-n+1)$ be the subset of $\overline{U}^{(m)}(mn+1, mn-n+1)$ in which each lattice path has j up or horizontal steps after its rightmost lowest point. Then $\{\overline{U}^{(m,j)}(mn + 1, mn - n + 1) : 1 \le j \le mn + 1\}$ is a partition of $\overline{U}^{(m)}(mn + 1, mn - n + 1)$ with $mn + 1$ parts. We will show that $\{\overline{U}^{(m,j)}(mn + 1, mn - n + 1) : 1 \le j \le mn + 1\}$ uniformly partitions the set $\overline{U}^{(m)}(mn + 1, mn - n + 1)$ and each $|\overline{U}^{(m,j)}(mn + 1, mn - n + 1)| = |\mathcal{R}^{(m)}_{n,0}(h, a, b)| = R^{(m)}_{n,0}$. Consequently

$$|\overline{U}^{(m)}(mn + 1, mn - n + 1)| = (mn + 1)R^{(m)}_{n,0}.$$

For instance, when $m = 2$ and $n = 1$, $|\overline{U}^{(2)}(3, 2)| = U^{(2)}_{1,0} + U^{(2)}_{1,1} = 3R^{(2)}_{1,0}$. Figure 4 shows the set $\overline{U}^{(2)}(3, 2)$ is partitioned into 3 blocks $\overline{U}^{(2,j)}(3, 2)$, $j = 1, 2, 3$, and $|\overline{U}^{(2,1)}(3, 2)| = |\overline{U}^{(2,2)}(3, 2)| = |\overline{U}^{(2,3)}(3, 2)| = R^{(2)}_{1,0} = 2$.

Theorem 5.5. *There is a bijection between the set* $\mathcal{R}^{(m)}_{n,0}(h, a, b)$ *and the set* $\overline{U}^{(m,j)}(mn + 1, mn - n + 1)$ *for* $j = 1, 2, \cdots, mn + 1$.

Proof. When $j = 1$, $\overline{U}^{(m,1)}(mn + 1, mn - n + 1)$ is the subset of $\overline{U}^{(m)}(mn + 1, mn - n + 1)$ in which each lattice path has 1 up or horizontal step after its rightmost lowest point. For each $\alpha \in \overline{U}^{(m,1)}(mn + 1, mn - n + 1)$, its last step must be an up step and the rightmost lowest point is the starting point of the last step. Hence $\alpha = \beta U$, where $U = (1, 1)$ is an up step and β is a path from $(0, 0)$ to $(mn, mn - n)$ staying on or above the line $y = \frac{m-1}{m}x$, i.e., $\beta \in \mathcal{R}^{(m)}_{n,0}(h, a, b)$. It follows that

$$\overline{U}^{(m,1)}(mn + 1, mn - n + 1) = \{\beta U; \beta \in V^{(m)}(n, 0) \text{ and } U = (1, 1)\},$$

and $|\overline{U}^{(m,1)}(mn + 1, mn - n + 1)| = R^{(m)}_{n,0}$. For any $\beta \in \mathcal{R}^{(m)}_{n,0}(h, a, b)$, define $\phi_0(\beta) = \beta U$, where $U = (1, 1)$ is an up step. Obviously, ϕ_0 is a bijection between $\mathcal{R}^{(m)}_{n,0}(h, a, b)$ and $\overline{U}^{(m,1)}(mn + 1, mn - n + 1)$.

We now proceed to give a bijective map $\phi_j : \overline{U}^{(m,j)}(mn + 1, mn - n + 1) \to \overline{U}^{(m,j+1)}(mn + 1, mn - n + 1)$ for any $1 \le j \le mn$.

Let $\alpha \in \overline{U}^{(m,j)}(mn + 1, mn - n + 1)$, i.e., α is a path from $(0, 0)$ to $(mn + 1, mn - n + 1)$ such that there are j up or horizontal steps after the rightmost lowest point of α and the last step of α is an up or horizontal step. Then the path α can be decomposed as

$$\alpha = \beta_0 S \beta_1 U \beta_2,$$

where S is the first appearance of up or horizontal step, U is the up step after the rightmost lowest point, β_0 is an initial section (possibly empty) consisting of vertical steps $V_1 = (0, -1)$ or $V_2 = (0, -2)$, β_1 is the remaining section (possibly empty) between the step S and the step U, β_2 is the terminal section which contains $j - 1$ up or horizontal steps. Now we define

$$\phi_j(\alpha) = \beta_1 U \beta_2 \beta_0 S.$$

According to the construction of the map ϕ_j, $\phi_j(\alpha) \in \overline{U}^{(m,j+1)}(mn + 1, mn - n + 1)$. An example of a path $\alpha \in \overline{U}^{(2,1)}(7, 4)$ and the corresponding $\phi_1(\alpha) \in \overline{U}^{(2,2)}(7, 4)$ are illustrated in Figure 3. The maps ϕ_i from $\overline{U}^{(2,i)}(3, 2)$ to $\overline{U}^{(2,i+1)}(3, 2)$ for $i = 1, 2$, are illustrated in Figure 4. □

From the above discussion, we obtain the following theorem.

Theorem 5.6. *Let* $\overline{U}^{(m)}(mn+1, mn-n+1)$ *be the set of all free generalized m-Schröder paths from* $(0,0)$ *to* $(mn+1, mn-n+1)$ *ending with an up or horizontal step, and let* $\overline{U}^{(m,j)}(mn+1, mn-n+1)$ *denote the subset of such paths with* j *up or horizontal steps after its rightmost lowest point. Then for* $n \geq 1$, *we have*

$$|\overline{U}^{(m,j)}(mn+1, mn-n+1)| = R^{(m)}_{n,0}, \ j = 1, 2, \cdots, mn+1.$$

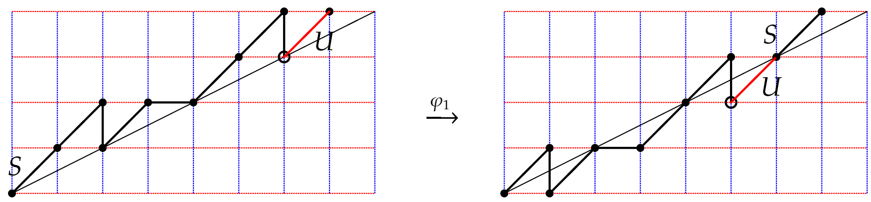

Figure 3: A path $\alpha \in \overline{U}^{(2,1)}(7,4)$ and $\phi_1(\alpha) \in \overline{U}^{(2,2)}(7,4)$

The classical Chung-Feller theorem can be restated as the special case of $a = 1$ and $h = b = 0$ in Theorem 5.6.

Corollary 5.7. *Let* $\overline{U}^{(m)}(mn+1, mn-n+1)$ *be the set of all free paths from* $(0,0)$ *to* $(mn+1, mn-n+1)$ *using up steps* $U = (1,1)$ *and vertical steps* $V_1 = (0,-1)$ *and ending with an up step, and let* $\overline{U}^{(m,j)}(mn+1, mn-n+1)$ *denote the subset of such paths with* j *up steps after its rightmost lowest point. Then for* $n \geq 1$, *we have*

$$|\overline{U}^{(m,j)}(mn+1, mn-n+1)| = \frac{1}{mn+1}\binom{(m+1)n}{n}, \ j = 1, 2, \cdots, mn+1.$$

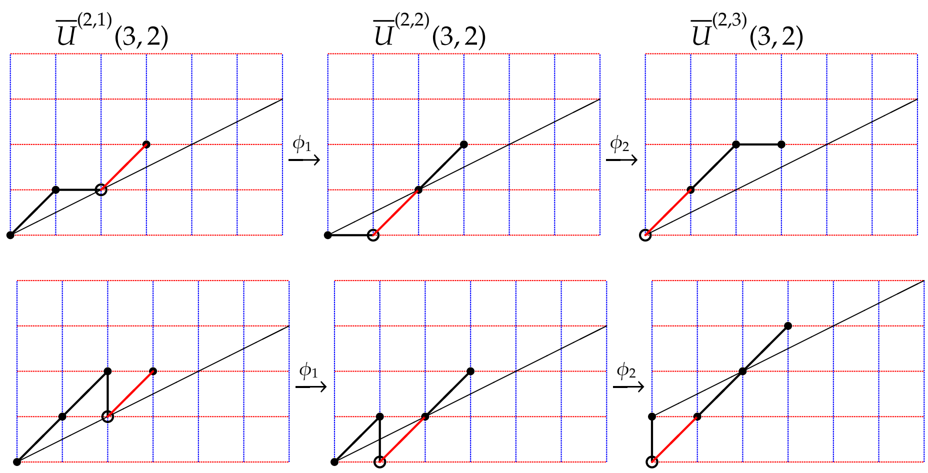

Figure 4: The set $\overline{U}^{(2)}(3,2)$ of free generalized m-Schröder paths from $(0,0)$ to $(3,2)$ ending with an up or horizontal step is uniformly partitioned into three blocks $\overline{U}^{(2,i)}(3,2)$ with $|\overline{U}^{(2,i)}(3,2)| = 2 = R^{(2)}_{1,0}$, $i = 1, 2, 3$.

Example 5.8. *Let $h = a = b = 1$. The begins of $G(1,1,1) = \left(\frac{1}{1-t-t^2}, \frac{t(1+t)}{1-t-t^2}\right)$ are given in* (11)*, and*

$$R^{(1)}(1,1,1) = (R_1(t), tR_1(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 7 & 4 & 1 & 0 & 0 & 0 & \cdots \\ 31 & 18 & 6 & 1 & 0 & 0 & \cdots \\ 154 & 90 & 33 & 8 & 1 & 0 & \cdots \\ 820 & 481 & 185 & 52 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$U^{(1)}(1,1,1) = \left(\frac{(tR_1(t))'}{1+tR_1(t)}, tR_1(t)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 16 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 95 & 29 & 7 & 1 & 0 & 0 & \cdots \\ 591 & 179 & 46 & 9 & 1 & 0 & \cdots \\ 3780 & 1140 & 303 & 67 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $R_1(t) = 1 + tR_1(t) + tR_1(t)^2 + t^2R_1(t)^3$.

It is easy to see that

$$(n+1)R^{(1)}_{n,0} = U^{(1)}_{n,0} + U^{(1)}_{n,1}. \tag{14}$$

$$R^{(2)}(1,1,1) = \left(R_2(t), tR_2(t)^2\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 11 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 81 & 45 & 10 & 1 & 0 & 0 & \cdots \\ 684 & 383 & 95 & 14 & 1 & 0 & \cdots \\ 6257 & 3519 & 925 & 161 & 18 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$U^{(2)}(1,1,1) = \left(\frac{\left(tR_2(t)^2\right)'}{R_2(t)(1+tR_2(t)^2)}, tR_2(t)^2\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 46 & 9 & 1 & 0 & 0 & 0 & \cdots \\ 475 & 92 & 13 & 1 & 0 & 0 & \cdots \\ 5161 & 995 & 154 & 17 & 1 & 0 & \cdots \\ 57727 & 11100 & 1083 & 232 & 21 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $R_2(t) = 1 + tR_2(t)^2 + tR_2(t)^3 + t^2R_2(t)^5$. The following identity holds

$$(2n+1)R^{(2)}_{n,0} = U^{(2)}_{n,0} + U^{(2)}_{n,1}. \tag{15}$$

Example 5.9. *Let $h = b = 1$ and $a = 0$. Then*

$$G(1,0,1) = \left(\frac{1}{1-t^2}, \frac{t(1+t)}{1-t^2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 4 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 3 & 6 & 7 & 4 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 9 & 13 & 11 & 5 & 1 & 0 & \cdots \\ 0 & 4 & 12 & 22 & 24 & 16 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{pmatrix},$$

$$U^{(1)}(1,0,1) = \left(\frac{B(t)}{1+tC(t)}, tC(t)\right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 2 & 1 & 0 & 0 & 0 & \cdots \\ 13 & 7 & 3 & 1 & 0 & 0 & \cdots \\ 46 & 24 & 11 & 4 & 1 & 0 & \cdots \\ 166 & 86 & 40 & 16 & 5 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$R^{(1)}(1,0,1) = (C(t), tC(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 5 & 5 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 14 & 14 & 9 & 4 & 1 & 0 & 0 & \cdots \\ 42 & 42 & 28 & 14 & 5 & 1 & 0 & \cdots \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $C(t) = \sum_{n=0}^{\infty} C_n t^n = \frac{1-\sqrt{1-4t}}{2t}$ *is the generating function of the Catalan numbers, and* $B(t) = \sum_{n=0}^{\infty} \binom{2n}{n} t^n = \frac{1}{\sqrt{1-4t}}$ *is the generating function of the central binomial coefficients. For these matrices, we have*

$$(n+1)R_{n,0}^{(1)} = U_{n,0}^{(1)} + U_{n,1}^{(1)}. \tag{16}$$

$$U^{(2)}(1,0,1) = \left(\frac{\mathcal{B}_3(t)}{(1-2t\mathcal{B}_3(t)^3)(1+t\mathcal{B}_3(t)^2)}, t\mathcal{B}_3(t)^2\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 11 & 4 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 62 & 22 & 6 & 1 & 0 & 0 & 0 & \cdots \\ 367 & 128 & 37 & 8 & 1 & 0 & 0 & \cdots \\ 2232 & 771 & 230 & 56 & 10 & 1 & 0 & \cdots \\ 13820 & 4744 & 1444 & 376 & 79 & 12 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$R^{(2)}(1,0,1) = \left(\mathcal{B}_3(t), t\mathcal{B}_3(t)^2\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 3 & 3 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 12 & 12 & 5 & 1 & 0 & 0 & 0 & \cdots \\ 55 & 55 & 25 & 7 & 1 & 0 & 0 & \cdots \\ 273 & 273 & 130 & 42 & 9 & 1 & 0 & \cdots \\ 1428 & 1428 & 700 & 245 & 63 & 11 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathcal{B}_3(t)$ is the generating function of the 3-Fuss-Catalan numbers. For these matrices, we have the following relation

$$(2n+1)R^{(2)}_{n,0} = U^{(2)}_{n,0} + U^{(2)}_{n,1}. \tag{17}$$

Acknowledgements
The authors thank the referees and editor for their valuable suggestions which improved the quality of this paper. This work was supported by the National Natural Science Foundation of China [Grant No. 12201155].

Declaration of competing interest
The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability
Data will be made available on request.

References

[1] P. Barry, *On the halves of a Riordan array and their antecedents*, Linear Algebra Appl. **582** (2019), 114–137.
[2] J. Bonin, L. Shapiro, R. Simion, *Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths*, J. Statist. Plann. Inference **34** (1993), 35–55.
[3] Y. M. Chen, *The Chung-Feller theorem revisited*, Discrete Math. **308** (2008), 1328–1329.
[4] G.-S. Cheon, H. Kim, L. W. Shapiro, *Combinatorics of Riordan arrays with identical A and Z sequences*, Discrete Math. **312** (2012), 2040–2049.
[5] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Co., Dordrecht, 1974.
[6] S. P. Eu, T. S. Fu and Y. N. Yeh, *Refined Chung-Feller theorems for lattice paths*, J. Combin. Theory Ser. A **112** (2005), 143–162.
[7] P. Flajolet, R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, Cambridge, 2009.
[8] S. Fu, Z. Lin, Y. Wang, *A combinatorial bijection on di-sk trees*, Electron. J. Combin. **28 (4)** (2021), P4.48.
[9] S. Fu, Z. Lin, J. Zeng, *On two unimodal descent polynomials*, Discrete Math. **341** (2018), 2616–2626.
[10] S. Fu, Y. L. Wang, *Bijective proofs of recurrences involving two Schröder triangles*, European J. Combin. **86** (2020), 103077.
[11] I. M. Gessel, *Schröder numbers, large and small*, Available at `http://people.brandeis.edu/~gessel/homepage/slides/schroder.pdf`, 2009.
[12] I. M. Gessel, *Lagrange inversion*, J. Combin. Theory Ser. A **144** (2016), 212–249.
[13] N. S. S. Gu, N. Y. Li, T. Mansour, 2-*Binary trees: Bijections and related issues*, Discrete Math. **308** (2008), 1209–1221.
[14] T. X. He, R. Sprugnoli, *Sequence characterization of Riordan arrays*, Discrete Math. **309** (2009) 3962–3974.
[15] T. X. He, *Riordan array approach to Lagrange inversion formulas and Chung-Feller theory*, Adv. Appl. Math. **144** (2023), 102154.
[16] S. Hong, S. Park, *Hybrid d-ary trees and their generalization*, Bull. Korean Math. Soc. **51 (1)** (2014), 229–235.
[17] M. Ito, *Generalized Schröder paths arising from a combinatorial interpretation of generalized Laurent bi-orthogonal polynomials*, Discrete Math. **348** (2025), 114230.
[18] J. Kim, D. Stanton, *Combinatorics of orthogonal polynomials of type R_I*, Ramanujan J. **61** (2023), 329–390
[19] S. C. Liu, Y. Wang, Y. N. Yeh, *Chung-Feller property in view of generating functions*, Electron. J. Combin. **18** (2011), P104.
[20] J. Ma, Y. N. Yeh, *Generalizations of Chung-Feller theorems*, Bull. Inst. Math. Acad. Sin. (NS) **4 (3)** (2009), 299–332.
[21] D. Merlini, D. G. Rogers, R. Sprugnoli, and M. C. Verri, *On some alternative characterizations of Riordan arrays*, Canad. J. Math. **49 (2)** (1997), 301–320.
[22] A. Nkwanta, L. W. Shapiro, *Pell walks and Riordan matrices*, Fibonacci Quart. **43 (2)** (2005), 170–180.
[23] J. M. Pallo, *On the listing and random generation of hybrid binary trees*, Int. J. Comput. Math. **50** (1994), 135–145.

[24] M. Pétréolle, A. D. Sokal, B.-X. Zhu, *Lattice paths and branched continued fractions: an infinite sequence of generalizations of the Stieltjes–Rogers and Thron–Rogers polynomials, with coefficientwise Hankel-total positivity*, Mem. Amer. Math. Soc. **291 (1450)** (2023), v+154 pp.
[25] J. L. Ramírez, V. F. Sirvent, *A Generalization of the k-Bonacci Sequence from Riordan Arrays*, Electron. J. Combin. **22 (1)** (2015), P1.38.
[26] N. J. A. Sloane, *On-line Encyclopedia of Integer Sequences (OEIS)*. Published electronically at http://oeis.org, 2024.
[27] L. W. Shapiro, S. Getu, W.-J. Woan, L. Woodson, *The Riordan Group*, Discrete Appl. Math. **34** (1991), 229–239.
[28] C. W. Song, *The Generalized Schröder Theory*, Electron. J. Combin. **12** (2005), R53.
[29] R. Sprugnoli, *Riordan arrays and combinatorial sums*, Discrete Math. **132** (1994), 267–290.
[30] R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge Univ. Press, Cambridge/New York, 1999.
[31] R. P. Stanley, *Hipparchus, Plutarch, Schröder and Hough*, Amer. Math. Monthly **104** (1997), 344–350.
[32] R. A. Sulanke, *Bijective reccurence concerning Schröder paths*, Electron. J. Combin. **5** (1998), R47.
[33] R. A. Sulanke, *Counting lattice paths by Narayana polynomials*, Electron. J. Combin. **7** (2000), R40.
[34] W. Woan, *Uniform partitions of lattice paths and Chung-Feller generalizations*, Amer. Math. Monthly **108 (6)** (2001), 556–559.
[35] L. Yang, S. L. Yang, *A Chung-Feller property for the generalized Schröder paths*, Discrete Math. **343** (2020), 111826.
[36] L. Yang, F. Y. Ren, S. L. Yang, *Some refined enumerations of hybrid binary trees*, Indian J. Pure Appl. Math. **55** (2024), 94–104.
[37] L. Yang, Y. Y. Zhang, S. L. Yang, *The halves of Delannoy matrix and Chung-Feller properties of the m-Schröder paths*, Linear Algebra Appl. **685** (2024), 138–161.
[38] S. L. Yang, Y. X. Xu, X. Gao, *On the half of a Riordan array*, Ars Combin. **133** (2017), 407–422.
[39] S. L. Yang, M. Y. Jiang, *The m-Schröder paths and m-Schröder numbers*, Discrete Math. **344** (2021), 112209.