Filomat 40:2 (2026), 739–750
https://doi.org/10.2298/FIL2602739R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Homological invariants of edge ideals of power graphs of finite groups

Bilal Ahmad Rather[a,*], Jianfeng Wang[a]

[a] *School of Mathematics and Statistics, Shandong University of Technology, Zibo 255049, China*

Abstract. This article examines the homological invariants, including Castelnuovo-Mumford regularity, projective dimension, and Betti numbers, of the edge ideals associated with the power graphs of integer modulo groups. We characterize the edge ideals of power graphs of group $\mathbb{Z}_n$ with 2-linear resolution and list all of their Betti numbers. We explicitly determine the projective dimension and extremal Betti numbers of power graphs of $\mathbb{Z}_n$. For the power graph of $\mathbb{Z}_n$, where n is the product of three distinct primes, the initial graded Betti numbers of its edge ideal are investigated alongside the Hilbert series. We present a general inequality for the Betti numbers and the regularity of edge ideals of power graphs of $\mathbb{Z}_n$.

1. Introduction

Given a polynomial ring $\mathbb{R} = \mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$ with intermediates $x_1, \ldots, x_n$ and standard degree grading, a set of finite simple graphs of order n and the quadratic square free monomial ideals in $\mathbb{R}$ have a one-to-one natural mapping. Given a simple finite graph G with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$, a *edge ideal* $I(G)$ associated with G and is defined as $I(G) = \{x_i x_j : \{x_i, x_j\} \in E(G)\} \subseteq \mathbb{R}$ (see, Villarreal [30]). The *edge ring* of G is the quotient ring $\mathbb{R}/I(G)$. According to the Hilbert-Syzygy theorem, the graded $\mathbb{R}$-module $\mathbb{R}/I(G)$ has a minimum $\mathbb{N}$-graded free resolution that is unique

$$0 \to \bigoplus_{j=\eta+1}^{s_\eta} \mathbb{R}(-j)^{\beta_{e,j}} \to \cdots \longrightarrow \bigoplus_{j=i+1}^{s_i} \mathbb{R}(-j)^{\beta_{i,j}} \longrightarrow \cdots \to \bigoplus_{j=2}^{s_1} \mathbb{R}(-j)^{\beta_{1,j}} \to \mathbb{R} \to \mathbb{R}/I(G) \to 0,$$

of length $\eta \le n$. The *projective dimension* η of $\mathbb{R}/I(G)$ is the length of the smallest graded free resolution of edge ring, which is expressed as $\mathrm{pd}(\mathbb{R}/I(G))$ (or, when G is understood, $\mathrm{pd}(G)$). The projective dimension reveals structural graph features like cycles and chordality. It is constrained above by the order n of G. Additionally, $\mathbb{R}(-j)$ is a graded free $\mathbb{R}$-module with rank one generated in degree j. The i-th graded *Betti number* of i-th syzygy module in degree j is denoted by $\beta_{i,j}(\mathbb{R}/I(G))$ (or $\beta_{i,j}(G)$, when there is no ambiguity). It is the number of generators with degree j in the i-th stage of the resolution. A Betti table contains the

2020 *Mathematics Subject Classification*. Primary 13D02; Secondary 13F55, 05E45, 05C25.
Keywords. Betti numbers; edge ideals; homological invariants; power graphs.
Received: 18 August 2025; Revised: 10 November 2025; Accepted: 19 November 2025
Communicated by Paola Bonacini
Jianfeng Wang is supported by National Natural Science Foundation of China (No. 12371353) and Special Fund for Taishan Scholars Project.
* Corresponding author: Bilal Ahmad Rather
Email addresses: bilalahmadrr@gmail.com (Bilal Ahmad Rather), jfwang@aliyun.com (Jianfeng Wang)
ORCID iDs: https://orcid.org/0000-0003-1381-0291 (Bilal Ahmad Rather), https://orcid.org/0000-0003-4676-8237 (Jianfeng Wang)

arrangement of the Betti numbers. The minimal free resolution of an edge ideal over a polynomial ring is described by its algebraic invariants $\beta_{i,j}(G)$. They play a key role in comprehending the algebraic and combinatorial characteristics of the graph related to the edge ideal. It is evident that $\beta_{i,j}(G) = 0$, for any $j < i + k$, if $I(G)$ is generated by elements of degree greater or equal to k. Therefore, for every $1 \leq i \leq \eta$, the graded Betti numbers of interest are $\beta_{i,i+k}(I(G))$. To represent the linear strand of the smallest resolution of $I(G)$, the numbers $\beta_{i,i+k}(G)$ for non-negative i count the number of linear syzygies that appear in the resolution [26]. The regularity of $I(G)$ is a crucial situation; if it is two, such edge rings (or graphs) are referred to as the Fröberg's characterization. The resolution in this case appears to be

$$0 \to \mathbb{R}(-\eta - 1)^{\beta_{\eta,j}} \longrightarrow \cdots \to \mathbb{R}(-3)^{\beta_{2,j}} \to \mathbb{R}(-2)^{\beta_{1,j}} \to \mathbb{R} \to \mathbb{R}/I(G) \to 0.$$

Finding the Betti numbers $\beta_{i,j}$ is often somewhat difficult, but for square-free monomial ideals like $I(G)$, the fundamental concept to explore $\beta_{i,j}(G)$ is Hochster's formula (see, [16, 26]), which is mentioned in the next section. However, because they are defined in terms of the dimension of reduced homology, it is still difficult to identify $\beta_{i,j}(G)$. Additionally, the Betti numbers $\beta_{i,j}(G)$ of fat forests are determined using the Hilbert series idea [13]. The numerous homological invariants of $I(G)$ that are encoded by the free resolution of the edge ideal $I(G)$ are closely related to the invariants of the underlying combinatorial graph G. Projective dimension, Castelnuovo-Mumford regularity (shortly regularity), and occasionally extremal Betti Numbers are some significant algebraic invariants related to $I(G)$. The regularity of $I(G)$ roughly quantifies the intricacy of ideals and modules. The regularity of $I(G)$ is expressed as

$$\operatorname{reg}^{\mathbb{K}}(I(G)) = \max\left\{j - i \,:\, \beta_{i,j}^{\mathbb{K}}(G) \neq 0\right\}.$$

The projective dimension of $I(G)$ is defined by

$$\operatorname{pd}^{\mathbb{K}}(I(G)) = \max\left\{i \,:\, \beta_{i,j}^{\mathbb{K}}(G) \neq 0 \text{ for some } j\right\}.$$

The gradual development of these invariants can be observed in [10, 11, 21, 30]. Fröberg [13] identified the Betti numbers and their Alexander duals for fat forests. Mohammadi and Moradi [22] presented resolutions for unmixed bipartite graphs. The Betti numbers of edge ideals of certain split graphs were examined by the authors in [29], [28] provides graded Betti numbers for a class of bipartite graphs. Both the underlying graph and the characteristic of field affect the Betti numbers, regularity, and projective dimension of $I(G)$ of a general graph G. However, in the current study, these homological invariants are not affected by the characteristic of field. Thus, using the shortened notations, we write $\beta_{i,j}^{\mathbb{K}}(\mathbb{R}/I(G)) = \beta_{i,j}(G)$, $\operatorname{reg}^{\mathbb{K}}(\mathbb{R}/I(G)) = \operatorname{reg}(G)$ and $\operatorname{pd}^{\mathbb{K}}(\mathbb{R}/I(G)) = \operatorname{pd}(G)$. A Betti number $\beta_{i,j} \neq 0$ of G is known as *extremal Betti number* if $\beta_{l,\ell} = 0$, for all $l \geq i, \ell \geq j+1$ and $\ell - l \geq j - i$, that is, if $\beta_{i,j}$ is the non-zero top left "corner" in a block of zeroes in the Macaulay "Betti" diagram. Extremal Betti numbers, which account for "notches" in the form of the minimal free resolution, are used to compute regularity. In this sense, extremal Betti numbers are a refinement of Mumford-Castelnuovo regularity. Bayer, Charalambous, and Popescu [3] investigated the extremal Betti numbers and their application to monomial ideals. Corso and Nagel [9] demonstrated that the edge rings of Ferrer's graphs are 2-linear in resolution. Chen [8] calculated the minimal resolution of all edge rings using 2-linear. Fröberg [12] addressed a conjecture about 2-linear resolution of edge rings.

In this contribution, we will demonstrate that the edge rings of a class of graphs derived from a power graph of integral modulo n group have 2-linear resolution when n is a prime power. We compute all of the graded Betti numbers that exist within its linear stand. Furthermore, we present the Beti numbers $\beta_{i,j}$ of a power graph of $\mathbb{Z}_n$ where n is the product of three primes and $j = i+1$. We show that the projective dimension of such graphs is $n - 1$, and the extremal Betti number is $\phi(n) + 1$, where ϕ represents Euler's function. As a result, we find enormous classes of edge rings with odd extremal Betti numbers. We characterize power graph of $\mathbb{Z}_n$ with regularity 2 and show that the regularity is at least 3 for remaining value of n. More information on the extremal Betti numbers of graded algebras may be found in [3, 20, 22–25].

All graph taken into consideration in this study are finite and simple. The graph $G = G(V(G), E(G))$ has the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and the edge set $E(G)$, which contains pairs of vertices $\{x_i, x_j\}$. An *induced* *subgraph* H of a graph G is a subgraph such that vertices u and v are adjacent in H if and only if they are so in G. We denote the induced subgraph on the vertex set V' by $G[V']$. The number of edges that are adjacent to a vertex v of G is its *degree* d_v. A complete graph, denoted by K_n, has every pair of distinct vertices adjacent. The complement $\overline{G}$ of a graph G with the same vertex set as $V(G)$ is a graph in which two distinct vertices of $\overline{G}$ are adjacent if and only if they are not in G. It is clear that the edge set $E(\overline{G}) = E(K_n) \setminus E(G)$. The complement of K_n is a completely disconnected graph, denoted as $\overline{K}_n$. A subset S of vertex set $V(G)$ is known as a *independent set* if the induced subgraph on set $G[S] \cong \overline{K}_{|S|}$, which means that S is a subset of pairwise non-adjacent vertices. A *clique* is a collection of pairwise adjacent vertices. C_n represents a cycle of order n. A graph G is said to be *chordal graph*, if C_3 is its only induced cycle. Equivalently, for $n \geq 4$, every induced cycle C_n has a chord (an edge that connects two vertices but is not part of the cycle itself). If $\overline{G}$ is chordal, then G is a co-chordal graph. A graph G is called *weakly chordal graph* if there is no induced chordless C_n in G and $\overline{G}$ for $n \geq 5$, meaning that each induced cycle in G and $\overline{G}$ is of length at most 4. Similarly, we denote vertex set as $V(G) = \{x_1, \dots, x_n\} = [x_n]$.

The article is organized as follows: Section 2 discusses simplicial complexes and Hochster's formula used for calculating the graded Betti numbers of edge ideal. In Section 3, we find the homological invariants of edge ideal of power graphs of finite cyclic group $\mathbb{Z}_n$, discuss their regularity, projective dimension, graded Betti numbers and Hilbert series.

2. Simplicial complexes and Hochster's formula

A *simplicial complex* $\Delta = \Delta(G)$ on vertex set $V(G) = \{x_1, x_2, \dots, x_n\}$ is a collection of its subsets such that each $\{x_i\} \in \Delta$ and if $F \in \Delta$, then each subset F' of F is in Δ. Roughly speaking Δ is closed under set inclusion operation. The maximal faces of Δ under inclusion are referred to as *facets*, while an element $F \in \Delta$ is known as *face*. If $|F| - 1 = \delta$, then $F \in \Delta$ is referred to as a δ*-dimensional face* (δ-face). The *dimension* of Δ, indicated by dim Δ, is defined as δ, where $\max\{|F| : F \in \Delta\} = \delta + 1$. The symbol *comp*$(\Delta)$ represents the number of connected components of Δ. If Δ' is a subset of simplicial complex Δ, it is called a subcomplex of Δ, which means that Δ' is a simplicial complex and $\Delta' \subseteq \Delta$. The induced subcomplex Δ_S associated with $S \subseteq V(\Delta)$ is defined as a simplicial complex $\Delta_S = \{F \in \Delta \mid F \subseteq S\}$. A simplicial subcomplex Δ' of a simplicial complex Δ is said to be full simplicial complex if every simplex in Δ whose vertices all belong to Δ' is also in Δ'. A simplicial complex Δ with only one facet is called a simplex, which means that every subset of the vertex set V is in Δ. The representation of a simplex of dimension δ (δ-simplex) is $\langle x_1, \dots, x_{\delta+1} \rangle$ or $\langle X \rangle$, where $X = [x_{\delta+1}]$. A simplicial complex $\Delta * \Delta'' = \{\alpha \cup \beta : \alpha \in \Delta, \beta \in \Delta''\}$ is the result of joining two simplicial complexes on different vertex sets, Δ and Δ''.

Consider the vertex set $[x_n]$ of a finite simple graph G. Next, the simplicial complex

$$\Delta(G) = \Big\{S : S \text{ is an independent subset of } [x_n]\Big\},$$

on $V(G)$ is called the *independent complex* of G. Similarly, a *clique complex* is defined on $V(G)$ of G. For a vertex set $[x_n]$ with a simplicial complex Δ, the squarefree monomial ideal I_Δ in $\mathbb{R} = \mathbb{K}[x_1, x_2, \dots, x_n]$ generated by all squarefree monomials $x_{i_1} x_{i_2} \dots x_{i_p}$ such that $\{x_{i_1}, x_{i_2}, \dots, x_{i_p}\}$ is not a face of Δ is known as *Stanley-Reisner ideal*, that is,

$$I_\Delta = \{x_{i_1} x_{i_2} \dots x_{i_p} \mid \{x_{i_1}, x_{i_2}, \dots, x_{i_p}\} \notin \Delta\} \subset \mathbb{R}.$$

The *Stanley-Reisner ring* of Δ is the quotient ring $\mathbb{K}[\Delta] = \mathbb{R}/I_\Delta$. On the other hand, we can attach a simplicial complex Δ on the vertex set $[x_n]$ such that $I = I_\Delta$ for any squarefree monomial ideal $I \subseteq \mathbb{R} = \mathbb{K}[x_1, x_2, \dots, x_n]$. Therefore, for an edge ideal $I(G)$ in a ring $\mathbb{R} = \mathbb{K}[x_1, x_2, \dots, x_n]$, the simplicial complex $\Delta(G)$ associated with a graph G on the set $[x_n]$ is expressed as

$$\Delta(G) = \Big\{\{x_{i_1} x_{i_2} \dots x_{i_p}\} \subseteq V : \{x_{i_1} x_{i_2} \dots x_{i_p}\} \text{ is a independent set}\Big\},$$

such that $I(G) = I_{\Delta(G)}$. That is, edge ideal $I(G)$ is same as the Stanley-Reisner $I_{\Delta(G)}$ of the independent complex $\Delta(G)$. An edge ring $\mathbb{R}/I(G)$ is equivalent to a Stanley-Reisner ring $\mathbb{R}/I_{\Delta(G)}$. We will now use the terms edge ring (edge ideal) and Stanley-Reisner (Stanley-Reisner ideal) interchangeably for a graph G. To calculate the graded Betti numbers of Stanley-Reisner ring $\mathbb{K}[\Delta]$, we recall an intriguing result attributed to Hochster [16] (also see, [26]). This formula relates the graded Betti numbers of I_Δ to the dimensions of the reduced homology of the simplicial complex Δ on $[x_n]$.

Theorem 2.1 ([16], Hochster's formula). *The graded Betti number $\beta_{i,j}$ of the Stanley-Reisner ring $\mathbb{K}[\Delta] = \mathbb{R}/I_\Delta$ in degree j is given by*

$$\beta_{i,j}(\mathbb{K}[\Delta]) = \sum_{\substack{S\subseteq V \\ |S|=j}} dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta_S;\mathbb{K}), \tag{1}$$

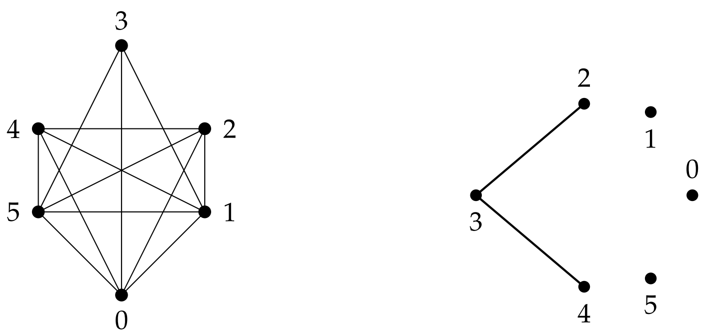
for each $i, j \geq 0$.

Clearly, the regularity of Stanley-Reisner ideal of a graph satisfies $\text{reg}(I_\Delta(G)) \geq 2$. Furthermore, $\text{reg}(\mathbb{R}/I(G)) = \text{reg}(G) = \text{reg}(I(G)) + 1$. The Fröberg's classification of edge ideals refers to the classification of the Stanley-Reisner ideal of graphs with regularity 2, that is, classification of edge ideals using linear resolutions [11], so that $\text{reg}(I(G)) \leq 2$ if and only if G is chordal. This type of characterisation, known as Fröberg's 2-linear resolution characterization of edge rings, is extremely valuable in algebraic combinatorics. Graphs with a regularity of at least 3 are yet to be fully characterized. For some advances on graphs with edge regularity 3, we refer to [10]; for other developments, see [13, 15, 32] and the references therein.

3. Homological invariants of edge ideals of power graphs of finite groups

Several types of graphs can be associated to algebraic structures like Cayley graph of a group, zero divisor graph of a ring [4], comaximal graphs of rings [27], commuting graphs of groups [5], power graphs [18] and several other classes of algebraic graphs [2].

The directed power graph of a semigroup Ω is a directed graph with vertex set Ω such that for $x, y \in \Omega$ there is an arc form x to y if and only if $x \neq y$ and $y^i = x$ for some positive integer i [18]. The undirected power graph $\mathcal{P}(\mathbb{G})$ of a group $\mathbb{G}$ has been defined in [6] as an undirected graph with vertex set as $\mathbb{G}$ and two vertices $x, y \in \mathbb{G}$ adjacent if and only if $x^i = y$ or $y^j = x$, for $2 \leq i, j \leq n$. First survey of power graph was carried by Abawajy, Kelarev and Chowdhury [1] and a recent one by Kumar, Selvaganesh, Cameron and Chelvam [17]. Figure 1 (a) shows the power graph of finite cyclic group $\mathbb{Z}_6$, and (b) Figure 1 show its independent simplicial complex.



(a) Power graph of $\mathbb{Z}_6$ (b) Independent simplicial complex of $\mathcal{P}(\mathbb{Z}_6)$

Figure 1: Power graphs of $\mathbb{Z}_6$ and its simplicial complex with 0 and 1-faces

Consider the power graph $\mathcal{P}(\mathbb{Z}_n)$ of the cyclic group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. It is easy to see that each element of $\mathbb{Z}_n$ can be written as integral modulo n of the identity element 0. So, 0 is adjacent to all other elements of $\mathcal{P}(\mathbb{Z}_n)$. Also, we note that the invertible elements $\alpha = \{a \in \mathbb{Z}_n : (a, n) = 1\}$ generate all elements of $\mathbb{Z}_n$ and there are $\phi(n)$ such elements, where ϕ is Euler function and (a, n) represents the highest common factor of a and n. Thus, it follows that $\phi(n)+1$ elements in $\mathbb{Z}_n$ generate all other elements and by definition of power graph they form a clique of size $\phi(n)+1$ in graph $\mathcal{P}(\mathbb{Z}_n)$. Next, we take into account other $n-\phi(n)-1$ non invertible elements of $\mathbb{Z}_n$. Mehranian, Gholami and Ashrafi [19] proved that $n-\phi(n)-1$ non invertible elements of $\mathbb{Z}_n$ form a special graph whose induced subgraphs are isomorphic to $K_{\phi(d_i)}$, where $d_i \notin \{1, n\}$ is a divisor of n. Furthermore, each vertex of $K_{\phi(d_i)}$ is adjacent to every vertex of $K_{\phi(d_j)}$ if d_i divides d_j. Also, in the same article, it is proved that the automorphism group of $\mathcal{P}(\mathbb{Z}_n)$ is $Aut(\mathcal{P}(\mathbb{Z}_n)) \cong S_{\phi(n)+1} \times \prod_i S_{\phi(i)}$, where i is a divisor of n other than 1 and n, and S_n is a symmetric group.

We will study the homological properties of power graphs of $\mathbb{Z}_n$ and characterize the value of n such that the Stanley-Reisner ideal has 2-linear resolution. Thereby, it contributes the combinatorial classification of Fröberg's 2-linear resolution of Stanley-Reisner ideal. Also, we discuss their regularity, projective dimension, Betti numbers including the extremal Betti number. With $j = i+1$, we have the following result presenting the extremal Betti number and the projective dimension of power graphs.

Theorem 3.1. *The extremal Betti number of $\mathcal{P}(\mathbb{Z}_n)$ is $\phi(n)+1$ and its projective dimension is $n-1$.*

Proof. Let $\Delta(\mathcal{P}(\mathbb{Z}_n)) = \Delta$ be the simplicial complex of $\mathcal{P}(\mathbb{Z}_n)$. Then by Hochster's formula (1), we have

$$\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n)) = \sum_{\substack{S\subseteq V(\mathcal{P}(\mathbb{Z}_n))\\ |S|=j}} \dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta_S; \mathbb{K}), \tag{2}$$

where $i, j \geq 0$. Also, note that Δ is not connected as there are $\phi(n)+1$ simplicial of dimension zero and other simplices of unknown dimension due to the fact that other part of power graph is not known. So, $j = i+1$ and with this information, $\beta_{i,j}(G)$ is non-zero and since the number of connected components in Δ is greater than 1. In particular there are $\phi(n)+2$ connected components as there are exactly $\phi(n)+1$, 0-dimensional facets in Δ and $\Delta_{V(\mathcal{P}(\mathbb{Z}_n))\setminus S}$ has only one connected component, where S is the set of non-zero non-invertible elements. Thus, the extremal Betti number is $\phi(n)+1$ and projective dimension is $n-1$. □

For $n = p^\alpha$, where p is prime and α is a positive integer, $\mathcal{P}(\mathbb{Z}_n)$ is complete graph, so its regularity is 2 and its only Betti numbers are given as

$$\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n)) = \begin{cases} i\binom{p^\alpha}{i+1} & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we discuss the homological invariants of $\mathcal{P}(\mathbb{Z}_n)$ when n is other than prime power. If $n = 2p$ where $p \geq 3$ is a prime. Then the component of $\overline{\mathcal{P}(\mathbb{Z}_n)}$ on non-zero non-unit elements of $\mathbb{Z}_n$ is a star graph of order $n-\phi(n)-1$ and remaining vertices are isolated. Thus, there is no induced cycle greater than order 4. So, it follows that $\mathcal{P}(\mathbb{Z}_n)$ is co-chordal and hence regularity of its edge ideal is 2. So, by Fröberg, the edge ideal of $I(\mathcal{P}(\mathbb{Z}_n))$ has two linear resolution and by applying Theorem 2.1, it can be proven that all its Betti numbers are given by

$$\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n)) = \begin{cases} i\left(\binom{p-1}{i+1} + \binom{p}{i+1} + \binom{p}{i}\right) + \sum_{\substack{a+b=i+1\\ a,b>1}} i\binom{p}{a}\binom{p-1}{b} \\ \quad + \sum_{\substack{a+b=i\\ a,b>1}} a\binom{p}{a}\binom{p-1}{b} & \text{for } j = i+1 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

For $n = 2 \cdot 3$, the Betti numbers of $\mathcal{P}(\mathbb{Z}_6)$ shown in Figure 1 are given below:

$$\beta_{1,2} = 1\binom{2}{2} + 1\binom{3}{2} + 1\binom{3}{1} + 1\binom{3}{1}\binom{2}{1} = 1 + 3 + 3 + 6 = 13,$$

$$\beta_{2,3} = 2\binom{3}{3} + 2\binom{3}{1}\binom{2}{2} + 2\binom{3}{2}\binom{2}{1} + \binom{3}{1}\binom{2}{1} = 2 + 3 \cdot 6 + 12 = 32$$

$$\beta_{3,4} = 3\binom{3}{3} + 3\binom{3}{2}\binom{2}{2} + 3\binom{3}{3}\binom{2}{1} + \binom{3}{1}\binom{2}{2} + 2\binom{3}{2}\binom{2}{1} = 33$$

$$\beta_{4,5} = 4\binom{3}{3}\binom{2}{2} + 2\binom{3}{2}\binom{2}{2} + 3\binom{3}{3}\binom{2}{1} = 4 + 2 \cdot 6 = 16, \beta_{5,6} = 3\binom{3}{3}\binom{2}{2} = 3.$$

By computer calculations using Macaulay 2.0 [14], the Betti number are same as above and are given as:

```
0  1  2  3  4 5
total: 1 13 32 33 16 3
0: 1  .  .  .  . .
1: . 13 32 33 16 3
```

Figure 2: Betti table of the minimal free resolution of $\mathbb{R}/I(\mathcal{P}(\mathbb{Z}_6))$.

When $n = \prod_{i=1}^{t} p_i^{\alpha_i}$ with $i \geq 2$, then the complement of induced subgraph of $\mathcal{P}(\mathbb{Z}_n)$ on $n - \phi(n) - 1$ elements always contains C_4 as an induced subgraph, since for any two divisors e_1 and e_2 with $(e_1, e_2) = 1$, their corresponding induced subgraphs are $\overline{K}_{e_1}$ and $\overline{K}_{e_2}$, and they form a complete bipartite graph $\mathcal{P}(\mathbb{Z}_n)$. Thus, the regularity of $I(\mathcal{P}(\mathbb{Z}_n))$ is at least 3. We make the above observations precise in the following result.

Theorem 3.2. *Let $I(\mathcal{P}(\mathbb{Z}_n))$ be the edge ideal of $\mathcal{P}(\mathbb{Z}_n)$. Then, we have*

$$reg(I(\mathcal{P}(\mathbb{Z}_n))) \begin{cases} = 2, & \text{if } n \text{ is a prime power or } n = 2p, \text{ where } p \text{ is prime} \\ \geq 3, & \text{otherwise.} \end{cases}$$

Next, we discus the regularity of $\mathcal{P}(\mathbb{Z}_n))$ for other values of n. We consider the following definitions: A *matching* in of G is a subset of $E(G)$ such that no two edges have a common vertex, that is, sort of subgraph consisting of pairwise disjoint edges only. If no two vertices belonging to any two given edges of a matching of G are adjacent, then such a matching is known as the *induced matching* (that is, the induced subgraph of a matching is induced matching). For an induced matching M of G, the cardinality of M is its size. The size β of the maximum induced matching in M is the induced matching number of G, denoted by ind(G). The induced matching of size β of G can be regarded as the βK_2 induced subgraph of G. For a graph G, the two disjoint edges $e_1 = \{y_i, y_j\}$, and $e_2 = \{z_i, z_j\}$ are said to be 3-disjoint if e_1 and e_2 are disjoint and their induced subgraph is $2K_2$. A collection of edges $\{e_1, \ldots, e_g\}$ is called pairwise 3-disjoint collection if e_i and e_j are 3-disjoint, for $1 \leq i, j \leq t$ with $i \neq j$. It is clear that the induced matching number of G is the largest cardinality of pairwise 3-disjoint subset of edges (see [31]). The following result relates the regularity of edge ring of G with its induced matching number.

Theorem 3.3 ([15]). *If G is a chordal graph, then $reg(G) = ind(G)$.*

The above result is even true for weakly chordal graphs (see [31]), as chordal graphs are weakly chordal.

For $n = pq$, with $2 < p < q$ are primes. The power graph of $G \cong \mathcal{P}(\mathbb{Z}_n)$ consist of $\phi(n) + 1$ vertices of degree $n - 1$, $p - 1$ vertices of degree $\phi(n) + p - 1$ and $q - 1$ vertices of degree $\phi(n) + q - 1$. So, structure of G is completely known with $V_1 = \{v \in V(\mathcal{P}(\mathbb{Z}_n)) : d_v = n - 1\}, V_2 = \{v \in V(\mathcal{P}(\mathbb{Z}_n)) : d_v = \phi(n) + p - 1\}$ and $V_3 = \{v \in V(\mathcal{P}(\mathbb{Z}_n)) : d_v = \phi(n) + q - 1\}$. Now, it is easy to see that $\overline{K}_{|V_2|,|V_3|}$ is the connected component of $\overline{\mathcal{P}(\mathbb{Z}_n)}$ and it implies that G is not co-chordal. Thus, regularity of $I(\mathcal{P}(\mathbb{Z}_n))$ is at least three. Let Ω be any subset of $V(\mathcal{P}(\mathbb{Z}_n))$. If Ω is a subset of V_i with $|\Omega| \geq 5$, then the induced subgraph $G[\Omega]$ cannot be a cycle, since for even cases $G[\Omega] \cong \overline{K}_{|\Omega|}$. If Ω is a subset of $V(\mathcal{P}(\mathbb{Z}_n))$ such that $\Omega \cap V_i \neq \emptyset$ and $\Omega \cap V_j \neq \emptyset$, then $G[\Omega]$ is $K_{|\Omega|}$. Again choosing Ω such that it intersects non trivially each V_i, then $G[\Omega]$ is always with a cord, since any two vertices in V_2 (or V_3) will be connected by an edge and are connected to all other vertices considered from V_1. Thus, $\mathcal{P}(\mathbb{Z}_n)$ is chordal and its regularity is same as the induced matching number. Now, $e_1 = \{u, v\}$ with $u, v \in V_2$ and $e_2 = \{w, x\}$ with $w, x \in V_3$ is an induced matching as $e_1 \cap e_2 = \emptyset$ and $G[e_1 \cup e_2] \cong 2K_2$. Thus the induced matching number of $\mathcal{P}(\mathbb{Z}_n)$ is 2, and the regularity of $I(\mathcal{P}(\mathbb{Z}_n)) = 3$, since reg$I(G) = $ reg$(G) + 1$. We make it precise in the following result.

Proposition 3.4. *The regularity of edge ideal of $\mathcal{P}(\mathbb{Z}_n)$ is* 3.

Let x_i be the vertices of V_1. Then the facets of the simplicial complex of edge ideal of $\mathcal{P}(\mathbb{Z}_n)$ are: 0-dimensional facets $\{x_i\}$ and facet $V_2 \cup V_2$. The following result gives the graded Betti numbers in the linear strand of $I(\mathcal{P}(\mathbb{Z}_n))$ with $j = i + 1$.

Theorem 3.5. *The Betti number of $I(\mathcal{P}(\mathbb{Z}_n))$ are given by*

$$\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n)) = i\left(\binom{\phi(p)}{i+1} + \binom{\phi(q)}{i+1} + \binom{\phi(n)+1}{i}\right) + \sum_{\substack{a+b=i+1\\ a,b>1}} i\binom{\phi(n)+1}{a}\binom{\phi(p)}{b}$$
$$+ \sum_{\substack{a+b=i+1\\ a,b>1}} i\binom{\phi(n)+1}{a}\binom{\phi(q)}{b} + \sum_{\substack{a+b+c=i+1\\ a,b,c>1}} a\binom{\phi(n)+1}{a}\binom{\phi(p)}{b}\binom{\phi(q)}{c}.$$

Proof. Let $\mathcal{P}(\mathbb{Z}_n)$ be the zero divisor graph of $\mathbb{Z}_n$ with $n = pq$. From the above observation, the vertex set of $V(\mathcal{P}(\mathbb{Z}_n))$ can be partitioned into mutually disjoint subsets V_i for $i = 1, 2, 3$. The induced subgraphs of V_i are $K_{\phi(n)+1}, K_{\phi(p)}$ and $K_{\phi(q)}$, respectively. Let $\Delta = \Delta(\mathcal{P}(\mathbb{Z}_n))$ be the simplicial complex of $\mathcal{P}(\mathbb{Z}_n)$. The facets of Δ are: 0-dimensional facets $\{x_i\}$ and facet $V_2 \cup V_2$. By using Hochster's formula (1), the initial Betti numbers can be found by the following formula

$$\beta_{i,i+1}(\mathcal{P}(\mathbb{Z}_n)) = \sum_{\substack{S\subseteq V(\mathcal{P}(\mathbb{Z}_n))\\ |S|=j}} \dim_{\mathbb{K}} \widetilde{H}_{j-(i+1)}(\Delta_S; \mathbb{K}). \tag{4}$$

With the above observations and $i - 1 = j$, above expression can be written as

$$\beta_{i,j}(\Gamma(\mathbb{Z}_n)) = \sum_{\substack{S\subseteq V_i\\ |S|=j}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) + \sum_{\substack{S\in S^*\\ |S|=j}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}), \tag{5}$$

where $S^* = \left\{W \subseteq V(\mathcal{P}(\mathbb{Z}_n)) : |W| - 1 = i,\ W \text{ non-trivially intersects } V_i\right\}$. We recall that $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = \text{comp}(\Delta_S) - 1$ for any subset S of $V(\mathcal{P}(\mathbb{Z}_n))$. So, any subset S of V_i will contribute $|S| - 1$ to $\beta_{i,i+1}(G)$. Thus, we have

$$\sum_{\substack{S\subseteq V_i\\ |S|=j}} \dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = i\binom{\phi(p)}{i+1} + i\binom{\phi(q)}{i+1} + i\binom{\phi(n)+1}{i}.$$

Now, for the other quantity of (5), we evaluate the non-zero contribution of other subsets of S^*. The possible consideration of subset S in are $V_1 \cup V_2$, $V_1 \cup V_3$ and $V_1 \cup V_2 \cup V_3$. The case $V_2 \cup V_3$ is ignored since its homotopy is connected and $\text{comp}(\Delta_S) - 1$ for any $S \in V_2 \cup V_3$. For the first case, let $S \subseteq V_1 \cup V_2$ be as subset such that $|S \cap V_1| = a$, and $|S \cap V_2| = b$, where a and b are positive integers and $|S| = i + 1$. As Δ_S consists of 0-dimensional subcomplex, so $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = \text{comp}(\Delta_S) - 1 = a + b + 1 - 1 = i$. Choosing a elements from V_1 and b elements from V_2, the contribution of such a subset to $\beta_{i,i+1}(G)$ is given as

$$\sum_{\substack{a+b=i+1\\ a,b>1}} i\binom{\phi(n)+1}{a}\binom{\phi(p)}{b}.$$

For the second case, let $S \subseteq V_1 \cup V_3$ be as subset such that $|S \cap V_1| = a$, and $|S \cap V_3| = b$, where a and b are positive integers and $|S| = i + 1$. As Δ_S consists of 0-dimensional simplexes, so $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = \text{comp}(\Delta_S) - 1 = a + b = i$ and its contributes to $\beta_{i,i+1}(G)$ is given as

$$\sum_{\substack{a+b=i+1\\ a,b>1}} i\binom{\phi(n)+1}{a}\binom{\phi(q)}{b}.$$

For the last case, let $S \subseteq V_1 \cup V_2 \cup V_3$ be as subset such that $|S \cap V_1| = a, |S \cap V_2| = b$ and $|S \cap V_3| = c$, where a, b and c are positive integers and $|S| = i + 1$. As Δ_S consists of 0-dimensional simplexes and other simplexes of $V_2 \cup V_3$, so $\dim_{\mathbb{K}} \widetilde{H}_0(\Delta_S; \mathbb{K}) = \text{comp}(\Delta_S) - 1 = a + 1 - 1 = a$, and the total contributions of such subsets to $\beta_{i,i+1}(G)$ is given by

$$\sum_{\substack{a+b+c=i+1\\ a,b,c>1}} a\binom{\phi(n)+1}{a}\binom{\phi(p)}{b}\binom{\phi(q)}{c}.$$

Therefore, substituting all these cases in Theorem 5, we obtain the required result. □

The proof of the expression related to all graded Betti numbers of $\mathcal{P}(\mathbb{Z}_{2p})$ given in (3) can be deduced from above result with some modifications in its proof.

If $\dim \Delta = d-1$, then the f-vector (face vector) of Δ is defined as $(f_{-1}, f_0, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces of Δ. Also, by convention f_{-1} counts the number of the empty face and is taken as $f_{-1} = 1$. The f-polynomial of Δ is defined as $f_\Delta(x) = \sum_{i=0}^{d} f_{i-1}x^{d-i}$. The h-polynomial of Δ is obtained from f-polynomial by replacing x by $x-1$ and we get corresponding h-vector. Thus,

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}, \tag{6}$$

where h-vector is $(h_0, h_1, \ldots, h_d)$. For example the f-vector of $\Delta(\overline{K}_3)$ is $(1,3,3,1)$, f-polynomial is x^3+3x^2+3x+1, h-vector is $(1,0,0,0)$ and h-polynomial is x^3.

Another type of h-polynomial (see, [7]) is given as

$$\sum_i h_i x^i = \sum_{i=1}^{d} f_{i-1}x^i(1-x)^{d-i}.$$

The h-vector in above h-polynomial is same as in (6). The coefficients of f-polynomial and h-polynomial are related by the following relation (see, [7])

$$h_i = \sum_{j=0}^{i}(-1)^{i-j}\binom{d-j}{i-j}f_{j-1}, \quad \text{and} \quad f_{i-1} = \sum_{j=0}^{i}\binom{d-j}{i-j}h_j, \tag{7}$$

where $0 \le i \le d$. Obviously, $f_{d-1} = \sum_{i=0}^{d} h_i$. The degree of h-polynomial is at most d.

The f-vector of Δ represents the number of faces in each dimension. It is a fundamental entity that offers information on the structure of Δ. The h-polynomial is a concise and informative technique to express the combinatorial structure of Δ. It converts complicated facial information (recorded by the f-vector) into an algebraic and combinatorial form. For the simplicial complex Δ on n vertices with f-vector $(f_{-1}, f_0, \ldots, f_{d-1})$ and h-vector $(h_0, h_1, \ldots, h_d)$, the Hilbert series (Hilbert-Poincaré) of Stanley–Reisner ring $\mathbb{K}[\Delta]$ of Δ is

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \sum_{i=0}^{d} \frac{f_{i-1}t^i}{(1-t)^i} = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

Since, G is a graph with n vertices, rewriting the above series as

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \sum_{i=0}^{d} \frac{f_{i-1}t^i}{(1-t)^i} = \frac{(1-t)^{n-d}(h_0 + h_1 t + \cdots + h_d t^d)}{(1-t)^n}.$$

Let $h'(t) = (1-t)^{n-d}(h_0 + h_1 t + \cdots + h_d t^d) = \sum_{i=1}^{n} h'_i$. Then the relation between h'_i and h_j (vectors) is given by the following expression

$$h'_i = \sum_{j=0}^{i}(-1)^{i-j}\binom{n-d}{i-j}h_j,$$

where $0 \leq i \leq n$ and h_j is given by (7). Thus, the Hilbert series of $\mathbb{K}[\Delta]$ can be written as

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \frac{1}{(1-t)^n} \sum_{i=0}^{n} h'_i t^i. \tag{8}$$

Let M be a graded finite $\mathbb{R}$-module ($\mathbb{R} = \mathbb{K}[x_1, x_2, \ldots, x_n]$) with finite projective dimension and graded free resolution (see [7], Lemma 4.1.13). Then the relation between Hilbert series and graded Betti numbers are given by the following expression

$$\mathcal{H}(M, t) = \frac{\sum_{i,j}(-1)^i \beta_{i,i+j} t^j}{(1-t)^n}. \tag{9}$$

By equating (8) and (9), we obtain the Betti numbers with the help of h and h' vectors.

Keeping in view the faces of $\Delta(\mathcal{P}(\mathbb{Z}_n))$, it follows that the f-vector of $\Delta(\mathcal{P}(\mathbb{Z}_n))$ is

$$f(\Delta(\mathcal{P}(\mathbb{Z}_n))) = \Big(1, n, \phi(n)\Big),$$

and its f-polynomial is $F_{\Delta(\mathcal{P}(\mathbb{Z}_n))}(x) = x^2 + nx + \phi(n)$. The h-polynomial is $H_{\Delta(\mathcal{P}(\mathbb{Z}_n))}(x) = x^2 + (n-2)x + \phi(n) + 1 - n$, and the corresponding h-vector of $\Delta(\mathcal{P}(\mathbb{Z}_n))$ is

$$h(\Delta(\mathcal{P}(\mathbb{Z}_n))) = \Big(1, n-2, \phi(n) + 1 - n\Big).$$

The Hilbert series is

$$\begin{aligned}\mathcal{H}(\mathbb{K}[\Delta(\mathcal{P}(\mathbb{Z}_n))], t) &= \frac{1 + (n-2)t + (\phi(n) + 1 - n)t^2}{(1-t)^2} \\ &= \frac{(1-t)^{n-2}\Big(1 + (n-2)t + (\phi(n) + 1 - n)t^2\Big)}{(1-t)^n}.\end{aligned} \tag{10}$$

From the above expression and definition of h'_i, we have

$$h'_{i+2} = \sum_{j=0}^{i+2} (-1)^{i+2-j} \binom{n-2}{i+2-j} h_j,$$

and with (7), $h_j = \sum_{b=0}^{j} (-1)^{j-b} \binom{2-b}{j-b} f_{b-1}$.

By (8) the Hilbert series of edge ring of $\mathcal{P}(\mathbb{Z}_n)$ is given as follows

$$\begin{aligned}\mathcal{H}(\mathbb{K}[\Delta(\mathcal{P}(\mathbb{Z}_n))], t) &= \frac{\sum_{i=0}^{n-1} \sum_{j=0}^{2} (-1)^i \beta_{i,i+j} t^{i+j}}{(1-t)^n} \\ &= \frac{1 + \sum_{i=1}^{n-1} (-1)^i \Big(\beta_{i,i+1} t^{i+1} + \beta_{i,i+2} t^{i+2}\Big)}{(1-t)^n}.\end{aligned} \tag{11}$$

By (10) and (11), we obtain

$$h'_{i+2} = (-1)^{i+1} \beta_{i+1,i+2}(G) + (-1)^i \beta_{i,i+2}(G).$$

It follows that $\beta_{i,i+2}(G) = (-1)^i h'_{i+2} + \beta_{i+1,i+2}(G)$. Now summing up all these entities, we have

$$\beta_{i,i+2}(G) = (-1)^i \sum_{j=0}^{i+2} (-1)^{i-j+2} \binom{n-2}{i+2-j} \sum_{b=0}^{j} (-1)^{j-b} \binom{2-b}{j-b} f_{b-1} + \beta_{i+1,i+2}(G),$$

where $\beta_{i+1,i+2}$ is given in Theorem 3.5 and f-vector as $(1, n, \phi(n))$. Which gives the complete list of Betti numbers of $I(\mathcal{P}(\mathbb{Z}_n))$ for $n = pq$ with primes $p < q$.

Next, we discuss the homological invariants of $\mathcal{P}(\mathbb{Z}_n)$ with $n = pqr$, where $p < q < r$ are primes. By the definition of $\mathcal{P}(\mathbb{Z}_n)$, there are $\phi(n) + 1$ vertices which are adjacent to all other vertices of $\mathcal{P}(\mathbb{Z}_n)$, we denote such a vertex set by V_1 and its cardinality by w. Also, there are other mutually disjoint subsets V_{d_i}, where d_i is a proper divisor of n. We note that d_i is in $\{p, q, r, pq, pr, qp\}$ and their corresponding cardinalities are denoted by n_i for $1 \le i \le 6$. Thus, each vertex of $K_{\phi(p)}$ is adjacent to every vertex of $K_{\phi(pq)}$ and $K_{\phi(pr)}$, each vertex of $K_{\phi(q)}$ is adjacent to every vertex of $K_{\phi(pq)}$ and $K_{\phi(qr)}$ and each vertex of $K_{\phi(r)}$ is adjacent to every vertex of $K_{\phi(pr)}$ and $K_{\phi(qr)}$. As, with $\mathcal{P}(\mathbb{Z}_n)$ for $n = pq$, it is easy to see that $\mathcal{P}(\mathbb{Z}_{pqr})$ is chordal graph and $\{e_1, e_2, e_3\}$ is its induced matching, where $e_1 = \{v_1, v_2\}, v_i \in V(K_{\phi(pq)})$, $e_2 = \{u_1, u_2\}, u_i \in V(K_{\phi(pr)})$ and $e_3 = \{w_1, w_2\}, w_i \in V(K_{\phi(qr)})$. Thus, regularity of edge ideal is $I(\mathcal{P}(\mathbb{Z}_n)) = 4$.

Proposition 3.6. *The regularity of edge ideal of $\mathcal{P}(\mathbb{Z}_n)$ with $2 \le p < q < r$ is* 4.

Next, we find its all Betti numbers with $j = j + 1$. The proof is very constructive. We first state result as below.

Theorem 3.7. *The Betti number of $I(\mathcal{P}(\mathbb{Z}_n))$ with $2 \le p < q < r$ and $j = i + 1$ are given by*

$$
\begin{aligned}
\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n)) = {} & i\binom{w}{i+1} + \sum_{j=1}^{6} i\binom{n_i}{i+1} + \sum_{\substack{a+b=i+1\\ a,b\ge 1}} \sum_{j=1}^{6} i\binom{w}{a}\binom{n_i}{b} + \sum_{\substack{a+b=i+1\\ a,b\ge 1}} i\binom{n_1}{a}\left(\binom{n_4}{b} + \binom{n_5}{b}\right)\\
& + \sum_{\substack{a+b=i+1\\ a,b\ge 1}} i\binom{n_2}{a}\left(\binom{n_4}{b} + \binom{n_4}{b}\right) + \sum_{\substack{a+b=i+1\\ a,b\ge 1}} i\binom{n_3}{a}\left(\binom{n_5}{b} + \binom{n_6}{b}\right) + \sum_{\substack{a+b+c=i+1\\ a,b,c\ge 1}} a\binom{w}{a}\left(\sum_{\substack{i<j\\ 1\le i<j\le 6}} \binom{n_i}{b}\binom{n_i}{b}\right)\\
& + \sum_{\substack{a+b+c=i+1\\ a,b,c>1}} \binom{n_1}{a}\left(c\binom{n_2}{b}\binom{n_4}{c} + a\binom{n_3}{b}\binom{n_5}{c} + c\binom{n_4}{b}\binom{n_5}{c}\right) + \sum_{\substack{a+b+c=i+1\\ a,b,c>1}} \binom{n_2}{a}\left(c\binom{n_4}{b}\binom{n_5}{c} + a\binom{n_3}{b}\binom{n_6}{c}\right)\\
& + \sum_{\substack{a+b+c=i+1\\ a,b,c>1}} a\binom{n_3}{a}\binom{n_5}{b}\binom{n_6}{c} + \sum_{\substack{a+b+c+d=i+1\\ a,b,c,d\ge 1}} a\binom{w}{a}\left(\binom{n_1}{b}(\binom{n_2}{c}\left(\sum_{i=3,5,6}\binom{n_i}{d} + d\binom{n_4}{d}\right) + \binom{n_3}{c}(\binom{n_4}{d}\right.\\
& + \binom{n_6}{d} + d\binom{n_5}{d})) + \left(\left(\binom{n_4}{c} + \binom{n_5}{c}\right)\binom{n_6}{d}\right)) + b\binom{n_1}{b}\binom{n_4}{c}\binom{n_5}{d} + \binom{n_2}{b}\\
& (\binom{n_3}{c}\left(\binom{n_4}{d} + \binom{n_5}{d} + d\binom{n_6}{d}\right) + \binom{n_4}{c}\binom{n_5}{d} + \binom{n_5}{c}\binom{n_6}{d}) + \binom{n_3}{b}\binom{n_4}{c}\\
& (\binom{n_5}{d} + \binom{n_6}{d}) + b\binom{n_6}{d}(\binom{n_2}{b}\binom{n_4}{c} + \binom{n_3}{b}\binom{n_5}{c}))\\
& + \sum_{\substack{a+b+c+d+e=i+1\\ a,b,c,d,e\ge 1}} \binom{w}{a}\left(\binom{n_1}{b}(\binom{n_2}{c}(\binom{n_3}{c}(\binom{n_5}{e} + \binom{n_6}{e}) + \binom{n_4}{d}(\binom{n_5}{e}\right.\\
& + \binom{n_6}{e}) + \binom{n_5}{d}\binom{n_6}{e}) + \binom{n_3}{c}(\binom{n_4}{d}(\binom{n_5}{e} + \binom{n_6}{e}) + \binom{n_5}{d}\binom{n_6}{e})\\
& + \binom{n_4}{c}\binom{n_5}{c}\binom{n_6}{c}) + \binom{n_2}{b}\binom{n_3}{c}\binom{n_4}{d}(\binom{n_5}{e} + \binom{n_6}{e}))\\
& + \sum_{\substack{a+b+c+d+e+f=i+1\\ a,b,c,d,e,f\ge 1}} a\binom{w}{a}\left(\binom{n_1}{b}\binom{n_2}{c}\binom{n_3}{c}(\binom{n_4}{e}\binom{n_5}{f} + \binom{n_4}{e}\binom{n_6}{f})\right)
\end{aligned}
$$

$$+\binom{n_4}{d}\binom{n_5}{e}\binom{n_6}{f}\Big(\binom{n_1}{b}\Big(\binom{n_2}{c}+\binom{n_3}{c}\Big)+\binom{n_2}{b}\binom{n_3}{c}\Big)\Big)+\sum_{\substack{a+b_1+\cdots+b_6=i+1\\ a,b_i\geq 1}} a\binom{w}{a}\prod_{i=1}^{6}\binom{n_i}{b_i}.$$

The computation of the above result are quite constructive, its proof is similar to that of Theorem 3.5, we sketch main ideas of its proof: We need to consider all subsets V_1 and V_{d_i} and there collection which makes a non-zero contribution to $\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n))$. Keeping in view simplicial complex $\Delta = \Delta(\mathcal{P}(\mathbb{Z}_n))$ and Theorem 2.1 and connected components comp(Δ_S) where S in any set of $V(\mathcal{P}(\mathbb{Z}_n))$ such that Δ_S contributes non trivially to initial Betti numbers. First consider $S \subseteq V_1$ (or V_{d_i}) and calculate $\dim_{\mathbb{K}}\widetilde{H}_0(\Delta_S;\mathbb{K})$ for Δ_S and their choices. Next, we take collection of two sets from V_1 (or V_{d_i}) and consider a set S intersecting two sets non-trivially, compute comp(Δ_S) and their total contribution to $\beta_{i,i+1}(\mathcal{P}(\mathbb{Z}_n))$. We ignore those sets S such that comp$(\Delta_S) = 1$, as it contributes nothing to $\beta_{i,i+1}(\mathcal{P}(\mathbb{Z}_n))$. Now, take collection of three sets among V_1 (or V_{d_i}) such that they intersect non-trivially a set S, then compute comp(Δ_S) and their total contribution to $\beta_{i,i+1}(\mathcal{P}(\mathbb{Z}_n))$. Similarly, we proceed with four (five, six, and seven) subsets from V_1 (and V_{d_i}), and consider a non empty subset intersecting non-trivially all these collection of subsets of $V(\mathcal{P}(\mathbb{Z}_n))$, check if comp$(\Delta_S) \geq 2$ (otherwise ignore,) choose common elements form S and V_1 (or V_{d_i}), calculate $\dim_{\mathbb{K}}\widetilde{H}_0(\Delta_S;\mathbb{K})$ and count their final contribution to $\beta_{i,i+1}(\mathcal{P}(\mathbb{Z}_n))$. Finally, summing all such cases, we obtain the required result as given in Theorem 3.7.

With the structure of $G \cong \mathcal{P}(\mathbb{Z}_n)$, the dimension of $\Delta = \Delta(\mathcal{P}(\mathbb{Z}_n))$ is 2. So, Δ consist of i-faces for $0 \leq i \leq 2$. Clearly, the number of 0-faces are n (order of G.) The number of 1-faces are the edges in Δ, which are $\phi(n)(\phi(n)+4)$, since the induced subgraph of non invertible elements of G from a subgraph C, where each vertex of $K_{\phi(d_i)}$ is adjacency to each vertex of $K_{\phi(d_{i+1})}$, and each vertex of $K_{\phi(d_1)}$ is adjacency to each vertex of $K_{\phi(d_6)}$, for $i = p, pr, r, qr, q, pq$. Thus, there are $\phi(n)$, 1-faces between V_{d_i} for $i = p, q, r$ and $\phi(n)^2$, 1-between vertices of V_{d_i} for $i \in \{pq, pr, qr\}$, $\phi(n)$, 1-faces between V_i and V_j for $(i, j) \in \{(p, qr), (q, pr), (r, pq)\}$. And the 2-face of Δ exists among (V_i, V_j, V_k) for $(i, j, k) \in \{(p, q, r), (pq, pr, qr)\}$ are $\phi(n)$ and $\phi(n)^2$, respectively. Thus, the f-vector is

$$\Big(1, n, \phi(n)\big(\phi(n)+4\big), \phi(n)\big(\phi(n)+1\big)\Big).$$

By using (7), it is easy to calculate the h-vector as

$$\Big(1, n-3, \phi(n)\big(\phi(n)+4\big)+3-2n, n-3\phi(n)-1\Big).$$

The Hilbert series is

$$\begin{aligned}\mathcal{H}(\mathbb{K}[\Delta(\mathcal{P}(\mathbb{Z}_n))], t) &= \frac{1+(n-3)t+\phi(n)\big(\phi(n)+4\big)+3-2n)t^2+(n-3\phi(n)-1)t^3}{(1-t)^3}\\ &= \frac{(1-t)^{n-3}\Big(1+(n-3)t+\phi(n)\big(\phi(n)+4\big)+3-2n)t^2+(n-3\phi(n)-1)t^3\Big)}{(1-t)^n}.\end{aligned}$$

Thus, from above results, it remain quite difficult to find all the graded Betti numbers of edge ideals of $\mathcal{P}(\mathbb{Z}_n)$, for other values of n. Next, we establish some inequalities in this regard. For the initial Betti numbers of $I(\mathcal{P}(\mathbb{Z}_n))$, we have the following result.

Theorem 3.8. *For $n = \prod_{i=1}^{t} n_i$ with proper divisor d_i for $1 \leq i \leq k$, the Betti number of $I(\mathcal{P}(\mathbb{Z}_n))$ satisfies*

$$\begin{aligned}\beta_{i,j}(\mathcal{P}(\mathbb{Z}_n)) \geq\ & i\Bigg(\binom{\phi(p)}{i+1}+\sum_{i=1}^{k}\binom{\phi(d_i)}{i+1}\Bigg)+\sum_{\substack{a+l_1+\cdots+l_k=i+1\\ a,l_i\geq 1}}\sum_{j=1}^{k} i\binom{\phi(n)+1}{a}\binom{\phi(d_i)}{l_i}\\ &+\sum_{\substack{a+l_1+\cdots+l_k=i+1\\ a,l_i>1}} a\binom{\phi(n)+1}{a}\prod_{i=1}^{k}\binom{\phi(d_k)}{l_i}\end{aligned}$$

with equality holding if and only if n is product of two distinct primes.

Conclusion

This paper investigates algebraic invariants associated with power graphs of cyclic groups, that is $\mathcal{P}(\mathbb{Z}_n)$ of cyclic group $\mathbb{Z}_n$. For the case where n factors into three distinct primes, we determine: (1) the projective dimension, (2) extremal Betti numbers, and (3) the initial graded Betti numbers of $\mathcal{P}(\mathbb{Z}_n)$, accompanied by its Hilbert series. Furthermore, we establish bounds relating the regularity of edge ideals to their Betti numbers for general power graphs. The results provide new insights into the homological properties of these graph-theoretic representations of finite cyclic groups. However, advancements for the other values of n of edge ideals of $\mathcal{P}(\mathbb{Z}_n)$ remains yet to be investigated, and is an open direction for future work.

References

[1] J. Abawajy, A. Kelarev and M. Chowdhury, Power graphs: A survey, *Electronic J. Graph Theory Appl.* **1**(2) (2013), 125–147.
[2] D. F. Anderson, T. Asir, A. Badawi and T. Tamizh Chelvam, *Graphs from rings*, Springer International Publishing, New York, 2021.
[3] D. Bayer, H. Charalambous and S. Popescu, Extremal Betti numbers and applications to monomial ideals, *J. Algebra* **221** (1999), 497–512.
[4] I. Beck, Coloring of a commutative rings, *J. Algebra* **116** (1988), 208–226.
[5] R. Brauer and K. A. Fowler, On groups of even order, *Ann. Math.* **62** (1955), 565–583.
[6] I. Chakrabarty, M. Ghosh and M. K. Sen, Undirected power graph of semigroups, *Semigroup Forum* **78** (2009), 410–426.
[7] W. Burns and H. Herzog, *Cohen Maculay Rings*, Cambridge Studies in Advanced Mathematics, Second Edition, Cambridge University Press, Cambridge (1998).
[8] R. Chen, Minimal free resolutions of linear edge ideals, *J. Algebra* **324**(12) (2010), 3591–3613.
[9] A. Corso and U. Nagel, Monomial and toric ideals associated to Ferrers graphs, *Trans. Amer. Math. Soc.* **361**(3) (2009), 1371–1395.
[10] O. Fernández-Ramos, P. Gimenez, Regularity 3 in edge ideals associated to bipartite graphs, *J. Algebraic Comb.* **39**(4) (2014), 919–937.
[11] R. Fröberg, On Stanley-Reisner rings, *Topics in Algebra*, Part 2 (Warsaw,1988), 57–70, Banach Center Publications, 26, Part 2, PWN, (1990) Warsaw.
[12] R. Fröberg, Solution to a conjecture on edge rings with 2-linear resolutions, *Comm. Algebra* **51**(4) (2023), 1447–1450.
[13] R. Fröberg, Betti numbers of fat forests and their Alexander dual, *J. Algebraic Comb.* **56** (2022), 1023–1030.
[14] D. Grayson and M. E. Stillman, Macaulay 2, a software system for research in algebraic geometry, Available at: https://faculty.math.illinois.edu/Macaulay2/.
[15] H. T. Hà and A. Van Taul, Monomial ideals, edge, ideals of hypergraphs, and their graded Betti numbers, *J. Algebraic Comb.* **27**(2) (2008), 215–245.
[16] M. Hochster, Cohen-Macaulay rings, combinatorics, and simplicial complexes, In Ring Theory II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla, Lecture Notes in Pure and Appl. Math., Vol. 26, New York, Dekker (1975), 171–223.
[17] A. Kumar, L. Selvaganesh, P. J. Cameron and T. T. Chelvam, Recent developments on the power graph of finite groups-a survey, *AKCE Int. J. Graphs Comb.* **18**(2) (2021), 65–94.
[18] A. V. Kelarev and S. J. Quinn, Directed graphs and combinatorial properties of semigroups, *J. Algebra* **251** (2002), 16–26.
[19] Z. Mehranian, A. Gholami and A. R. Ashrafi, A note on the power graph of a finite group, *Int. J. Group Theory* **5** (2016), 1–10.
[20] J. Herzog and T. Hibi, *Monomial Ideals* Graduate Texts in Mathematics, Vol. 260, Springer London 2010.
[21] S. Jacques, Betti Numbers of Graph Ideals, PhD Thesis, Univesity of Sheffield, (2004), https://arxiv.org/abs/math/0410107.
[22] F. Mohammadi and S. Moradi, Resolution of unmixed bipartite graphs, *Bull. Korean Math. Soc.* **52** (2015) 977–986.
[23] B. A. Rather, Betti numbers of edge ideals of some graphs with application to graphs assigned to groups, *Filomat* **38**(6) (2024), 2185–2204.
[24] B. A. Rather, M. Imran and A. Diene, The linear strand of edge ideals of comaximal graphs of commutative rings, *Comm. Algebra* **52**(4) (2023), 1486–1500.
[25] B. A. Rather, M. Imran and S. Pirzada, Linear strand of edge ideals of zero divisor graphs of the ring $\mathbb{Z}_n$, *Comm. Algebra* **52**(12) (2024), 5069–5085.
[26] M. Roth, A. V. Tyul, On the linear strand of an edge ideal, *Comm. Algebra* **35** (2007), 821–832.
[27] R. K. Shamra and S. M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra* **176**(1) (1995), 124–127.
[28] P. Singh, N. Gupta and S. Anand, Graded Betti numbers of some bipartite graphs, *J. Algebra Appl.* **25**(02) (2026), Art. No. 2550324.
[29] P. Singh and R. Verma, Betti numbers of edge ideals of some split graphs, *Comm. Algebra* **48**(12) (2020), 5026–5037.
[30] R. H. Villarreal, Rees algebras of edge ideals, *Comm. Algebra* **23**(9) (1995), 3513–3524.
[31] R. Woodroofe, Matchings, coverings and Castelnuovo-Mumford regularity, *J. Commut. Algebra* **6**(2) (2014), 287–304.
[32] X. Zheng, Resolutions of facet ideals, *Comm. Algebra* **32**(6) (2004), 2301–2324.