Filomat 40:2 (2026), 763–772
https://doi.org/10.2298/FIL2602763M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some results related to positive semidefinite matrices by means of Cauchy-Schwarz inequality

Mahmut Furkan Meniz[a], Ramazan Türkmen[b,*], Sever S. Dragomir[c,d]

[a]*Selcuk University, Institute of Science, Department of Mathematics, Konya, Turkey*
[b]*Selçuk University, Faculty of Science, Department of Mathematics, Konya, Turkey*
[c]*Department of Mathematical and Physical Sciences, La Trobe University, Plenty Road, Bundoora, Melbourne VIC 3086, Australia*
https://scholars.latrobe.edu.au/sdragomir
[d]*Mathematical Sciences, School of Science, RMIT University, GPO Box 2476V, Melbourne, Victoria 3001, Australia*
https://academics.rmit.edu.au/silvestru-dragomir

Abstract. In this study, the relationship between the Cauchy-Schwarz inequality and matrix norms and singular values is discussed. First, the classical form of the Cauchy-Schwarz inequality is recalled, and its extended versions in matrix space are examined. Then, the connections between singular values and unitarily invariant norms are addressed, and various inequalities are derived in this context. In particular, the majorization properties of singular values for Hermitian and positive semidefinite matrices are investigated.

1. Introduction

Let $M_n(\mathbb{C})$ be the space of all $n \times n$ complex matrices. For an arbitrary matrix A, let $|A| \equiv (A^*A)^{1/2}$. Here, A^* is the conjugate transpose of A, and the square root of the eigenvalue of A^*A is called the singular value of matrix A. Symbolically, $s_i(A) = \sqrt{\lambda_i(A^*A)} = \lambda_i^{1/2}(A^*A) = \lambda_i(A^*A)^{1/2} = \lambda_i(|A|)$, meaning $s_i(A) = \lambda_i(|A|)$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. When we arrange the components of this vector in descending order, we obtain $x^{\downarrow} = \left(x_1^{\downarrow}, x_2{}^{\downarrow}, \ldots, x_n^{\downarrow}\right)$, where $x_1^{\downarrow} \geq x_2^{\downarrow} \geq \ldots \geq x_n^{\downarrow}$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, if for $k = 1, 2, \ldots, n$;

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow}$$

then the vector x is said to be weakly majorized by y, denoted as $x \prec_w y$.
If $x \prec_w y$ and also

2020 *Mathematics Subject Classification*. Primary 15A42; Secondary 15A48.
Keywords. Cauchy-Schwarz Inequality, Singular Values, Majorization, Unitarily Invariant Norms.
Received: 19 August 2025; Revised: 18 October 2025; Accepted: 13 November 2025
Communicated by Fuad Kittaneh
* Corresponding author: Ramazan Türkmen
Email addresses: menizfurkan1@gmail.com (Mahmut Furkan Meniz), rturkmen@selcuk.edu.tr (Ramazan Türkmen), sever.dragomir@vu.edu.au (Sever S. Dragomir)
ORCID iDs: https://orcid.org/0009-0000-1129-1171 (Mahmut Furkan Meniz), https://orcid.org/0000-0002-5503-1862 (Ramazan Türkmen), https://orcid.org/0000-0003-2902-6805 (Sever S. Dragomir)

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

then x is said to be majorized by y, denoted as $x \prec y$. Thus, majorization includes weak majorization. To use majorization, $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$ is arranged in descending order as $s_1(A) \geq s_2(A) \geq \ldots \geq s_n(A)$. If A is a Hermitian matrix, its eigenvalues are real and can be ordered as $\lambda_1(A) \geq \ldots \geq \lambda_n(A)$. For Hermitian matrices A and B, if $A - B \geq 0$, then $B \leq A$. Hence, $\lambda_j(B) \leq \lambda_j(A) \quad 1 \leq j \leq n$. This results is known as Weyl's Monotonicity Principle.

- For an arbitrary matrix A, $A^*A \geq 0$.
- For $A \geq 0$, $X^*AX \geq 0$.
- If $A, B \geq 0$, then $A + B \geq 0$.
- $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0 \Rightarrow \begin{pmatrix} A & -B \\ -B^* & C \end{pmatrix} \geq 0, \begin{pmatrix} C & B \\ B^* & A \end{pmatrix} \geq 0$. Additionally;

The direct sum of matrices $A, B \in M_n(\mathbb{C})$ is defined as $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and denoted by $A \oplus B$.

A norm $\|.\|$ on $M_n(\mathbb{C})$ is called a unitarily invariant norm if for all $A \in M_n(\mathbb{C})$ and unitary matrices $U, V \in M_n(\mathbb{C})$,

$$\|UAV\| = \|A\|$$

holds. Unitarily invariant norms are monotonically increasing functions of singular values.
Let H be an inner product space. For $x, y \in H$,

$$|\langle x, y \rangle| \leq \|x\|\|y\|$$

is known as the classical Cauchy-Schwarz Inequality.
Let a and b be positive numbers. For $0 \leq v \leq 1$,

$$H_v(a, b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2}$$

is called the Heinz Mean. The Heinz Mean lies between the geometric mean and the arithmetic mean, and can be written as

$$\sqrt{ab} \leq H_v(a, b) \leq \frac{a + b}{2}.$$

Bhatia and Davis [6] generalized this to matrices. For positive semidefinite matrices,

$$\left\|A^{1/2}B^{1/2}\right\| \leq \|H_v(A, B)\| \leq \left\|\frac{A + B}{2}\right\|$$

holds.
Let A be a Hermitian matrix. If for all $x \in \mathbb{C}^n$,

$$\langle Ax, x \rangle \geq 0$$

then A is called a positive semidefinite matrix. (If $\langle Ax, x \rangle > 0$ for $x \neq 0$, then A is called a positive definite matrix.) Moreover, for any $x, y \in \mathbb{C}^n$ and $A \geq 0$, the following inequality:

$$|\langle Ax, y \rangle|^2 \leq \langle Ax, x \rangle \langle Ay, y \rangle$$

is a generalized form of the Cauchy-Schwarz inequality, (See [11], p. 221).
For a positive semidefinite matrix A,

$$\|Ax\|^2 \le \|A\|\langle Ax, x\rangle.$$

In 1952, Kato [12] showed that for any matrix $T \in M_n(\mathbb{C})$, $x, y \in \mathbb{C}^n$ and $\alpha \in [0,1]$,

$$|\langle Tx, y\rangle|^2 \le \langle (T^*T)^{\alpha} x, x\rangle\langle (TT^*)^{1-\alpha} y, y\rangle.$$

In 1994, Furuta [10] showed that for any $T \in M_n(\mathbb{C})$ $x, y \in \mathbb{C}^n, \alpha, \beta \in [0,1]$, and $\alpha + \beta \ge 1$,

$$|\langle T|T|^{\alpha+\beta-1} x, y\rangle|^2 \le \langle |T|^{2\alpha} x, x\rangle\langle |T^*|^{2\beta} y, y\rangle.$$

More detailed and good works on singular values inequalities have been discussed recently in [1],[3],[4],[5], [14],[15],[17].

In this paper, we generalize some inequalities involving singular values and norm inequalities for 2×2 positive semidefinite block matrices in a different perspective.

LEMMAS

In this section, we list some results that will be used in our further considerations. The first lemma was mentioned by Kittaneh [13].

Lemma 1.1. *Let $A, B, C \in M_n(\mathbb{C})$, with $A, B \ge 0$. Then*

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix}$$

is a positive if and only if

$$|\langle Cx, y\rangle|^2 \le \langle Ax, x\rangle\langle By, y\rangle,$$

for all $x, y \in \mathbb{C}^n$.

This is an extension of the Cauchy-Schwarz inequality.

Lemma 1.2. *[20] (von Neumann): Let $A, B \in M_{m\times n}$. Then*

$$s(A) \prec_w s(B) \Leftrightarrow \|A\| \le \|B\|,$$

for all unitarily invariant matrix-vector norms $\|.\|$ on $\in M_{m\times n}$.

Y.Tao proved the following lemma, which is used in many studies:

Lemma 1.3. *[18] Let $A, C \in M_n(\mathbb{C})$ and $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \ge 0$. Then*

$$2s_j(B) \le s_j\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \quad j = 1, 2, \dots, n.$$

W.Audeh and F.Kittaneh showed the following lemma in [2].

Lemma 1.4. *Let $A, B, C \in M_n(\mathbb{C})$ such that $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \ge 0$. Then*

$$s_j(B) \le s_j(A \oplus C) \quad j = 1, 2 \dots, n.$$

F.Zhang proved the following lemma, which is used in many studies:

Lemma 1.5. *[20] Let H and K be $n \times n$ Hermitian matrices. Then*

$$\begin{pmatrix} H & K \\ K & H \end{pmatrix} \ge 0 \Leftrightarrow \pm K \le H \Rightarrow |s(K)| \prec_w s(H) \Rightarrow \|K\| \le \|H\|.$$

Burqan and F.Kittaneh showed the following lemma in [9].

Lemma 1.6. *Let* $A, B, C \in M_n(\mathbb{C})$ *and* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \geq 0.$ *Then*

$$s_j\left(B + B^*\right) \leq s_j((A + C) \oplus (A + C)) \quad j = 1, 2 \ldots, n$$

and

$$\|B + B^*\| \leq \|A + C\|.$$

R.Bhatia and F.Kittaneh proved the following lemma in [7].

Lemma 1.7. *For all* $n \times n$ *matrices* A *and* B*;*

$$2s_j\left(A^*B\right) \leq s_j\left(AA^* + BB^*\right) \quad j = 1, 2, \ldots, n.$$

Lemma 1.8. *[19] Let* A *and* B *be positive matrices. Then*

$$s(A \oplus A) \prec_w s((A + B) \oplus (A - B)).$$

Lemma 1.9. *[16] Let* M, N *be square matrices of the same size. Then*

$$\begin{pmatrix} M & K \\ K^* & N \end{pmatrix} \geq 0 \Rightarrow s(K) \prec_w \frac{1}{2}\{\lambda(M) + \lambda(N)\}.$$

Lemma 1.10. *[19] Let* $x, y, z \in \mathbb{R}^n$*. If* $2x \prec y + z$*, then* $(x, x) \prec (y, z)$*.*

Lemma 1.11. *[8] Let* X, Y *be Hermitian matrices and* $\pm Y \leq X$*. Then*

$$s_j(Y) \leq s_j(X \oplus X) \quad 1 \leq j \leq n.$$

T.Furuta has proved the following lemma in [10].

Lemma 1.12. *Let* $A, B \geq 0$ *and* $0 \leq \alpha \leq 1$*. Then*

$$s_j\left(A^\alpha B^{1-\alpha}\right) \leq s_j(\alpha A + (1 - \alpha)B) \quad 1 \leq j \leq n.$$

2. Main Results

Theorem 2.1. *Let* A *and* B *be positive semidefinite matrices. For all* $x, y \in \mathbb{C}^n$ *and* $\alpha \in [0, 1]$*, the following inequality holds:*

$$\left|\left\langle\left(A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha\right)x, y\right\rangle\right|^2 \leq \left\langle\left(A^{2\alpha} + A^{2-2\alpha}\right)x, x\right\rangle\left\langle\left(B^{2-2\alpha} + B^{2\alpha}\right)y, y\right\rangle.$$

Proof. Let $X = \begin{pmatrix} A^\alpha & B^{1-\alpha} \\ A^{1-\alpha} & B^\alpha \end{pmatrix}$. Then $X^* = \begin{pmatrix} A^\alpha & A^{1-\alpha} \\ B^{1-\alpha} & B^\alpha \end{pmatrix}$. We have

$$\begin{aligned} X^*X &= \begin{pmatrix} A^\alpha & A^{1-\alpha} \\ B^{1-\alpha} & B^\alpha \end{pmatrix}\begin{pmatrix} A^\alpha & B^{1-\alpha} \\ A^{1-\alpha} & B^\alpha \end{pmatrix} \\ &= \begin{pmatrix} A^{2\alpha} + A^{2-2\alpha} & A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha \\ B^{1-\alpha}A^\alpha + B^\alpha A^{1-\alpha} & B^{2-2\alpha} + B^{2\alpha} \end{pmatrix} \geq 0. \end{aligned}$$

From Lemma 1.1, we obtain;

$$\left|\left\langle\left(A^\alpha B^{1-\alpha} + A^{1-\alpha}B^\alpha\right)x, y\right\rangle\right|^2 \leq \left\langle\left(A^{2\alpha} + A^{2-2\alpha}\right)x, x\right\rangle\left\langle\left(B^{2-2\alpha} + B^{2\alpha}\right)y, y\right\rangle$$

which proves the statement.

Corollary 2.2. *For $\alpha = 1/2$,*

$$\left|\left\langle 2\left(\left(A^{1/2}+B^{1/2}\right)\right)x,y\right\rangle\right|^2 \leq \langle (2A)x,x\rangle\langle (2B)y,y\rangle$$

is obtained. Specifically, if A is replaced by A^2 and B is replaced by B^2, we get

$$|\langle (A+B)x,y\rangle|^2 \leq \left\langle A^2x,x\right\rangle\left\langle B^2y,y\right\rangle.$$

In addition, using properties of inner product, it is easily seen that

$$\begin{aligned}|\langle (A+B)x,y\rangle|^2 &\leq \langle Ax,A^*x\rangle\langle By,B^*y\rangle\\ &= \langle Ax,Ax\rangle\langle By,By\rangle\\ &= \|Ax\|^2\,\|By\|^2.\end{aligned}$$

From different perspective, we have the following result as well.

Theorem 2.3. *Let $A,B\geq 0$. For all $x,y\in\mathbb{C}^n$ and $\alpha\in[0,1]$. Then*

$$\left|\left\langle \left(A^{\alpha}B^{1-\alpha}\right)x,y\right\rangle\right|^2 \leq \left\langle A^{2\alpha}x,x\right\rangle\left\langle B^{2-2\alpha}y,y\right\rangle.$$

Proof. Let $X=\begin{pmatrix}A^{\alpha} & B^{1-\alpha}\\ 0 & 0\end{pmatrix}$. Then $X^*=\begin{pmatrix}A^{\alpha} & 0\\ B^{1-\alpha} & 0\end{pmatrix}$.

$$X^*X\geq 0 \Rightarrow \begin{pmatrix}A^{\alpha} & 0\\ B^{1-\alpha} & 0\end{pmatrix}\begin{pmatrix}A^{\alpha} & B^{1-\alpha}\\ 0 & 0\end{pmatrix}=\begin{pmatrix}A^{2\alpha} & A^{\alpha}B^{1-\alpha}\\ B^{1-\alpha}A^{\alpha} & B^{2-2\alpha}\end{pmatrix}\geq 0.$$

From Lemma 1.1, we obtain

$$\left|\left\langle \left(A^{\alpha}B^{1-\alpha}\right)x,y\right\rangle\right|^2 \leq \left\langle A^{2\alpha}x,x\right\rangle\left\langle B^{2-2\alpha}y,y\right\rangle.$$

Corollary 2.4. *For $\alpha = 1/2$,*

$$\left|\left\langle \left(A^{1/2}B^{1/2}\right)x,y\right\rangle\right|^2 \leq \langle Ax,x\rangle\langle By,y\rangle.$$

Theorem 2.5. *Let $A,B\geq 0$ and $\alpha\in[0,1]$. Then*

$$s_j\left(A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha}\right)\leq s_j\left(\left(A^{2\alpha}+B^{2-2\alpha}\right)\oplus\left(A^{2\alpha}+B^{2-2\alpha}\right)\right).$$

Proof. Let $X=\begin{pmatrix}A^{\alpha} & B^{1-\alpha}\\ 0 & 0\end{pmatrix}$. Then $X^*=\begin{pmatrix}A^{\alpha} & 0\\ B^{1-\alpha} & 0\end{pmatrix}$.

$X^*X=\begin{pmatrix}A^{\alpha} & 0\\ B^{1-\alpha} & 0\end{pmatrix}\begin{pmatrix}A^{\alpha} & B^{1-\alpha}\\ 0 & 0\end{pmatrix}=\begin{pmatrix}A^{2\alpha} & A^{\alpha}B^{1-\alpha}\\ B^{1-\alpha}A^{\alpha} & B^{2-2\alpha}\end{pmatrix}\geq 0$. Thus

$$\begin{pmatrix}A^{2\alpha}+B^{2-2\alpha} & A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha}\\ A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha} & A^{2\alpha}+B^{2-2\alpha}\end{pmatrix}\geq 0.$$

By Lemma 1.6,

$$s_j\left(A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha}\right)\leq s_j\left(\left(A^{2\alpha}+B^{2-2\alpha}\right)\oplus\left(A^{2\alpha}+B^{2-2\alpha}\right)\right)$$

is obtained.

Corollary 2.6. *From Lemma 1.2,*

$$\left\|A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha}\right\|\leq\left\|A^{2\alpha}+B^{2-2\alpha}\right\|.$$

For $\alpha = 1/2$,

$$\left\|A^{1/2}B^{1/2}+B^{1/2}A^{1/2}\right\|\leq\|A+B\|.$$

Specifically, if A is replaced by A^2 and B is replaced by B^2,

$$\|AB + BA\| \leq \left\|A^2 + B^2\right\|$$

holds.

Theorem 2.7. *Let $A \in M_n(\mathbb{C})$ and $\alpha, \beta \in [0,1]$, $\alpha + \beta \geq 1$. Then*

$$s_j\left(A|A|^{\alpha+\beta-1}\right) \leq s_j\left(|A|^{2\alpha} \oplus |A|^{2\beta}\right).$$

Proof. Observe that

$$\begin{pmatrix} |A|^{2\alpha} & A|A|^{\alpha+\beta-1} \\ |A^*|^{\alpha+\beta-1} A^* & |A|^{2\beta} \end{pmatrix} \geq 0.$$

It is clear that this matrix is positive semidefinite due to Furuta's result [10]. Therefore, by Lemma 1.3,

$$2s_j\left(A|A|^{\alpha+\beta-1}\right) \leq \begin{pmatrix} |A|^{2\alpha} & A|A|^{\alpha+\beta-1} \\ |A^*|^{\alpha+\beta-1} A^* & |A|^{2\beta} \end{pmatrix}$$

is obtained. Also, using Lemma 1.4,

$$\begin{pmatrix} |A|^{2\alpha} & A|A|^{\alpha+\beta-1} \\ |A^*|^{\alpha+\beta-1} A^* & |A|^{2\beta} \end{pmatrix} \leq 2s_j\left(|A|^{2\alpha} \oplus |A|^{2\beta}\right)$$

can be written. Connecting these two inequalities; we get

$$2s_j\left(A|A|^{\alpha+\beta-1}\right) \leq \begin{pmatrix} |A|^{2\alpha} & A|A|^{\alpha+\beta-1} \\ |A^*|^{\alpha+\beta-1} A^* & |A|^{2\beta} \end{pmatrix} \leq 2s_j\left(|A|^{2\alpha} \oplus |A|^{2\beta}\right).$$

That is;

$$2s_j\left(A|A|^{\alpha+\beta-1}\right) \leq 2s_j\left(|A|^{2\alpha} \oplus |A|^{2\beta}\right).$$

Rearranging this inequality, as result

$$s_j\left(A|A|^{\alpha+\beta-1}\right) \leq s_j\left(|A|^{2\alpha} \oplus |A|^{2\beta}\right)$$

is obtained.

Theorem 2.8. *Let A and B be positive semidefinite matrices and $\alpha \in [0,1]$. Then*

$$2s_j(A \oplus B) \leq s_j\left(\left(A^{2\alpha} + A^{2-2\alpha}\right) \oplus \left(B^{2\alpha} + B^{2-2\alpha}\right)\right) \textit{ for } \; j = 1, 2, \ldots, n.$$

Proof. Let $X = \begin{pmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & A^{1-\alpha} \\ B^{1-\alpha} & 0 \end{pmatrix}$. Then $X^* = \begin{pmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{pmatrix}$ and $Y^* = \begin{pmatrix} 0 & B^{1-\alpha} \\ A^{1-\alpha} & 0 \end{pmatrix}$. Hence

$$X^*Y = \begin{pmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{pmatrix}\begin{pmatrix} 0 & A^{1-\alpha} \\ B^{1-\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}.$$

Since $s_j(A) = s_j(A^*) = s_j(|A^*|)$, $s_j(X^*Y) = s_j(A \oplus B)$. On the other hand,

$$XX^* = \begin{pmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{pmatrix}\begin{pmatrix} A^\alpha & 0 \\ 0 & B^\alpha \end{pmatrix} = \begin{pmatrix} A^{2\alpha} & 0 \\ 0 & B^{2\alpha} \end{pmatrix},$$

$$YY^* = \begin{pmatrix} 0 & A^{1-\alpha} \\ B^{1-\alpha} & 0 \end{pmatrix}\begin{pmatrix} 0 & B^{1-\alpha} \\ A^{1-\alpha} & 0 \end{pmatrix} = \begin{pmatrix} A^{2-2\alpha} & 0 \\ 0 & B^{2-2\alpha} \end{pmatrix}$$

and also

$$XX^* + YY^* = \begin{pmatrix} A^{2\alpha} & 0 \\ 0 & B^{2\alpha} \end{pmatrix} + \begin{pmatrix} A^{2-2\alpha} & 0 \\ 0 & B^{2-2\alpha} \end{pmatrix}$$
$$= \begin{pmatrix} A^{2\alpha} + A^{2-2\alpha} & 0 \\ 0 & B^{2\alpha} + B^{2-2\alpha} \end{pmatrix} \geq 0.$$

By Lemma 1.7,

$$2s_j(A \oplus B) \leq s_j\left(\left(A^{2\alpha} + A^{2-2\alpha}\right) \oplus \left(B^{2\alpha} + B^{2-2\alpha}\right)\right) for \quad j = 1, 2, \ldots, n$$

is obtained.

Corollary 2.9. *From Lemma 1.2,*

$$\|A \oplus B\| \leq \frac{\left\|\left(A^{2\alpha} + A^{2-2\alpha}\right) \oplus \left(B^{2\alpha} + B^{2-2\alpha}\right)\right\|}{2}$$

is found. For $\alpha = 1$,

$$\|A \oplus B\| \leq \frac{\left\|\left(A^2 + I\right) \oplus \left(B^2 + I\right)\right\|}{2}$$

is obtained.

Theorem 2.10. *Let $A \in M_n(\mathbb{C})$ and $\alpha \in [0, 1]$. Then*

$$s_j(A) \leq s_j\left((A^*A)^\alpha \oplus (AA^*)^{1-\alpha}\right).$$

Proof. Observe that

$$\begin{pmatrix} (A^*A)^\alpha & A \\ A^* & (AA^*)^{1-\alpha} \end{pmatrix} \geq 0.$$

We know that this matrix is positive semidefinite due to Kato's result [12]. From Lemma 1.3,

$$2s_j(A) \leq s_j\begin{pmatrix} (A^*A)^\alpha & A \\ A^* & (AA^*)^{1-\alpha} \end{pmatrix}$$

can be derived. Also, using Lemma 1.4,

$$s_j\begin{pmatrix} (A^*A)^\alpha & A \\ A^* & (AA^*)^{1-\alpha} \end{pmatrix} \leq 2s_j\left((A^*A)^\alpha \oplus (AA^*)^{1-\alpha}\right)$$

is obtained. Connecting these two inequalities;

$$2s_j(A) \leq 2s_j\left((A^*A)^\alpha \oplus (AA^*)^{1-\alpha}\right)$$
$$s_j(A) \leq s_j\left((A^*A)^\alpha \oplus (AA^*)^{1-\alpha}\right)$$

is obtained.

Theorem 2.11. *Let $A, B \geq 0$ and $\alpha \in [0, 1]$. Then*

$$s_j\left(A^\alpha B^{1-\alpha} + B^{1-\alpha}A^\alpha\right) \leq s_j\left(\left(A^{2\alpha} + B^{2-2\alpha}\right) \oplus \left(A^{2\alpha} + B^{2-2\alpha}\right)\right) for \quad j = 1, 2, \ldots, n.$$

Proof. Let $X = \begin{pmatrix} A^\alpha & 0 \\ B^{1-\alpha} & 0 \end{pmatrix}$. Then $X^* = \begin{pmatrix} A^\alpha & B^{1-\alpha} \\ 0 & 0 \end{pmatrix}$.

$$XX^* = \begin{pmatrix} A^\alpha & 0 \\ B^{1-\alpha} & 0 \end{pmatrix}\begin{pmatrix} A^\alpha & B^{1-\alpha} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{2\alpha} & A^\alpha B^{1-\alpha} \\ B^{1-\alpha}A^\alpha & B^{2-2\alpha} \end{pmatrix} \geq 0.$$

From Lemma 1.8 and Lemma 1.9,

$$s_j\Big(A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha}\Big) \le \lambda_j\Big((A^{2\alpha}+B^{2-2\alpha})\oplus(A^{2\alpha}+B^{2-2\alpha})\Big) = s_j\Big((A^{2\alpha}+B^{2-2\alpha})\oplus(A^{2\alpha}+B^{2-2\alpha})\Big).$$

Since it is known that $x \le y$ (component-wise) $\Rightarrow x \prec_w y$;

$$s\left(A^{\alpha}B^{1-\alpha}+B^{1-\alpha}A^{\alpha}\right) \prec_w s\left(\left(A^{2\alpha}+B^{2-2\alpha}\right)\oplus\left(A^{2\alpha}+B^{2-2\alpha}\right)\right)$$

is obtained.

Corollary 2.12. *For $\alpha = 1/2$,*

$$s\left(A^{1/2}B^{1/2}+B^{1/2}A^{1/2}\right) \prec_w s((A+B)\oplus(A+B)),$$

and if A is replaced by A^2 and B is replaced by B^2;

$$s(AB+BA) \prec_w s\left(\left(A^2+B^2\right)\oplus\left(A^2+B^2\right)\right)$$

is found. This inequality was obtained by Bhatia and Kittaneh (see, [9], Proposition 6.2).

Theorem 2.13. *Let $A, B \ge 0$ and $\alpha \in [0,1]$. Then*

$$2s_j(A+B) \le s_j\left(A^{2\alpha}+A^{2-2\alpha}+B^{2\alpha}+B^{2-2\alpha}\oplus 0\right).$$

Proof. Let $X = \begin{pmatrix} A^{\alpha} & A^{1-\alpha} \\ B^{1-\alpha} & B^{\alpha} \end{pmatrix}$. Then

$$X^*X = \begin{pmatrix} A^{2\alpha}+B^{2-2\alpha} & A+B \\ A+B & A^{2-2\alpha}+B^{2\alpha} \end{pmatrix}$$

and

$$2s_j(A+B) \le s_j\begin{pmatrix} A^{2\alpha}+B^{2-2\alpha} & A+B \\ A+B & A^{2-2\alpha}+B^{2\alpha} \end{pmatrix}$$

is obtained.

Corollary 2.14. *From Lemma 1.2,*

$$2\|A+B\| \le \left\|\begin{pmatrix} A^{2\alpha}+B^{2-2\alpha} & A+B \\ A+B & A^{2-2\alpha}+B^{2\alpha} \end{pmatrix}\right\|$$

is found.

Theorem 2.15. *Let $A, B \ge 0$. For $0 \le \alpha \le 1$ and r is a positive integer. Then*

$$2s_j\left(A^{\alpha}\left(A^{2\alpha}+B^{2-2\alpha}\right)^r B^{1-\alpha}\right) \le s_j\left(A^{2\alpha}+B^{2-2\alpha}\right)^{r+1}.$$

Proof. For any matrix X, the polar decomposition is $X = UP$. Then,

$$(XX^*)^{r+1} = X(X^*X)^r X^*$$

can be easily seen. For $X = \begin{pmatrix} A^{\alpha} & 0 \\ B^{1-\alpha} & 0 \end{pmatrix}$,

$$(XX^*)^{r+1} = X(X^*X)^r X^* = \begin{pmatrix} A^{\alpha}(A^{2\alpha}+B^{2-2\alpha})^r A^{\alpha} & A^{\alpha}(A^{2\alpha}+B^{2-2\alpha})^r B^{1-\alpha} \\ B^{1-\alpha}(A^{2\alpha}+B^{2-2\alpha})^r A^{\alpha} & B^{1-\alpha}(A^{2\alpha}+B^{2-2\alpha})^r B^{1-\alpha} \end{pmatrix}$$

is obtained. Moreover, $(XX^*)^{r+1}$ and $(X^*X)^{r+1}$ are unitarily equivalent, and their singular values are equal. Using Lemma 1.11, seen that

$$\begin{aligned} 2\, s_j\Big(A^{\alpha}(A^{2\alpha}+B^{2-2\alpha})^{*}B^{1-\alpha}\Big) &\le s_j\Big((XX^*)^{r+1}\Big) \\ &= s_j\Big((X^*X)^{r+1}\Big) \\ &= s_j\Big((A^{2\alpha}+B^{2-2\alpha})^{r+1}\Big). \end{aligned}$$

Corollary 2.16. *For $\alpha = 1/2$, we get*

$$s_j\left(A^{1/2}(A+B)^r B^{1/2}\right) \le s_j(A+B)^{r+1}, 1 \le j \le n.$$

This inequality was obtained by Bhatia and Kittaneh (see,e.g.,[18]).

Theorem 2.17. *Let A and B be positive semidefinite matrices. For all $x, y \in \mathbb{C}^n$ and $\alpha \in [0,1]$. Then*

$$|\langle Ax, y\rangle|^2 \le \left\langle A^{2\alpha}x, x\right\rangle \left\langle A^{2-2\alpha}y, y\right\rangle.$$

Proof. Let $X = \begin{pmatrix} A^{\alpha} & A^{1-\alpha} \\ 0 & 0 \end{pmatrix}$. Then $X^* = \begin{pmatrix} A^{\alpha} & 0 \\ A^{1-\alpha} & 0 \end{pmatrix}$. Thus,

$$X^*X = \begin{pmatrix} A^{\alpha} & 0 \\ A^{1-\alpha} & 0 \end{pmatrix}\begin{pmatrix} A^{\alpha} & A^{1-\alpha} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A^{2\alpha} & A \\ A & A^{2-2\alpha} \end{pmatrix}.$$

From Lemma 1.1,

$$\begin{aligned} |\langle Ax, y\rangle|^2 &\le \langle A^{2\alpha}x, x\rangle\langle A^{2-2\alpha}y, y\rangle \\ &= \langle A^{\alpha}x, A^{\alpha}x\rangle\langle A^{1-\alpha}y, A^{1-\alpha}y\rangle \\ &\le \|A^{\alpha}x\|^2\|A^{1-\alpha}y\|^2 \\ &\le \|A^{\alpha}x\|\|A^{1-\alpha}y\|. \end{aligned}$$

In particular, for $\alpha = 1$ and $A = I$;

$$|\langle x, y\rangle| \le \|x\|\|y\|$$

is obtained. This inequality is the known Cauchy-Schwarz Inequality.

Theorem 2.18. *Let $A \in M_n(\mathbb{C})$ and $\alpha \in [0,1]$. Then*

$$s\,(A + A^*) \prec_w s\left((A^*A)^{\alpha} + (AA^*)^{1-\alpha}\right).$$

Proof. Let

$$M = \begin{pmatrix} (A^*A)^{\alpha} & A \\ A^* & (AA^*)^{1-\alpha} \end{pmatrix} \ge 0,$$

and

$$N = \begin{pmatrix} (AA^*)^{1-\alpha} & A^* \\ A & (A^*A)^{\alpha} \end{pmatrix} \ge 0.$$

So

$$M + N = \begin{pmatrix} (A^*A)^{\alpha} + (AA^*)^{1-\alpha} & A + A^* \\ A + A^* & (A^*A)^{\alpha} + (AA^*)^{1-\alpha} \end{pmatrix} \ge 0.$$

From Lemma 1.5,

$$\pm(A + A^*) \le (A^*A)^{\alpha} + (AA^*)^{1-\alpha}$$

$$s(A + A^*) \prec_w s\left((A^*A)^{\alpha} + (AA^*)^{1-\alpha}\right)$$

is obtained.

Corollary 2.19. *From Lemma 1.2,*

$$\|A + A^*\| \leq \left\|(A^*A)^{\alpha} + (AA^*)^{1-\alpha}\right\|$$

is found.

Theorem 2.20. *Let* $\begin{pmatrix} M & K \\ K^* & N \end{pmatrix} \geq 0$ *and* M, N *be square matrices of the same size.*

$$(s(K), s(K)) \prec_w (\lambda(M), \lambda(N)).$$

Proof. If $\begin{pmatrix} M & K \\ K^* & N \end{pmatrix} \geq 0$, then using Lemma 1.9, we get

$$s(K) \prec_w \frac{1}{2}\{\lambda(M) + \lambda(N)\}.$$

By Lemma 1.10,

$$(s(K), s(K)) \prec_w (\lambda(M), \lambda(N))$$

is obtained.

Corollary 2.21. *Using Lemma 1.2 and since* $\|A \oplus B\| = max\{\|M\|, \|N\|\}$*; we get*

$$\|K\| \leq \|M \oplus N\|.$$

Acknowledgements

The authors is grateful to the anonymous referee and editors for for careful reading and useful comments.

References

[1] A. Al-Natoor, A. Burqan, M. A. Amleh, C. Conde, *Some singular value inequalities for matrices*, J. Math. Inequal. **18** (2024), 911–919.
[2] W. Audeh, F. Kittaneh, *Singular value inequalities for compact operators*, Linear Algebra Appl. **437** (2012), 2516–2522.
[3] W. Audeh, *Norm inequalities and applications*, Int. J. Math. Comput. Sci. **16** (2021), 809–817.
[4] W. Audeh, *Singular value inequalities with applications*, J. Math. Comput. Sci. **24** (2022), 323–329.
[5] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.
[6] R. Bhatia, C. Davis, *More matrix forms of the arithmetic-geometric mean inequality*, SIAM J. Matrix Anal. Appl. **14** (1993), 132–136.
[7] R. Bhatia, F. Kittaneh, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl. **11** (1990), 272–277.
[8] R. Bhatia, F. Kittaneh, *The matrix arithmetic-geometric mean inequality revisited*, Linear Algebra Appl. **428** (2008), 2177–2191.
[9] A. Burqan, F. Kittaneh, *Singular value and norm inequalities associated with* 2×2 *positive semidefinite block matrices*, Electron. J. Linear Algebra **32** (2017), 116–124. doi:10.13001/1081-3810.3442.
[10] T. Furuta, *An extension of the Heinz-Kato theorem*, Proc. Amer. Math. Soc. **120** (1994), 785–787.
[11] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space*, Wiley, New York, 1969.
[12] T. Kato, *Notes on some inequalities for linear operators*, Math. Ann. **125** (1952), 208–212.
[13] F. Kittaneh, *Notes on some inequalities for Hilbert space operators*, Publ. Res. Inst. Math. Sci. **24** (1988), 283–293.
[14] F. Kittaneh, *Singular values of compact operators via operator matrices*, Math. Inequal. Appl. **27** (2024), 1–15.
[15] F. Kittaneh, H. R. Moradi, M. Sababheh, *Further bounds for singular values with applications*, Linear Multilinear Algebra, to appear (2025).
[16] R. B. Papat, *Majorization and singular values*, Linear Multilinear Algebra **21** (1987), 211–214.
[17] M. Sababheh, C. Conde, H. R. Moradi, *On the Matrix Cauchy-Schwarz Inequality*, Oper. Matrices **17** (2023), 525–538. doi:10.7153/oam-2023-17-34.
[18] Y. Tao, *More results on singular value inequalities of matrices*, Linear Algebra Appl. **416** (2006), 724–729.
[19] R. Turkmen, V. E. Paksoy, F. Zhang, *Some inequalities of majorization type*, Linear Algebra Appl. **437** (2012), 1305–1316.
[20] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer, New York, 2010.
[21] J. Zhao, Q. Jiang, *A note on "Remarks on some inequalities for positive semidefinite matrices and questions for Bourin"*, J. Math. Inequal. **13** (2019), 747–752. doi:10.7153/jmi-2019-13-51.