Filomat 40:2 (2026), 773–797
https://doi.org/10.2298/FIL2602773L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Moduli of continuity of functions in Hölder's class and solution of Van der Pol-Duffing equations by Legendre wavelet

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Abstract. In this paper, Legendre wavelet is considered. The convergence analysis of Legendre wavelet series for functions in the Hölder's class is studied. Two new moduli of continuity and two estimators of functions in Hölder's class, using Legendre wavelets, have been determined. These moduli of continuity and estimators are novel, sharper and among the best possible in wavelet analysis. The Van der Pol-Duffing equation can be expressed physically in three ways: single well, double well, and double hump. The Legendre wavelet collocation approach is introduced to solve the Van der Pol-Duffing equations, and their solutions are obtained by this technique as well as Runge-Kutta method (ODE45). These solutions are compared and it is observed that the absolute errors between exact and Legendre wavelet solution are less than the absolute errors between exact and ODE45 solution of Van Der Pol-Duffing equation. Hence, the proposed method is more effective and accurate than ODE45 method.

1. Introduction

Wavelet theory has found applications in a variety of fields in recent decades. The approximation of functions belonging to a particular class using wavelet methods has been discussed by many researchers, such as Devore [7], Morlet [12], Meyer [11], and Debnath [6]. The approximation by an orthogonal family of functions is widely used in science and engineering. The sine-cosine functions, block-pulse functions, Legendre, Laguerre, and Chebyshev wavelet sets of functions are the most commonly used orthogonal functions. The orthonormal wavelets provide bases for many important spaces. Aside from their traditional applications in signal and image processing, the wavelet basis has received attention for numerical solutions of integral and fractional order differential equations. Wavelets are new tools for solving differential equations [10], estimating moduli of continuity [2], and function approximation. In the Hölder's class, wavelets help in the most accurate representation of functions. Many wavelets are well-known, including the Haar wavelet, Fibonacci wavelet, and Wilson wavelet, etc. One of the simplest wavelets types used in wavelet analysis is the Haar wavelet. Due to its simplicity and wide range of applications, it is employed in the solution of both integral and differential equations. There is a non-smooth character in the Haar wavelet due to this reason, estimating the moduli of continuity and the approximation of smooth function by it is a challenging task. By using Legendre wavelets, this flaw is almost eliminated, also moduli of continuity

2020 *Mathematics Subject Classification*. Primary 42C40; Secondary 65T60, 65L10, 65L60.
Keywords. Legendre wavelet, Modulus of continuity, Wavelet approximation, Hölder's class, Van der Pol-Duffing equations.
Received: 16 March 2023; Accepted: 21 July 2025
Communicated by Miodrag Spalević
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and function approximations are estimated with more accuracy. Sripathy [15] discussed the wavelet-based approximation for solving differential equations.

Steffens [16] was the first to investigate the moduli of continuity of functions. To the best of our knowledge, no work has been done on the modulus of continuity of a function in Hölder's class using the Legendre wavelet. In order to conduct a more in-depth investigation in this area, this paper determines the moduli of continuity and approximation of functions in Hölder's class.

Galerkin, collocation, and other well-known techniques can be used to solve a variety of linear and nonlinear differential equations. The wavelet technique can solve these equations in very efficient and appropriate ways. This encourages us to consider the Legendre wavelets for solving differential equations. Furthermore, Güner et al. [8] proposed Legendre collocation method for solving differential equations.

In recent decades, there has been a significant rise in research on chaos and chaotic systems, which are non-linear in nature. This study focuses on the dynamics of non-linear oscillators, specifically the Van der Pol and Duffing oscillators [9]. The Van der Pol oscillator also explains how a pacemaker handles the abnormal heart rate of the human heart and how the entire cardiac system can be designed to function as a Van der Pol oscillator. The most commonly used chaotic oscillator for detecting weak signals is the Duffing oscillator. Even though the Duffing oscillator is widely used for weak signal detection, it is also used to solve physical, engineering, & even biological problems. The majority of scientific issues are intrinsically nonlinear. Except a limited number of these problems, most of them do not have exact solution. The Analytical perturbation approach and numerical methodology are used to solve some of them. The main purpose of this paper is to obtain the numerical solution of Van der Pol-Duffing equations [17] by using Legendre wavelet collocation method. This equation has no exact solution and can not be efficiently evaluated using other numerical techniques. The main characteristic of this technique is that it reduces the problem to a system of algebraic equations.

This paper is organized as follows: Section 2 is introductory, in which the importance of moduli of continuity, Legendre wavelet, and approximation of functions in Hölder's class, along with related literature, are studied. Section 3 covers the investigation of the convergence analysis of the Legendre wavelet series and certain lemmas that will be required in the subsequent sections. In section 4, the theorem concerning the moduli of continuity of $f - S_{2^{k-1},M}(f)$ has been established, and its detailed proof is also discussed in this section. In section 5, the process of solving the problem of the Van der Pol-Duffing equations using the Legendre collocation method has been introduced and provides several numerical examples related to the Van der Pol-Duffing equations in various physical situations (single well, double well & double hump) to demonstrate the accuracy of the proposed method. Finally, the main conclusions are summarized in section 6.

2. Definitions and Preliminaries

2.1. Legendre Wavelets and its properties

Wavelets constitute a family of functions constructed from dialation and translation of a single function $\Psi \in L^2(\mathbb{R})$, called mother wavelet. Let

$$\Psi_{b,a}(t) = |a|^{\frac{-1}{2}}\Psi\left(\frac{t-b}{a}\right), \ a \neq 0 \qquad \text{(Daubechies [5])}.$$

If we restrict the values of dialation and translation parameter to $a = a_0^{-k}, b = (2n-1)b_0a_0{}^{-k}, a_0 > 1, b_0 > 0$ respectively, the following family of discrete wavelets is constructed:

$$\Psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\Psi\left(a_0^k t - (2n-1)b_0\right).$$

Now, taking $a_0 = 2$, $\Psi(t) = \sqrt{\frac{2m+1}{2\ell}} P_m(\frac{t}{\ell})$, and $b_0 = 1$, the Legendre wavelet $\Psi(k,n,m,t)$, generally denoted by $\Psi^{(L)}_{n,m}(t)$, over the interval $[0,\ell)$, is defined by:

$$\Psi^{(L)}_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2m+1}{2\ell}} P_m(\frac{2^k t}{\ell} - 2n + 1), & \frac{n-1}{2^{k-1}}\ell \leq t < \frac{n}{2^{k-1}}\ell\,; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

where $n = 1, 2, ..., 2^{k-1}$, k is the level of resolution and can assume any fixed positive integer value, $m = 0, 1, 2, ...M$ represents the degree of Legendre polynomial $P_m(t)$ [1, 13]. Here M can assume any fixed positive integer value, and t is the normalized time. In above definition, the coefficient $\sqrt{\frac{2m+1}{2\ell}}$ is used for orthonormality, and the Legendre polynomials $P_m(t)$ of degree m can be determined with the aid of following recurrence formulae:

$$P_0(t) = 1,\ P_1(t) = t,\ P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right) t\, P_m(t) - \left(\frac{m}{m+1}\right) P_{m-1}(t),\ m \geq 1. \tag{2}$$

The set $\{P_m(t) : m = 0, 1, 2, 3, ...\}$ in the Hilbert space $L^2[-1,1]$ is an orthogonal set. The orthogonality of Legendre polynomials on the interval $[-1,1]$ implies that

$$\langle P_m, P_n\rangle = \int_{-1}^{1} P_m(t)\overline{P_n(t)}dt = \frac{2}{2n+1}\delta_{m,n} = \begin{cases} \frac{2}{2n+1}, & m = n\,; \\ 0, & m \neq n. \end{cases} \tag{3}$$

Furthermore, the set of wavelets $\Psi^{(L)}_{n,m}$ makes an orthonormal basis of $L^2[0,\ell)$, *i.e.*

$$\langle \Psi^{(L)}_{n,m}, \Psi^{(L)}_{n',m'}\rangle = \int_0^{\ell} \Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n',m'}(t)dt = \delta_{n,n'}\delta_{m,m'}, \tag{4}$$

where $\delta_{n,n'}$ denotes the Kronecker delta function defined by

$$\delta_{n,n'} = \begin{cases} 1, & n = n'\,; \\ 0, & \text{otherwise.} \end{cases}$$

The Legendre polynomials have some interesting properties that are highly useful in the study of convergence analysis of Legendre wavelet series and in the error analysis of function approximation using Legendre wavelet.

(i) If $-1 < t < 1$ and n is any positive integer, then

$$|P_n(t)| < 1 \quad \text{and} \quad |P_n(t)| < \left\{\frac{\pi}{2n(1-t^2)}\right\}^{\frac{1}{2}}.$$

(ii)
$$P_m(t) = \frac{1}{2m+1}\frac{d}{dt}\left(P_{m+1}(t) - P_{m-1}(t)\right).$$

(iii)
$$\int_0^1 P_n(t)dt = \begin{cases} \frac{(-1)^{\frac{n-1}{2}}(n-1)!}{2^n(\frac{n+1}{2})!(\frac{n-1}{2})!}, & n \text{ is odd;} \\ 0, & n \text{ is even.} \end{cases}$$

(iv)
$$P_n(0) = \begin{cases} \frac{(-1)^{\frac{n}{2}} n!}{2^n\left((\frac{n}{2})!\right)^2}, & n \text{ is even}\,; \\ 0\,, & n \text{ is odd.} \end{cases}$$

To simplify the factorials that appear in properties (*iii*) and (*iv*), we used Stirling's approximation $n! = n^n e^{-n}\sqrt{2\pi n}$ for large value of n.

2.2. Legendre wavelet expansion and approximation of function

The function $f \in L^2[0,\ell)$ is expressed in the Legendre wavelet series as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t) \tag{5}$$

$$\text{where } c_{n,m} = \langle f, \Psi^{(L)}_{n,m}\rangle = \int_0^{\ell} f(t)\Psi^{(L)}_{n,m}(t) \tag{6}$$

The $(2^{k-1}, M+1)^{th}$ partial sums of series (5) is given by

$$S_{2^{k-1},M}(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi^{(L)}_{n,m}(t) \tag{7}$$

in which C and $\Psi^{(L)}(t)$ are $2^{k-1}(M+1)$ vectors of the form

$$C^T = [c_{1,0}, c_{1,1}, ...c_{1,M}, c_{2,0}, c_{2,1}..., c_{2,M}, ..., c_{2^{k-1},0}, ..., c_{2^{k-1},M}] \text{ and}$$

$$\Psi^{(L)}(t) = [\Psi^{(L)}_{1,0}, \Psi^{(L)}_{1,1}, ..., \Psi^{(L)}_{1,M}, \Psi^{(L)}_{2,0}, \Psi^{(L)}_{2,1}, ..., \Psi^{(L)}_{2,M}, ..., \Psi^{(L)}_{2^{k-1},0}, ..., \Psi^{(L)}_{2^{k-1},M}]^T$$

The Legendre wavelet approximation $E_{2^{k-1},M}(f)$ of a function $f \in L^2[0,\ell)$ by $(2^{k-1}, M+1)^{th}$ partial sums $S_{2^{k-1},M}(f)$ of its Legendre wavelet series is given by

$$E_{2^{k-1},M}(f) = \min_{S_{2^{k-1},M}(f)} \|f - S_{2^{k-1},M}(f)\|_2, \tag{8}$$

where,

$$\|f\|_2 = \left(\int_0^{\ell} |f(t)|^2\, dt\right)^{\frac{1}{2}}$$

If $E_{2^{k-1}}(f) \to 0$ as $k, M \to \infty$ then $E_{2^{k-1},M}(f)$ is called the best approximation of f of order $(2^{k-1}, M+1)$ (Zygmund [18]).

2.3. Modulus of continuity and function of generalized Hölder's class

The Modulus of continuity of a function $f \in L^2[0,\ell)$ is defined as

$$\begin{aligned} W(f,\delta) &= \sup_{0<h\leq\delta} \|f(\cdot + h) - f(\cdot)\|_2 \\ &= \sup_{0<h\leq\delta}\left(\int_0^{\ell} |f(t+h) - f(t)|^2 dt\right)^{\frac{1}{2}}. \end{aligned} \tag{9}$$

It is remarkable to note that $W(f,\delta)$ is a non-decresing function of δ and $W(f,\delta) \to 0$ as $\delta \to 0^+$ (Chui [3]).

A function f is said to be in generalized Hölder's class $H^{\alpha,\phi}[0,\ell)$ of order α, $0 < \alpha \leq 1$, if f satisfies

$$f(x+t) - f(x) = O\left(\phi(|t|)|t|^{\alpha}\right), \quad \forall x, t, x+t \in [0,\ell) \quad \text{(G.Das [4])} \tag{10}$$

where $\phi(t)$ is positive, monotonic increasing function of t such that $|t|^{\alpha}\phi(t) \to 0$ as $t \to 0^+$. It should be noted that if $\phi(t)$ = constant, then the generalised Hölder's class $H^{\alpha,\phi}[0,\ell)$ corresponds to the known Hölder's class $H^{\alpha}[0,\ell)$.

3. Convergence analysis of Legendre wavelet series

In this section, we show that the Legendre wavelet expansion of a function in the generalized Hölder's class converges uniformly to the function. For this the following lemma has been deduced.

Lemma 3.1. If the Legendre wavelet expansion of a function f in the generalized Hölder's class $H^{\alpha,\phi}[0,\ell)$ converges uniformly, then this wavelet expansion converges to the function f.

Proof of lemma.
Assuming that the Legendre wavelet expansion of $f \in H^{\alpha,\phi}[0,\ell)$ uniformly converges to a function g, i.e.

$$g(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t) \tag{11}$$

where $c_{n,m} = \langle f, \Psi_{n,m}\rangle$. Since the series (11) is uniformly convergent to $g(t)$ on the interval $[0,\ell]$, $g(t)$ is integrable, and the series (11) is term-by-term integrable. Additionally, since every Legendre wavelet basis $\Psi^{(L)}_{n,m}(t)$ is intragrable, $g(t)\Psi_{n,m}(t)$ is also integrable, as the product of two integrable functions is integrable. From (11) it follows that

$$\begin{aligned}
\langle g, \Psi_{r,s}\rangle = \int_0^{\ell} g(t)\Psi_{r,s}(t)dt &= \int_0^{\ell}\left\{\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t)\right\}\Psi^{(L)}_{r,s}(t)dt \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\left\{\int_0^{\ell}\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{r,s}(t)dt\right\} \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\delta_{m,s}\delta_{n,r} \\
&= c_{r,s} = \langle f, \Psi_{r,s}\rangle
\end{aligned}$$

$$\text{Thus } \langle g, \Psi_{n,m}\rangle = \langle f, \Psi_{n,m}\rangle \text{ for } n = 1,2,3.....\quad m = 0,1,2,3,......$$

Finally, equation (11) represents the Legendre wavelet expansion of functions f and g with the same wavelet coefficients. Therefore, $g = f$ on $[0,\ell)$.

Theorem 3.2. In the Hölder's class $H^{\alpha,\phi}[0,\ell)$, a function $f(t)$ can be expanded as an infinite sum of Legendre wavelets series, and the series converges uniformly to $f(t)$, i.e.

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t)$$

Proof of theorem 3.2 From (6)

$$\begin{aligned}
c_{n,m} = \langle f, \Psi^{(L)}_{n,m}\rangle &= \int_{\frac{(n-1)\ell}{2^{(k-1)}}}^{\frac{n\ell}{2^{(k-1)}}} f(x)\Psi^{(L)}_{n,m}(x)dx \\
&= \sqrt{\frac{2m+1}{2\ell}}2^{\frac{k}{2}}\int_{\frac{(n-1)\ell}{2^{(k-1)}}}^{\frac{n\ell}{2^{(k-1)}}} f(x)P_m\left(\frac{2^k x}{\ell} - 2n + 1\right)dx \\
&= \sqrt{\frac{2m+1}{2\ell}}2^{\frac{k}{2}}\int_{-1}^{1} f\left(\frac{(t+2n-1)\ell}{2^k}\right)P_m(t)\frac{\ell dt}{2^k}, \quad \frac{2^k x}{\ell} - 2n + 1 = t
\end{aligned}$$

$$\text{Let, } C = \frac{\sqrt{2m+1}\sqrt{\ell}}{2^{\frac{k+1}{2}}}$$

$$c_{n,m} = \frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big[\int_{-1}^{1} f\left(\frac{(t+2n-1)\ell}{2^k}\right)P_m(t)dt - f\Big(\frac{(2n-1)\ell}{2^k}\Big)\int_{-1}^{1} P_m(t)dt\Big]$$

$$= \frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big[\int_{-1}^{1}\Big(f\left(\frac{(t+2n-1)\ell}{2^k}\right) - f\Big(\frac{(2n-1)\ell}{2^k}\Big)\Big)P_m(t)dt\Big],$$

$$= \frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big[\int_{-1}^{1} O\Big(\phi\left(\left|\frac{\ell t}{2^k}\right|\right)\left|\frac{\ell t}{2^k}\right|^{\alpha}\Big)P_m(t)dt\Big] \qquad \text{by eqn. (2.10)}$$

$$= \frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\int_{0}^{1} O(t^{\alpha})P_m(t)dt + \int_{0}^{1} O(t^{\alpha})P_m(-t)dt\Big]$$

$$\Big|c_{n,m}\Big| \le \frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\Big|\int_{0}^{1} O(t^{\alpha})P_m(t)dt\Big| + \Big|\int_{0}^{1} (-1)^m O(t^{\alpha})P_m(t)dt\Big|\Big]$$

$$\le \frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\Big|\int_{0}^{1} O(t^{\alpha})P_m(t)dt\Big| + \Big|\int_{0}^{1} O(t^{\alpha})P_m(t)dt\Big|\Big]$$

$$\le C_f\frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\Big|\int_{0}^{1} t^{\alpha}P_m(t)dt\Big|\Big], \qquad \text{here } C_f \text{ is some constant.}$$

$$\Big|c_{n,m}\Big| \le C_f\frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\Big|\int_{0}^{1} t^{\alpha}P_m(t)dt\Big|\Big] \tag{12}$$

$$\int_0^1 t^{\alpha}P_m(t)dt = \int_0^1 t^{\alpha}\frac{1}{2m+1}\frac{d}{dt}\Big(P_{m+1}(t) - P_{m-1}(t)\Big)dt$$

$$= \frac{1}{2m+1}\Big[\Big(t^{\alpha}(P_{(m+1)}(t) - P_{(m-1)}(t))\Big)_0^1 - \int_0^1 \alpha t^{\alpha-1}(P_{(m+1)}(t) - P_{(m-1)}(t))dt\Big]$$

$$= \frac{-\alpha}{2m+1}\int_0^1 t^{\alpha-1}\Big(P_{(m+1)}(t) - P_{(m-1)}(t)\Big)dt$$

$$= \frac{-\alpha}{2m+1}\int_0^1 t^{\alpha-1}\Big(P_{(m+1)}(t) - \frac{(m+1)}{m}P_{(m+1)}(t) - \frac{(2m+1)}{m}tP_m(t)$$

$$= \frac{-\alpha}{2m+1}\int_0^1 t^{\alpha-1}\Big(\frac{(2m+1)}{m}P_{(m+1)}(t) - \frac{(2m+1)}{m}tP_m(t)\Big)dt$$

$$= \frac{-\alpha}{m}\int_0^1 t^{\alpha-1}\Big(P_{(m+1)}(t) - tP_m(t)\Big)dt$$

$$\begin{aligned}\left|\int_0^1 t^{\alpha}P_m(t)dt\right| &\leq \left|\frac{-\alpha}{m}\int_0^1 t^{\alpha-1}\Big(P_{(m+1)}(t)-tP_m(t)\Big)dt\right| \\ &\leq \frac{\alpha}{m}\Big(\int_0^1 t^{\alpha-1}\Big|P_{(m+1)}(t)\Big|dt+\int_0^1 t^{\alpha}\Big|P_m(t)\Big|dt\Big) \\ &\leq \frac{\alpha}{m}\Big(\int_0^1 t^{\alpha-1}\sqrt{\frac{\pi}{2(m+1)(1-t^2)}}dt+\int_0^1 t^{\alpha}\sqrt{\frac{\pi}{2m(1-t^2)}}dt\Big) \\ &= \frac{\alpha}{m}\Big(\sqrt{\frac{\pi}{2(m+1)}}\int_0^1\frac{t^{\alpha-1}}{\sqrt{1-t^2}}dt+\sqrt{\frac{\pi}{2m}}\int_0^1\frac{t^{\alpha}}{\sqrt{1-t^2}}dt\Big) \\ &= \frac{\alpha}{m}\sqrt{\frac{\pi}{2m}}\Big(\int_0^1\frac{t^{\alpha-1}}{\sqrt{1-t^2}}dt+\int_0^1\frac{t^{\alpha}}{\sqrt{1-t^2}}dt\Big) \\ &= \frac{\alpha}{m}\sqrt{\frac{\pi}{2m}}\Big(\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)}{2\,\Gamma\left(\frac{\alpha+1}{2}\right)}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{2\,\Gamma\left(\frac{\alpha+2}{2}\right)}\Big) \end{aligned}\tag{13}$$

From equation(12),

$$\begin{aligned}\left|c_{n,m}\right| &\leq C_f\frac{\sqrt{(2m+1)\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\frac{\alpha}{m}\sqrt{\frac{\pi}{2m}}\Big(\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)}{2\,\Gamma\left(\frac{\alpha+1}{2}\right)}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{2\,\Gamma\left(\frac{\alpha+2}{2}\right)}\Big)\Big] \\ &\leq C_f\alpha\sqrt{\pi\ell}\Big(\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)}{2\,\Gamma\left(\frac{\alpha+1}{2}\right)}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{2\,\Gamma\left(\frac{\alpha+2}{2}\right)}\Big)\frac{\sqrt{(2m+1)}}{m\sqrt{2m}2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\end{aligned}$$

$$\left|c_{n,m}\right|^2 \leq L_f\,\frac{1}{2^{(k+1)}}\Big(\frac{1}{m^2}+\frac{1}{2m^3}\Big)\Big(\frac{\ell}{2^k}\Big)^{2\alpha}\phi^2\Big(\frac{\ell}{2^k}\Big)\tag{14}$$

Here $L_f = C_f^2\alpha^2\pi\ell\Big(\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}\right)}{2\,\Gamma\left(\frac{\alpha+1}{2}\right)}+\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+1}{2}\right)}{2\,\Gamma\left(\frac{\alpha+2}{2}\right)}\Big)^2$

$$\left|c_{n,m}\right|^2 \leq L_f\,\frac{\ell^{2\alpha}}{n^{(1+2\alpha)}}\Big(\frac{1}{m^2}+\frac{1}{2m^3}\Big)\phi^2\Big(\frac{\ell}{2^k}\Big),\ \forall\, m\geq 1 \quad \text{as} \quad n\leq 2^{k-1}.\tag{15}$$

For $m=0$ equation (12) becomes

$$\begin{aligned}\left|c_{n,0}\right| &\leq C_f\frac{\sqrt{\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\Big[\Big|\int_0^1 t^{\alpha}P_0(t)dt\Big|\Big] \\ &= C_f\frac{\sqrt{\ell}}{2^{\frac{k+1}{2}}}\Big(\frac{\ell}{2^k}\Big)^{\alpha}\phi\Big(\frac{\ell}{2^k}\Big)\frac{1}{(\alpha+1)}\end{aligned}$$

$$\left|c_{n,0}\right|^2 \leq C_f^2\frac{\ell^{2\alpha+1}}{(\alpha+1)}\phi\Big(\frac{\ell}{2^k}\Big)\frac{1}{n^{(1+2\alpha)}}\tag{16}$$

Since $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^a m^b}$ is convergent $\iff a>1\ \&\ b>1$.
Hence $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|c_{n,m}\right|^2$ is convergent as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|c_{n,m}\right|^2 \leq L_f\,\ell^{2\alpha}\,\phi^2\Big(\frac{\ell}{2^k}\Big)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{n^{(1+2\alpha)}}\Big(\frac{1}{m^2}+\frac{1}{2m^3}\Big)<\infty \quad \text{as } 0<\alpha\leq 1\tag{17}$$

Also $\sum_{n=1}^{\infty}\left|c_{n,0}\right|^2$ is convergent as

$$\sum_{n=1}^{\infty}\left|c_{n,0}\right|^2 \leq C_f^2\frac{\ell^{2\alpha+1}}{(\alpha+1)}\phi\Big(\frac{\ell}{2^k}\Big)\sum_{n=1}^{\infty}\frac{1}{n^{(1+2\alpha)}}<\infty\,,\text{as }\ 0<\alpha\leq 1\tag{18}$$

From equation (17) & (18), $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left|c_{n,m}\right|^2$ is convergent because

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\left|c_{n,m}\right|^2=\sum_{n=1}^{\infty}\left|c_{n,0}\right|^2+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left|c_{n,m}\right|^2<\infty. \tag{19}$$

Let $(S_{2^{k-1},M}f)(t)$ denote the $(2^{k-1},M)^{th}$ partial sums of series (5) i.e.

$$(S_{2^{k-1},M})(f)(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\,\Psi_{n,m}^{(L)}(t)$$

Now to prove that $\left\{(S_{2^{k-1},M}f)(t)\right\}_{M\in\mathbb{N}}$ is a cauchy sequence in $L^2[0,\ell)$.
For $M>N$,

$$\begin{aligned}
\|(S_{(2^{k-1},M)}f)-(S_{2^{k-1},N}f)\|_2^2&=\left\|\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\,\Psi_{n,m}^{(L)}(t)-\sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{N-1}c_{n',m'}\,\Psi_{n',m'}^{(L)}(t)\right\|_2^2\\
&=\left\|\sum_{n=1}^{2^{k-1}}\left(\sum_{m=0}^{N-1}+\sum_{m=N}^{M-1}\right)c_{n,m}\,\Psi_{n,m}^{(L)}(t)-\sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{N-1}c_{n',m'}\,\Psi_{n',m'}^{(L)}(t)\right\|_2^2\\
&=\left\|\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{M-1}c_{n,m}\,\Psi_{n,m}^{(L)}(t)\right\|_2^2\\
&=\left\langle\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{M-1}c_{n,m}\,\Psi_{n,m}^{(L)}(t)\ ,\sum_{n'=1}^{2^{k-1}}\ \sum_{m'=N}^{M-1}c_{n',m'}\,\Psi_{n',m'}^{(L)}(t)\right\rangle\\
&=\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{M-1}|c_{n,m}|^2
\end{aligned}$$

Since $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}\left|c_{n,m}\right|^2$ is convergent series which implies

$$\|(S_{(2^{k-1},M)}f)-(S_{2^{k-1},N}f)\|_2^2=\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{M-1}|c_{n,m}|^2\ \to\ 0\text{ as }M,N\ \to\ \infty.$$

This implies that $\left\{(S_{2^{k-1},M}f)\right\}_{M\in\mathbb{N}}$ is a cauchy sequence in $L^2[0,\ell)$. Since $L^2[0,\ell)$ is a Banach space, the cauchy sequence $\left\{(S_{2^{k-1},M}f)\right\}_{M\in\mathbb{N}}$ converges to a function g (say). by lemma 3.1, $g=f$.

Theorem 3.3. If $f(t)$ is the exact solution of the Van Der Pol-Duffing differential equation, then its Legendre wavelet solution $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\,\Psi_{n,m}^{(L)}(t)$ converges to the exact solution $f(t)$ as $M\to\infty$.

Proof Let $(S_{2^{k-1},M}f)(t)$ denote the $(2^{k-1},M)^{th}$ partial sums of series $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}c_{n,m}\,\Psi_{n,m}^{(L)}(t)$ i.e.

$$(S_{2^{k-1},M})(f)(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\,\Psi_{n,m}^{(L)}(t)$$

Now to prove that $\left\{(S_{2^{k-1},M}f)(t)\right\}_{M\in\mathbb{N}}$ is a cauchy sequence in $L^2[0,1)$.

For $M > N$,

$$\begin{aligned}
\|(S_{(2^{k-1},M)}f) - (S_{2^{k-1},N}f)\|_2^2 &= \left\| \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, \Psi^{(L)}_{n,m}(t) - \sum_{n'=1}^{2^{k-1}} \sum_{m'=0}^{N-1} c_{n',m'}\, \Psi^{(L)}_{n',m'}(t) \right\|_2^2 \\
&= \left\| \sum_{n=1}^{2^{k-1}} \left(\sum_{m=0}^{N-1} + \sum_{m=N}^{M-1} \right) c_{n,m}\, \Psi^{(L)}_{n,m}(t) - \sum_{n'=1}^{2^{k-1}} \sum_{m'=0}^{N-1} c_{n',m'}\, \Psi^{(L)}_{n',m'}(t) \right\|_2^2 \\
&= \left\| \sum_{n=1}^{2^{k-1}} \sum_{m=N}^{M-1} c_{n,m}\, \Psi^{(L)}_{n,m}(t) \right\|_2^2 \\
&= \left\langle \sum_{n=1}^{2^{k-1}} \sum_{m=N}^{M-1} c_{n,m}\, \Psi^{(L)}_{n,m}(t) \;,\; \sum_{n'=1}^{2^{k-1}} \sum_{m'=N}^{M-1} c_{n',m'}\, \Psi^{(L)}_{n',m'}(t) \right\rangle \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=N}^{M-1} |c_{n,m}|^2
\end{aligned}$$

By means of Bessel's inequality, we found that $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} \left|c_{n,m}\right|^2$ is convergent series, which implies

$$\|(S_{(2^{k-1},M)}f) - (S_{2^{k-1},N}f)\|_2^2 = \sum_{n=1}^{2^{k-1}} \sum_{m=N}^{M-1} |c_{n,m}|^2 \;\to\; 0 \text{ as } M, N \;\to\; \infty.$$

This means that $\left\{(S_{2^{k-1},M}f)\right\}_{M\in\mathbb{N}}$ is a Cauchy sequence in $L^2[0,\ell)$. Since $L^2[0,\ell)$ is a Banach space, the cauchy sequence $\left\{(S_{2^{k-1},M}f)\right\}_{M\in\mathbb{N}}$ converges to a function $g(t)$ (say). To prove that $g = f$, let p and q be arbitrary constants, where $p = 1, 2, ..., 2^{k-1}$, and $q = 0, 1, 2, ...$

$$\begin{aligned}
\left\langle f(t) - g(t) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle &= \left\langle f(t) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle - \left\langle g(t) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle \\
&= c_{p,q} - \left\langle g(t) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle \\
&= c_{p,q} - \left\langle \lim_{M\to\infty} (S_{2^{k-1},M}f) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle \\
&= c_{p,q} - \lim_{M\to\infty} \left\langle (S_{2^{k-1},M}f) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle \\
&= c_{p,q} - \lim_{M\to\infty} \left\langle \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{M-1} c_{p,q}\, \Psi^{(L)}_{p,q}(t) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle \\
&= c_{p,q} - c_{p,q} = 0.
\end{aligned}$$

$$\left\langle f(t) - g(t) \;,\; \Psi^{(L)}_{p,q}(t) \right\rangle = 0 \;\forall\, p, q \text{ as } p, q \text{ are arbitrary}$$
$$\implies g(t) = f(t)$$

thus $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\, \Psi^{(L)}_{n,m}(t)$ converges to $f(t)$ as $M \to \infty$.

4. Modulus of continuity and Error estimation

In this paper, the following theorem has been proved.

Theorem 4.1. If a function $f \in H^{\alpha,\phi}[0,\ell)$ and its Legendre wavelet expansion be

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi_{n,m}^{(L)}(t)$$

having $(2^{k-1}, M+1)^{th}$ partial sums

$$\left(S_{2^{k-1},M}(f)\right)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi_{n,m}^{(L)}(t)$$

Then the Moduli of continuity of $f - \left(S_{2^{k-1},M}(f)\right)$ satisfies

$$\textbf{(i)} \qquad W\left(\left(f - S_{2^{k-1},0}f\right), \frac{\ell}{2^k}\right) = \sup_{0<h\leq \frac{\ell}{2^k}} \| \left(f - S_{2^{k-1},0}f\right)(\cdot + h) - \left(f - S_{2^{k-1},0}f\right)(\cdot)\| = O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\ \ell^{\alpha}}{2^{(k-1)\alpha}}\right)$$

$$\textbf{(ii)} \qquad W\left(\left(f - S_{2^{k-1},M}f\right), \frac{\ell}{2^k}\right) = \sup_{0<h\leq \frac{\ell}{2^k}} \| \left(f - S_{2^{k-1},M}f\right)(\cdot + h) - \left(f - S_{2^{k-1},M}f\right)(\cdot)\| = O\left(\frac{\phi(\frac{\ell}{2^{k}})\ \ell^{\alpha}}{2^{k\alpha}\sqrt{M+1}}\right), \quad M \geq 1.$$

Proof of theorem 4.1

(**i**) By dividing the interval $[0,\ell)$ into the 2^{k-1} number of subintervals as $\left[\frac{(n-1)\ell}{2^{k-1}}, \frac{n\ell}{2^{k-1}}\right)$, $n = 1, 2, ..., 2^{k-1}$. The error between $f(t)$ and its Legendre wavelet expansion in interval $\left[\frac{(n-1)\ell}{2^{k-1}}, \frac{n\ell}{2^{k-1}}\right)$, is given by

$$\begin{aligned}
e_n(t) &= f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}}, \frac{n\ell}{2^{k-1}}\right)}(t) - c_{n,0}\Psi_{n,0}^{(L)}(t) \\
&= f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}}, \frac{n\ell}{2^{k-1}}\right)}(t) - \langle f, \Psi_{n,0}^{(L)}\rangle\Psi_{n,0}^{(L)}(t) \\
&= f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}}, \frac{n\ell}{2^{k-1}}\right)}(t) - \left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} f(x)\Psi_{n,0}^{(L)}(x)(x)dx\right)\Psi_{n,0}^{(L)}(t) \\
&= f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}}, \frac{n\ell}{2^{k-1}}\right)}(t) - \frac{2^{\frac{k-1}{2}}}{\sqrt{\ell}}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} f(x)dx\right)\frac{2^{\frac{k-1}{2}}}{\sqrt{\ell}} \\
&= \frac{2^{k-1}}{\ell}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} f(t)dx\right) - \frac{2^{k-1}}{\ell}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} f(x)dx\right) \\
&= \frac{2^{k-1}}{\ell}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} (f(t) - f(x))\, dx\right).
\end{aligned}$$

$$\text{Then, } |e_n(t)| \leq \frac{2^{k-1}}{\ell}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} |(f(t)-f(x))|dx\right)$$

$$= \frac{2^{k-1}}{\ell}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} O\Big(\phi(|t-x|)|t-x|^{\alpha}\Big)dx\right)$$

$$\leq \frac{2^{k-1}}{\ell}\left(\int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} O\left(\phi\left(\frac{\ell}{2^{k-1}}\right)\left|\frac{\ell}{2^{k-1}}\right|^{\alpha}\right)dx\right)$$

$$= \frac{2^{k-1}}{\ell} O\left(\phi\left(\frac{\ell}{2^{k-1}}\right)\left|\frac{\ell}{2^{k-1}}\right|^{\alpha}\right)\frac{\ell}{2^{k-1}}$$

$$|e_n(t)| \leq C_f\, \phi\left(\frac{\ell}{2^{k-1}}\right)\frac{\ell^{\alpha}}{2^{(k-1)\alpha}}\ , \qquad C_f \text{ is constant}$$

$$\|e_n\|_2^2 = \int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} |e_n(t)|^2 dt$$

$$\leq C_f^2\, \phi^2\left(\frac{\ell}{2^{k-1}}\right)\frac{\ell^{2\alpha}}{2^{2(k-1)\alpha}} \int_{\frac{(n-1)\ell}{2^{k-1}}}^{\frac{n\ell}{2^{k-1}}} dt$$

$$\|e_n\|_2^2 \leq C_f^2\, \phi^2\left(\frac{\ell}{2^{k-1}}\right)\frac{\ell^{2\alpha+1}}{2^{(k-1)(2\alpha+1)}}$$

The error between $f(t)$ and its Legendre wavelet expansion in interval $[0,\ \ell)$, is given by

$$\begin{aligned}
f(t) - \Big(S_{2^{k-1},0}(f)\Big)(t) &= f(t)\chi_{\left[0,\frac{\ell}{2^{k-1}}\right)\cup,\dots\dots,\cup\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)\cup,\dots\dots,\cup\left[\frac{(2^{k-1}-1)\ell}{2^{k-1}},\ell\right)}(t) - \Big(S_{2^{k-1},0}(f)\Big)(t) \\
&= f(t)\chi_{\left[0,\frac{\ell}{2^{k-1}}\right)}(t) + f(t)\chi_{\left[\frac{\ell}{2^{k-1}},\frac{2\ell}{2^{k-1}}\right)}(t) + f(t)\chi_{\left[\frac{2\ell}{2^{k-1}},\frac{3\ell}{2^{k-1}}\right)}(t) + \dots \\
&\quad + f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)} + \dots + f(t)\chi_{\left[\frac{(2^{k-1}-1)\ell}{2^{k-1}},\ell\right)}(t) - \Big(S_{2^{k-1},0}(f)\Big)(t) \\
&= \sum_{n=1}^{2^{k-1}} f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t) - \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}(t) \\
&= \sum_{n=1}^{2^{k-1}} \Big(f(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t) - c_{n,0}\psi_{n,0}(t)\Big) \\
&= \sum_{n=1}^{2^{k-1}} e_n(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t)
\end{aligned}$$

$$
\begin{aligned}
\|f-\left(S_{2^{k-1},0}(f)\right)\|_2^2 &= \int_0^{\ell} |f(t)-\left(S_{2^{k-1},0}(f)\right)(t)|^2 dt\\
&= \int_0^{\ell}\left(\sum_{n=1}^{2^{k-1}} e_n(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t)\right)^2 dt\\
&= \int_0^{\ell}\left(\sum_{n=1}^{2^{k-1}}\left(e_n(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t)\right)^2\right)dt\\
&+ \sum\sum_{1\leqslant\, n\neq n'\leq\, 2^{k-1}}\int_0^{\ell}\left(e_n(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t)\right)\left(e_{n'}(t)\chi_{\left[\frac{(n'-1)\ell}{2^{k-1}},\frac{n'\ell}{2^{k-1}}\right)}(t)\right)dt\\
&= \int_0^{\ell}\left(\sum_{n=1}^{2^{k-1}}\left(e_n(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t)\right)^2\right)dt\\
&= \sum_{n=1}^{2^{k-1}}\int_0^{\ell}\left(e_n(t)\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}(t)\right)^2 dt\\
&= \sum_{n=1}^{2^{k-1}}\|e_n\chi_{\left[\frac{(n-1)\ell}{2^{k-1}},\frac{n\ell}{2^{k-1}}\right)}\|_2^2\\
&\leq \sum_{n=1}^{2^{k-1}} C_f^2\,\phi^2\left(\frac{\ell}{2^{k-1}}\right)\frac{\ell^{2\alpha+1}}{2^{(k-1)(2\alpha+1)}} = C_f^2\,\phi^2\left(\frac{\ell}{2^{k-1}}\right)\frac{\ell^{2\alpha+1}}{2^{(k-1)(2\alpha+1)}}\times 2^{k-1}\\
\|f-\left(S_{2^{k-1},0}(f)\right)\|_2 &\leq C_f\,\phi\left(\frac{\ell}{2^{k-1}}\right)\frac{\ell^{\alpha+\frac{1}{2}}}{2^{(k-1)\alpha}}\\
\|f-\left(S_{2^{k-1},0}(f)\right)\|_2 &= O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\,\ell^{\alpha}}{2^{(k-1)\alpha}}\right)
\end{aligned}
$$

$$
\begin{aligned}
W\left(\left(f-S_{2^{k-1},0}f\right),\frac{\ell}{2^k}\right) &= \sup_{0<h\leq\frac{1}{2^k}}\|\left(f-S_{2^{k-1},0}f\right)(t+h)-\left(f-S_{2^{k-1},0}f\right)(t)\|_2\\
&\leq \|\left(f-S_{2^{k-1},0}f\right)\|_2+\|\left(f-S_{2^{k-1},0}f\right)\|_2\\
&= 2\|\left(f-S_{2^{k-1},0}f\right)\|_2\\
&= 2.O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\,\ell^{\alpha}}{2^{(k-1)\alpha}}\right)\\
&= O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\,\ell^{\alpha}}{2^{(k-1)\alpha}}\right)
\end{aligned}
$$

$$
W\left(\left(f-S_{2^{k-1},0}f\right),\frac{\ell}{2^k}\right) = O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\,\ell^{\alpha}}{2^{(k-1)\alpha}}\right).
$$

(ii) Consider

$$\begin{aligned}
f(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{m,n}(t)\\
f(t)-S_{2^{k-1},M}(f)(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi^{(L)}_{n,m}(t)\\
&= \left(\sum_{n=1}^{2^{k-1}} + \sum_{n=2^{k-1}+1}^{\infty}\right)\sum_{m=0}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t) - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi^{(L)}_{n,m}(t),\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t), \text{ by definition of } \Psi^{(L)}_{n,m}\\
(f(t)-S_{2^{k-1},M}(f)(t))^2 &= \left(\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}\Psi^{(L)}_{n,m}(t)\right)^2\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} c^2_{n,m}\left(\Psi^{(L)}_{n,m}(t)\right)^2\\
&+ \sum_{n=1}^{2^{k-1}}\sum\sum_{M+1\le m\neq m'\le\infty} c_{n,m}c_{n,m'}\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n,m'}(t)\\
&+ \sum\sum_{1\leqslant\ n\neq n'\le\ 2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}c_{n',m}\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n',m}(t)\\
&+ \sum\sum_{1\leqslant\ n\neq n'\le\ 2^{k-1}}\sum\sum_{M+1\le m\neq m'\le\infty} c_{n,m}c_{n',m'}\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n',m'}(t)\\
\|f-S_{2^{k-1},M}(f)\|_2^2 &= \int_0^1 |f(t)-S_{2^{k-1},M}(f)(t)|^2dt\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} |c_{n,m}|^2\int_0^1 |\Psi^{(L)}_{n,m}(t)|^2dt\\
&+ \sum_{n=1}^{2^{k-1}}\sum\sum_{M+1\le m\neq m'\le\infty} c_{n,m}c_{n',m'}\int_0^1\left(\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n',m'}(t)\right)dt\\
&+ \sum\sum_{1\leqslant\ n\neq n'\le\ 2^{k-1}}\sum_{m=M+1}^{\infty} c_{n,m}c_{n',m'}\int_0^1\left(\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n',m'}(t)\right)dt\\
&+ \sum\sum_{1\leqslant\ n\neq n'\le\ 2^{k-1}}\sum\sum_{M+1\le m\neq m'\le\infty} c_{n,m}c_{n',m'}\int_0^1\left(\Psi^{(L)}_{n,m}(t)\Psi^{(L)}_{n',m'}(t)\right)dt\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty} |c_{n,m}|^2, \quad \text{by orthonormality of } \{\Psi^{(L)}_{n,m}\}.\\
\|f-S_{2^{k-1},M}(f)\|_2^2 &\le \sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}\left[L_f\,\frac{1}{2^{(k+1)}}\left(\frac{1}{m^2}+\frac{1}{2m^3}\right)\left(\frac{\ell}{2^k}\right)^{2\alpha}\phi^2\left(\frac{\ell}{2^k}\right)\right]\\
&= L_f\,\frac{1}{2^{(k+1)}}\left(\frac{\ell}{2^k}\right)^{2\alpha}\phi^2\left(\frac{\ell}{2^k}\right)\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}\left(\frac{1}{m^2}+\frac{1}{2m^3}\right)\\
&\le L_f\,\frac{2}{2^{(k+1)}}\left(\frac{\ell}{2^k}\right)^{2\alpha}\phi^2\left(\frac{\ell}{2^k}\right)\sum_{n=1}^{2^{k-1}}\sum_{m=M+1}^{\infty}\frac{1}{m^2}
\end{aligned}$$

$$\leq L_f \frac{2}{2^{(k+1)}}\left(\frac{\ell}{2^k}\right)^{2\alpha}\phi^2\left(\frac{\ell}{2^k}\right)\sum_{n=1}^{2^{k-1}}\left[\frac{1}{(M+1)^2}+\int_{M+1}^{\infty}\frac{1}{m^2}dm\right]$$

$$\leq L_f \frac{2}{2^{(k+1)}}\left(\frac{\ell}{2^k}\right)^{2\alpha}\phi^2\left(\frac{\ell}{2^k}\right)\sum_{n=1}^{2^{k-1}}\left[\frac{1}{(M+1)}+\left(\frac{-1}{m}\right)_{M+1}^{\infty}\right]$$

$$= L_f \frac{2}{2^{(k+1)}}\left(\frac{\ell}{2^k}\right)^{2\alpha}\phi^2\left(\frac{\ell}{2^k}\right)\left[\frac{1}{(M+1)}+\frac{1}{M+1}\right]\times 2^{k-1}$$

$$\|f-S_{2^{k-1},M}(f)\|_2 \leq \sqrt{L_f}\left(\frac{\ell}{2^k}\right)^{\alpha}\phi\left(\frac{\ell}{2^k}\right)\frac{1}{\sqrt{M+1}}$$

$$= O\left(\frac{\phi\left(\frac{\ell}{2^k}\right)\ell^{\alpha}}{2^{k\alpha}\sqrt{M+1}}\right)$$

$$W\left(\left(f-S_{2^{k-1},M}f\right),\frac{1}{2^k}\right) = \sup_{0<h\leq\frac{1}{2^k}}\|\left(f-S_{2^{k-1},M}f\right)(t+h)-\left(f-S_{2^{k-1},M}f\right)(t)\|_2$$

$$\leq \|\left(f-S_{2^{k-1},0}f\right)\|_2+\|\left(f-S_{2^{k-1},M}f\right)\|_2$$

$$= 2\|\left(f-S_{2^{k-1},M}f\right)\|_2$$

$$= O\left(\frac{\phi\left(\frac{\ell}{2^k}\right)\ell^{\alpha}}{2^{k\alpha}\sqrt{M+1}}\right).$$

Thus, this theorem is completely established.
The following corollaries are derived from theorem 4.1.

Corollary 4.2. If a function $f \in H^{\alpha,\phi}[0,\ell)$, then the Legendre wavelet approximation $E_{2^{k-1},M}(f)$ of f by $S_{2^{k-1},M}(f)$ is given by

$$(i)\qquad \left(E_{2^{k-1},0}f\right)=\min\|f-(S_{2^{k-1},0}f)\|_2=\min\|f-\sum_{n=1}^{2^{k-1}}c_{n,0}\Psi_{n,0}^{(L)}(t)\|_2=O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\ \ell^{\alpha}}{2^{(k-1)\alpha}}\right)$$

$$(ii)\qquad \left(E_{2^{k-1},M}f\right)=\min\|f-(S_{2^{k-1},M}f)\|_2=\min\|f-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M}c_{n,m}\Psi_{n,m}^{(L)}(t)\|_2=O\left(\frac{\phi(\frac{\ell}{2^k})\ \ell^{\alpha}}{2^{(k\alpha)}\sqrt{M+1}}\right),\ M\geq 1$$

The proof of corollary 4.2 can be developed in parallel to the proof of theorem 4.1, independently.

Remark

If the function $f \in H^{\alpha,\phi}[0,\ell)$, then the moduli of continuity $W\left(\left(f-S_{2^{k-1},M}f\right),\frac{1}{2^k}\right)=O\left(\frac{\phi(\frac{\ell}{2^k})\ \ell^{\alpha}}{2^{(k\alpha)}\sqrt{M+1}}\right)$ and the error of approximation $\left(E_{2^{k-1},M}f\right)=O\left(\frac{\phi(\frac{\ell}{2^k})\ \ell^{\alpha}}{2^{(k\alpha)}\sqrt{M+1}}\right)\to 0$ as $k\to\infty$, $M\to\infty$. Hence $W\left(\left(f-S_{2^{k-1},M}f\right),\frac{1}{2^k}\right)$ and $E_{2^{k-1},M}(f)$ are best possible modulus of continuity and approximation of functions respectively in wavelet analysis. It is also observed that

$$W\left(\left(f-S_{2^{k-1},M}f\right),\frac{1}{2^k}\right)\leq 2\,E_{2^{k-1},M}(f).$$

Hence the modulus of continuity is sharper than the approximation of function in $H^{\alpha,\phi}[0,\ell)$ by Legendre wavelet method.

5. Algorithm of the Legendre collocation method for solving Van der Pol-Duffing oscillators

The Van der Pol-Duffing oscillator is defined by the following non-linear equation [14]:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - \mu(1-y^2)\frac{\mathrm{d}y}{\mathrm{d}t} + \alpha y + \beta y^3 = F\cos(\omega t), \;\; y(0)=1, \;\; y'(0)=0 \tag{20}$$

Here, y represent the displacement from the equilibrium position, while F and ω are the amplitude and frequency of the external excitation, respectively. $\mu > 0$ stands for the damping parameter of the system, β represents the nonlinear stiffness parameter, α is a system parameter, and t denotes time. The Van der Pol–Duffing equation can be represented in three basic physical situations: single well ($\alpha > 0, \beta > 0$), double well ($\alpha < 0, \beta > 0$), and double hump ($\alpha > 0, \beta < 0$).
According to theorem 3.2, the Legendre wavelet solution is given by:

$$y(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi^{(L)}_{n,m}(t) \tag{21}$$

This solution converges to the exact solution of equation (20). The Legendre collocation method is used to determine the unknown coefficients $c_{n,m}$. A total of $2^{k-1}(M+1)$ conditions exists for the determination of these coefficients:

$$c_{1,0},\ c_{1,1}, ..., c_{1,M},\ c_{2,0},\ c_{2,1} ...., c_{2,M}, ..., c_{2^{k-1},0},\ c_{2^{k-1},1}, ..., c_{2^{k-1},M}.$$

Equation (20) is subject to two initial conditions, which yield the following two conditions:

$$y(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi^{(L)}_{n,m}(0) = 1, \qquad y'(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \left(\Psi^{(L)}_{n,m}\right)'(0) = 0 \tag{22}$$

It is important to note that an additional $2^{k-1}(M+1)-2$ conditions are needed to determine all the unknown coefficients. These extra conditions can be obtained using equations (20) and (21). Put the value of $y(t)$ from equation (21) into equation (20), we obtain:

$$\begin{aligned}&\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi''_{n,m}(t) - \mu\left[1 - \left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi_{n,m}(t)\right)^2\right]\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi'_{n,m}(t)\right)\\ &+\alpha\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi_{n,m}(t)\right) + \beta\left(\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{n,m} \Psi_{n,m}(t)\right)^3 = Fcos(\omega t)\end{aligned} \tag{23}$$

By evaluating the above equation at discreate points, also known as collocation points:

$$t_j = \frac{j-0.5}{2^{k-1}(M+1)} \;\; j = 3, 4, ..., 2^{k-1}(M+1)$$

$2^{k-1}(M+1)-2$ equations are obtained. By equations (22) and (23), we have systems of $2^{k-1}(M+1)$ equations that can be solved for $c_{n,m}$.

5.1. *Numerical examples*

Several numerical examples are presented to demonstrate the effectiveness and applicability of the Legendre collocation method in solving the Van der Pol–Duffing equations.

Example 1. If $\alpha = \beta = \mu = 0$ and $F = \omega = 1$, then equation (20) becomes

$$y'' + y = \cos(t), \quad y(0) = 1, \; y'(0) = 0 \tag{24}$$

having the exact solution $y(t) = \cos(t) + \frac{1}{2}t\sin(t)$.

$$\begin{aligned}
|y(t_1) - y(t_2)| &= |\cos(t_1) - \cos(t_2)| + \frac{1}{2}|t_1\sin(t_1) - t_2\sin(t_2)| \\
&= |2\sin\left(\frac{t_1+t_2}{2}\right)\sin\left(\frac{t_2-t_1}{2}\right)| + \frac{1}{2}|t_1\sin(t_1) - t_2\sin(t_2)| \\
&\leq 2\frac{|t_2-t_1|}{2} + \frac{1}{2}|t_1\sin(t_1) - t_2\sin(t_2)|
\end{aligned}$$

Using the Lagrange mean value theorem for the function $t\sin(t)$ on $[0, 1]$:

$$\begin{aligned}
|t_1\sin(t_1) - t_2\sin(t_2)| &\leq 2|t_1 - t_2|, \quad t_1, t_2 \in [0,1] \\
|y(t_1) - y(t_2)| &\leq |t_2 - t_1| + |t_1 - t_2| \\
&= 2|t_1 - t_2|
\end{aligned}$$

Hence, $y \in H^1[0, 1)$. According to lemma 3.1, $y(t)$ can be expressed as

$$y(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M} c_{n,m}\Psi_{n,m}(t) \tag{25}$$

Applying the Legendre collocation method for $k = 1$ and $M = 20$, the corresponding values of $c_{n,m}$ are given by

$$\begin{aligned}
c_{1,0} &= 0.99205532427760837602799995998, & c_{1,1} &= -0.0091183250840296521061684247 89 \\
c_{1,2} &= -0.0049635340500583859151827 07, & c_{1,3} &= -0.00140534950186329744152028 7345 \\
c_{1,4} &= -0.000146682519146904362604734, & c_{1,5} &= 0.0000092225730394209363879330674 \\
c_{1,6} &= 0.67306073739688893304867 57 \times 10^{-6}, & c_{1,7} &= -0.209586793646632357510679 \times 10^{-7} \\
c_{1,8} &= -0.116410596836768205488319 \times 10^{-8}, & c_{1,9} &= 0.244885246127825201342168 8 \times 10^{-10} \\
c_{1,10} &= 0.109593596772150759405244 6 \times 10^{-11}, & c_{1,11} &= -0.174327361631027744432 6 \times 10^{-13}
\end{aligned}$$

$$\begin{aligned}
c_{1,12} &= -0.703053491884161084353806 \times 10^{-15}, & c_{1,13} &= 0.413314963912699572960 85 \times 10^{-16} \\
c_{1,14} &= -0.232727695570334355686669 \times 10^{-16}, & c_{1,15} &= 0.135712770565024774580 72 \times 10^{-16} \\
c_{1,16} &= -0.903633004248579591954796 \times 10^{-17}, & c_{1,17} &= 0.389296295244636756454 93 \times 10^{-17} \\
c_{1,18} &= -0.242763969004858296177242 \times 10^{-17}, & c_{1,19} &= 0.572698143877291770625 75 \times 10^{-18} \\
c_{1,20} &= -0.000000000000000000345901357511573028704934408407 07
\end{aligned}$$

By substituting the Legendre wavelet coefficients $c_{n,m}$ from above into equation (25), the explicit form of the approximate solution to equation (24) is obtained. A comparison between the exact solution, the Legendre wavelet solution based on the Legendre collocation method, and the solution obtained by ODE45 method for different values of t is shown in table 1.1. It has been demonstrated that the solution obtained by the Legendre wavelet method is superior to the ODE45 solution and is nearly identical to the exact solution. This demonstrates the efficiency of this technique.

Variable (t)	Exact solution	Legendre solution M=20	Solution by ODE 45
0	1	1	1
0.1	0.999995836110367	0.999995836110335	0.999995836118480
0.2	0.999933510920748	0.999933510920679	0.999933510938151
0.3	0.999664520124807	0.999664520124702	0.999664520146955
0.4	0.998944662464615	0.998944662464475	0.998944662481346
0.5	0.997438946541424	0.997438946541250	0.997438946537228
0.6	0.994728356928189	0.994728356927983	0.994728356882647
0.7	0.990318377817680	0.990318377817445	0.990318377706047
0.8	0.983649145706974	0.983649145706712	0.983649145500912
0.9	0.974107077603032	0.974107077602744	0.974107077271467
1	0.961037798272088	0.961037798271778	0.961037797782178

Table1.1 : Comparison between exact and Legendre solutions for different values of t.

Variable (t)	Absolute error = |Exact solution-Legendre solution|	Absolute error = |Exact solution-ODE45 solution|
0	0	0
0.1	$0.032085445411667 \times 10^{-12}$	$0.008112954752448 \times 10^{-9}$
0.2	$0.068833827526760 \times 10^{-12}$	$0.017402967955604 \times 10^{-9}$
0.3	$0.104805053524615 \times 10^{-12}$	$0.022147950140550 \times 10^{-9}$
0.4	$0.139888101102770 \times 10^{-12}$	$0.016730949958799 \times 10^{-9}$
0.5	$0.173638881051374 \times 10^{-12}$	$0.004195976899268 \times 10^{-9}$
0.6	$0.205502281858116 \times 10^{-12}$	$0.045541903581636 \times 10^{-9}$
0.7	$0.235367281220533 \times 10^{-12}$	$0.111632925126059 \times 10^{-9}$
0.8	$0.262789789928775 \times 10^{-12}$	$0.206062056307132 \times 10^{-9}$
0.9	$0.287880830285303 \times 10^{-12}$	$0.331564997679834 \times 10^{-9}$
1.0	$0.309863246172881 \times 10^{-12}$	$0.489910001455485 \times 10^{-9}$

Table1.2 : Absolute error of the exact solution with Legendre solution and ODE45 solution.

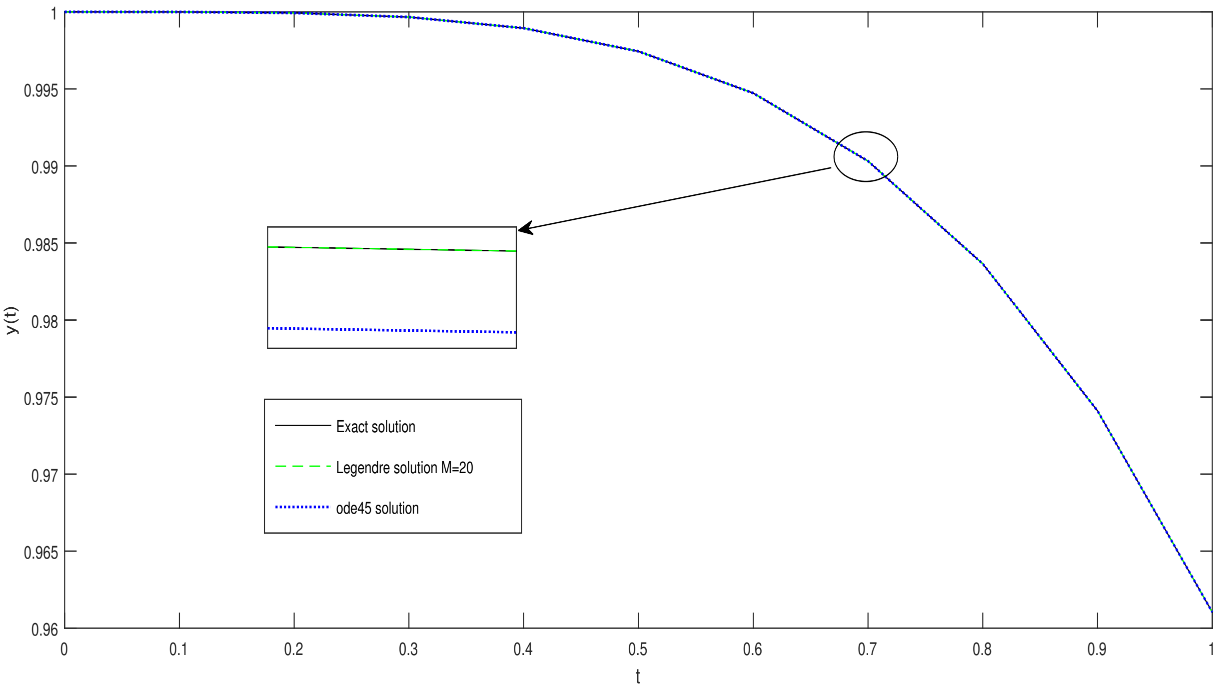

Fig.1. The graphs of exact, Legendre and ODE45 solutions for different values of t.

Example 2. (Single well) If $\alpha = 0.5,\ \beta = 0.5,\ \mu = 0.1,\ F = 0.5$ and $\omega = 0.79$, then equation (20) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 0.1(1-y^2)\frac{\mathrm{d}y}{\mathrm{d}t} + 0.5y + 0.5y^3 = 0.5\cos(0.79t),\quad y(0)=1,\ \ y'(0)=0 \tag{26}$$

Using Legendre collocation method for $k = 1$ and $M = 3$, the values of $c_{n,m}$ are given by

$$c_{1,0} = 0.917912628801809328850300014658,\qquad c_{1,1} = -0.0689502830760488894322843335 13$$

$$c_{1,2} = -0.015593255852576403429037 4414,\qquad c_{1,3} = 0.000933739387577275038411347904 53$$

Appling Legendre collocation method for $k = 1$ and $M = 5$, the values of $c_{n,m}$ are given by

$$c_{1,0} = 0.921678946190289593586930421 3269,\qquad c_{1,1} = -0.0665116043290528470185665688275$$

$$c_{1,2} = -0.0156625013809733956058255154 00,\qquad c_{1,3} = 0.000779476509441346728775392985859$$

$$c_{1,4} = 0.0000549891781070669928853568 1768,\qquad c_{1,5} = -0.0000118575568165822652818695886$$

By substituting the Legendre wavelet coefficients $c_{n,m}$ from above into equation (25), the explicit form of the approximate solution to equation (26) is obtained.

Variable (t)	Legendre solution M=3	Solution by ODE 45	Legendre solution M=5
0	1	1	1
0.1	0.997216221376900	0.997502849384376	0.997510682936606
0.2	0.989062520884312	0.990045163226671	0.990068054994725
0.3	0.975835351587300	0.977725786077155	0.977764780595462
0.4	0.957831166550928	0.960702463426295	0.960756916220771
0.5	0.935346418840263	0.939183087567294	0.939252017908430
0.6	0.908677561520369	0.913415056073070	0.913497248747009
0.7	0.878121047656311	0.883673496319420	0.883767486370842
0.8	0.843973330313154	0.850249110913629	0.850353430455005
0.9	0.806530862555964	0.813436335485702	0.813549710210282
1	0.766090097449804	0.773522390112571	0.773642991878140

Table2 : Comparison between ODE45 and Legendre solutions for different values of M.

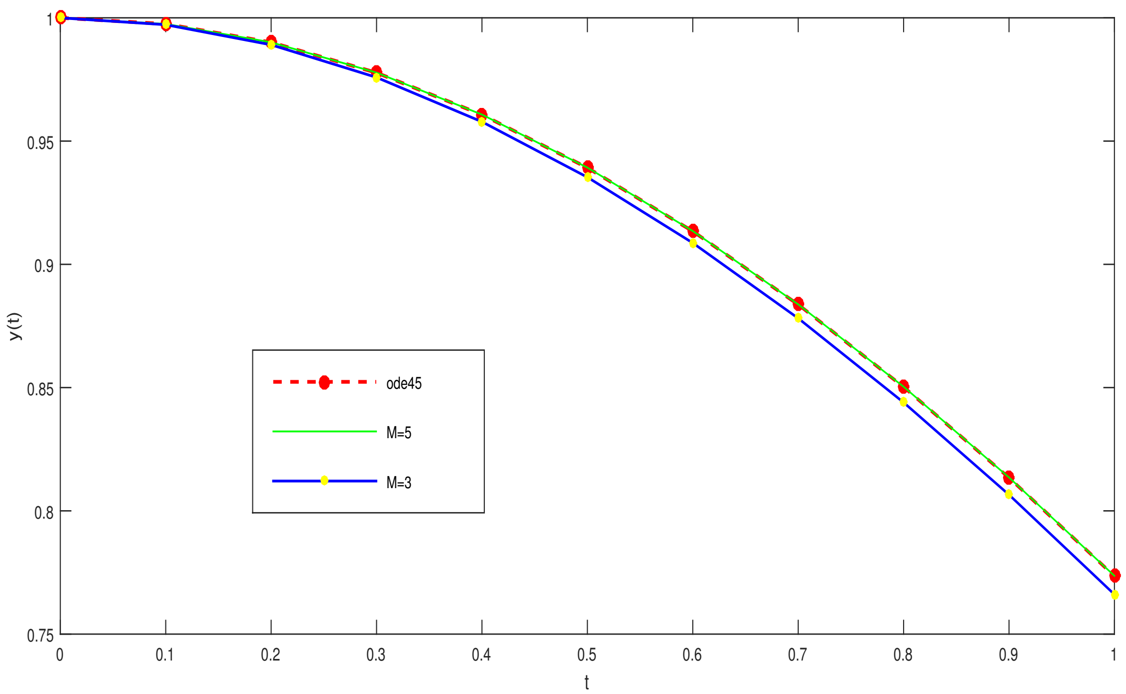

Fig.2. The graphs of ODE45 and Legendre solutions for different values of M.

Example 3. (Double hump) Now, solve equation (20) for $\alpha = 0.5$, $\beta = -0.5$, $\mu = 0.1$, $F = 0.5$, and $\omega = 0.79$. Using Legendre collocation method for $k = 1$ and $M = 3$, the values of $c_{n,m}$ are given by

$$c_{1,0} = 1.0781646925258586553056300349, \qquad c_{1,1} = 0.069169330214733968813691083886 8$$

$$c_{1,2} = 0.0193845949587195074178153953, \qquad c_{1,3} = 0.00064449379852740816444110332 72$$

Appling Legendre collocation method for $k = 1$ and $M = 5$, the values of $c_{n,m}$ are given by

$c_{1,0} = 1.0845329125207945374279619 3$, $c_{1,1} = 0.07394045810297756333153892829 3$

$c_{1,2} = 0.01993416216743432386684610$, $c_{1,3} = 0.00047506116245111011758461578 7$

$c_{1,4} = 0.00009263184185320123695211$, $c_{1,5} = 0.00001791757233098035679548615 9$

Variable (t)	Legendre solution M=3	Solution by ODE 45	Legendre solution M=5
0	1	1	1
0.1	1.002123268635160	1.002500774247598	1.002440554227072
0.2	1.008629488165635	1.010012363126603	1.009822871205626
0.3	1.019723279028916	1.022563047075262	1.022221731630769
0.4	1.035609261662494	1.040202595777685	1.039704691501554
0.5	1.056492056503860	1.063007463804532	1.062350052502435
0.6	1.082576283990506	1.091088970289386	1.090264832384714
0.7	1.114066564559923	1.124604856306245	1.123602735347982
0.8	1.151167518649602	1.163774818150023	1.162582122421577
0.9	1.194083766697034	1.208900893902217	1.207503981846024
1	1.243019929139710	1.260393969535203	1.258769899454484

Table3 : Comparison between ODE45 and Legendre solutions for different values of M.

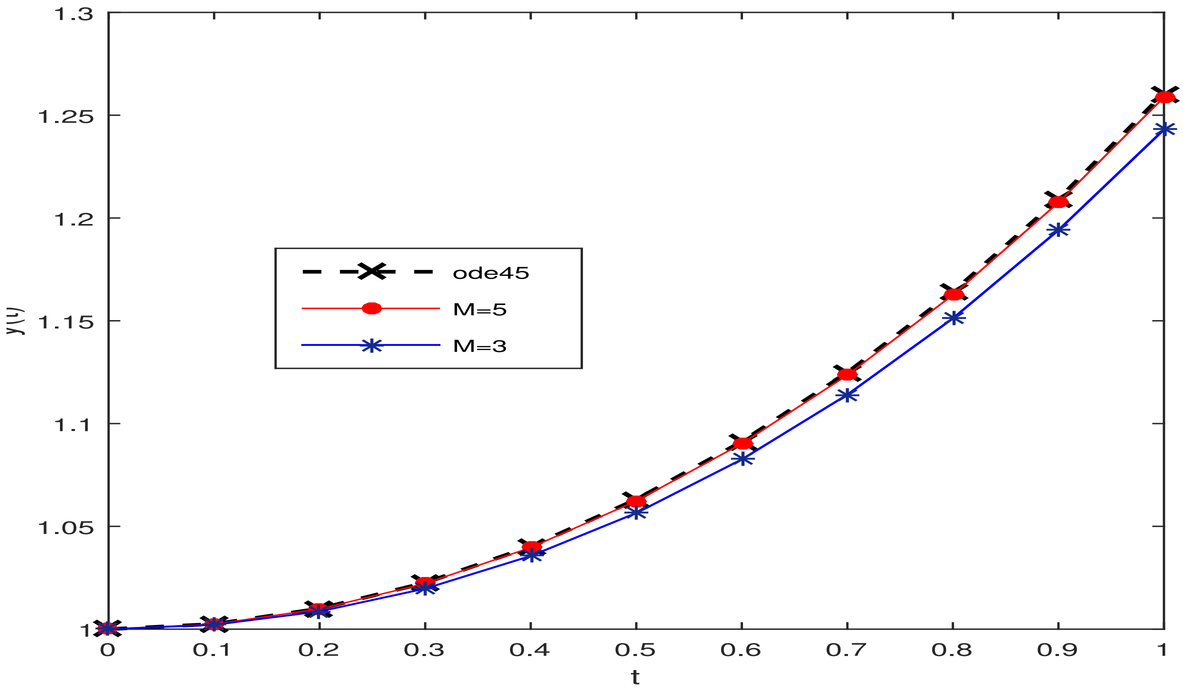

Fig.3. The graphs of ode45 and Legendre solution's for different values of M.

Example 4.(Double well) Now solve equation (20) for $\alpha = -0.5,\ \beta = 0.5,\ \mu = 0.1,\ F = 0.5$, and $\omega = 0.79$.

Using Legendre collocation method for $k = 1$ and $M = 3$, the values of $c_{n,m}$ are given by

$c_{1,0} = 1.09067625886858968147817956 2$, $c_{1,1} = 0.07360343785956907006094296038 2$

$c_{1,2} = 0.01391831817436061784837647 2$, $c_{1,3} = -0.00214923055005250696105663178$

Appling Legendre collocation method for $k = 1$ and $M = 5$, the values of $c_{n,m}$ are given by

$c_{1,0} = 1.07597237810426219279909332260\,5$, $c_{1,1} = 0.063742950068503095619024761446\,6$
$c_{1,2} = 0.013996785103445973489713273379$, $c_{1,3} = -0.00140387969251223641532546235\,0$
$c_{1,4} = -0.000193552125921031494723863\,9$, $c_{1,5} = -0.000000671442147438563522044492$

Variable (t)	Legendre solution M=3	Solution by ODE 45	Legendre solution M=5
0	1	1	1
0.1	1.003459510606984	1.002496602038700	1.002476075109690
0.2	1.013383136064287	1.009945356556822	1.009880085668257
0.3	1.029088516826441	1.022221704543159	1.022104466066525
0.4	1.049893293347976	1.039114504665607	1.038946120687192
0.5	1.075115106083421	1.060322231572470	1.060105750483720
0.6	1.104071595487307	1.085448985298767	1.085187179559220
0.7	1.136080402014165	1.114001011140708	1.113696681745338
0.8	1.170459166118523	1.145384622684424	1.145042307181143
0.9	1.206525528254913	1.178906589155956	1.178533208892013
1	1.243597128877864	1.213778144111941	1.213378969368522

Table4 : Comparison between ODE45 and Legendre solutions for different values of M.

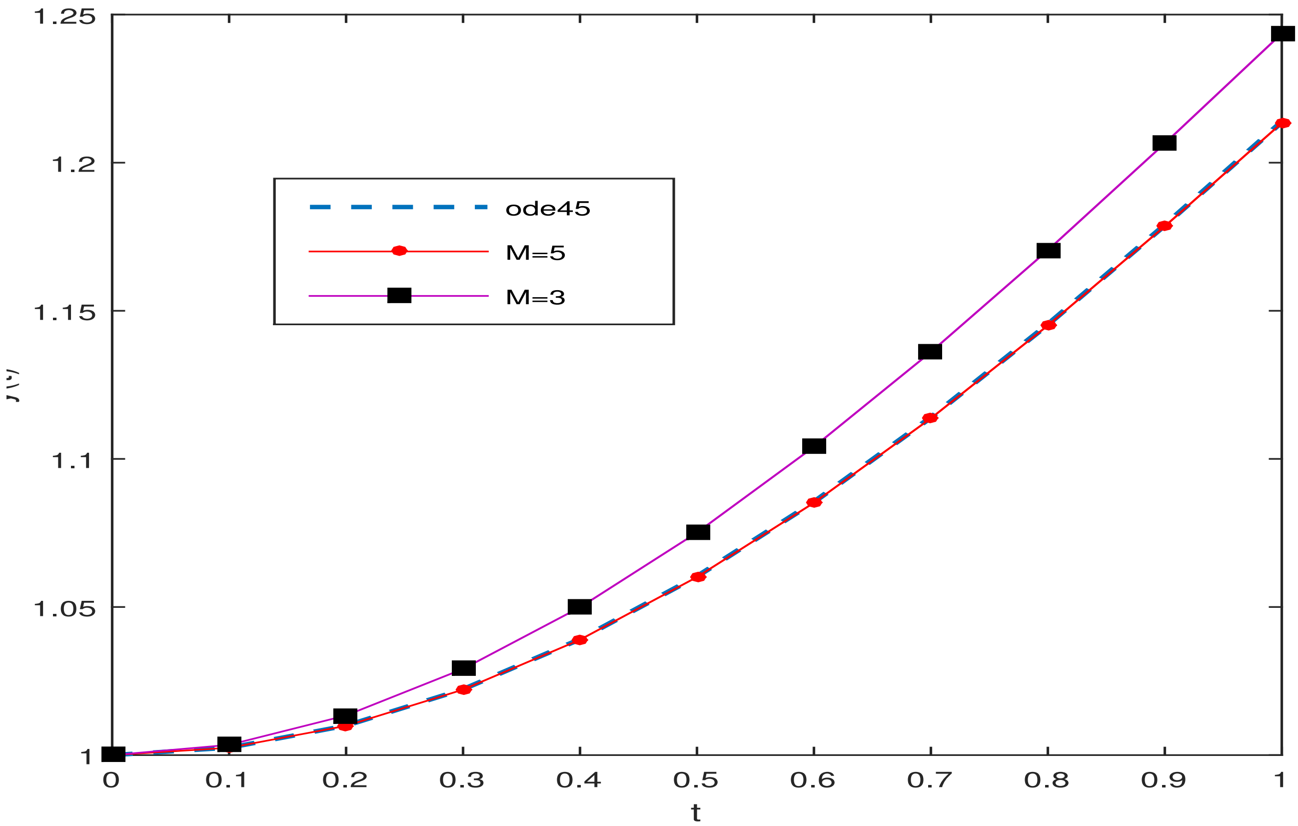

Fig.4. The graphs of ODE45 and Legendre solutions for different values of M.

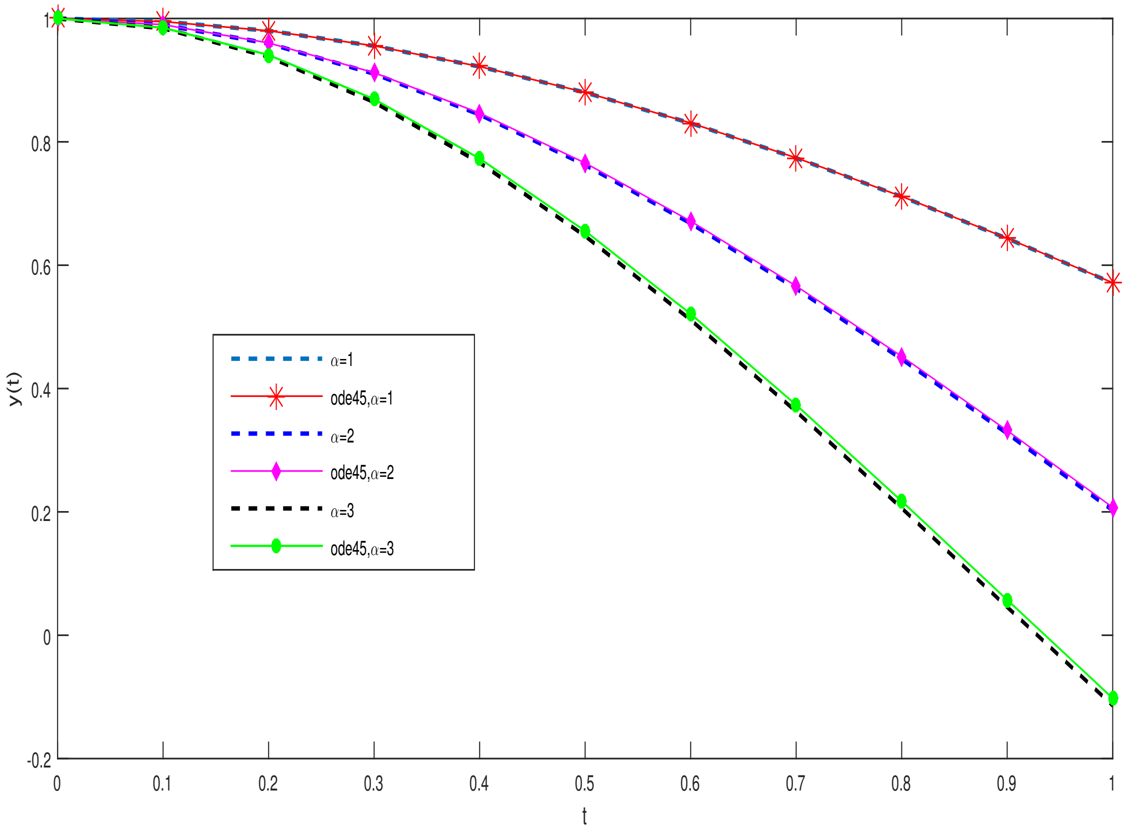

Fig.5. The graphs of Legendre solutions for $\beta = F = 0.5$, $\mu = 0.1$, $\omega = 0.79$, $M = 5$.

6. Conclusions

6.1. Conclusions

(i) In theorem 4.1, the moduli of continuity have been computed and given by:

$$W\left(\left(f - S_{2^{k-1},0}f\right), \frac{\ell}{2^k}\right) = O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\ \ell^\alpha}{2^{(k-1)\alpha}}\right) \to 0 \quad \text{as} \quad k \to \infty,$$

$$W\left(\left(f - S_{2^{k-1},M}f\right), \frac{\ell}{2^k}\right) = O\left(\frac{\phi(\frac{\ell}{2^k})\ \ell^\alpha}{2^{k\alpha}\sqrt{M+1}}\right) \to 0 \text{ as } k \to \infty, M \to \infty.$$

(ii) By corollary 1, $\quad E_{2^{k-1},0}(f) = O\left(\frac{\phi(\frac{\ell}{2^{k-1}})\ \ell^\alpha}{2^{(k-1)\alpha}}\right) \to 0 \quad \text{as} \quad k \to \infty,$

$$E_{2^{k-1},M}(f) = O\left(\frac{\phi(\frac{\ell}{2^k})\ \ell^\alpha}{2^{k\alpha}\sqrt{M+1}}\right) \to 0 \quad \text{as} \quad k \to \infty, M \to \infty.$$

Thus, $W\left(\left(f - S_{2^{k-1},0}f\right), \frac{\ell}{2^k}\right)$, $W\left(\left(f - S_{2^{k-1},M}f\right), \frac{\ell}{2^k}\right)$, $E_{2^{k-1},0}(f)$, and $E_{2^{k-1},M}(f)$ are the best possible estimators in wavelet analysis.

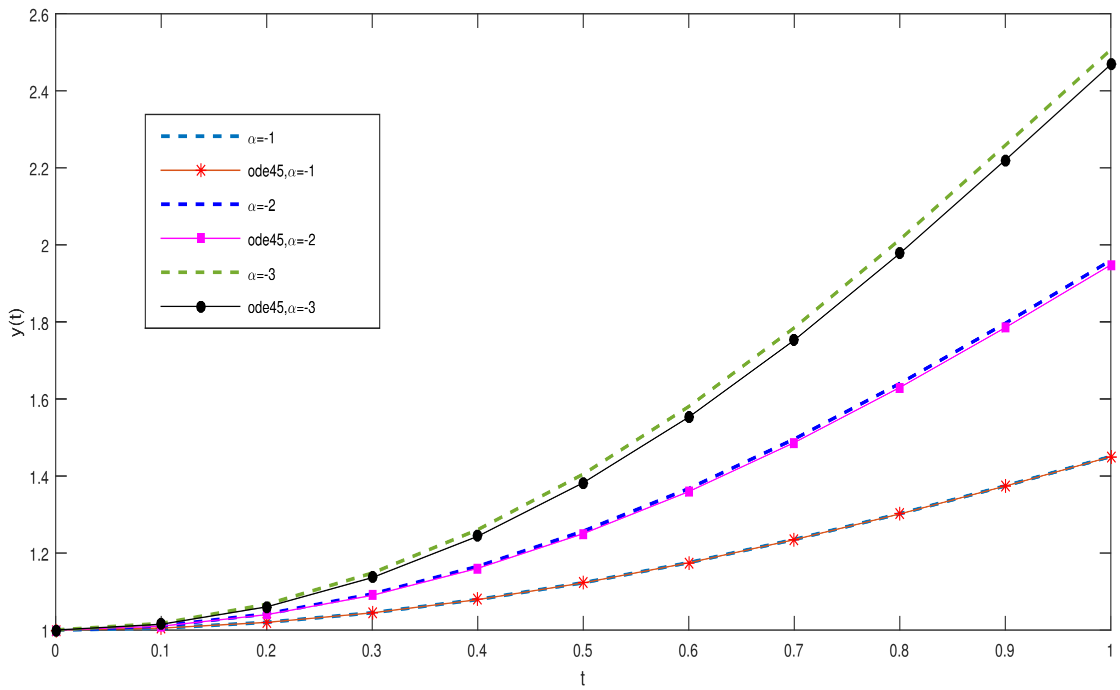

Fig.6. The graphs of Legendre solutions for $\beta = F = 0.5$, $\mu = 0.1$, $\omega = 0.79$, $M = 5$.

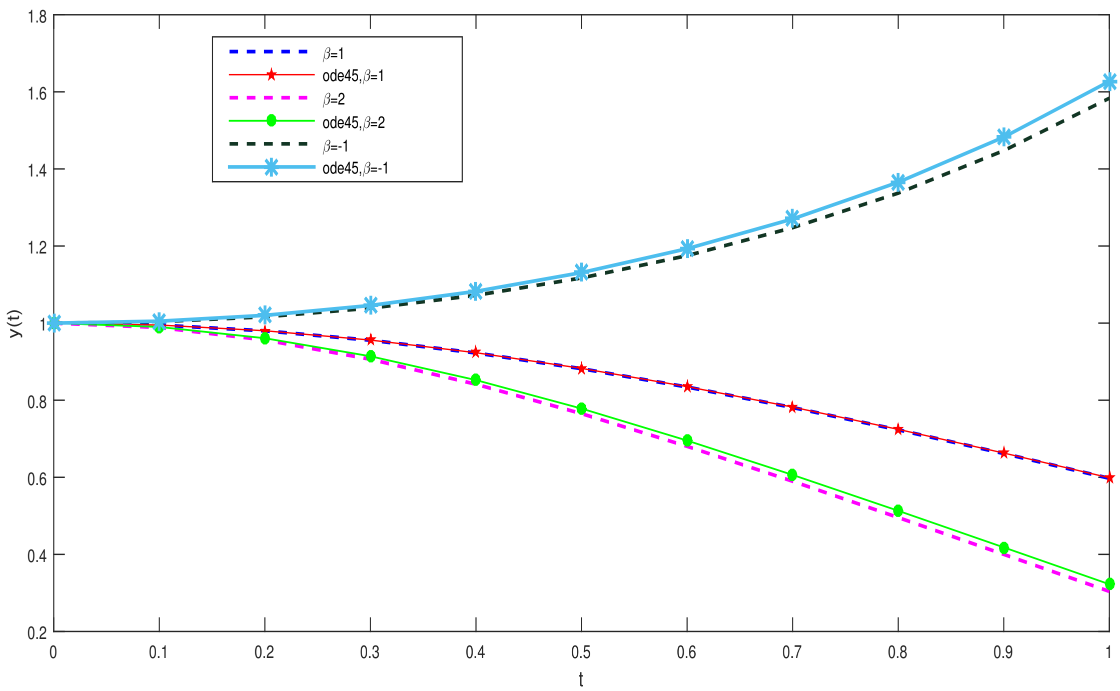

Fig.7. The graphs of Legendre solution's for $\alpha = F = 0.5$, $\mu = 0.1$, $\omega = 0.79$, $M = 5$.

(iii) From theorem 4.1 and corollary 4.2, it is observed that:

$$W\left(\left(f - S_{2^{k-1},0}f\right), \frac{\ell}{2^k}\right) \leq 2\, E_{2^{k-1},0}(f),$$
$$W\left(\left(f - S_{2^{k-1},M}f\right), \frac{\ell}{2^k}\right) \leq 2\, E_{2^{k-1},M}(f).$$

Hence, the moduli of continuity $W\left(\left(f - S_{2^{k-1},0}f\right), \frac{\ell}{2^k}\right)$ and $W\left(\left(f - S_{2^{k-1},M}f\right), \frac{\ell}{2^k}\right)$ are superior and more precise than the approximations $E_{2^{k-1},0}(f)$ and $E_{2^{k-1},M}(f)$, respectively.

(iv) Equation (20) is solved using the Legendre collocation technique for specific values of the α and β parameters. Tables 1 to 4 display some values of y for the three primary physical scenarios. Figures 1 to 7 compare and plot the results of the numerical solution based on the ODE45 technique with those obtained using the Legendre collocation method. The comparison with numerical data demonstrates that the solution achieved through the Legendre collocation method exhibits a very high level of accuracy, which is deemed acceptable. Moreover, the accuracy of the solution increases as the degree of the polynomial (M) increases. After demonstrating the efficiency of the Legendre collocation method as a powerful analytical technique, the effects of the constant parameter α & β on the response is shown in figures 1 to 7. Using the derived Legendre wavelet solutions, diagrams for the single well case ($\alpha = 0.5, \beta = 0.5$), double well case ($\alpha = -0.5, \beta = 0.5$), and double hump case ($\alpha = 0.5, \beta = -0.5$) of the Van der Pol oscillator are presented in figures 2, 4, and 3, respectively. A key advantage of the proposed Legendre collocation technique is its ability to provide solutions for all possible values of the constant parameters α and β.

Acknowledgments

Shyam Lal, one of the authors, is thankful to DST - CIMS for encouragement to this work. Deepak Kumar Singh, one of the authors, is grateful to U.G.C. (India) for providing financial assistance in the form of UGC Non-NET Fellowship for his research work.

Data Availability Statement

No Data associated in the manuscript.

References

References

[1] N. Ali,, M. Ullah, M. Sajid, S.U. Khan, Application of the Legendre wavelets method to the parallel plate flow of a third grade fluid and forced convection in a porous duct, Eur. Phys. J. Plus, 132(2017), Article No. 133.
[2] A. G. Babenko, Y.V. Kryakin, P. T. Staszak, Special moduli of continuity and the constant in the jackson–stechkin theorem, Constr. Approx, 38(2013), 339–364.
[3] C.K. Chui, An Introduction to Wavelets, Wavelet analysis and its applications, Academic Press, USA, Vol. 1, 1992.
[4] G. Das, T. Ghosh, , B.K. Ray, Degree of approximation of functions by their Fourier series in the generalized Hölder metric, Proc. Indian Acad. Sci. (Math. Sci.), 106(1996), 139–153.
[5] I. Daubechies, Ten Lectures on Wavelets, SIAM, Philadelphia, PA, 1992.
[6] L. Debnath, Wavelet transform and their applications, Birkhäuser, Boston, MA, 2002.
[7] R.A. Devore, Nonlinear approximation, Acta Numer., Cambridge University Press, Cambridge, 7(1998), 51-150.
[8] A. Güner, Y. Salih, Legendre collocation method for solving nonlinear differential equations, Math. Comput. Appl., 18 (2013), 521-530.
[9] A. Kimiaeifar, , A.R. Saidi, G.H. Bagheri, M. Rahimpour, D.G. Domairry, Analytical solution for Van Der Pol–Duffing oscillators, Chaos Solit. Fract., 42(2009), 2660-2666.
[10] S. Lal, P. Kumari, Wavelet approximation of functions belonging to generalized Lipschitz class by extended Haar wavelet series and its application in the solution of bessel differential equation using Haar operational matrix, Int. J. Appl. Comput. Math, 5(2019), Article No. 67.
[11] Y. Meyer, Wavelets: Their past and their future, in Progress in Wavelet Analysis and Applications, Frontieres, 33(1993), 9-18.

[12] J. Morlet, G. Arens, E. Fourgeau, D. Giard, Wave propagation and sampling theory, Part II: Sampling theory and complex waves, Geophysics, 47(1982), 222-236.
[13] S. Nemati, P.M. Lima, S. Sedaghat, Legendre wavelet collocation method combined with the Gauss–Jacobi quadrature for solving fractional delay-type integro-differential equations, Appl. Numer. Math., 149(2020), 99-112.
[14] P. V. Ramana, B. K. Prasad, Modified adomian decomposition method for Van der Pol equations , Int. J. Non-Linear Mech., 65(2014), 121-132.
[15] B. Sripathy, P. Vijayaraju, G. Hariharan, Chebyshev wavelet based approximation method to some non-linear differential equations arising in engineering, Int. J. Math. Anal., 9(2015), 993–1010.
[16] K.G. Steffens, The History of Approximation Theory: From Euler to Bernstein, Birkhäuser, Boston, 2006.
[17] Z. Zhihong, Y. Shaopu, Application of van Der Pol–Duffing oscillator in weak signal detection, Comput. Electr. Eng., 41(2015), 1-8.
[18] A. Zygmund, Trigonometric Series, Vol. I (2nd edition), Cambridge University Press, Cambridge, 1959.