Filomat 40:2 (2026), 751–762
https://doi.org/10.2298/FIL2602751R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Explicit values and vanishing coefficients for Ramanujan's continued fractions of order twenty with applications

Shraddha Rajkhowa[a], Nipen Saikia[a,*]

[a] *Department of Mathematics, Rajiv Gandhi University, Rono Hills, Doimukh-791112, Arunachal Pradesh, India*

Abstract. Two q-continued fractions of order twenty are obtained from a general continued fraction identity of Ramanujan. Some theta-function identities are established for the two continued fractions and are used to prove general theorems for the explicit evaluation of the continued fractions. Some partition theoretic results and vanishing coefficients arising from q-products of the continued fractions are also offered.

1. Introduction

For complex numbers σ and q, define the q-product as

$$(\sigma; q)_\infty := \prod_{t=0}^{\infty} \left(1 - \sigma q^t\right), \qquad |q| < 1. \tag{1}$$

For simplicity, we will write

$$(\sigma_1; q)_\infty (\sigma_2; q)_\infty (\sigma_3; q)_\infty \cdots (\sigma_m; q)_\infty = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_m; q)_\infty .$$

Ramanujan's general theta-function $\mathfrak{f}(g, h)$ [4, p. 34] is

$$\mathfrak{f}(g, h) = \sum_{t=-\infty}^{\infty} g^{t(t+1)/2} h^{t(t-1)/2}, \qquad |gh| < 1. \tag{2}$$

Jacobi's triple product identity [4, p. 35, Entry 19] is

$$\mathfrak{f}(g, h) = (-g, -h, gh; gh)_\infty = (-g; gh)_\infty (-h; gh)_\infty (gh; gh)_\infty. \tag{3}$$

2020 *Mathematics Subject Classification*. Primary 11A55; Secondary 11F27, 11P84.
Keywords. Continued fraction, theta-function, explicit value, colour partition, vanishing coefficient.
Received: 13 August 2025; Accepted: 21 November 2025
Communicated by Paola Bonacini
* Corresponding author: Nipen Saikia
Email addresses: shraddha.rajkhowa@rgu.ac.in (Shraddha Rajkhowa), nipennak@yahoo.com (Nipen Saikia)
ORCID iDs: https://orcid.org/0009-0008-9591-1734 (Shraddha Rajkhowa), https://orcid.org/0000-0002-0288-8731 (Nipen Saikia)

Three specific instances of $\mathfrak{f}(g,h)$ [4, p. 36, Entry 22 (i)-(iii)] are

$$\phi(q) := \mathfrak{f}(q,q) = \sum_{t=-\infty}^{\infty} q^{t^2}, \tag{4}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{t=0}^{\infty} q^{t(t+1)/2} \tag{5}$$

and

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{t=-\infty}^{\infty} (-1)^t q^{t(3t-1)/2}, \tag{6}$$

respectively. Ramanujan in his second notebook [14, p. 299] recorded a continued fraction of order eight, known as the Ramanujan-Göllnitz-Gordon continued fraction defined by

$$H(q) := q^{1/2}\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = q^{1/2}\frac{\mathfrak{f}(-q,-q^7)}{\mathfrak{f}(-q^3,-q^5)} = \cfrac{q^{1/2}}{1+q+\cfrac{q^2}{1+q^3+\cfrac{q^4}{1+q^5+\cdots}}}, \quad |q|<1. \tag{7}$$

Göllnitz [8], Gordon [9] and Andrews [1] also independently rediscovered and proved (7). An alternative proof of (7) was also given by Ramanathan [13]. Ramanujan also offered the following theta-function identity [14, p. 299] for $H(q)$:

$$\frac{1}{H(q)} - H(q) = \frac{\phi(q^2)}{q^{1/2}\psi(q^4)}. \tag{8}$$

Proof of (8) can be found in [4, p. 221]. Chan and Huang [7] found many identities involving the continued fraction $H(q)$ and evaluated explicitly $H(e^{-\pi\sqrt{n}/2})$ for several positive integers n. Baruah and Saikia [3] established some general theorems for explicit evaluations of $H(q)$ and evaluated some values.

Apart from particular q-continued fractions, Ramanujan also recorded some general continued fraction identities in his notebooks. For example, Ramanujan gave the following general continued fraction identity [4, p. 24, Entry 12]: let g, h and q are complex numbers with $|gh|<1$ and $|q|<1$, or that $g = h^{2l+1}$ for some integer l, then

$$\frac{(g^2q^3,h^2q^3;q^4)_\infty}{(g^2q,h^2q;q^4)_\infty} = \cfrac{1}{1-gh+\cfrac{(g-hq)(h-gq)}{(1-gh)(q^2+1)+\cfrac{(g-hq^3)(h-gq^3)}{(1-gh)(q^4+1)+\cdots}}}. \tag{9}$$

Replacing q by q^5 in (9), setting $\{g=q, h=q^4\}$ and $\{g=q^2, h=q^3\}$, and then simplifying using (3) and the results that $\{(q^{23};q^{20})_\infty = (q^3;q^{20})_\infty/(1-q^3)\}$ and $\{(q^{21};q^{20})_\infty = (q;q^{20})_\infty/(1-q)\}$, we derive following two continued fractions of order twenty as

$$T_1(q) := q\frac{(q^3,q^{17};q^{20})_\infty}{(q^7,q^{13};q^{20})_\infty} = q\frac{\mathfrak{f}(-q^3,-q^{17})}{\mathfrak{f}(-q^7,-q^{13})} = \cfrac{q(1-q^3)}{(1-q^5)+\cfrac{q^5(1-q^2)(1-q^8)}{(1-q^5)(1+q^{10})+\cfrac{q^5(1-q^{12})(1-q^{18})}{(1-q^5)(1+q^{20})+\cdots}}} \tag{10}$$

and

$$T_2(q) := q^2\frac{(q,q^{19};q^{20})_\infty}{(q^9,q^{11};q^{20})_\infty} = q^2\frac{\mathfrak{f}(-q,-q^{19})}{\mathfrak{f}(-q^9,-q^{11})} = \cfrac{q^2(1-q)}{(1-q^5)+\cfrac{q^5(1-q^4)(1-q^6)}{(1-q^5)(1+q^{10})+\cfrac{q^5(1-q^{14})(1-q^{16})}{(1-q^5)(1+q^{20})+\cdots}}}, \tag{11}$$

respectively.

We establish theta-function identities and modular relations for $T_1(q)$ and $T_2(q)$ in Sect. 2. Sect. 3 is devoted to proving general theorems for explicit evaluation of the two continued fractions. In Sect. 4, we derive some partition-theoretic results from the theta-function identities of $T_1(q)$ and $T_2(q)$ using colour partition of integer. Finally, we offer vanishing coefficient results arising from the q-products of the continued fractions in Sect. 5.

2. Theta-function identities and modular relations for $T_1(q)$ and $T_2(q)$

This section is devoted to proving some theta-function identities and modular relations for the continued fractions $T_1(q)$ and $T_2(q)$.

Theorem 2.1. *We have*

(i) $T_1^{-1}(q) \pm T_1(q) = \dfrac{\phi(\mp q^5)\mathfrak{f}(\pm q^2, \pm q^8)}{q\psi(q^{10})\mathfrak{f}(-q^3, -q^7)}$,

(ii) $T_2^{-1}(q) \pm T_2(q) = \dfrac{\phi(\mp q^5)\mathfrak{f}(\pm q^4, \pm q^6)}{q^2\psi(q^{10})\mathfrak{f}(-q, -q^9)}$,

(iii) $\left(T_1^{-1}(q) - T_1(q)\right)\left(T_2^{-1}(q) - T_2(q)\right) = \dfrac{\phi^2(q^5)\psi(q)}{q^3\psi^2(q^{10})\psi(q^5)}$,

(iv) $\left(T_1^{-1}(q) + T_1(q)\right)\left(T_2^{-1}(q) + T_2(q)\right) = \dfrac{\phi^2(-q^5)\phi(-q^{10})}{q^3\psi^2(q^{10})\psi(q^5)\chi(-q)\chi(-q^2)}$,

(v) $\left(T_1^{-1}(q) - T_1(q)\right) + \left(T_2^{-1}(q) - T_2(q)\right) = \dfrac{\phi(q^5)\mathfrak{f}(q, q^4)\mathfrak{f}(-q^2, -q^3)}{q^2\psi(q^{10})\chi(-q)f(q^5)f(-q^{20})}$,

(vi) $\left(T_1^{-1}(q) + T_1(q)\right) + \left(T_2^{-1}(q) + T_2(q)\right) = \dfrac{\phi(-q^5)f(q)}{q^2\psi(q^{10})\chi(-q)f(-q^{20})}$.

Proof. From (10), we obtain

$$\sqrt{T_1^{-1}(q)} - \sqrt{T_1(q)} = \frac{\mathfrak{f}(-q^7, -q^{13}) - q\mathfrak{f}(-q^3, -q^{17})}{\sqrt{q\mathfrak{f}(-q^3, -q^{17})\mathfrak{f}(-q^7, -q^{13})}}. \tag{12}$$

From [4, p. 46, Entry 30 (ii) and (iii)], we note that

$$\mathfrak{f}(g, h) = \mathfrak{f}(g^3h, gh^3) + g\,\mathfrak{f}(h/g, g^5h^3). \tag{13}$$

Setting $\{g = -q, h = q^4\}$ and $\{g = q, h = -q^4\}$ in (13), we obtain

$$\mathfrak{f}(-q, q^4) = \mathfrak{f}(-q^7, -q^{13}) - q\mathfrak{f}(-q^3, -q^{17}) \tag{14}$$

and

$$\mathfrak{f}(q, -q^4) = \mathfrak{f}(-q^7, -q^{13}) + q\mathfrak{f}(-q^3, -q^{17}), \tag{15}$$

respectively. Employing (14) in (12), we find that

$$\sqrt{T_1^{-1}(q)} - \sqrt{T_1(q)} = \frac{\mathfrak{f}(-q, q^4)}{\sqrt{q\mathfrak{f}(-q^3, -q^{17})\mathfrak{f}(-q^7, -q^{13})}}. \tag{16}$$

Similarly, from (10) and applying (15), we deduce that

$$\sqrt{T_1^{-1}(q)} + \sqrt{T_1(q)} = \frac{\mathfrak{f}(q, -q^4)}{\sqrt{q\mathfrak{f}(-q^3, -q^{17})\mathfrak{f}(-q^7, -q^{13})}}. \tag{17}$$

Combining (16) and (17), we arrive at

$$T_1^{-1}(q) - T_1(q) = \frac{\mathfrak{f}(-q,q^4)\mathfrak{f}(q,-q^4)}{q\mathfrak{f}(-q^3,-q^{17})\mathfrak{f}(-q^7,-q^{13})}. \tag{18}$$

Again, from [4, p. 46, Entry 30 (i),(iv)], we note that

$$\mathfrak{f}(g,gh^2)\mathfrak{f}(h,g^2h) = \mathfrak{f}(g,h)\psi(gh) \tag{19}$$

and

$$\mathfrak{f}(g,h)\mathfrak{f}(-g,-h) = \mathfrak{f}(-g^2,-h^2)\phi(-gh). \tag{20}$$

Setting $\{g = -q^3, h = -q^7\}$ in (19) and $\{g = -q, h = q^4\}$ in (20), we obtain

$$\mathfrak{f}(-q^3,-q^{17})\mathfrak{f}(-q^7,-q^{13}) = \mathfrak{f}(-q^3,-q^7)\psi(q^{10}) \tag{21}$$

and

$$\mathfrak{f}(-q,q^4)\mathfrak{f}(q,-q^4) = \mathfrak{f}(-q^2,-q^8)\phi(q^5), \tag{22}$$

respectively. Employing (21) and (22) in (18), we obtain

$$T_1^{-1}(q) - T_1(q) = \frac{\phi(q^5)\mathfrak{f}(-q^2,-q^8)}{q\psi(q^{10})\mathfrak{f}(-q^3,-q^7)}. \tag{23}$$

Squaring (17), we obtain

$$T_1^{-1}(q) + T_1(q) = \frac{\mathfrak{f}^2(q,-q^4)}{q\mathfrak{f}(-q^3,-q^{17})\mathfrak{f}(-q^7,-q^{13})} - 2. \tag{24}$$

From [4, p. 46, Entry 30 (v),(vi)], we note that

$$\mathfrak{f}^2(g,h) = \mathfrak{f}(g^2,h^2)\phi(gh) + 2g\mathfrak{f}(h/g,g^3h)\psi(g^2h^2). \tag{25}$$

Setting $g = q$ and $h = -q^4$, we obtain

$$\mathfrak{f}^2(q,-q^4) = \mathfrak{f}(q^2,q^8)\phi(-q^5) + 2q\mathfrak{f}(-q^3,-q^{17})\mathfrak{f}(-q^7,-q^{13}). \tag{26}$$

Employing (21) and (26) in (24) and simplifying, we arrive at

$$T_1^{-1}(q) + T_1(q) = \frac{\phi(-q^5)\mathfrak{f}(q^2,q^8)}{q\psi(q^{10})\mathfrak{f}(-q^3,-q^7)}. \tag{27}$$

Combining (23) and (27), we complete the proof of (i).
Proof of (ii) is identical to the proof of (i), so we omit.
Again, from (i) and (ii), we have

$$\left(T_1^{-1}(q) - T_1(q)\right)\left(T_2^{-1}(q) - T_2(q)\right) = \frac{\phi^2(q^5)\mathfrak{f}(-q^2,-q^8)\mathfrak{f}(-q^4,-q^6)}{q^3\psi^2(q^{10})\mathfrak{f}(-q^3,-q^7)\mathfrak{f}(-q,-q^9)}. \tag{28}$$

From [4, p. 258, Entry 9(vii)], we have

$$\mathfrak{f}(-q,-q^4)\mathfrak{f}(-q^2,-q^3) = f(-q)f(-q^5), \tag{29}$$

$$\mathfrak{f}(q,q^4)\mathfrak{f}(q^2,q^3) = \frac{\phi(-q^5)f(-q^5)}{\chi(-q)} \tag{30}$$

and

$$\mathfrak{f}(q,q^9)\mathfrak{f}(q^3,q^7) = \chi(q)f(-q^5)f(-q^{20}). \tag{31}$$

Replacing q by q^2 in (29) and replacing q by $-q$ in (31) and then using in (28), we obtain

$$\left(T_1^{-1}(q) - T_1(q)\right)\left(T_2^{-1}(q) - T_2(q)\right) = \frac{\phi^2(q^5)f(-q^2)f(-q^{10})}{q^3\psi^2(q^{10})\chi(-q)f(q^5)f(-q^{20})}. \tag{32}$$

Again, by q-series manipulation one can easily see that

$$\chi(-q) = \frac{f(-q)}{f(-q^2)}, \quad \text{and} \quad f(q) = \frac{f^3(-q^2)}{f(-q)f(-q^4)}. \tag{33}$$

Simplifying (32) with the help of (33), we arrive at (iii). Employing the same procedure and using (30), we arrive at (iv). Again, employing (i) and (ii), we have

$$\left(T_1^{-1}(q) - T_1(q)\right) + \left(T_2^{-1}(q) - T_2(q)\right) = \frac{\phi(q^5)\left[q\mathfrak{f}(-q^2,-q^8)\mathfrak{f}(-q,-q^9) + \mathfrak{f}(-q^4,-q^6)\mathfrak{f}(-q^3,-q^7)\right]}{q^2\psi(q^{10})\mathfrak{f}(-q^3,-q^7)\mathfrak{f}(-q,-q^9)}. \tag{34}$$

From [4, p. 45, Entry 29], we have

$$\mathfrak{f}(a,b)\mathfrak{f}(c,d) = \mathfrak{f}(ac,bd)\mathfrak{f}(ad,bc) + a\mathfrak{f}(b/c,ac^2d)\mathfrak{f}(b/d,acd^2); \quad for \quad ab = cd. \tag{35}$$

Setting $\{a = -q^2, b = -q^8, c = -q, d = -q^9\}$ and $\{a = -q^4, b = -q^6, c = -q^3, d = -q^7\}$ in (35), we obtain

$$\mathfrak{f}(-q^2,-q^8)\mathfrak{f}(-q,-q^9) = \mathfrak{f}(q^3,q^{17})\mathfrak{f}(q^{11},q^9) - q^2\mathfrak{f}(q^7,q^{13})\mathfrak{f}(q^{-1},q^{21}) \tag{36}$$

and

$$\mathfrak{f}(-q^4,-q^6)\mathfrak{f}(-q^3,-q^7) = \mathfrak{f}(q^7,q^{13})\mathfrak{f}(q^{12},q^8) - q^4\mathfrak{f}(q^3,q^{17})\mathfrak{f}(q^{-1},q^{21}), \tag{37}$$

respectively. Using (36), (37) in (34) and replacing q by $-q$ in (31) and then employing in (34), we arrive at

$$\left(T_1^{-1}(q) - T_1(q)\right) + \left(T_2^{-1}(q) - T_2(q)\right) = \frac{\phi(q^5)\left(\mathfrak{f}(q^7,q^{13}) + q\mathfrak{f}(q^3,q^{17})\right)\left(\mathfrak{f}(q^9,q^{11}) - q^2\mathfrak{f}(q,q^{19})\right)}{q^2\psi(q^{10})\chi(-q)f(q^5)f(-q^{20})}, \tag{38}$$

where we have used the fact that $\mathfrak{f}(q^{-1},q^{21}) = q^{-1}f(q,q^{19})$.
Setting $\{g = q, h = q^4\}$ and $\{g = -q^2, h = -q^3\}$ in (13), we obtain

$$\mathfrak{f}(q,q^4) = \mathfrak{f}(q^7,q^{13}) + q\mathfrak{f}(q^3,q^{17}) \tag{39}$$

and

$$\mathfrak{f}(-q^2,-q^3) = \mathfrak{f}(q^9,q^{11}) - q^2\mathfrak{f}(q,q^{19}), \tag{40}$$

respectively. Employing (39) and (40) in (38), we arrive at (v). Proof of (vi) is similar to the proof of (v), so omitted. □

Theorem 2.2. *For any non-negative integer n, we have*

(i) $T_1^n(q)T_1^n(-q) = \begin{cases} T_1^n(q^2), & if \quad n \equiv 0 \pmod 2 \\ -T_1^n(q^2), & if \quad n \equiv 1 \pmod 2. \end{cases}$

(ii) $T_2^n(q)T_2^n(-q) = T_2^n(q^2)$.

Proof. From (10), we note that

$$T_1^n(q)T_1^n(-q) = (-1)^n q^{2n} \frac{\mathfrak{f}^n(-q^3,-q^{17})}{\mathfrak{f}^n(-q^7,-q^{13})} \times \frac{\mathfrak{f}^n(q^3,q^{17})}{\mathfrak{f}^n(q^7,q^{13})}. \tag{41}$$

Setting $\{g = q^3, h = q^{17}\}$ and $\{g = q^7, h = q^{13}\}$ in (20), we find that

$$\mathfrak{f}(q^3,q^{17})\mathfrak{f}(-q^3,-q^{17}) = \mathfrak{f}(-q^6,-q^{34})\phi(-q^{20}) \tag{42}$$

and

$$\mathfrak{f}(q^7,q^{13})\mathfrak{f}(-q^7,-q^{13}) = \mathfrak{f}(-q^{14},-q^{26})\phi(-q^{20}), \tag{43}$$

respectively. Employing (42) and (43) in (41), we obtain

$$T_1^n(q)T_1^n(-q) = (-1)^n q^{2n} \frac{\mathfrak{f}^n(-q^6,-q^{34})}{\mathfrak{f}^n(-q^{14},-q^{26})} = (-1)^n T_1^n(q^2). \tag{44}$$

Noting the fact that n is even if $n \equiv 0 \pmod 2$ and odd if $n \equiv 1 \pmod 2$ in (44), we complete the proof of (i). Proof of (ii) is identical to the proof of (i), so we omit. □

3. Explicit evaluations of $T_1(q)$ and $T_2(q)$

In this section, we establish general theorems to find explicit values of $T_1(q)$ and $T_2(q)$ with the help of the explicit values of the continued fractions $I(q)$ and $J(q)$ of order ten established in [12]:

$$I(q) := q^{3/4}\frac{(q,q^9;q^{10})_\infty}{(q^4,q^6;q^{10})_\infty} = q^{3/4}\frac{\mathfrak{f}(-q,-q^9)}{\mathfrak{f}(-q^4,-q^6)} = \cfrac{q^{3/4}(1-q)}{(1-q^{5/2}) + \cfrac{q^{5/2}(1-q^{3/2})(1-q^{7/2})}{(1-q^{5/2})(1+q^5) + \cfrac{q^{5/2}(1-q^{13/2})(1-q^{17/2})}{(1-q^{5/2})(1+q^{10}) + \cdots}}} \tag{45}$$

and

$$J(q) := q^{1/4}\frac{(q^2,q^8;q^{10})_\infty}{(q^3,q^7;q^{10})_\infty} = q^{1/4}\frac{\mathfrak{f}(-q^2,-q^8)}{\mathfrak{f}(-q^3,-q^7)} = \cfrac{q^{1/4}(1-q^2)}{(1-q^{5/2}) + \cfrac{q^{5/2}(1-q^{1/2})(1-q^{9/2})}{(1-q^{5/2})(1+q^5) + \cfrac{q^{5/2}(1-q^{11/2})(1-q^{19/2})}{(1-q^{5/2})(1+q^{10}) + \cdots}}}. \tag{46}$$

We will use the parameter $s_{4,n}$, which is the particular case, $k = 4$ of the parameter $s_{k,n}$ defined by Berndt [6, p. 9, (4.7)]:

$$s_{4,n} = \frac{f(q)}{\sqrt{2}q^{1/8} f(-q^4)}, \qquad q = e^{-\pi\sqrt{n}/2}, \tag{47}$$

where n is a positive real number. It is useful to note that Baruah and Saikia [3] proved the following formula for the explicit evaluation of $H(q)$ [3, p. 275, (3.5)]:

$$\frac{1}{H(e^{-\pi\sqrt{n}/4})} - H(e^{-\pi\sqrt{n}/4}) = 2s_{4,n}^2. \tag{48}$$

Baruah and Saikia [3] calculated many explicit values of the parameter $s_{4,n}$ to evaluate explicit values of $H(q)$ by appealing to (48). In [12], authors proved some general theorems for the explicit evaluation of $I(q)$ and $J(q)$ and evaluated some explicit values. For example, they evaluated

$$I^2(e^{-\pi\sqrt{2/5}}) = \left(-1 + \sqrt{2(5-2\sqrt{5})}\right) / \sqrt{5-2\sqrt{5}}, \tag{49}$$

$$J^2(e^{-\pi\sqrt{2/5}}) = 5\left(-1+\sqrt{2(5-2\sqrt{5})}\right)/\left(5-2\sqrt{5}\right)^{3/2}, \tag{50}$$

$$I^2(e^{-\pi/\sqrt{5}}) = \frac{\left(-5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}+\sqrt{10(20+8\sqrt{5}-5\sqrt{2(1+\sqrt{5})}-3\sqrt{10(1+\sqrt{5})})}\right)}{2\sqrt{10\left(5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}\right)}}, \tag{51}$$

and

$$J^2(e^{-\pi/\sqrt{5}}) = \frac{\sqrt{5}\left(-5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}+\sqrt{10(20+8\sqrt{5}-5\sqrt{2(1+\sqrt{5})}-3\sqrt{10(1+\sqrt{5})})}\right)}{\sqrt{2}\left(5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}\right)^{3/2}}. \tag{52}$$

In the next theorem, we give general formulas for explicit evaluation of $T_1(q)$ and $T_2(q)$.

Theorem 3.1. *We have*

$$(i)\quad T_1^{-1}(e^{-\pi\sqrt{n}/2}) - T_1(e^{-\pi\sqrt{n}/2}) = 2s_{4,25n}^2 J(e^{-\pi\sqrt{n}/2}), \tag{53}$$

$$(ii)\quad T_2^{-1}(e^{-\pi\sqrt{n}/2}) - T_2(e^{-\pi\sqrt{n}/2}) = 2s_{4,25n}^2 I^{-1}(e^{-\pi\sqrt{n}/2}). \tag{54}$$

Proof. Employing (8) and (46) in Theorem 2.1(i), we obtain

$$T_1^{-1}(q) - T_1(q) = \left(\frac{1}{H(q^{5/2})} - H(q^{5/2})\right)J(q). \tag{55}$$

Setting $q = e^{-\pi\sqrt{n}/2}$ in (55), we obtain

$$T_1^{-1}(e^{-\pi\sqrt{n}/2}) - T_1(e^{-\pi\sqrt{n}/2}) = \left(\frac{1}{H(e^{-\pi\sqrt{25n}/4})} - H(e^{-\pi\sqrt{25n}/4})\right)J(e^{-\pi\sqrt{n}/2}). \tag{56}$$

Employing (48) (replace n by $25n$) in (56), we arrive at (i). Proof of (ii) follows identically from Theorem 2.1 (ii). □

Remark 3.2. *From Theorem 3.1, one can easily find the explicit values of $T_1(e^{-\pi\sqrt{n}/2})$ and $T_2(e^{-\pi\sqrt{n}/2})$ with the help of the parameters $s_{4,n}$ and explicit values of $I(q)$ and $J(q)$. For example, taking $n = 8/5$, employing the value $s_{4,40} = 2^{-3/4}(1+\sqrt{5})^{3/4}(2+3\sqrt{2}+\sqrt{5})^{1/4}$ from [3] and then employing* (50) *and* (49) *in Theorem 3.1 (i) and (ii), respectively, we obtain*

$$T_1(e^{-\pi\sqrt{2/5}}) = \frac{1}{4(5-2\sqrt{5})^{3/4}} -5\sqrt{2\cdot w} - \sqrt{10\cdot w} + 4\cdot\sqrt{z} \tag{57}$$

and

$$T_2(e^{-\pi\sqrt{2/5}}) = \frac{4\sqrt{-1+\sqrt{10-4\sqrt{5}}}}{4\cdot\sigma + (5-2\sqrt{5})^{1/4}(1+\sqrt{5})\sqrt{14+6(\sqrt{2}+\sqrt{5}+\sqrt{10})}}, \tag{58}$$

where

$$w = -7-3\sqrt{2}-3\sqrt{5}-3\sqrt{10}+\left(3\sqrt{10}+7\sqrt{2}+6+6\sqrt{5}\right)\sqrt{5-2\sqrt{5}},$$

$$z = -45 - 30\sqrt{2} - 20\sqrt{5} - 15\sqrt{10} + \left(20\sqrt{10} + 45\sqrt{2} + 65 + 20\sqrt{5}\right)\sqrt{5 - 2\sqrt{5}}$$

and

$$\sigma = -1 + \left(3\sqrt{10} + 7\sqrt{2} + 9 + 4\sqrt{5}\right)\sqrt{5 - 2\sqrt{5}}.$$

Again, taking $n = 4/5$, employing the value $s_{4,20} = 2^{-1/2}(1+\sqrt{5})^{3/8}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/4}$ from [3] and then employing (52) *and* (51) *in Theorem 3.1 (i) and (ii), respectively, we obtain*

$$T_1(e^{-\pi/\sqrt{5}}) = \frac{1}{4}\Big(-5^{1/4}\frac{2^{3/4}(1+\sqrt{5})^{3/4}}{\left(5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}\right)^{3/4}}\sqrt{(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})})\cdot A}$$
$$+\sqrt{16+\frac{2\sqrt{10}(1+\sqrt{5})^{3/2}(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})})\cdot A}{\left(5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}\right)}}\Big) \tag{59}$$

and

$$T_2(e^{-\pi/\sqrt{5}}) = \frac{1}{2}\Big(-2^{3/4}(1+\sqrt{5})^{3/2}(5+\sqrt{5}-\sqrt{10(1+\sqrt{5})})^{1/4}\sqrt{A^{-1}(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})})}$$
$$+\sqrt{4+\left(2(1+\sqrt{5})^{3/2}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)\sqrt{10\left(5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}\right)}\right)\cdot A^{-1}}\Big), \tag{60}$$

where

$$A = -5+\sqrt{5}-\sqrt{10(1+\sqrt{5})}+\sqrt{10\left(20+8\sqrt{5}-(5+3\sqrt{5})\sqrt{2(1+\sqrt{5})}\right)}.$$

Theorem 3.3. *We have*

(*i*) $T_1^2(-q) = -T_1^2(q^2)T_1^{-2}(q)$,

(*ii*) $T_2(-q) = -T_2(q^2)T_2^{-1}(q)$.

Proof. Setting $n = 2$ in Theorem 2.2 (i), and $n = 1$ in (ii), we arrived at (i) and (ii), respectively. □

From the above theorem, we can easily evaluate the explicit values of $T_1(-q)$ and $T_2(-q)$ for any values of q, if we know the explicit values of $T_1(q)$, $T_1(q^2)$, $T_2(q)$ and $T_2(q^2)$.

4. Some partition-theoretic results

In this section, we show that colour partition identities can be obtained from the theta-function identities established in Theorem 2.1 using colour partition of integer. As example, we deduce two partition-theoretic identities from the theta-function identities of the continued fraction $T_1(q)$. One can also obtain similar identities from the other theta-function identities of Theorem 2.1. First, we give the definition colour partition of a positive integer n and its generating function.

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. A part in a partition of n is said to have r colours if each part has r copies and all of them are viewed as distinct objects. For any positive integer n and r, let $C_r(n)$ denote the number of partitions of n with each part has r distinct colours. For example, if each part of partition of 3 has 2 colours, say white (indicated by the suffix w) and black (indicated by the suffix b), then the number of 2 colour partition of 3 is 10, namely

3_w, 3_b, 2_w+1_w, 2_w+1_b, 2_b+1_w, 2_b+1_b, $1_w+1_w+1_w$, $1_w+1_w+1_b$, $1_w+1_b+1_b$, $1_b+1_b+1_b$. The generating function of $C_r(n)$ is given by

$$\sum_{n=0}^{\infty} C_r(n)q^n = \frac{1}{(q;q)_\infty^r}. \tag{61}$$

For positive integers s, m and r, the quotient

$$\frac{1}{(q^s;q^m)_\infty^r} \tag{62}$$

is the generating function of the number of partitions of n with parts congruent to s modulo m and each part has r colours. For example,

$$\frac{1}{(q^{s_1};q^m)_\infty^r(q^{s_2};q^m)_\infty^r} = \frac{1}{(q^{s_1},q^{s_2};q^m)_\infty^r} \tag{63}$$

is the generating function of the number of partitions of positive integer with parts congruent to s_1 or s_2 modulo m and each part has r distinct colours. For convenience, we use the notation

$$(q^{r\pm};q^t)_\infty := (q^r,q^{t-r};q^t)_\infty, \tag{64}$$

where r and t are positive integers and $r < t$. It is also useful to note here that

$$\phi(-q) = \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{65}$$

Theorem 4.1. *Let* $S_1(n) :=$ *number of partitions of n into parts* $\equiv \pm2, \pm3, \pm8$ *or* $\pm10 \pmod{20}$ *such that the parts* $\equiv \pm3$ *and* $\pm10 \pmod{20}$ *have 2 colours.*
Let $S_2(n) :=$ *number of partitions of n into parts* $\equiv \pm2, \pm7, \pm8$ *or* $\pm10 \pmod{20}$ *such that parts* $\equiv \pm7$ *and* $\pm10 \pmod{20}$ *have 2 colours.*
Let $S_3(n) :=$ *number of partitions of n into parts* $\equiv \pm3, \pm5$ *and* $\pm7 \pmod{20}$ *with 2 colours.*
Then for any integer $n \geq 2$,

$$S_1(n) - S_2(n-2) - S_3(n) = 0.$$

Proof. *Employing* (4), (5) *and* (10) *in* (23), *we obtain*

$$\frac{(q^{7\pm};q^{20})_\infty}{(q^{3\pm};q^{20})_\infty} - q^2\frac{(q^{3\pm};q^{20})_\infty}{(q^{7\pm};q^{20})_\infty} - \frac{(q^{2\pm,8\pm};q^{20})_\infty(q^{10\pm};q^{20})_\infty^2}{(q^{3\pm,7\pm};q^{20})_\infty(q^{5\pm};q^{20})_\infty^2} = 0. \tag{66}$$

Dividing (66) *by* $(q^{2\pm,3\pm,7\pm,8\pm};q^{20})_\infty(q^{10\pm},q^{20})_\infty^2$, *we obtain*

$$\frac{1}{(q^{3\pm,10\pm};q^{20})_\infty^2(q^{2\pm,8\pm};q^{20})_\infty} - \frac{q^2}{(q^{7\pm,10\pm};q^{20})_\infty^2(q^{2\pm,8\pm};q^{20})_\infty} - \frac{1}{(q^{3\pm,5\pm,7\pm};q^{20})_\infty^2} = 0. \tag{67}$$

The above quotients of (67) *represent the generating functions for* $C_1(n)$, $C_2(n)$ *and* $C_3(n)$, *respectively. Hence,* (67) *is equivalent to*

$$\sum_{n=0}^{\infty} S_1(n)q^n - q^2\sum_{n=0}^{\infty} S_2(n)q^n - \sum_{n=0}^{\infty} S_3(n)q^n = 0, \tag{68}$$

where we set $S_1(0) = S_2(0) = S_3(0) = 1$. *Equating coefficients of* q^n *on both sides of* (68), *we arrive at the desired result.* □

To verify Theorem 4.1, we construct the following table with $n = 8$:

$S_1(8) = 5$	$S_2(6) = 1$	$S_3(8) = 4$
8	$2+2+2$	$5_r + 3_r$
$3_r + 3_r + 2$		$5_r + 3_g$
$3_r + 3_g + 2$		$5_g + 3_r$
$3_g + 3_g + 2$		$5_g + 3_g$
$2+2+2+2$		

Theorem 4.2. *Let $X_1(n)$:= the number of partitions of n into parts $\equiv \pm 3, \pm 4, \pm 5$ or $\pm 10 \pmod{20}$ such that the parts $\equiv \pm 3$ and $\pm 5 \pmod{20}$ have 2 colours.*
Let $X_2(n)$:= the number of partitions of n into parts $\equiv \pm 4, \pm 5, \pm 7$ or $\pm 10 \pmod{20}$ such that parts $\equiv \pm 5$ and $\pm 7 \pmod{20}$ have 2 colours.
Let $X_3(n)$:= the number of partitions of n into parts $\equiv \pm 2, \pm 3, \pm 7$ or $\pm 8 \pmod{20}$ such that parts $\equiv \pm 3$ and $\pm 7 \pmod{20}$ have 2 colours.
Then for any integer $n \geq 2$,

$$X_1(n) + X_2(n-2) - X_3(n) = 0.$$

Proof. *Employing* (5), (10) *and* (65) *in* (27), *employing the same procedure, we obtain*

$$\frac{1}{(q^{3\pm,5\pm};q^{20})_\infty^2 (q^{4\pm,10\pm};q^{20})_\infty} + \frac{q^2}{(q^{5\pm,7\pm};q^{20})_\infty^2 (q^{4\pm,10\pm};q^{20})_\infty} = \frac{1}{(q^{3\pm,7\pm};q^{20})_\infty^2 (q^{2\pm,8\pm};q^{20})_\infty}. \tag{69}$$

The above quotients of (69) *represent the generating functions for $X_1(n)$, $X_2(n)$ and $X_3(n)$, respectively. Hence,* (69) *is equivalent to*

$$\sum_{n=0}^{\infty} X_1(n)q^n + q^2 \sum_{n=0}^{\infty} X_2(n)q^n - \sum_{n=0}^{\infty} X_3(n)q^n = 0, \tag{70}$$

where we set $X_1(0) = X_2(0) = X_3(0) = 1$. Equating coefficients of q^n on both sides of (70), *we arrive at the desired result.* □

To verify Theorem 4.2, we construct the following table with $n = 7$:

$X_1(7) = 2$	$X_2(5) = 2$	$X_3(7) = 4$
$4 + 3_r$	5_r	7_r
$4 + 3_g$	5_g	7_g
		$3_r + 2 + 2$
		$3_g + 2 + 2$

5. Vanishing Coefficients

Recently, many authors have studied vanishing coefficients in the arithmetic progressions of several q-series expansions. One can see [2], Hirschhorn [10] and references there in for details. In this section, we offer vanishing coefficient results obtain from the q-series expansion of the continued fractions $T_1(q)$, $T_2(q)$ and their reciprocals.

Theorem 5.1. *If*

$$T_1^*(q) := q^{-1}T_1(q) = \frac{\mathfrak{f}(-q^3,-q^{17})}{\mathfrak{f}(-q^7,-q^{13})} = \sum_{n=0}^{\infty} s_n q^n \quad \text{and} \quad \frac{1}{T_1^*(q)} = \sum_{n=0}^{\infty} s'_n q^n ,$$

then

$$(i) \quad s_{10n+2} = 0 \quad \text{and} \quad (ii) \quad s'_{10n+4} = 0.$$

Proof. J. M. Laughlin [11] stated the following p-dissection formula

$$\frac{(q^t,q^t,q^{r+s},q^{t-r-s};q^t)_\infty}{(q^s,q^{t-s},q^r,q^{t-r};q^t)_\infty} = \sum_{j=0}^{p-1} q^{jr}\frac{(q^{pt},q^{pt},q^{pr+s+jt},q^{(p-j)t-pr-s};q^{pt})_\infty}{(q^{jt+s},q^{(p-j)t-s},q^{pr},q^{(t-r)p};q^{pt})_\infty}, \tag{71}$$

where all of the powers of q in each of the infinite products on the right hand side must be multiples of p and the integer r must satisfy $gcd(r,p)=1$. Now, setting $t=20$, $r=7$, $s=10$ and $p=5$ in (71), we obtain

$$\frac{(q^{20},q^{20},q^{17},q^3;q^{20})_\infty}{(q^{10},q^{10},q^7,q^{13};q^{20})_\infty} = \frac{(q^{100};q^{100})^2_\infty}{(q^{35},q^{65};q^{100})_\infty}\Big[\frac{(q^{45},q^{55};q^{100})_\infty}{(q^{10},q^{90};q^{100})_\infty} + q^7\frac{(q^{65},q^{35};q^{100})_\infty}{(q^{30},q^{70};q^{100})_\infty}$$
$$+q^{14}\frac{(q^{85},q^{15};q^{100})_\infty}{(q^{50},q^{50};q^{100})_\infty} + q^{21}\frac{(q^{105},q^{-5};q^{100})_\infty}{(q^{70},q^{30};q^{100})_\infty} + q^{28}\frac{(q^{125},q^{-25};q^{100})_\infty}{(q^{90},q^{10};q^{100})_\infty}\Big]. \tag{72}$$

Multiplying both sides of (72) by $(q^{10};q^{20})^2_\infty/(q^{20};q^{20})^2_\infty$ and then simplifying, we obtain

$$\sum_{n=0}^{\infty} s_n q^n = \frac{(q^{10};q^{20})^2_\infty(q^{100};q^{100})^2_\infty}{(q^{20};q^{20})^2_\infty(q^{35},q^{65};q^{100})_\infty}\Big[\frac{(q^{45},q^{55};q^{100})_\infty}{(q^{10},q^{90};q^{100})_\infty} + q^7\frac{(q^{65},q^{35};q^{100})_\infty}{(q^{30},q^{70};q^{100})_\infty}$$
$$+q^{14}\frac{(q^{85},q^{15};q^{100})_\infty}{(q^{50},q^{50};q^{100})_\infty} + q^{21}\frac{(q^{105},q^{-5};q^{100})_\infty}{(q^{70},q^{30};q^{100})_\infty} + q^{28}\frac{(q^{125},q^{-25};q^{100})_\infty}{(q^{90},q^{10};q^{100})_\infty}\Big]. \tag{73}$$

Extracting the terms involving q^{5n+2} from (73), dividing by q^2 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} s_{5n+2} q^n = q\frac{(q^2;q^4)^2_\infty(q^{20};q^{20})^2_\infty}{(q^4;q^4)^2_\infty(q^6,q^{14};q^{20})_\infty}. \tag{74}$$

Since the right hand side of (74) contains no term involving q^{2n}, extracting the terms involving q^{2n} in (74), we arrive at (i).

Again, setting $t=20$, $r=3$, $s=10$ and $p=5$ in (71), we obtain

$$\frac{(q^{20},q^{20},q^{13},q^7;q^{20})_\infty}{(q^{10},q^{10},q^3,q^{17};q^{20})_\infty} = \frac{(q^{100};q^{100})^2_\infty}{(q^{15},q^{85};q^{100})_\infty}\Big[\frac{(q^{25},q^{75};q^{100})_\infty}{(q^{10},q^{90};q^{100})_\infty} + q^3\frac{(q^{45},q^{55};q^{100})_\infty}{(q^{30},q^{70};q^{100})_\infty} + q^6\frac{(q^{65},q^{35};q^{100})_\infty}{(q^{50},q^{50};q^{100})_\infty}$$
$$+q^9\frac{(q^{85},q^{15};q^{100})_\infty}{(q^{70},q^{30};q^{100})_\infty} + q^{12}\frac{(q^{105},q^{-5};q^{100})_\infty}{(q^{90},q^{10};q^{100})_\infty}\Big]. \tag{75}$$

Multiplying both sides of (75) by $(q^{10};q^{20})^2_\infty/(q^{20};q^{20})^2_\infty$ and then simplifying, we obtain

$$\sum_{n=0}^{\infty} s'_n q^n = \frac{(q^{10};q^{20})^2_\infty(q^{100};q^{100})^2_\infty}{(q^{20};q^{20})^2_\infty(q^{15},q^{85};q^{100})_\infty}\Big[\frac{(q^{25},q^{75};q^{100})_\infty}{(q^{10},q^{90};q^{100})_\infty} + q^3\frac{(q^{45},q^{55};q^{100})_\infty}{(q^{30},q^{70};q^{100})_\infty} + q^6\frac{(q^{65},q^{35};q^{100})_\infty}{(q^{50},q^{50};q^{100})_\infty}$$
$$+q^{21}\frac{(q^{85},q^{15};q^{100})_\infty}{(q^{70},q^{30};q^{100})_\infty} + q^{12}\frac{(q^{105},q^{-5};q^{100})_\infty}{(q^{90},q^{10};q^{100})_\infty}\Big]. \tag{76}$$

Extracting the terms involving q^{5n+4} from (76), dividing by q^4 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} s'_{5n+4}q^n = q\frac{(q^2;q^4)_\infty^2(q^{20};q^{20})_\infty^2}{(q^4;q^4)_\infty^2(q^6,q^{14};q^{20})_\infty}. \tag{77}$$

Since the right hand side of (77) contains no term involving q^{2n}, extracting the terms involving q^{2n} in (77), we arrive at (ii). □

The proof of the next theorem is similar to the proof of Theorem 5.1, so we omit the proof here.

Theorem 5.2. *If*

$$T_2^*(q) := q^{-2}T_2(q) = \frac{\mathfrak{f}(-q,-q^{19})}{\mathfrak{f}(-q^9,-q^{11})} = \sum_{n=0}^{\infty} c_n q^n \quad \textit{and} \quad \frac{1}{T_2^*(q)} = \sum_{n=0}^{\infty} c'_n q^n\ ,$$

then

$$(i)\quad c_{10n+5} = 0 \quad \textit{and} \quad (ii)\quad c'_{10n+9} = 0.$$

References

[1] G. E. Andrews, *On q-difference equations for certain well-poised basic hypergeometric series*, Quart. J. Math. (Oxford) **19** (1968), 433–447.

[2] N. D. Baruah and M. Kaur, *Some results on vanishing coefficients in infinite product expansions*, Ramanujan J. **53** (2020), 551–568.

[3] N. D. Baruah and N. Saikia, *Explicit evaluations of Ramanujan-Göllnitz-Gordon continued fraction*, Monatsh Math. **154**(4) (2008), 271–288.

[4] B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.

[5] B. C. Berndt, *Ramanujan's Notebooks, Part V*, Springer-Verlag, New York, 1998.

[6] B. C. Berndt, *Flowers which we cannot yet see growing in Ramanujan's garden of hypergeometeric series, elliptic functions, and q's, in Special Functions 2000: Current Perspective and Future Directions* (J. Bustoz, M. E. H. Ismail, and S. K. Suslov, eds.), Kluwer, Dordrecht, (2001), 61–85.

[7] H. H. Chan and S.-S. Huang, *On the Ramanujan-Göllnitz-Gordon continued fraction*, Ramanujan J. **1** (1997), 75–90.

[8] H. Göllnitz, *Partitionen mit Differenzenbedingungen*, J. Reine Angew. Math. **225** (1967), 154–190.

[9] B. Gordon, *Some continued fractions of Rogers-Ramanujan's type*, Duke Math. **16** (1985), 695–724.

[10] M. D. Hirschhorn, *Two remarkable q-series expansions*, Ramanujan J. **49**(2) (2018), 451–463.

[11] J. M. Laughlin, *Some observations on Lambert series, vanishing coefficients and dissections of infinite products and series*. https://doi.org/10.48550/arXiv.1906.11978

[12] S. Rajkhowa and N. Saikia, *Some results on Ramanujan's continued fractions of order ten and applications*, Indian J. Pure Appl. Math. **56**(1) (2025),13–37.

[13] K. G. Ramanathan, *On Ramanujan's continued fraction*, Acta Arith. **43** (1984), 209–226.

[14] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.