Filomat 40:2 (2026), 415–439
https://doi.org/10.2298/FIL2602415B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Fractional midpoint inequalities for co-ordinated (s_1, s_2)-convex functions: A new perspective

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Abstract. In this study, some Hermite–Hadamard type inequalities are investigated for differentiable co-ordinated (s_1, s_2)-convex functions in the second sense. In particular, generalizations of midpoint-type inequalities are established on rectangular domains in the plane. Additionally, several new inequalities are derived for the cases of Riemann–Liouville and k−Riemann–Liouville fractional integrals by considering special cases of the main results.

1. Introduction

Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamic models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined, enables us to obtain more realistic approaches. One of the most important applications of the fractional integrals is the Hermite-Hadamard integral inequality, see [1, 2, 6, 9, 11, 12, 14, 18–21].

Hermite sent a letter in which he mentioned an inequality obtained by showing convex functions to Mathesis magazine in 1881 and this study was published with the file extension "Mathesis 3 p.82, 1883". Unfortunately, Hermite's main work is often mentioned anonymously. Later, similar results were mentioned in a study conducted by Hadamard in 1893. This important inequality has taken its place in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D10, 26D15, 26B15, 26B25.
Keywords. Midpoint inequality, Co-ordinated (s_1, s_2)-convex functions, Generalized fractional integral.
Received: 26 May 2025; Revised: 19 October 2025; Accepted: 08 November 2025
Communicated by Dragan S. Djordjević
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where $f : I \to \mathbb{R}$ is a convex mapping defined on the interval $I \subseteq \mathbb{R}$ and $a, b \in I$, with $a < b$.

In recent years, remarkable number of papers have been investigated to trapezoid and midpoint type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1), respectively. For instance, Dragomir and Agarwal first established to trapezoid inequalities in the case of convex functions in the [7] and Kirmacı first investigated to midpoint inequalities to the case of convex functions in the [13]. Iqbal et al. presented some fractional midpoint type inequalities for convex functions in [10]. On the other hand, Dragomir established Hermite-Hadamard inequalities in the case of co-ordinated convex mappings in [6]. The midpoint and trapezoid type inequalities for co-ordinated convex functions were presented in the papers [15] and [21], respectively. Moreover, some fractional midpoint type inequalities for co-ordinated convex functions were presented in the paper [22].

In [20], Sarikaya and Ertuğral first investigated new fractional integrals which are called generalized fractional integrals. Moreover, several trapezoids and midpoint type inequalities for generalized fractional integrals are proved by the authors. In addition to these, Turkay et al. described the generalized fractional integrals in the case of functions with two variables. These authors investigated Hermite-Hadamard inequalities for this kind of fractional integrals in [23]. For more information about these type of inequalities, we refer to [4, 5, 16].

In this paper, we get some inequalities similar to the Hermite-Hadamard inequality on a rectangle from the plane $\mathbb{R}^2$.

For our work to be done in two-dimensional space, we need the following definition:

Definition 1.1. *Let $\Delta =: [a, b] \times [c, d]$ in $\mathbb{R}^2$. A mapping $g : \Delta \to \mathbb{R}$ is said to be co-ordinated convex in the bidimensional interval Δ, if the following inequality holds:*

$$\begin{aligned} & g(tx + (1-t)y, sz + (1-s)w) \\ \leq \; & tsg(x, z) + t(1-s)g(x, w) + s(1-t)g(y, z) + (1-t)(1-s)g(y, w) \end{aligned}$$

for all $(x, z), (y, w) \in \Delta$ and $t, s \in [0, 1]$.

Definition 1.2. *A function $g : \Delta \to \mathbb{R}$ will be called s-convex in the second sense on Δ if the following inequality*

$$g(tx + (1-t)y, tz + (1-t)w) \leq t^s g(x, z) + (1-t)^s g(y, w)$$

is valid for all $t, s \in [0, 1]$ and $(x, z), (y, w) \in \Delta$.

Modifications to the case of convex and s-convex functions on Δ, which are also known as co-ordinated convex and co-ordinated $s-$convex functions on Δ, respectively, were introduced by Dragomir [15], Sarikaya [21] and Latif [14].

In the paper [17, 21], clasical definition in the case of co-ordinated $s-$convex functions in the second sense can be stated as follows:

Definition 1.3. *A function $g : \Delta \to \mathbb{R}^2$ is called co-ordinated s-convex in the second sense on Δ if the following inequality holds:*

$$\begin{aligned} g(tx + (1-t)y, \lambda z + (1-\lambda)w) \leq \; & t^s \lambda^s g(x, z) + t^s (1-\lambda)^s g(x, w) \\ & + (1-t)^s \lambda^s g(y, z) + (1-t)^s (1-\lambda)^s g(y, w) \end{aligned} \tag{2}$$

for all $t, \lambda \in [0, 1]$ and $(x, z), (y, w) \in \Delta$, and for fixed $s \in (0, 1]$.

Let us consider $s = 1$ in inequality (2). Then, the function g is said to be co-ordinated convex on Δ. If the inequality (2) is in reversed order, then g will be called a co-ordinated $s-$concave in the second sense on Δ.

Definition 1.4. *A function $g : \Delta \to \mathbb{R}^2$ is said to be co-ordinated (s_1, s_2)-convex in the second sense on Δ if the following inequality*

$$\begin{aligned} g(tx + (1-t)y, \lambda z + (1-\lambda)w) \leq \; & t^{s_1} \lambda^{s_2} g(x, z) + t^{s_1} (1-\lambda)^{s_2} g(x, w) \\ & + (1-t)^{s_1} \lambda^{s_2} g(y, z) + (1-t)^{s_1} (1-\lambda)^{s_2} g(y, w) \end{aligned}$$

is valid for all $t, \lambda \in [0, 1]$ and $(x, z), (y, w) \in \Delta$, and for fixed $s_1, s_2 \in (0, 1]$.

The Hermite-Hadamard inequality for co-ordinated convex functions is proved by Dragomir in [6], and it is stated as follows:

Theorem 1.5. *Suppose that $g : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then the below sharp inequalities hold:*

$$\begin{aligned} g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b g\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d g\left(\frac{a+b}{2}, y\right)dy\right] \\ &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)\,dydx \\ &\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b g(x,c)dx + \frac{1}{b-a}\int_a^b g(x,d)dx \right. \\ &\quad \left. + \frac{1}{d-c}\int_c^d g(a,y)dy + \frac{1}{d-c}\int_c^d g(b,y)dy\right] \\ &\leq \frac{g(a,c)+g(a,d)+g(b,c)+g(b,d)}{4}. \end{aligned} \tag{3}$$

In [20], Sarıkaya and Ertuğral described the new left-sided and right-sided generalized fractional integrals as the follows:

$$\begin{aligned} I_{a^+}^{\varphi} g(x) &= \int_a^x \frac{\varphi(x-t)}{x-t} g(t)dt, \ x > a \\ I_{b^-}^{\varphi} g(x) &= \int_a^b \frac{\varphi(t-x)}{t-x} g(t)dt, \ x < b, \end{aligned}$$

respectively, where $\varphi : [0,\infty) \to [0,\infty)$ is a function which satisfies $\int_0^1 \frac{\varphi(t)}{t}dt < \infty$.

Furthermore, they noticed that these generalized fractional integrals may contain some types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integral, conformable fractional integral, etc., with some special choices.

Inspired by this definition, Turkay et al. give the following definitions:

Definition 1.6. *[23] Let $g \in L_1(\Delta)$. The Generalized Riemann–Liouville integrals $I_{a^+,c^+}^{\varphi,\psi} g(x,y)$, $I_{a^+,d^-}^{\varphi,\psi} g(x,y)$, $I_{b^-,c^+}^{\varphi,\psi} g(x,y)$, $I_{b^-,d^-}^{\varphi,\psi} g(x,y)$ are defined by*

$$I_{a^+,c^+}^{\varphi,\psi} g(x,y) = \int_a^x \int_c^y \frac{\varphi(x-t)}{x-t}\frac{\psi(y-s)}{y-s} g(t,s)dsdt, \ x > a, \ y > c, \tag{4}$$

$$I_{a^+,d^-}^{\varphi,\psi} g(x,y) = \int_a^x \int_y^d \frac{\varphi(x-t)}{x-t}\frac{\psi(s-y)}{s-y} g(t,s)dsdt, \ x > a, \ y < d, \tag{5}$$

$$I_{b^-,c^+}^{\varphi,\psi} g(x,y) = \int_x^b \int_c^y \frac{\varphi(t-x)}{t-x}\frac{\psi(y-s)}{y-s} g(t,s)dsdt, \ x < b, \ y > c, \tag{6}$$

$$I_{b^-,d^-}^{\varphi,\psi} g(x,y) = \int_x^b \int_y^d \frac{\varphi(t-x)}{t-x}\frac{\psi(s-y)}{s-y} g(t,s)dsdt, \ x < b, \ y < d, \tag{7}$$

where $\varphi, \psi : [0,\infty) \to [0,\infty)$ functions which satisfy $\int_0^1 \frac{\varphi(t)}{t}dt < \infty$ and $\int_0^1 \frac{\psi(s)}{s}ds < \infty$, respectively.

In this definition, known fractional integrals can be obtained by some special choices. For example;

i) If we take $\varphi(t) = t$ and $\psi(s) = s$, then the operators (4)-(7) transform into the the Riemann integrals.

ii) If we take $\varphi(t) = \frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^{\beta}}{\Gamma(\beta)}$, then for $\alpha, \beta > 0$ the operators (4), (5), (6) and (7) transform into the the Riemann-Liouville integrals ([19]) respectively as the following:

$$J^{\alpha,\beta}_{a^+,c^+}g(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y (x-t)^{\alpha-1}(y-s)^{\beta-1}g(t,s)dsdt,\ x>a,\ y>c,$$

$$J^{\alpha,\beta}_{a^+,d^-}g(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d (x-t)^{\alpha-1}(s-y)^{\beta-1}g(t,s)dsdt,\ x>a,\ y<d,$$

$$J^{\alpha,\beta}_{b^-,c^+}g(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y (t-x)^{\alpha-1}(y-s)^{\beta-1}g(t,s)dsdt,\ x<b,\ y>c,$$

and

$$J^{\alpha,\beta}_{b^-,d^-}g(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d (t-x)^{\alpha-1}(s-y)^{\beta-1}g(t,s)dsdt,\ x<b,\ y<d,$$

along with Γ called Gamma function.

iii) If we take $\varphi(t) = \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\frac{\beta}{k}}}{k\Gamma_k(\beta)}$, then for $\alpha, \beta, k > 0$ then the operators (4), (5), (6) and (7) transform into the the Riemann-Liouville $k-$fractional integrals ([9]) respectively as the following:

$${}_kI^{\alpha,\beta}_{a^+,c^+}g(x,y) = \frac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_a^x\int_c^y (x-t)^{\frac{\alpha}{k}-1}(y-s)^{\frac{\beta}{k}-1}g(t,s)dsdt,\ x>a,\ y>c,$$

$${}_kI^{\alpha,\beta}_{a^+,d^-}g(x,y) = \frac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_a^x\int_y^d (x-t)^{\frac{\alpha}{k}-1}(s-y)^{\frac{\beta}{k}-1}g(t,s)dsdt,\ x>a,\ y<d,$$

$${}_kI^{\alpha,\beta}_{b^-,c^+}g(x,y) = \frac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_x^b\int_c^y (t-x)^{\frac{\alpha}{k}-1}(y-s)^{\frac{\beta}{k}-1}g(t,s)dsdt,\ x<b,\ y>c,$$

and

$${}_kI^{\alpha,\beta}_{b^-,d^-}g(x,y) = \frac{1}{\Gamma_k(\alpha+1)\Gamma_k(\beta+1)}\int_x^b\int_y^d (t-x)^{\frac{\alpha}{k}-1}(s-y)^{\frac{\beta}{k}-1}g(t,s)dsdt,\ x<b,\ y<d,$$

along with Γ_k called $k-$Gamma function.

Similarly, with some special choices, many known fractional integrals can be obtained.

Meanwhile, Sarikaya and Ertuğral [20] gave the following theorem which is about Herimite-Hadamard inequality for the generalized fractional integral operators.

Theorem 1.7. *Let $g : [a,b] \to \mathbb{R}$ be convex function on $[a,b]$ with $a < b$, then the following inequalities for the generalized fractional integral hold:*

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Omega(1)}\left[I^{\varphi}_{a^+}g(b) + I^{\varphi}_{b^-}g(a)\right] \leq \frac{g(a)+g(b)}{2}$$

where $\Omega(x)$ defined by

$$\Omega(x) = \int_0^x \frac{\varphi((b-a)t)}{t}dt.$$

Turkay et al prove the following Hermite-Hadamrd inequality for co-ordinated convex functions.

Theorem 1.8. *[23] Let $g : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& g\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{4\Omega(1)}\left[I_{a^+}^{\varphi} g\left(b, \frac{c+d}{2}\right) + I_{b^-}^{\varphi} g\left(a, \frac{c+d}{2}\right)\right] \\
& + \frac{1}{4\Psi(1)}\left[I_{c^+}^{\psi} g\left(\frac{a+b}{2}, d\right) + I_{d^-}^{\psi} g\left(\frac{a+b}{2}, c\right)\right] \\
\leq\ & \frac{1}{4\Omega(1)\Psi(1)}\left[I_{a^+,c^+}^{\varphi,\psi} g(b,d) + I_{a^+,d^-}^{\varphi,\psi} g(b,c) + I_{b^-,c^+}^{\varphi,\psi} g(a,d) + I_{b^-,d^-}^{\varphi,\psi} g(a,c)\right] \\
\leq\ & \frac{1}{8\Omega(1)}\left[I_{a^+}^{\varphi} g(b,c) + I_{a^+}^{\varphi} g(b,d) + I_{b^-}^{\varphi} g(a,c) + I_{b^-}^{\varphi} g(a,d)\right] \\
& + \frac{1}{8\Psi(1)}\left[I_{c^+}^{\psi} g(a,d) + I_{c^+}^{\psi} g(b,d) + I_{d^-}^{\psi} g(a,c) + I_{d^-}^{\psi} g(b,c)\right] \\
\leq\ & \frac{g(a,b) + g(a,c) + g(b,c) + g(b,d)}{4},
\end{aligned}$$

where $\varphi, \psi : [0,\infty) \to [0,\infty)$ are defined in Definition 1.6 and the mapping $\Omega, \Phi : [0,1] \to \mathbb{R}$ are defined by

$$\begin{aligned}
\Omega(x) &= \int_0^x \frac{\varphi((b-a)t)}{t} dt, \\
\Phi(x) &= \int_0^x \frac{\psi((d-c)s)}{s} ds.
\end{aligned}$$

On the other hand, Ertuğral et al. proved the following new version of Hermite-Hadamard inequality for generalized fractional inegrals.

Theorem 1.9. *[8] Let $g : [a,b] \to \mathbb{R}$ be a function with $a < b$ and $g \in L_1[a,b]$. If g is a convex function on $[a,b]$, then we have the following inequalities for generalized fractional integral operators:*

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Lambda(1)}\left[I_{a^+}^{\varphi} g\left(\frac{a+b}{2}\right) + I_{b^-}^{\varphi} g\left(\frac{a+b}{2}\right)\right] \leq \frac{g(a)+g(b)}{2} \tag{8}$$

where the mapping $\Lambda : [0,1] \to \mathbb{R}$ defined by

$$\Lambda(x) = \int_0^x \frac{\varphi(\frac{b-a}{2}t)}{t} dt.$$

In the same paper, the authors gave the following lemma and by this lemma they proved the several midpoint type inequalities.

Lemma 1.10. *[8] Let $g : [a,b] \to \mathbb{R}$ be differentiable function on (a,b) with $a < b$. If $g' \in L[a,b]$, then we have the following inequalities for generalized fractional integral operators:*

$$\begin{aligned}
& \frac{1}{2\Lambda(1)}\left[I_{a^+}^{\varphi} g\left(\frac{a+b}{2}\right) + I_{b^-}^{\varphi} g\left(\frac{a+b}{2}\right)\right] - g\left(\frac{a+b}{2}\right) \\
=\ & \frac{b-a}{4\Delta(0)}\left[\int_0^1 \Delta(t) f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt - \int_0^1 \Delta(t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt\right]
\end{aligned}$$

where the mapping $\Delta(t)$ is defined by

$$\Delta(x) = \int_x^1 \frac{\varphi(\frac{b-a}{2}t)}{t} dt.$$

Theorem 1.11. *[8] Let $g : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with $a < b$. If $|g'|$ is convex function, then we have the following inequalities for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{2\Lambda(1)}\left[I^{\varphi}_{a^+}g\left(\frac{a+b}{2}\right)+I^{\varphi}_{b^-}g\left(\frac{a+b}{2}\right)\right]-g\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{4\Delta(0)}\left(\int_0^1|\Delta(t)|\,dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

Theorem 1.12. *[8] Let $g : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with $a < b$. If $|g'|^q$, $q > 1$ is convex function, then we have the following inequalities for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{2\Lambda(1)}\left[I^{\varphi}_{a^+}g\left(\frac{a+b}{2}\right)+I^{\varphi}_{b^-}g\left(\frac{a+b}{2}\right)\right]-g\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{2^{2/q}\Delta(0)}\left(\int_0^1|\Delta(t)|^p\,dt\right)^{1/p}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.13. *[8] Let $g : [a, b] \to \mathbb{R}$ be differentiable function on (a, b) with $a < b$. If $|g'|^q$, $q > 1$ is convex function, then we have the following inequalities for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{2\Lambda(1)}\left[I^{\varphi}_{a^+}g\left(\frac{a+b}{2}\right)+I^{\varphi}_{b^-}g\left(\frac{a+b}{2}\right)\right]-g\left(\frac{a+b}{2}\right)\right| \\
\leq\ & \frac{b-a}{2^{2+1/q}\Lambda(1)}\left(\int_0^1|\Delta(t)|\,dt\right)^{1-1/q} \\
&\times\left[(B_1\left|f'(a)\right|^q+B_2\left|f'(b)\right|^q)^{1/q}+(B_2\left|f'(a)\right|^q+B_1\left|f'(b)\right|^q)^{1/q}\right]
\end{aligned}$$

where the constants B_1 and B_2 are defined by

$$B_1=\int_0^1|\Delta(t)|(1-t)dt \text{ and } B_2=\int_0^1|\Delta(t)|(1+t)dt.$$

Motivated the previous results, the main objective of this study is to establish new midpoint-type inequalities for partial differentiable functions by employing generalized fractional integral operators in two variables. In particular, we aim to extend classical results by utilizing a generalized framework based on fractional calculus, and to develop novel inequalities involving mixed partial derivatives of differentiable functions. This approach not only broadens the applicability of fractional integral techniques but also enhances the theoretical understanding of midpoint approximations in the context of coordinated convexity and multidimensional analysis.

2. Midpoint Type Inequalities for Co-ordinated Concex Fnction

In this section, firstly we need to give a lemma for partial differentiable functions which will help us to prove our main theorems. Then, we present some midpoint type inequalities which are the generalization of those given in earlier works.

Lemma 2.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable function on Δ. If $\frac{\partial^2 f}{\partial s \partial t} \in L_1(\Delta)$, then we have the following identity for generalized fractional integral operators:*

$$\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.$$

$$
\begin{aligned}
& +I_{b^-,c^+}^{\varphi,\psi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{b^-,d^-}^{\varphi,\psi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A \\
= \quad & \frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)} \\
& \times\Bigg[\int_0^1\int_0^1 [\Lambda(1)-\Lambda(t)]\,[\Psi(1)-\Psi(s)]\,\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt \\
& -\int_0^1\int_0^1 [\Lambda(1)-\Lambda(t)]\,[\Psi(1)-\Psi(s)]\,\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt \\
& -\int_0^1\int_0^1 [\Lambda(1)-\Lambda(t)]\,[\Psi(1)-\Psi(s)]\,\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt \\
& +\int_0^1\int_0^1 [\Lambda(1)-\Lambda(t)]\,[\Psi(1)-\Psi(s)]\,\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\Bigg]
\end{aligned} \tag{9}
$$

where

$$
\begin{aligned}
A \quad = \quad & \frac{1}{2\Psi(1)}\left[I_{c^+}^{\psi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{d^-}^{\psi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right] \\
& +\frac{1}{2\Lambda(1)}\left[I_{a^+}^{\varphi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{b^-}^{\varphi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]
\end{aligned} \tag{10}
$$

and and the mapping $\Lambda,\Psi:[0,1]\to\mathbb{R}$ *are defined by*

$$
\begin{aligned}
\Lambda(x) \quad &= \quad \int_0^x \frac{\varphi(\frac{b-a}{2}t)}{t}dt, \\
\Psi(x) \quad &= \quad \int_0^x \frac{\psi(\frac{d-c}{2}s)}{s}ds.
\end{aligned} \tag{11}
$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
I_1 \quad = \quad & \int_0^1\int_0^1 [\Lambda(1)-\Lambda(t)]\,[\Psi(1)-\Psi(s)]\,\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt \\
= \quad & \frac{4}{(b-a)(d-c)}\Lambda(1)\Psi(1)f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& -\frac{4}{(b-a)(d-c)}\Psi(1)\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{c+d}{2}\right)dt \\
& -\frac{4}{(b-a)(d-c)}\Lambda(1)\int_0^1 \frac{\psi\left(\frac{d-c}{2}s\right)}{s} f\left(\frac{a+b}{2},\frac{1-s}{2}c+\frac{1+s}{2}d\right)ds \\
& +\frac{4}{(b-a)(d-c)}\int_0^1\int_0^1 \frac{\varphi\left(\frac{b-a}{2}t\right)}{t}\frac{\psi\left(\frac{d-c}{2}s\right)}{s} f\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt \\
= \quad & \frac{4}{(b-a)(d-c)}\Lambda(1)\Psi(1)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{4}{(b-a)(d-c)}\Psi(1)I_{b^-}^{\varphi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\
& -\frac{4}{(b-a)(d-c)}\Lambda(1)I_{d^-}^{\psi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{4}{(b-a)(d-c)}I_{b^-,d^-}^{\varphi,\psi} f\left(\frac{a+b}{2},\frac{c+d}{2}\right),
\end{aligned} \tag{12}
$$

$$
I_2 \quad = \quad \int_0^1\int_0^1 [\Lambda(1)-\Lambda(t)]\,[\Psi(1)-\Psi(s)]\,\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt \tag{13}
$$

$$\begin{aligned}
&= -\frac{4}{(b-a)(d-c)}\Lambda(1)\Psi(1)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{4}{(b-a)(d-c)}\Psi(1)I_{b^-}^{\varphi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad+\frac{4}{(b-a)(d-c)}\Lambda(1)I_{c^+}^{\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{4}{(b-a)(d-c)}I_{b^-,c^+}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)
\end{aligned}$$

and similarly we get

$$\begin{aligned}
I_3 &= \int_0^1\int_0^1[\Lambda(1)-\Lambda(t)][\Psi(1)-\Psi(s)]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt \qquad (14)\\
&= -\frac{4}{(b-a)(d-c)}\Lambda(1)\Psi(1)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{4}{(b-a)(d-c)}\Psi(1)I_{a^+}^{\varphi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad+\frac{4}{(b-a)(d-c)}\Lambda(1)I_{d^-}^{\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{4}{(b-a)(d-c)}I_{a^+,d^-}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \int_0^1\int_0^1[\Lambda(1)-\Lambda(t)][\Psi(1)-\Psi(s)]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt \qquad (15)\\
&= \frac{4}{(b-a)(d-c)}\Lambda(1)\Psi(1)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{4}{(b-a)(d-c)}\Psi(1)I_{a^+}^{\varphi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad-\frac{4}{(b-a)(d-c)}\Lambda(1)I_{c^+}^{\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+\frac{4}{(b-a)(d-c)}I_{a^+,c^+}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right).
\end{aligned}$$

By the equalities (12)-(15), we have

$$\begin{aligned}
&I_1-I_2-I_3+I_4 \qquad (16)\\
&= \frac{16}{(b-a)(d-c)}\Lambda(1)\Psi(1)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad-\frac{8}{(b-a)(d-c)}\Psi(1)\left[I_{a^+}^{\varphi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{b^-}^{\varphi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&\quad-\frac{8}{(b-a)(d-c)}\Lambda(1)\left[I_{c^+}^{\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{d^-}^{\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&\quad+\frac{4}{(b-a)(d-c)}\left[\begin{array}{c} I_{a^+,c^+}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{b^-,c^+}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ +I_{a^+,d^-}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I_{b^-,d^-}^{\varphi,\psi}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\end{array}\right].
\end{aligned}$$

Multiplying the both sides of (16) by $\frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)}$, we obtain (9), which completes the proof. □

Corollary 2.2. *Under the hypotheses of Lemma 2.1, if we take $\varphi(t)=t$ and $\psi(s)=s$, then we have*

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_1\\
&= \frac{(b-a)(d-c)}{16}\\
&\quad\times\left[\int_0^1\int_0^1(1-t)(1-s)\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\right.\\
&\quad-\int_0^1\int_0^1(1-t)(1-s)\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\
&\quad-\int_0^1\int_0^1(1-t)(1-s)\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt
\end{aligned}$$

$$+\int_0^1\int_0^1(1-t)(1-s)\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\Bigg]$$

where

$$A_1=\frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2},s\right)ds+\frac{1}{b-a}\int_a^b f\left(t,\frac{c+d}{2}\right)dt. \tag{17}$$

Remark 2.3. *Under the hypotheses of Lemma 2.1, if we choose $\varphi(t)=\frac{t^{\alpha}}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^{\beta}}{\Gamma(\beta)}$, then we get*

$$\begin{aligned}
&\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[\begin{array}{c} J^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ +J^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\end{array}\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_2\\
=\ &\frac{(b-a)(d-c)}{16}\\
&\times\Bigg[\int_0^1\int_0^1[1-t^{\alpha}]\left[1-s^{\beta}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\
&-\int_0^1\int_0^1[1-t^{\alpha}]\left[1-s^{\beta}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\
&-\int_0^1\int_0^1[1-t^{\alpha}]\left[1-s^{\beta}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\
&+\int_0^1\int_0^1[1-t^{\alpha}]\left[1-s^{\beta}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
A_2\ =\ &\frac{2^{\beta-1}\Gamma(\beta+1)}{(d-c)^{\beta}}\left[J^{\psi}_{c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\psi}_{d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&+\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\varphi}_{a^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\varphi}_{b^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right].
\end{aligned} \tag{18}$$

Corollary 2.4. *Under assumption of Lemma 2.1 with $\varphi(t)=\frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\beta/k}}{k\Gamma_k(\beta)}$, we obtain*

$$\begin{aligned}
&\frac{2^{\frac{\alpha}{k}-1}2^{\frac{\beta}{k}-1}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}}\left[{}_kI^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.+{}_kI^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_3\\
=\ &\frac{(b-a)(d-c)}{16}\\
&\times\Bigg[\int_0^1\int_0^1\left[1-t^{\alpha/k}\right]\left[1-s^{\beta/k}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\
&-\int_0^1\int_0^1\left[1-t^{\alpha/k}\right]\left[1-s^{\beta/k}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\\
&-\int_0^1\int_0^1\left[1-t^{\alpha/k}\right]\left[1-s^{\beta/k}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)dsdt\\
&+\int_0^1\int_0^1\left[1-t^{\alpha/k}\right]\left[1-s^{\beta/k}\right]\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)dsdt\Bigg],
\end{aligned}$$

where

$$A_3 = \frac{2^{\frac{\beta}{k}-1}\Gamma_k(\beta+k)}{(d-c)^{\beta/k}}\left[{}_kI^{\beta}_{c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\beta}_{d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right] + \frac{2^{\frac{\alpha}{k}-1}\Gamma_k(\alpha+k)}{(b-a)^{\alpha/k}}\left[{}_kI^{\alpha}_{a^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha}_{b^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]. \tag{19}$$

Theorem 2.5. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be partial differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$ is co-ordinated $(s_1,s_2)-$convex function, then we have the following midpoint inequality for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\le\ & \frac{(b-a)(d-c)}{2^{s_1+s_2+4}\Lambda(1)\Psi(1)}\left(\int_0^1\left((1-t)^{s_1}+(1+t)^{s_1}\right)(\Lambda(1)-\Lambda(t))\,dt\right)\left(\int_0^1\left((1-s)^{s_2}+(1+s)^{s_2}\right)(\Psi(1)-\Psi(s))\,ds\right)\\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where the mapping $\Lambda(t)$, $\Psi(s)$ are defined by (11) and A is defined as in (10).

Proof. From Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\le\ & \frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)}\\
&\times\left[\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|dsdt\right.\\
&+\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|dsdt\\
&+\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|dsdt\\
&\left.+\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|dsdt\right].
\end{aligned} \tag{20}$$

By using the co-ordinated $(s_1,s_2)-$ convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|$, we have

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|\\
\le\ & \frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\
&+\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \qquad (22)\\
\leq\ & \frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\
&+\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right| \qquad (23)\\
\leq\ & \frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\
&+\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right| \qquad (24)\\
\leq\ & \frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|\\
&+\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|.
\end{aligned}$$

By substituting the inequalities (21)-(24) in (20), we have

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\leq\ & \frac{(b-a)(d-c)}{2^{s_1+s_2+4}\Lambda(1)\Psi(1)}\left(\int_0^1(\Lambda(1)-\Lambda(t))\left((1-t)^{s_1}+(1+t)^{s_1}\right)dt\right)\\
&\times\left(\int_0^1(\Psi(1)-\Psi(s))\left((1-s)^{s_2}+(1+s)^{s_2}\right)ds\right)\\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right].
\end{aligned}$$

This completes the proof. □

Remark 2.6. *Under the hypotheses of Theorem 2.5, if we take $s_1 = s_2 = 1$, then we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\leq\ & \frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)}\left(\int_0^1(\Lambda(1)-\Lambda(t))\,dt\right)\left(\int_0^1(\Psi(1)-\Psi(s))\,ds\right)
\end{aligned}$$

$$\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],$$

where A is defined as in (10).

Corollary 2.7. *Let us consider $\varphi(t)=t$ and $\psi(s)=s$ for all $(s,t)\in\Delta$ in Theorem 2.5. Let us also consider $s_1=s_2=r$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_1\right|\\
\leq\;&\frac{(b-a)(d-c)}{2^{2r+4}}\left(\frac{1}{r+2}+\chi_1\right)^2\\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where A_1 is defined as in (17) and

$$\chi_1=\int_0^1(1-t)(1+t)^r dt=\int_0^1(1-s)(1+s)^r ds.$$

Remark 2.8. *Under the hypotheses of Corollary 2.7, if we take $r=1$, then we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_1\right|\\
\leq\;&\frac{(b-a)(d-c)}{64}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where A_1 is defined as in (17). This results is the same as ones proved by Latif and Dragomir in [15, Theorem 2].

Corollary 2.9. *In Theorem 2.5, if we assign $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(s,t)\in\Delta$ and if we choose $s_1=s_2=r$, then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+J^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_2\right|\\
\leq\;&\frac{(b-a)(d-c)}{2^{2r+4}}\chi_2\phi_2\\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where A_2 is defined as in (18) and

$$\begin{aligned}
\chi_2&=\int_0^1(1-t^\alpha)\left[(1+t)^r+(1-t)^r\right]dt,\\
\phi_2&=\int_0^1(1-s^\beta)\left[(1+s)^r+(1-s)^r\right]ds.
\end{aligned}$$

Remark 2.10. *Under the hypotheses of Corollary 2.9, if we choose $r = 1$, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}}\left[J^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+J^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_2\right|\\
\leq\ &\frac{\alpha\beta(b-a)(d-c)}{16(\alpha+1)(\beta+1)}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where A_2 is defined as in (18).

Corollary 2.11. *In Theorem 2.5, if we assign $\varphi(t)=\varphi(t)=\frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$ and $\psi(s)=\frac{s^{\beta/k}}{k\Gamma_k(\beta)}$ for all $(s,t)\in\Delta$ and if we choose $s_1=s_2=r$, then, we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}2^{\frac{\beta}{k}-1}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}}\left[{}_kI^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+{}_kI^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_3\right|\\
\leq\ &\frac{(b-a)(d-c)}{2^{2r+4}}\chi_3\phi_3\\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where A_3 is defined as in (19) and

$$\begin{aligned}
\chi_3 &= \int_0^1\left(1-t^{\alpha/k}\right)\left[(1+t)^r+(1-t)^r\right]dt,\\
\phi_3 &= \int_0^1\left(1-s^{\beta/k}\right)\left[(1+s)^r+(1-s)^r\right]ds.
\end{aligned}$$

Remark 2.12. *Under assumption of Corollary 2.11 with $r = 1$, we get the following midpoint type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\frac{\alpha}{k}-1}2^{\frac{\beta}{k}-1}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}}\left[{}_kI^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+{}_kI^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_3\right|\\
\leq\ &\frac{\alpha\beta(b-a)(d-c)}{16(\alpha+k)(\beta+k)}\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|\right],
\end{aligned}$$

where A_3 is defined as in (19).

Theorem 2.13. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be partial differentiable function on Δ. If $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$, $q>1$, is co-ordinated (s_1,s_2) −convex function, then we have the following midpoint inequality for generalized fractional integral operators:*

$$\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right. \tag{25}$$

$$+ I^{\varphi,\psi}_{b^-,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + I^{\varphi,\psi}_{b^-,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\Bigg] + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\Bigg|$$

$$\leq \frac{(b-a)(d-c)}{2^{4+\frac{s_1+s_2}{q}}\left[(s_1+1)(s_2+1)\right]^{1/q}\Lambda(1)\Psi(1)}\left(\int_0^1\int_0^1 |\Lambda(1)-\Lambda(t)|^p\,|\Psi(1)-\Psi(s)|^p\,dsdt\right)^{1/p}$$

$$\times\left\{\begin{bmatrix} \left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + (2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ +(2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + (2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{bmatrix}^{1/q}\right.$$

$$+\begin{bmatrix} (2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ +(2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + (2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{bmatrix}^{1/q}$$

$$+\begin{bmatrix} (2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + (2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ +\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + (2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{bmatrix}^{1/q}$$

$$\left.+\begin{bmatrix} (2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + (2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ +(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{bmatrix}^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$. *Here, A is defined as in (10).*

Proof. By using the well-known Hölder inequality and co-ordinated (s_1,s_2)−convexity of $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ in (20), we obtain

$$\Bigg|\frac{1}{4\Lambda(1)\Psi(1)}\Bigg[I^{\varphi,\psi}_{a^+,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + I^{\varphi,\psi}_{a^+,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \tag{26}$$

$$+ I^{\varphi,\psi}_{b^-,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + I^{\varphi,\psi}_{b^-,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\Bigg] + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A\Bigg|$$

$$\leq \frac{1}{16\Lambda(1)\Psi(1)}$$

$$\times\Bigg[\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|^p\,|[\Psi(1)-\Psi(s)]|^p\,dsdt\right)^{1/p}$$

$$\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b, \frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right)^{1/q}$$

$$+\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|^p\,|[\Psi(1)-\Psi(s)]|^p\,dsdt\right)^{1/p}$$

$$\times\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b, \frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right)^{1/q}$$

$$+\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|^p\,|[\Psi(1)-\Psi(s)]|^p\,dsdt\right)^{1/p}$$

$$\left(\int_0^1\int_0^1 \left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b, \frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right)^{1/q}$$

$$+\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|^p\,|[\Psi(1)-\Psi(s)]|^p\,dsdt\right)^{1/p}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right)^{1/q}\Bigg].$$

Since $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is co-ordinated (s_1,s_2) −convex, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt \qquad (27)\\
\leq\;&\int_0^1\int_0^1\Bigg[\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\
&+\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\Bigg]dsdt\\
=\;&\frac{1}{2^{s_1+s_2}(s_1+1)(s_2+1)}\left[\begin{array}{c}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +(2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt \qquad (28)\\
\leq\;&\frac{1}{2^{s_1+s_2}(s_1+1)(s_2+1)}\left[\begin{array}{c}(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +(2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]
\end{aligned}$$

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt \qquad (29)\\
\leq\;&\frac{1}{2^{s_1+s_2}(s_1+1)(s_2+1)}\left[\begin{array}{c}(2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt \qquad (30)\\
\leq\;&\frac{1}{2^{s_1+s_2}(s_1+1)(s_2+1)}\left[\begin{array}{c}(2^{s_1+1}-1)(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{s_1+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +(2^{s_2+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right].
\end{aligned}$$

By substituting the inequalities (27)-(30) in (26), we obtain the desired inequalty in (25). □

Remark 2.14. *Under the hypotheses of Theorem 2.13, if we choose $s_1 = s_2 = 1$, we obtain the following midpoint type inequality*

$$\begin{aligned}
&\Bigg|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \qquad (31)\\
&\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\Bigg|\\
\leq\;&\frac{(b-a)(d-c)}{16^{1+1/q}\Lambda(1)\Psi(1)}\left(\int_0^1\int_0^1|\Lambda(1)-\Lambda(t)|^p\,|\Psi(1)-\Psi(s)|^p\,dsdt\right)^{1/p}
\end{aligned}$$

$$\times\left\{\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\right.$$
$$+\left[3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$
$$+\left[3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$
$$\left.+\left[9\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Here, A is defined by (10).

Corollary 2.15. *Let us consider $\varphi(t)=t$ and $\psi(s)=s$ for all $(s,t)\in\Delta$ in Theorem 2.13. Let us also consider $s_1=s_2=r$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_1\right|$$
$$\leq\frac{(b-a)(d-c)}{2^{4+\frac{2r}{q}}(r+1)^{2/q}(p+1)^{2/p}}$$
$$\times\left\{\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\right.$$
$$+\left[(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$
$$+\left[(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$
$$\left.+\left[(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\right\},$$

where A_1 is defined as in (17).

Corollary 2.16. *Under the hypotheses of Corollary 2.15, if we choose $r=1$, we obtain the following midpoint type inequality*

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_1\right|$$
$$\leq\frac{(b-a)(d-c)}{16^{1+1/q}(p+1)^{2/p}}$$
$$\times\left\{\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\right.$$
$$+\left[3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$
$$+\left[3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$

$$+\left[9\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\Bigg\},$$

where A_1 is defined as in (17).

Corollary 2.17. *Let us consider $\varphi(t)=\frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s)=\frac{s^\beta}{\Gamma(\beta)}$ for all $(s,t)\in\Delta$ in Theorem 2.13. Let us also consider $s_1=s_2=r$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.+J^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_2\Bigg|\\
\le\ & \frac{(b-a)(d-c)}{2^{4+\frac{2r}{q}}(r+1)^{2/q}}\left(\frac{B\left(\frac{1}{\alpha},p+1\right)B\left(\frac{1}{\beta},p+1\right)}{\alpha\beta}\right)^{1/p}\\
&\times\Bigg\{\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\Bigg\},
\end{aligned}$$

where A_1 is defined as in (17).

Corollary 2.18. *Under the hypotheses of Corollary 2.17, if we choose $r=1$, then we get the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\Bigg|\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.+J^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+J^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_2\Bigg|\\
\le\ & \frac{(b-a)(d-c)}{16^{1+1/q}}\left(\frac{B\left(\frac{1}{\alpha},p+1\right)B\left(\frac{1}{\beta},p+1\right)}{\alpha\beta}\right)^{1/p}\\
&\times\Bigg\{\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[9\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\Bigg\},
\end{aligned}$$

where A_2 is defined as in (18).

Corollary 2.19. *Let us consider $\varphi(t) = \frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\beta/k}}{k\Gamma_k(\beta)}$ for all $(s,t) \in \Delta$ in Theorem 2.13. Let us also consider $s_1 = s_2 = r$, then, we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left| \frac{2^{\frac{\alpha}{k}-1} 2^{\frac{\beta}{k}-1} \Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}} \left[{}_kI^{\alpha,\beta}_{a^+,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_kI^{\alpha,\beta}_{a^+,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.\right. \\
&\left.\left. + {}_kI^{\alpha,\beta}_{b^-,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_kI^{\alpha,\beta}_{b^-,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right] + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A_3 \right| \\
\leq\ & \frac{(b-a)(d-c)}{2^{4+\frac{2r}{q}}(r+1)^{2/q}} \left(\frac{k^2 B\left(\frac{k}{\alpha}, p+1\right) B\left(\frac{k}{\beta}, p+1\right)}{\alpha\beta} \right)^{1/p} \\
&\times \left\{ \left[\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + (2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + (2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + (2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \right. \\
&+ \left[(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + (2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + (2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \\
&+ \left[(2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + (2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + (2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \\
&\left. + \left[(2^{r+1}-1)^2\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + (2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + (2^{r+1}-1)\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \right\},
\end{aligned}$$

where A_3 is defined as in (19).

Corollary 2.20. *Under assumption of Corollary 2.19 with $r = 1$, we obtain the following midpoint type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left| \frac{2^{\frac{\alpha}{k}-1} 2^{\frac{\beta}{k}-1} \Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}} \left[{}_kI^{\alpha,\beta}_{a^+,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_kI^{\alpha,\beta}_{a^+,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.\right. \\
&\left.\left. + {}_kI^{\alpha,\beta}_{b^-,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_kI^{\alpha,\beta}_{b^-,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right] + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A_3 \right| \\
\leq\ & \frac{(b-a)(d-c)}{16^{1+1/q}} \left(\frac{k^2 B\left(\frac{k}{\alpha}, p+1\right) B\left(\frac{k}{\beta}, p+1\right)}{\alpha\beta} \right)^{1/p} \\
&\times \left\{ \left[\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + 3\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + 3\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + 9\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \right. \\
&+ \left[3\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + 9\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + 3\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \\
&+ \left[3\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + 9\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + 3\left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \\
&\left. + \left[9\left|\frac{\partial^2 f}{\partial s \partial t}(a,c)\right|^q + 3\left|\frac{\partial^2 f}{\partial s \partial t}(a,d)\right|^q + 3\left|\frac{\partial^2 f}{\partial s \partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s \partial t}(b,d)\right|^q \right]^{1/q} \right\},
\end{aligned}$$

where A_3 is defined as in (19).

Theorem 2.21. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be partial differentiable function on Δ . If $\left|\frac{\partial^2 f}{\partial s \partial t}\right|^q$, $q \geq 1$, is co-ordinated (s_1, s_2) −convex function, then we have the following midpoint inequality for generalized fractional integral operators:*

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\leq\;&\frac{(b-a)(d-c)}{2^{4+\frac{s_1+s_2}{q}}\Lambda(1)\Psi(1)}\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}\\
&\times\left[B_1\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+B_2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+B_3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+B_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[B_2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+B_1\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+B_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+B_3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[B_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+B_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+B_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+B_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[B_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+B_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+B_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+B_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}
\end{aligned}$$

where A is defined by (10) and the constants B_1, B_2, B_3 and B_4 are defined by

$$\begin{aligned}
B_1 &= \int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,(1-t)^{s_1}(1-s)^{s_2}dsdt,\\
B_2 &= \int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,(1-t)^{s_1}(1+s)^{s_2}dsdt,\\
B_3 &= \int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,(1+t)^{s_1}(1-s)^{s_2}dsdt,\\
B_4 &= \int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,(1+t)^{s_1}(1+s)^{s_2}dsdt.
\end{aligned}$$

Proof. The case of $q = 1$ is obvious from Theorem 2.5.

For $q > 1$ we proceed as follows. By using well-known power mean inequality (20), we obtain

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\leq\;&\frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)}\\
&\times\left\{\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}\right.\\
&\times\left[\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right]^{1/q}\\
&+\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}
\end{aligned}$$

$$\times\left[\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1-t}{2}a+\frac{1+t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right]^{1/q}$$
$$+\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}$$
$$\times\left[\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1-s}{2}c+\frac{1+s}{2}d\right)\right|^q dsdt\right]^{1/q}$$
$$+\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}$$
$$\left.\times\left[\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left|\frac{\partial^2 f}{\partial s\partial t}\left(\frac{1+t}{2}a+\frac{1-t}{2}b,\frac{1+s}{2}c+\frac{1-s}{2}d\right)\right|^q dsdt\right]^{1/q}\right\}$$

Since the function $\left|\frac{\partial^2 f}{\partial s\partial t}\right|^q$ is co-ordinated (s_1,s_2) −convex, we obtain

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\leq\;&\frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)}\left(\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}\\
&\times\left\{\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\right.\\
&\times\left(\left[\begin{array}{l}\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]dsdt\right)^{1/q}\\
&+\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\\
&\times\left(\left[\begin{array}{l}\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]dsdt\right)^{1/q}\\
&+\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\\
&\times\left(\left[\begin{array}{l}\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]dsdt\right)^{1/q}\\
&+\int_0^1\int_0^1 |[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\\
&\left.\times\left(\left[\begin{array}{l}\frac{(1+t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{(1+t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\frac{(1-t)^{s_1}(1-s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{(1-t)^{s_1}(1+s)^{s_2}}{2^{s_1+s_2}}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]dsdt\right)^{1/q}\right\}
\end{aligned}$$

which completes the proof. □

Remark 2.22. *Under the hypotheses of Theorem 2.21, if we choose $s_1 = s_2 = 1$, we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{4\Lambda(1)\Psi(1)}\left[I^{\varphi,\psi}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right.\\
&\left.\left.+I^{\varphi,\psi}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+I^{\varphi,\psi}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A\right|\\
\leq\;&\frac{(b-a)(d-c)}{16\Lambda(1)\Psi(1)}\left(\int_0^1\int_0^1|[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\,dsdt\right)^{1-1/q}\\
&\times\left[\mathfrak{B}_1\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\mathfrak{B}_2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\mathfrak{B}_3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\mathfrak{B}_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[\mathfrak{B}_2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\mathfrak{B}_1\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\mathfrak{B}_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\mathfrak{B}_3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[\mathfrak{B}_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\mathfrak{B}_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\mathfrak{B}_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\mathfrak{B}_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[\mathfrak{B}_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\mathfrak{B}_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\mathfrak{B}_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\mathfrak{B}_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}
\end{aligned}$$

where A is defined by (10) and the constants $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ and $\mathfrak{B}_4$ are defined by

$$\begin{aligned}
\mathfrak{B}_1&=\int_0^1\int_0^1|[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left(\frac{1-t}{2}\frac{1-s}{2}\right)dsdt,\\
\mathfrak{B}_2&=\int_0^1\int_0^1|[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left(\frac{1-t}{2}\frac{1+s}{2}\right)dsdt,\\
\mathfrak{B}_3&=\int_0^1\int_0^1|[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left(\frac{1+t}{2}\frac{1-s}{2}\right)dsdt,\\
\mathfrak{B}_4&=\int_0^1\int_0^1|[\Lambda(1)-\Lambda(t)]|\,|[\Psi(1)-\Psi(s)]|\left(\frac{1+t}{2}\frac{1+s}{2}\right)dsdt.
\end{aligned}$$

Corollary 2.23. *Let us consider $\varphi(t) = t$ and $\psi(s) = s$ for all $(s,t) \in \Delta$ in Theorem 2.21. Let us also consider $s_1 = s_2 = r$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_1\right|\\
\leq\;&\frac{(b-a)(d-c)}{2^{4+\frac{2r}{q}}}\left(\frac{1}{4}\right)^{1-1/q}\\
&\times\left[\frac{1}{(r+2)^2}\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\chi_4^2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{1}{(r+2)^2}\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\chi_4^2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\chi_4^2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\frac{1}{(r+2)^2}\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}\\
&+\left[\chi_4^2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\frac{1}{r+2}\chi_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\frac{1}{(r+2)^2}\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}
\end{aligned}$$

where A_1 is defined by (17) and the constant χ_4 is defined by

$$\chi_4 = \int_0^1 (1-s)(1+s)^r ds = \int_0^1 (1-t)(1+t)^r dt.$$

Corollary 2.24. *Under the hypotheses of Corollary 2.23, if we choose $r = 1$, we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(t,s)\,dsdt + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A_1\right| \\
\leq\;& \frac{(b-a)(d-c)}{4^{3-1/q}(36)^{1/q}} \\
&\times\left[\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + 2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + 2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + 4\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q} \\
&+\left[2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + 4\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + 2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q} \\
&+\left[2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + 4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + 2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q} \\
&+\left[4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + 2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + 2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}
\end{aligned}$$

where A_1 is defined as in (17).

Corollary 2.25. *Let us consider $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}$ and $\psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ for all $(s,t) \in \Delta$ in Theorem 2.21. Let us also consider $s_1 = s_2 = r$. Then, we have the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^\alpha(d-c)^\beta}\left[J^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + J^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\right. \\
&\left.\left.+ J^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right) + J^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right] + f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - A_2\right| \\
\leq\;& \frac{(b-a)(d-c)}{2^{4+\frac{s_1+s_2}{q}}}\left(\frac{\alpha\beta}{(\alpha+1)(\beta+1)}\right)^{1-1/q} \\
&\times\left[\omega_1\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \omega_2\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \omega_3\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \omega_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q} \\
&+\left[\omega_2\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \omega_1\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \omega_4\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \omega_3\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q} \\
&+\left[\omega_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \omega_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \omega_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \omega_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q} \\
&+\left[\omega_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \omega_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \omega_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \omega_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}
\end{aligned}$$

where A_2 is defined by (18) and the constants ω_1, ω_2, ω_3 and ω_4 are defined by

$$\omega_1 = \int_0^1\int_0^1 (1-t^\alpha)(1-s^\beta)(1-t)^r(1-s)^r dsdt,$$

$$
\begin{aligned}
\omega_2 &= \int_0^1 \int_0^1 (1-t^{\alpha})(1-s^{\beta})(1-t)^r(1+s)^r dsdt, \\
\omega_3 &= \int_0^1 \int_0^1 (1-t^{\alpha})(1-s^{\beta})(1+t)^r(1-s)^r dsdt, \\
\omega_4 &= \int_0^1 \int_0^1 (1-t^{\alpha})(1-s^{\beta})(1+t)^r(1+s)^r dsdt.
\end{aligned}
$$

Corollary 2.26. *Under the hypotheses of Corollary 2.25, if we take $r=1$, the following midpoint type inequality for Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left| \frac{2^{\alpha-1}2^{\beta-1}\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha}(d-c)^{\beta}} \left[J^{\alpha,\beta}_{a^+,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J^{\alpha,\beta}_{a^+,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.\right. \\
&\left.\left. + J^{\alpha,\beta}_{b^-,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + J^{\alpha,\beta}_{b^-,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A_2 \right| \\
\leq\ & \frac{\alpha\beta(b-a)(d-c)}{16^{1+1/q}(\alpha+1)(\beta+1)} \\
&\times \left[\begin{array}{c} \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ + \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{array} \right]^{1/q} \\
&+ \left[\begin{array}{c} \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ + \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{array} \right]^{1/q} \\
&+ \left[\begin{array}{c} \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q \\ + \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{array} \right]^{1/q} \\
&+ \left[\begin{array}{c} \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \left(\frac{3\alpha+5}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \\ \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{3\beta+5}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \left(\frac{\alpha+3}{\alpha+2}\right)\left(\frac{\beta+3}{\beta+2}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \end{array} \right]^{1/q}
\end{aligned}
$$

where A_2 is defined as in (18).

Corollary 2.27. *Let us consider $\varphi(t) = \frac{t^{\alpha/k}}{k\Gamma_k(\alpha)}$ and $\psi(s) = \frac{s^{\beta/k}}{k\Gamma_k(\beta)}$ for all $(s,t) \in \Delta$ in Theorem 2.21. Let us also consider $s_1 = s_2 = r$. Then, we have the following midpoint type inequality for k-Riemann-Liouville fractional integrals*

$$
\begin{aligned}
&\left| \frac{2^{\frac{\alpha}{k}-1}2^{\frac{\beta}{k}-1}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}} \left[{}_kI^{\alpha,\beta}_{a^+,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_kI^{\alpha,\beta}_{a^+,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right.\right. \\
&\left.\left. + {}_kI^{\alpha,\beta}_{b^-,c^+} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_kI^{\alpha,\beta}_{b^-,d^-} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A_3 \right| \\
\leq\ & \frac{(b-a)(d-c)}{2^{4+\frac{s_1+s_2}{q}}} \left(\frac{\alpha\beta}{(\alpha+k)(\beta+k)}\right)^{1-1/q} \\
&\times \left[\varpi_1 \left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \varpi_2 \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \varpi_3 \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \varpi_4 \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \right]^{1/q} \\
&+ \left[\varpi_2 \left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q + \varpi_1 \left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q + \varpi_4 \left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q + \varpi_3 \left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q \right]^{1/q}
\end{aligned}
$$

$$+\left[\varpi_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\varpi_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\varpi_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\varpi_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$
$$+\left[\varpi_4\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\varpi_3\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q+\varpi_2\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\varpi_1\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\right]^{1/q}$$

where A_3 is defined by (19) and the constants ϖ_1, ϖ_2, ϖ_3 and ϖ_4 are defined by

$$\begin{aligned}
\varpi_1 &= \int_0^1\int_0^1 (1-t^{\alpha/k})(1-s^{\beta/k})(1-t)^r(1-s)^r dsdt,\\
\varpi_2 &= \int_0^1\int_0^1 (1-t^{\alpha/k})(1-s^{\beta/k})(1-t)^r(1+s)^r dsdt,\\
\varpi_3 &= \int_0^1\int_0^1 (1-t^{\alpha/k})(1-s^{\beta/k})(1+t)^r(1-s)^r dsdt,\\
\varpi_4 &= \int_0^1\int_0^1 (1-t^{\alpha/k})(1-s^{\beta/k})(1+t)^r(1+s)^r dsdt.
\end{aligned}$$

Corollary 2.28. *Under assumption of Corollary 2.27 with $r=1$, we get the following midpoint type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}
&\Bigg|\frac{2^{\frac{\alpha}{k}-1}2^{\frac{\beta}{k}-1}\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}{(b-a)^{\alpha/k}(d-c)^{\beta/k}}\left[{}_kI^{\alpha,\beta}_{a^+,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{a^+,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right.\\
&\left.+{}_kI^{\alpha,\beta}_{b^-,c^+}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_kI^{\alpha,\beta}_{b^-,d^-}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]+f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-A_3\Bigg|\\
\leq\;& \frac{\alpha\beta(b-a)(d-c)}{16^{1+1/q}(\alpha+k)(\beta+k)}\\
&\times\left[\begin{array}{c}\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\left(\frac{3\alpha+5k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left(\frac{3\alpha+5k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]^{1/q}\\
&+\left[\begin{array}{c}\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\left(\frac{3\alpha^2+5\alpha k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left(\frac{3\alpha^2+5\alpha k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]^{1/q}\\
&+\left[\begin{array}{c}\left(\frac{3\alpha+5k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left(\frac{3\alpha+5k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]^{1/q}\\
&+\left[\begin{array}{c}\left(\frac{3\alpha+5k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,c)\right|^q+\left(\frac{3\alpha+5k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(a,d)\right|^q\\ +\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{3\beta+5k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,c)\right|^q+\left(\frac{\alpha+3k}{\alpha+2k}\right)\left(\frac{\beta+3k}{\beta+2k}\right)\left|\frac{\partial^2 f}{\partial s\partial t}(b,d)\right|^q\end{array}\right]^{1/q}
\end{aligned}$$

where A_3 is defined by (19).

3. Concluding Remarks

In this paper, we have established several new Hermite–Hadamard type inequalities for differentiable co-ordinated (s_1, s_2)-convex functions in the second sense. Our main results extend classical midpoint-type inequalities to the framework of fractional calculus on rectangular domains in $\mathbb{R}^2$. By employing the Riemann–Liouville and k-Riemann–Liouville fractional integrals, we have derived generalized inequalities that provide deeper insights into the behavior of co-ordinated convex functions under fractional integration. These findings not only generalize known results in the literature but also offer a unified approach that includes various classical inequalities as special cases. Furthermore, the techniques and inequalities presented in this study can be applied to a broader class of problems in analysis and optimization, particularly those involving fractional differential equations and integral inequalities. Future research may focus on extending these results to other classes of generalized convex functions, such as (h, m)-convex or preinvex functions, and exploring applications in applied mathematics and mathematical physics.

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