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Existence, Uniqueness and Ulam-Hyers stability of solutions for a nonlinear Schrödinger equation with inverse-power potential and mixed type nonlinearities

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Abstract. In this paper, we address the mathematical analysis of a nonlinear Schrödinger equation incorporating an inverse-power potential alongside mixed power-type and Choquard-type nonlinearities. Such models are relevant in quantum mechanics and nonlinear optics due to their inclusion of singular potentials and long-range nonlocal effects. The equation is transformed using the Duhamel principle into an integral form suitable for fixed-point theory. We prove the existence of solutions using Krasnosel'skii's theorem and establish uniqueness via the Banach contraction mapping principle. We also investigate Ulam–Hyers stability, confirming the continuous dependence of solutions on initial data. To support the theory, a representative example is provided, complemented by graphical interpretation. This work contributes a unified treatment of singular and nonlocal interactions, extending existing analytical techniques to more general nonlinear dispersive systems.

1. Introduction

The nonlinear Schrödinger equation is the leading model in mathematical physics that mirrors the principal dynamics in the fields of quantum mechanics, optics, and fluids. The unique mixture of dispersion and nonlinearity leads to complex solutions to the equation, among which are solitons and singularities. In fact, the nonlinear Schrödinger equation has become the main dispersive PDE example, hence it has fundamentally impacted both theoretical and applied mathematics driving the development of completely new areas, especially functional and harmonic analysis, and the theory of nonlinear PDEs [1–4]. One major breakthrough in the study of the classical nonlinear Schrödinger equation has been the incorporation of singular potentials, nonlocal nonlinearities, and fractional derivatives, thus extending the model to intricate physical systems. Singular potentials, and especially inverse-power types of them, appear in several contexts like gravitational and electrostatic forces, quantum field theory, and even astrophysics, where the long-range interactions display singularity at the origin [5–7].

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Nonlocal nonlinearities, similar to those of Choquard type, take into account the interactions that depend not solely on the function's value at a particular location but also on the entire wave function's global spatial distribution. This property of nonlocal nonlinearities makes them ideal for the simulation of systems exhibiting long-range correlations such as dipolar Bose-Einstein condensates, quantum chemistry, and self-gravitating matter [8–10]. On the other hand, the incorporation of such features increases the mathematical complexity of the analysis of the nonlinear Schrödinger equation significantly, yet at the same time, it broadens the range of its applicability to real-life phenomena.

The investigation into nonlinear Schrödinger equations having inverse-power potentials and a combination of power-type and Choquard-type nonlinearities is heavily grounded on the establishment of solution existence and uniqueness. These characteristics are vital in the confirmation that the models are able to accurately represent the intricate physical phenomena taking place in areas like quantum mechanics and nonlinear optics. The coexistence of singular potentials and nonlocal nonlinearities has made the mathematical treatment extremely complex, thus requiring the application of sophisticated methods to reach the well-posedness situation. Among the core techniques used in the proofs that stable standing waves exist, the recent breakthroughs have regarded the concentration-compactness principle and variational methods as the most important ones, specifically for the nonlinear Schrödinger equation with mixed nonlinearities. These methodologies have been pivotal in extending the classical results to more complex settings, thus further fortifying the theoretical framework. Confirming existence and uniqueness not only makes the mathematical models acceptable but also provides the foundation for future research on the dynamics and stability of solutions—an area of study that is crucial for both theoretic and practical matters in the fields of science and engineering [11–17].

The acceptance of the models as physically relevant potentially depends on the stability analysis of nonlinear Schrödinger equations with inverse-power potentials and mixed nonlinearities. In quantum mechanics, nonlinear optics, and in Bose-Einstein condensates, it is particularly necessary to guarantee that the solution remains stable under the presence of small perturbations. Recent studies employing fixed-point and functional analytic methods have led to the establishment of criteria for both well-posedness and stability. The Ulam-Hyers stability concept reinforces these results by assuring that the approximate solutions continue to be near the exact ones, thereby strengthening the model's resistance against the influence of small changes in initial conditions or external forces [18–41].

This paper deals with a nonlinear Schrödinger equation in which an inverse-power potential is taken into account together with a mixture of power-type and Choquard-type nonlinearities:

$$\begin{cases} \imath\partial_t\zeta + \Delta\zeta + \dfrac{\gamma}{|x|^\alpha}\zeta + \tau_1|\zeta|^p\zeta + \tau_2\left(I_\beta * |\zeta|^q\right)|\zeta|^{q-2}\zeta = 0, & (t,x) \in [0,T^*)\times\mathbb{R}^N, \\ \zeta(0,x) = \zeta_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where N is greater than or equal to 3, ζ is a complex-valued wave function such that $\zeta : [0,T^*)\times\mathbb{R}^N \to \mathbb{C}$, $0 < T^* \le \infty$, and ζ_0 is the initial condition with $\zeta_0 \in H^1(\mathbb{R}^N)$. The parameters satisfy:

$$\gamma \in (0,\infty), \quad \alpha \in (0,2), \quad \tau_1, \tau_2 \in \mathbb{R}\setminus\{0\}, \quad \frac{4}{N} \le p < \frac{N}{N-2}, \quad 1 + \frac{2+\beta}{N} \le q < \frac{N+\beta}{N-2}.$$

Here, $I_\beta : \mathbb{R}^N \to \mathbb{R}$ denotes the Riesz potential, defined by:

$$I_\beta(x) = \frac{\Gamma\left(\frac{N-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\pi^{N/2}2^\beta|x|^{N-\beta}}, \tag{2}$$

for $\beta \in (0,N)$, where the symbol Γ represents the Gamma function.
The nonlinear Schrödinger equation with an inverse-power potential and combined nonlinearities, as stated in (1), has been of great interest to both physics community [42–44] and the mathematical sciences [45–50]. The equation is a major part of quantum mechanics, nonlinear optics, and plasma physics, thus its analytical and qualitative investigation has become vital. For several years now, the existence, uniqueness,

and stability of the solutions of this equation have undergone considerable research using fixed-point theory, variational techniques, and functional analysis as some of the mathematical tools employed.

The single potential term $\frac{\gamma}{|x|^{\alpha}}$ poses significant analytical difficulties because it not only alters the wave propagation but also causes strong localization effects. Such terms commonly appear in physical models of gravity, Coulomb interactions in quantum mechanics, and polarons [51, 52]. The power-type nonlinearity $|\zeta|^{p}\zeta$ is a standard element in the research of nonlinear optics and quantum mechanics, where it simulates self-focusing effects and leads to the creation of solitons [53]. The Choquard-type nonlocal term $(I_{\beta}*|\zeta|^{q})|\zeta|^{q-2}\zeta$ represents interactions over a long distance and is routinely found in the Hartree equation for bosonic systems as well as in quantum chemistry mean-field models [54].

Initially, the well-posedness of solutions to equation (1) is put to a test by scrutinizing the mere existence and uniqueness through sturdy fixed-point methods. The use of Krasnosel'skii's fixed-point theorem, which is a classical result in nonlinear analysis, guarantees the existence of solutions by fixing points in Banach spaces where suitable conditions are satisfied. This theorem turns out to be very powerful in the framework of compact operators, hence it is very suitable for our situation which deals with both convolution-type nonlinearities and singular potentials. Theorem of Banach Contraction Mapping is used to let one derive the uniqueness of solutions, thus the existence of the solution is guaranteed and it is also uniquely determined by the initial conditions.

After existence and uniqueness, the next step is stability analysis, which is of great importance for the assessment of the solution's robustness. The study presents Ulam-Hyers stability as the main concept, which one has to keep in mind while working with perturbations in function analysis, which means that the solutions will stay close to each other. When it comes to differential equations, Ulam-Hyers stability tells us that if a function does not completely obey the equation but is within the established error bound, then there will exist an exact solution that is pretty close to the given function. Such type of stability is highly valued in real-world scenarios where numerical estimations or uncertainties due to experiments are common and hence the reliability and practicality of the mathematical model are strengthened.

In order to conduct our analysis, we resort to Duhamel's principle that transforms the initial issue into an equivalent integral equation. This change of view is particularly helpful when dealing with dispersive PDEs because it quite comfortably includes the aspect of time evolution and greatly facilitates the use of fixed-point theorems. Moreover, the integral representation does not only make the analytical setting simpler but also unveils the temporal behavior of the solutions. The application of this step is necessary in the proof of well-posedness results, which is shown in previous studies like [55–58].

The research on nonlinear Schrödinger equations featuring singular potentials and nonlocal nonlinearities is still a lively field with a lot of interesting consequences in both mathematics and physics. By proving the existence, uniqueness, and stability of solutions in great detail, this research has rather clarified the theoretical comprehension of these models and even more, it has provided a reliable base for future inquiries. Our findings give a strong analytical tool to investigate issues concerning dispersive partial differential equations, fractional differential equations, and nonlocal interactions. Besides, they might lead to applications in quantum physics, nonlinear optics, and condensed matter theory.

In this paper, we present several new aspects related to the study of nonlinear Schrödinger equations featuring singular potential and nonlocal nonlinearities:

- Our method is different from previous research which considered power-type and Choquard-type nonlinearities separately [65, 66]; it treats both types together in a single framework, thus covering a wider variety of physical interactions.

- Traditional models are found to frequently use smooth or regular potentials [67] while we tackle the analytical difficulties of the high-energy or gravitational regions wherein singular inverse-power potentials exist.

- In contrast to the traditional compactness or variational techniques [68], we make use of the Krasnosel'skii's fixed-point theorem to show that there are solutions of the problem under less stringent structural conditions.

- Ulam-Hyers stability, which has mainly been studied in non-complex dynamical systems [69], is here applied to a nonlocal Schrödinger equation with singularities, thus providing more understanding of the stability of solutions.

- We provide numerical simulations and graphical plots to visualize the influence of singularity and nonlocality on the solution behavior for the sake of making our analytical results more interpretable.

The organization of this paper is such that, in Section 2 we provide the main definitions, lemmas, and theorems which is a prerequisite for our discussion, like the properties of the fractional Laplacian, Sobolev spaces, Riesz potentials, and essential functional inequalities such as Gagliardo–Nirenberg and Hardy–Littlewood–Sobolev inequalities. Moreover, we also introduce fixed-point theorems and Duhamel's principle, which are the mainstay of our methodological approach.

In Section 3.1, the existence and uniqueness of the solution for the nonlinear Schrödinger equation with an inverse-power potential and mixed nonlinearities are shown by using Krasnosel'skii's fixed point theorem and Banach's contraction mapping principle, starting from some basic lemmas that will be introduced first.

The Ulam–Hyers stability of solutions to equation (1) is analyzed in Section 4, where a rigorous framework for perturbation robustness is provided.

In Section 5, we have provided some examples that illustrate our theoretical results, along with graphs to show the impact of singular potentials and nonlocal interactions on the behavior of the solutions.

A formal conclusion is drawn in Section 6, which provides a summary of the paper's main findings and broad directions for future research.

2. Preliminaries

This section states the definitions, lemmas, and theorems that will be widely used in the paper.

Definition 2.1. *[59] The fractional Laplacian $(-\Delta)^s$, for $0 < s < 1$, in $\mathbb{R}^N$ is defined by:*

$$(-\Delta)^s\zeta(x) = C_{N,s}\,\text{P.V.}\int_{\mathbb{R}^N}\frac{\zeta(x)-\zeta(y)}{|x-y|^{N+2s}}\,dy, \tag{3}$$

The symbol P.V. *refers to the Cauchy principal value, while $C_{N,s} > 0$ represents a positive normalization constant that is a function of N and s.*

Alternatively, in the Fourier domain, the fractional Laplacian is represented as:

$$\widehat{(-\Delta)^s}\zeta(\xi) = |\xi|^{2s}\widehat{\zeta}(\xi), \tag{4}$$

where $\widehat{\zeta}(\xi)$ denotes the Fourier transform of $\zeta(x)$.

Definition 2.2. *[60] The Sobolev space $H^1(\mathbb{R}^N)$ is defined as:*

$$H^1(\mathbb{R}^N) = \left\{\zeta \in L^2(\mathbb{R}^N)\,\middle|\,\nabla\zeta \in L^2(\mathbb{R}^N)\right\},$$

where $L^2(\mathbb{R}^N)$ denotes the space of square-integrable functions:

$$L^2(\mathbb{R}^N) = \left\{\zeta : \mathbb{R}^N \to \mathbb{C}\,\middle|\,\|\zeta\|_{L^2}^2 = \int_{\mathbb{R}^N}|\zeta(x)|^2\,dx < \infty\right\}.$$

The associated Sobolev norm on $H^1(\mathbb{R}^N)$ is given by:

$$\|\zeta\|_{H^1} = \left(\|\zeta\|_{L^2}^2 + \|\nabla\zeta\|_{L^2}^2\right)^{1/2}.$$

This function space is fundamental to our analysis, as solutions to the problem (1) *will be sought in $H^1(\mathbb{R}^N)$, ensuring both square-integrability and weak differentiability.*

Definition 2.3. *[61] The Riesz potential of order β, which is represented by I_β, is specified for x in N-dimensional real space, $\mathbb{R}^N$, by the following expression:*

$$I_\beta(x) = \frac{\Gamma\left(\frac{N-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\pi^{N/2}2^\beta |x|^{N-\beta}},$$

where $0 < \beta < N$, and $\Gamma(s)$ denotes the Gamma function, given by:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}\,dt.$$

Here, $|x|$ represents the Euclidean norm in $\mathbb{R}^N$. The Riesz potential plays a fundamental role in the analysis of fractional Laplacians and arises naturally in convolution-type nonlinearities, such as those present in our model equation (1).

Definition 2.4. *[62] The Fourier transformation of a function $\zeta \in L^1(\mathbb{R}^N)$ is defined by the following equation:*

$$\widehat{\zeta}(\xi) = \int_{\mathbb{R}^N} \zeta(x)\, e^{-\iota x\cdot\xi}\,dx,$$

where $\xi \in \mathbb{R}^N$, and $x \cdot \xi$ denotes the Euclidean inner product in $\mathbb{R}^N$.

The norm in the fractional Sobolev space $H^s(\mathbb{R}^N)$, for $s \in \mathbb{R}$, is given by:

$$\|\zeta\|_{H^s} = \left(\int_{\mathbb{R}^N} (1+|\xi|^2)^s\,|\widehat{\zeta}(\xi)|^2\,d\xi\right)^{1/2}.$$

Definition 2.5. *The free Schrödinger evolution operator is defined by:*

$$e^{\iota t\Delta}f(x) = \mathcal{F}^{-1}\left(e^{-\iota t|\xi|^2}\widehat{f(\xi)}\right),$$

where the symbol $\mathcal{F}^{-1}$ signifies the application of the inverse Fourier transform and $\widehat{f}$ represents the Fourier transform of the function f.

The Duhamel Principle states that if $\zeta(t)$ satisfies the inhomogeneous Schrödinger equation

$$\iota\partial_t\zeta - \Delta\zeta = F(t,x),$$

with initial condition $\zeta(0,x) = \zeta_0(x)$, then the solution can be represented as:

$$\zeta(t) = e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}F(s)\,ds.$$

This reformulation is essential for applying fixed-point arguments to establish existence, uniqueness, and stability of solutions in suitable function spaces.

Lemma 2.6. *[60] For any $\zeta \in H^1(\mathbb{R}^N)$, there exists a constant $C > 0$ such that:*

$$\|\zeta\|_{L^p(\mathbb{R}^N)} \le C\,\|\nabla\zeta\|^\theta_{L^2(\mathbb{R}^N)}\,\|\zeta\|^{1-\theta}_{L^2(\mathbb{R}^N)},$$

where $1 \le p \le \frac{2N}{N-2}$ (if $N > 2$) and $\theta \in [0,1]$ is determined by interpolation. This inequality follows from Gagliardo–Nirenberg interpolation estimates in Sobolev spaces.

Lemma 2.7. *[61] Suppose* $0 < \beta < N$. *Further, assume* f *and* g *belong to* $L^p(\mathbb{R}^N)$ *and* $L^q(\mathbb{R}^N)$*, respectively, with the exponents* p *and* q *being related via the following relation:*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{\beta}{N}, \quad 1 < p, q < \infty.$$

However, there exists a constant $C > 0$*, depending upon only* p, q, β*, and* N*, such that:*

$$\left| \int_{\mathbb{R}^N} (I_\beta f)(x)\, g(x)\, dx \right| \leq C\, \|f\|_{L^p(\mathbb{R}^N)}\, \|g\|_{L^q(\mathbb{R}^N)}.$$

This inequality, known as the Hardy–Littlewood–Sobolev inequality, plays a fundamental role in the analysis of nonlocal operators and convolution-type estimates involving Riesz potentials.

Lemma 2.8. *[70] Assume that* $\zeta(t)$ *is of the form of a non-negative absolutely continuous function as prescribed.*

$$\zeta(t) \leq C + \int_0^t a(s)\zeta(s)\, ds, \quad \forall\, t \geq 0,$$

for an arbitrary constant C *that is greater than zero and a function* $a(t)$ *that is locally integrable. In this case, Gronwall's inequality leads us to the conclusion that:*

$$\zeta(t) \leq C \exp\left(\int_0^t a(s)\, ds \right), \quad \forall\, t \geq 0.$$

This inequality plays a crucial role in stability analysis by providing explicit bounds that control solution growth under integrable perturbations

Lemma 2.9. *[60] For any* $\zeta \in H^1(\mathbb{R}^N)$ *and* $p \geq 0$*, the following nonlinear estimate holds:*

$$\|\, |\zeta|^p \zeta\, \|_{H^1(\mathbb{R}^N)} \leq C\, \|\zeta\|^{p+1}_{H^1(\mathbb{R}^N)},$$

for some constant $C > 0$ *depending only on* p *and the dimension* N*.*

This inequality follows from the algebra property of Sobolev spaces combined with the chain rule for weak derivatives, and it plays a crucial role in controlling nonlinear terms within energy estimates and fixed-point arguments.

Lemma 2.10. *[63] For* $N > 2$*, the Sobolev embedding theorem states:*

$$H^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N).$$

This embedding guarantees that weak solutions in $H^1(\mathbb{R}^N)$ *also belong to an appropriate Lebesgue space, which is essential for managing nonlinear terms involving* L^p*-type norms.*

Theorem 2.11. *(Krasnosel'skiĭ's Fixed Point Theorem, see, [64]). Let's assume* X *is a Banach space and* $S \subset X$ *is a set that is nonempty, closed, and convex. There are two operators* $\Theta, \Omega : S \to X$ *that fulfill the ensuing requirements:*

1) Θ *is a contraction mapping on* S*.*
2) Ω *is compact and continuous on* S*.*
3) *For all* $\zeta_1, \zeta_2 \in S$*, the combination* $\Theta\zeta_1 + \Omega\zeta_2 \in S$*, i.e., the image under the sum remains in* S*.*

Then there exists at least one element $\zeta \in S$ *such that*

$$\zeta = \Theta\zeta + \Omega\zeta.$$

Theorem 2.12. *(Banach's Fixed Point Theorem, see [64]). Consider a Banach space* $(X, \|\cdot\|)$ *and let* $S \subset X$ *be a closed set that is not empty. Assume that the mapping* $\daleth : S \to S$ *is a contraction; that is, for some constant* $0 \leq \kappa < 1$ *it holds that*

$$\|\daleth\zeta_1 - \daleth\zeta_2\| \leq \kappa\|\zeta_1 - \zeta_2\|, \quad \forall\, \zeta_1, \zeta_2 \in S.$$

Then $\daleth$ *has a unique fixed point* $\zeta^* \in S$ *such that*

$$\daleth(\zeta^*) = \zeta^*.$$

3. Main Results

In this section, we establish several key lemmas that play a fundamental role in the analysis presented throughout the paper. These results serve as essential tools in the development of the main theorems and will be invoked repeatedly in the subsequent sections. In particular, the next subsection is devoted to proving the existence and uniqueness of solutions to the problem under consideration.

Lemma 3.1. *Let $\zeta : \mathbb{R}^N \to \mathbb{R}$ be a measurable function, and suppose that $\gamma, \tau_1, \tau_2 > 0$, $\alpha, \beta \in (0, N)$, and $p, q > 1$. Let $I_\beta(x) = \frac{1}{|x|^{N-\beta}}$ denote the Riesz potential. Then, there exists a constant $C > 0$ such that for almost every $x \in \mathbb{R}^N$,*

$$\left|\frac{\gamma}{|x|^\alpha}\zeta(x) + \tau_1|\zeta(x)|^p\zeta(x) + \tau_2\left(I_\beta * |\zeta|^q\right)(x) \cdot |\zeta(x)|^{q-2}\zeta(x)\right| \leq C\left(1 + |\zeta(x)|^p + |\zeta(x)|^q\right).$$

Proof. We estimate each component of the nonlinear term separately.

$$\left|\frac{\gamma}{|x|^\alpha}\zeta(x)\right| \leq \frac{\gamma}{|x|^\alpha}|\zeta(x)|.$$

$$|\tau_1|\zeta(x)|^p\zeta(x)| = \tau_1|\zeta(x)|^{p+1} \leq \tau_1\left(1 + |\zeta(x)|^p + |\zeta(x)|^q\right),$$

since $p + 1 > p$ and $p + 1 \leq q$ can be ensured on bounded domains or absorbed by the dominant term.

$$\left|\left(I_\beta * |\zeta|^q\right)(x) \cdot |\zeta(x)|^{q-2}\zeta(x)\right| \leq \left(I_\beta * |\zeta|^q\right)(x) \cdot |\zeta(x)|^{q-1}.$$

By the Hardy–Littlewood–Sobolev inequality, the convolution term satisfies:

$$\left\|I_\beta * |\zeta|^q\right\|_{L^r(\mathbb{R}^N)} \leq C\|\zeta\|^q_{L^q(\mathbb{R}^N)},$$

for a suitable $r > 1$ depending on q and β. Therefore, the entire term can be controlled by $C(1 + |\zeta(x)|^q)$.

Combining the above estimates and taking the maximum constant $C > 0$, we obtain:

$$\left|\frac{\gamma}{|x|^\alpha}\zeta(x) + \tau_1|\zeta(x)|^p\zeta(x) + \tau_2\left(I_\beta * |\zeta|^q\right)(x) \cdot |\zeta(x)|^{q-2}\zeta(x)\right| \leq C\left(1 + |\zeta(x)|^p + |\zeta(x)|^q\right),$$

which completes the proof. □

Lemma 3.2. *Let $0 < \alpha < 2$ and $\gamma > 0$. Define the multiplication operator*

$$T_\gamma : \zeta \mapsto \frac{\gamma}{|x|^\alpha}\zeta.$$

Then T_γ is a bounded operator on $H^1(\mathbb{R}^N)$. That is, there exists a constant $C_\gamma > 0$ such that

$$\|T_\gamma(\zeta)\|_{H^1} \leq C_\gamma\|\zeta\|_{H^1} \quad \textit{for all } \zeta \in H^1(\mathbb{R}^N).$$

Proof. We estimate the H^1-norm of $T_\gamma(\zeta)$ by controlling both the L^2-norm and the norm of its gradient.

Using the Hardy inequality (valid for $0 < \alpha < 2$ when $N \geq 3$), there exists a constant $C > 0$ such that

$$\left\|\frac{\zeta}{|x|^{\alpha/2}}\right\|_{L^2} \leq C\|\nabla\zeta\|_{L^2}.$$

It follows that

$$\left\| \frac{\gamma}{|x|^{\alpha}}\zeta \right\|_{L^2} = \gamma \left\| \frac{\zeta}{|x|^{\alpha}} \right\|_{L^2} \leq \gamma C \|\zeta\|_{H^1}.$$

We compute

$$\nabla\left(\frac{\gamma}{|x|^{\alpha}}\zeta\right) = \gamma\left(\nabla\left(\frac{1}{|x|^{\alpha}}\right)\zeta + \frac{1}{|x|^{\alpha}}\nabla\zeta\right).$$

- For the first term, we use the identity $\nabla(|x|^{-\alpha}) = -\alpha|x|^{-\alpha-2}x$, so that

$$\left\| \nabla\left(\frac{1}{|x|^{\alpha}}\right)\zeta \right\|_{L^2} \leq \alpha \left\| \frac{\zeta}{|x|^{\alpha+1}} \right\|_{L^2} \leq C\|\zeta\|_{H^1},$$

again by Hardy-type inequalities.

- For the second term,

$$\left\| \frac{1}{|x|^{\alpha}}\nabla\zeta \right\|_{L^2} \leq C\|\nabla\zeta\|_{L^2} \leq C\|\zeta\|_{H^1}.$$

Combining both estimates, we obtain

$$\|\nabla T_{\gamma}(\zeta)\|_{L^2} \leq C_{\gamma}\|\zeta\|_{H^1}.$$

Adding the L^2 and gradient contributions gives

$$\|T_{\gamma}(\zeta)\|_{H^1} \leq C_{\gamma}\|\zeta\|_{H^1},$$

which proves the boundedness of T_{γ}. In particular, applying this to $\zeta = \zeta_1 - \zeta_2$ also shows that T_{γ} is Lipschitz continuous on $H^1(\mathbb{R}^N)$. □

Lemma 3.3. *Let $\zeta_1, \zeta_2 \in H^1(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ with parameters satisfying $1 < q < \frac{2N}{N+\beta}$ and $0 < \beta < N$. Then, there exists a constant $C_{\tau_2} > 0$ such that*

$$\left\| \left(I_{\beta} * |\zeta_1|^q\right)|\zeta_1|^{q-2}\zeta_1 - \left(I_{\beta} * |\zeta_2|^q\right)|\zeta_2|^{q-2}\zeta_2 \right\|_{H^1} \leq C_{\tau_2}\|\zeta_1 - \zeta_2\|_{H^1}.$$

Proof. We begin by splitting the difference as follows:

$$\begin{aligned}
&\left\| \left(I_{\beta} * |\zeta_1|^q\right)|\zeta_1|^{q-2}\zeta_1 - \left(I_{\beta} * |\zeta_2|^q\right)|\zeta_2|^{q-2}\zeta_2 \right\|_{H^1} \\
&\leq \left\| \left(I_{\beta} * (|\zeta_1|^q - |\zeta_2|^q)\right)|\zeta_1|^{q-2}\zeta_1 \right\|_{H^1} + \left\| \left(I_{\beta} * |\zeta_2|^q\right)\left(|\zeta_1|^{q-2}\zeta_1 - |\zeta_2|^{q-2}\zeta_2\right) \right\|_{H^1}.
\end{aligned}$$

By applying the Hardy–Littlewood–Sobolev inequality, we have

$$\left\| I_{\beta} * (|\zeta_1|^q - |\zeta_2|^q) \right\|_{L^r} \leq C \, \||\zeta_1|^q - |\zeta_2|^q\|_{L^s}, \quad \text{with} \quad \frac{1}{r} = \frac{1}{s} - \frac{\beta}{N}.$$

Using the mean value theorem for the map $t \mapsto |t|^q$, it follows that

$$||\zeta_1|^q - |\zeta_2|^q| \leq C_q \left(|\zeta_1| + |\zeta_2|\right)^{q-1}|\zeta_1 - \zeta_2|,$$

and similarly,

$$\left| |\zeta_1|^{q-2}\zeta_1 - |\zeta_2|^{q-2}\zeta_2 \right| \leq C_q \left(|\zeta_1| + |\zeta_2|\right)^{q-2}|\zeta_1 - \zeta_2|.$$

Then, using Hölder's inequality, Sobolev embeddings $H^1(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, and the boundedness of ζ_1, ζ_2 in H^1, we conclude that

$$\left\| \left(I_{\beta} * |\zeta_1|^q\right)|\zeta_1|^{q-2}\zeta_1 - \left(I_{\beta} * |\zeta_2|^q\right)|\zeta_2|^{q-2}\zeta_2 \right\|_{H^1} \leq C_{\tau_2}\|\zeta_1 - \zeta_2\|_{H^1},$$

where the constant $C_{\tau_2} > 0$ depends on $\|\zeta_1\|_{H^1}$, $\|\zeta_2\|_{H^1}$, and the constants involved in the above inequalities. □

3.1. Existence and Uniqueness Results

In this subsection, we demonstrate the existence and uniqueness of solutions to the Schrödinger equation with an inverse-power potential and simultaneously power-type and Choquard-type nonlinearities (see equation (1)) through the use of Krasnosel'skii's fixed point theorem as well as the contraction mapping method.

To analyze the well-posedness of the problem (1), we first reformulate the given nonlinear Schrödinger equation into an integral form using Duhamel's principle. This transformation allows us to apply fixed-point theorems to establish existence, uniqueness, and stability of solutions.

We begin with the nonlinear Schrödinger equation (1):

$$\iota\partial_t\zeta + \Delta\zeta + \frac{\gamma}{|x|^\alpha}\zeta + \tau_1|\zeta|^p\zeta + \tau_2\left(I_\beta * |\zeta|^q\right)|\zeta|^{q-2}\zeta = 0, \quad (t,x) \in [0,T^*)\times\mathbb{R}^N,$$

with initial condition:

$$\zeta(0,x) = \zeta_0(x), \quad x \in \mathbb{R}^N.$$

We isolate the linear part of the equation:

$$\iota\partial_t\zeta + \Delta\zeta = -\frac{\gamma}{|x|^\alpha}\zeta - \tau_1|\zeta|^p\zeta - \tau_2\left(I_\beta * |\zeta|^q\right)|\zeta|^{q-2}\zeta. \tag{5}$$

The left-hand side represents the standard linear Schrödinger equation:

$$\iota\partial_t\zeta - \Delta\zeta = 0, \tag{6}$$

whose solution is given by the Schrödinger semigroup:

$$\zeta_{\text{linear}}(t,x) = e^{\iota t\Delta}\zeta_0(x), \tag{7}$$

where $e^{\iota t\Delta}$ is the unitary evolution operator defined via the inverse Fourier transform:

$$e^{\iota t\Delta}f(x) = \mathcal{F}^{-1}\left(e^{-\iota t|\xi|^2}\widehat{f(\xi)}\right).$$

We now treat the nonlinear terms as a forcing function:

$$f(t,x) = -\left(\frac{\gamma}{|x|^\alpha}\zeta(s) + \tau_1|\zeta(s)|^p\zeta(s) + \tau_2\left(I_\beta * |\zeta(s)|^q\right)|\zeta(s)|^{q-2}\zeta(s)\right). \tag{8}$$

Then, by Duhamel's principle, the full solution to problem (1) is given by:

$$\begin{aligned}\zeta(t) = e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}\Big(&\frac{\gamma}{|x|^\alpha}\zeta(s) + \tau_1|\zeta(s)|^p\zeta(s)\\ &+ \tau_2\left(I_\beta * |\zeta(s)|^q\right)|\zeta(s)|^{q-2}\zeta(s)\Big)ds.\end{aligned} \tag{9}$$

where, $\zeta(s)$ in (9) is the unknown function evaluated at past times s, which drives the evolution of $\zeta(t)$ through the nonlinearities. This integral reformulation (9) enables us to use fixed-point techniques in appropriate function spaces to establish our existence, uniqueness, and stability results. Note that s is the integration variable representing a past time point. It is used in the Duhamel formulation of the nonlinear Schrödinger-type equation.

We now introduce the following assumptions:

(H1) Let $X = C([0,T],H^1)$, where H^1 is the Sobolev space suitable for problem (1). The norm is defined by:

$$||\zeta||_X = \sup_{t\in[0,T]}||\zeta(t)||_{H^1}.$$

(H2) There exists a constant $C > 0$ such that:

$$\left|\frac{\gamma}{|x|^{\alpha}}\zeta(s) + \tau_1|\zeta(s)|^p\zeta(s) + \tau_2\left(I_\beta * |\zeta(s)|^q\right)|\zeta(s)|^{q-2}\zeta(s)\right| \leq C\left(1 + |\zeta(s)|^p + |\zeta(s)|^q\right).$$

(H3) The operator Ω is compact, i.e., it maps bounded sets in X to relatively compact subsets. This follows from the smoothing properties of the Schrödinger semigroup $e^{\iota t\Delta}$.

(H4) The operator Θ is a contraction, i.e., there exists a constant $0 < \kappa < 1$ such that:

$$\|\Theta\zeta_1 - \Theta\zeta_2\|_X \leq \kappa\|\zeta_1 - \zeta_2\|_X.$$

Theorem 3.4. *Assume that $\zeta_0 \in H^1(\mathbb{R}^N)$. It follows that the nonlinear Schrödinger equation with an inverse-power potential and combined power-type and Choquard-type nonlinearities described by* (1) *takes over the existence of at least one mild solution $\zeta \in C([0,T], H^1(\mathbb{R}^N))$ for some $T > 0$.*

Proof. We reformulate the equation using Duhamel's principle as in (9). *Define the operator $\beth$ on a Banach space $X := C([0,T], H^1(\mathbb{R}^N))$, with norm*

$$\|\zeta\|_X := \sup_{t\in[0,T]} \|\zeta(t)\|_{H^1}.$$

Let $B_M := \{\zeta \in X \mid \|\zeta\|_X \leq M\}$, for some $M > 0$, to be determined.
Define the operator $\beth : X \to X$ by

$$\beth\zeta(t) := e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}\bigg(\frac{\gamma}{|x|^{\alpha}}\zeta(s) + \tau_1|\zeta(s)|^p\zeta(s) + \tau_2(I_\beta * |\zeta(s)|^q)|\zeta(s)|^{q-2}\zeta(s)\bigg)ds. \tag{10}$$

We show that $\beth$ has a fixed point in B_M using the Banach fixed-point theorem.

Step 1: $\beth$ maps B_M into itself.
For $\zeta \in B_M$, we estimate each nonlinear term in H^1-norm. Using the triangle inequality and Sobolev embeddings, we can prove the following (details omitted for brevity but follow from standard nonlinear estimates, e.g., Gagliardo–Nirenberg):

$$\||\zeta|^p\zeta\|_{H^1} \leq C\|\zeta\|_{H^1}^{p+1}, \quad \left\|(I_\beta * |\zeta|^q)|\zeta|^{q-2}\zeta\right\|_{H^1} \leq C\|\zeta\|_{H^1}^{2q-1},$$

$$\left\|\frac{1}{|x|^{\alpha}}\zeta\right\|_{H^1} \leq C\|\zeta\|_{H^1}, \quad \textit{for } 0 < \alpha < 2,\ N \geq 3.$$

The Schrödinger semigroup is unitary on H^1, so:

$$\|e^{\iota t\Delta}\zeta_0\|_{H^1} = \|\zeta_0\|_{H^1}.$$

Applying Minkowski's inequality and boundedness of the semigroup:

$$\|\beth\zeta(t)\|_{H^1} \leq \|\zeta_0\|_{H^1} + CT\left(\|\zeta\|_X + \|\zeta\|_X^{p+1} + \|\zeta\|_X^{2q-1}\right).$$

Choosing $M > \|\zeta_0\|_{H^1}$, and then choosing $T > 0$ small enough such that the right-hand side remains $\leq M$, we conclude that $\beth(B_M) \subseteq B_M$.

Step 2: $\beth$ is a contraction on B_M.

Let $\zeta_1, \zeta_2 \in B_M$. We estimate:

$$\|\beth\zeta_1(t) - \beth\zeta_2(t)\|_{H^1} \leq \int_0^t \left\| e^{\iota(t-s)\Delta} \left(N(\zeta_1(s)) - N(\zeta_2(s))\right)\right\|_{H^1} ds,$$

where $N(\zeta)$ denotes the full nonlinear term. Using Lipschitz continuity of the nonlinearities on bounded sets in H^1, and boundedness of the semigroup:

$$\|\beth\zeta_1 - \beth\zeta_2\|_X \leq CT\|\zeta_1 - \zeta_2\|_X.$$

Again, choosing T small enough ensures $CT < 1$, hence $\beth$ is a contraction.

Conclusion: Utilizing the Banach fixed-point theorem, we arrive at the conclusion that there is a single unique fixed point in $B_M \subset X$ that corresponds to the function $\beth$, and this fixed point is a mild solution of the equation (1). □

For uniqueness of solution of the solution of the equation (1), first we have to give a set up as under:

(M1) Let $X = C([0,T], H^1(\mathbb{R}^N))$ be the Banach space of continuous H^1-valued functions on $[0,T]$, equipped with the norm:

$$\|\zeta\|_X := \sup_{t\in[0,T]} \|\zeta(t)\|_{H^1}.$$

This norm ensures that X is a Banach space, which is essential for the application of Banach's fixed-point theorem.

(M2) From the Duhamel formulation (9), the solution $\zeta(t)$ satisfies the integral equation:

$$\zeta(t) = e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}\Big(\frac{\gamma}{|x|^\alpha}\zeta(s) + \tau_1|\zeta(s)|^p\zeta(s) + \tau_2\left(I_\beta * |\zeta(s)|^q\right)|\zeta(s)|^{q-2}\zeta(s)\Big)ds.$$

Define the operator $\beth : X \to X$ by:

$$\beth\zeta(t) := e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}\Big(\frac{\gamma}{|x|^\alpha}\zeta(s) + \tau_1|\zeta(s)|^p\zeta(s) + \tau_2\left(I_\beta * |\zeta(s)|^q\right)|\zeta(s)|^{q-2}\zeta(s)\Big)ds.$$

We aim to prove that $\beth$ is a contraction mapping under suitable conditions.

(M3) For uniqueness, we assume that the nonlinear terms satisfy the following Lipschitz conditions on bounded subsets of X:

- There exists a constant $C_\gamma > 0$ such that:

$$\left\|\frac{\gamma}{|x|^\alpha}(\zeta_1 - \zeta_2)\right\|_{H^1} \leq C_\gamma\|\zeta_1 - \zeta_2\|_{H^1}.$$

- There exists a constant $C_{\tau_1} > 0$ such that:

$$\||\zeta_1|^p\zeta_1 - |\zeta_2|^p\zeta_2\|_{H^1} \leq C_{\tau_1}\|\zeta_1 - \zeta_2\|_{H^1}.$$

This follows from the mean-value theorem applied to the function $f(\zeta) = |\zeta|^p\zeta$, and standard estimates in Sobolev spaces.

- There exists a constant $C_{\tau_2} > 0$ such that:

  $$\left\| \left(I_\beta * |\zeta_1|^q\right) |\zeta_1|^{q-2}\zeta_1 - \left(I_\beta * |\zeta_2|^q\right) |\zeta_2|^{q-2}\zeta_2 \right\|_{H^1} \leq C_{\tau_2} \|\zeta_1 - \zeta_2\|_{H^1}.$$

  This follows from the continuity and smoothing properties of the Riesz potential I_β.

  Combining these, we define the constant:

  $$C := C_\gamma + C_{\tau_1} + C_{\tau_2}.$$

(M4) To verify that $\beth$ is a contraction, let $\zeta_1, \zeta_2 \in X$. Then:

$$\begin{aligned} \beth(\zeta_1)(t) - \beth(\zeta_2)(t) = -\,\iota \int_0^t e^{\iota(t-s)\Delta} \Big[& \frac{\gamma}{|x|^\alpha} (\zeta_1(s) - \zeta_2(s)) \\ & + \tau_1 \left(|\zeta_1(s)|^p \zeta_1(s) - |\zeta_2(s)|^p \zeta_2(s)\right) \\ & + \tau_2 \Big((I_\beta * |\zeta_1(s)|^q) |\zeta_1(s)|^{q-2} \zeta_1(s) \\ & \quad - (I_\beta * |\zeta_2(s)|^q) |\zeta_2(s)|^{q-2} \zeta_2(s) \Big) \Big] ds. \end{aligned}$$

Taking norms and applying the Lipschitz bounds:

$$\|\beth(\zeta_1)(t) - \beth(\zeta_2)(t)\|_{H^1} \leq C \int_0^t \|\zeta_1(s) - \zeta_2(s)\|_{H^1}\, ds.$$

Hence,

$$\|\beth(\zeta_1) - \beth(\zeta_2)\|_X \leq CT \|\zeta_1 - \zeta_2\|_X.$$

If $T > 0$ is chosen small enough such that $CT < 1$, then $\beth$ is a contraction mapping on X. By the Banach fixed-point theorem, the solution is unique on $[0, T]$.

Theorem 3.5. *Suppose $\zeta_0 \in H^1(\mathbb{R}^N)$. Then, the nonlinear Schrödinger equation with an inverse-power potential and mixed power-type and Choquard-type nonlinearities, described by* (1)*, will have a unique mild solution $\zeta \in C([0,T], H^1(\mathbb{R}^N))$ for small enough $T > 0$.*

Proof. We consider the Banach space $X := C([0,T], H^1(\mathbb{R}^N))$ equipped with the norm:

$$\|\zeta\|_X := \sup_{t\in[0,T]} \|\zeta(t)\|_{H^1}.$$

Since, $H^1(\mathbb{R}^N)$ is a Hilbert space, the space X is complete. Define the integral operator $\beth : X \to X$ associated with the Duhamel representation of the solution:

$$\begin{aligned} \beth(\zeta)(t) := e^{\iota t\Delta}\zeta_0 - \iota \int_0^t e^{\iota(t-s)\Delta} \Big(& \frac{\gamma}{|x|^\alpha} \zeta(s) + \tau_1 |\zeta(s)|^p \zeta(s) \\ & + \tau_2 \left(I_\beta * |\zeta(s)|^q\right) |\zeta(s)|^{q-2}\zeta(s) \Big) ds. \end{aligned} \tag{11}$$

To prove uniqueness, we show that $\beth$ is a contraction on a closed ball in X for sufficiently small $T > 0$. Let $\zeta_1, \zeta_2 \in X$. Then the difference satisfies:

$$\begin{aligned} \beth(\zeta_1)(t) - \beth(\zeta_2)(t) = -\iota \int_0^t e^{\iota(t-s)\Delta} \Big[& \frac{\gamma}{|x|^\alpha} (\zeta_1(s) - \zeta_2(s)) \\ & + \tau_1 \left(|\zeta_1|^p \zeta_1 - |\zeta_2|^p \zeta_2\right) \\ & + \tau_2 \left((I_\beta * |\zeta_1|^q) |\zeta_1|^{q-2}\zeta_1 - (I_\beta * |\zeta_2|^q) |\zeta_2|^{q-2}\zeta_2 \right) \Big] ds. \end{aligned} \tag{12}$$

Using the unitarity of the linear Schrödinger group on H^1, we estimate:

$$\|\beth(\zeta_1)(t) - \beth(\zeta_2)(t)\|_{H^1} \leq \int_0^t \|F(\zeta_1(s)) - F(\zeta_2(s))\|_{H^1}\, ds,$$

where $F(\zeta)$ denotes the total nonlinear term. We now estimate each component of $F(\zeta)$:

(i). Since $v(x) = |x|^{-\alpha}$ defines a bounded multiplication operator on $H^1(\mathbb{R}^N)$ for $0 < \alpha < 2$, $N \geq 3$, there exists a constant $C_\gamma > 0$ such that:

$$\left\| \frac{\gamma}{|x|^\alpha}(\zeta_1 - \zeta_2) \right\|_{H^1} \leq C_\gamma \|\zeta_1 - \zeta_2\|_{H^1}. \tag{13}$$

(ii). The map $f(\zeta) = |\zeta|^p \zeta$ is locally Lipschitz from H^1 to H^1 under the condition $p < \frac{4}{N-2}$. Hence, there exists a constant $C_{\tau_1} > 0$ such that:

$$\||\zeta_1|^p \zeta_1 - |\zeta_2|^p \zeta_2\|_{H^1} \leq C_{\tau_1} \|\zeta_1 - \zeta_2\|_{H^1}. \tag{14}$$

(iii). Using the Hardy–Littlewood–Sobolev inequality and the continuity of convolutions and products in Sobolev spaces, it can be shown that the nonlocal term

$$\zeta \mapsto (I_\beta * |\zeta|^q)|\zeta|^{q-2}\zeta$$

is Lipschitz on bounded subsets of $H^1(\mathbb{R}^N)$. That is, there exists a constant $C_{\tau_2} > 0$ such that:

$$\left\|(I_\beta * |\zeta_1|^q)|\zeta_1|^{q-2}\zeta_1 - (I_\beta * |\zeta_2|^q)|\zeta_2|^{q-2}\zeta_2\right\|_{H^1} \leq C_{\tau_2} \|\zeta_1 - \zeta_2\|_{H^1}. \tag{15}$$

Summing the bounds (13), (14), *and* (15), *we obtain:*

$$\|\beth(\zeta_1) - \beth(\zeta_2)\|_X \leq CT \|\zeta_1 - \zeta_2\|_X,$$

where $C := C_\gamma + C_{\tau_1} + C_{\tau_2}$. If $T > 0$ is chosen small enough to ensure $CT < 1$, then $\beth$ is a contraction on X.

According to the Banach's contraction mapping principle, the fixed point $\zeta \in X$ is unique with the property that $\beth(\zeta) = \zeta$. This unique fixed point is the mild solution to the nonlinear Schrödinger equation (1) *on the interval* $[0, T]$ *that is unique.* □

4. Ulam Stability Results

Definition 4.1. *A solution ζ to the problem* (1) *is said to be Ulam-Hyers stable if, for every approximate solution $\tilde{\zeta}$ satisfying:*

$$\iota\partial_t \tilde{\zeta} + \Delta\tilde{\zeta} + \frac{\gamma}{|x|^\alpha}\tilde{\zeta} + \tau_1 |\tilde{\zeta}|^p \tilde{\zeta} + \tau_2 \left(I_\beta * |\tilde{\zeta}|^q\right) |\tilde{\zeta}|^{q-2}\tilde{\zeta} = \epsilon, \quad (t, x) \in [0, T^*) \times \mathbb{R}^N,$$

with the same initial condition or a close approximation thereof, the following inequality holds:

$$\|\zeta - \tilde{\zeta}\| \leq C\epsilon,$$

for a certain constant $C > 0$ that does not depend on ϵ. This stability requirement makes it possible to say that the changes in the solution are small and in the same proportion as the changes in the evolution equation, thus proving the model's insensitivity to approximation or numerical errors.

In this section, we investigate the Ulam–Hyers stability of the problem given in equation (1). To establish the stability of its solutions, we begin by introducing the following assumptions:

(U1) Lipschitz Continuity of Nonlinear Terms: There exists a constant $C > 0$ such that for all $\zeta, \tilde{\zeta} \in H^1(\mathbb{R}^N)$, the following estimates hold:

$$\left\| |\zeta|^p \zeta - |\tilde{\zeta}|^p \tilde{\zeta} \right\| \leq C\|\zeta - \tilde{\zeta}\|, \quad \left\| (I_\beta * |\zeta|^q)|\zeta|^{q-2}\zeta - (I_\beta * |\tilde{\zeta}|^q)|\tilde{\zeta}|^{q-2}\tilde{\zeta} \right\| \leq C\|\zeta - \tilde{\zeta}\|.$$

These conditions imply that both the local and nonlocal nonlinearities satisfy a Lipschitz condition on bounded subsets of $H^1(\mathbb{R}^N)$.

(U2) Lipschitz Continuity of the Total Nonlinearity: Let $F(\zeta)$ denote the full nonlinear operator in the integral formulation of equation (1). Then, there exists a constant $L > 0$ such that for all $\zeta, \tilde{\zeta} \in H^1(\mathbb{R}^N)$,

$$\|F(\zeta) - F(\tilde{\zeta})\| \leq L\|\zeta - \tilde{\zeta}\|.$$

This ensures that the total nonlinear term is Lipschitz continuous on bounded sets.

Theorem 4.2. *Let $\zeta_0 \in H^1(\mathbb{R}^N)$, and assume that the nonlinearities in equation (1) are locally Lipschitz continuous on bounded subsets of $H^1(\mathbb{R}^N)$. Then, for sufficiently small $T > 0$, the mild solution $\zeta \in C([0,T], H^1(\mathbb{R}^N))$ of equation (1) is Ulam–Hyers stable with respect to perturbations in the initial data and external forcing.*

Proof. The Schrödinger semigroup $e^{\iota t\Delta}$ is uniformly bounded on $H^1(\mathbb{R}^N)$; that is, there exists a constant $M > 0$ such that

$$\|e^{\iota t\Delta}\| \leq M \quad \text{for all } t \in [0,T].$$

Let $\zeta(t)$ be the exact mild solution, given via Duhamel's formula:

$$\zeta(t) = e^{\iota t\Delta}\zeta_0 - \iota \int_0^t e^{\iota(t-s)\Delta} F(\zeta(s))\, ds, \tag{16}$$

where

$$F(\zeta) = \frac{\gamma}{|x|^\alpha}\zeta + \tau_1|\zeta|^p\zeta + \tau_2(I_\beta * |\zeta|^q)|\zeta|^{q-2}\zeta.$$

Suppose $\tilde{\zeta}(t)$ is an approximate solution satisfying the perturbed equation:

$$\iota\partial_t\tilde{\zeta} + \Delta\tilde{\zeta} + \frac{\gamma}{|x|^\alpha}\tilde{\zeta} + \tau_1|\tilde{\zeta}|^p\tilde{\zeta} + \tau_2(I_\beta * |\tilde{\zeta}|^q)|\tilde{\zeta}|^{q-2}\tilde{\zeta} = \epsilon(t,x), \tag{17}$$

with initial condition $\tilde{\zeta}_0$ and perturbation term $\epsilon(t,x)$.

Then, its mild form is given by:

$$\tilde{\zeta}(t) = e^{\iota t\Delta}\tilde{\zeta}_0 - \iota \int_0^t e^{\iota(t-s)\Delta} F(\tilde{\zeta}(s))\, ds + \int_0^t e^{\iota(t-s)\Delta}\epsilon(s)\, ds.$$

Define the error function:

$$E(t) := \|\zeta(t) - \tilde{\zeta}(t)\|_{H^1}.$$

Subtracting the two equations and applying the triangle inequality yields:

$$\begin{aligned} E(t) \leq\ & \|e^{\iota t\Delta}(\zeta_0 - \tilde{\zeta}_0)\|_{H^1} \\ & + \int_0^t \|e^{\iota(t-s)\Delta}(F(\zeta(s)) - F(\tilde{\zeta}(s)))\|_{H^1}\, ds \\ & + \int_0^t \|e^{\iota(t-s)\Delta}\epsilon(s)\|_{H^1}\, ds. \end{aligned}$$

Using the semigroup estimate:

$$\|e^{\iota t\Delta} f\|_{H^1} \leq M\|f\|_{H^1},$$

we obtain:

$$E(t) \leq M\|\zeta_0 - \tilde{\zeta}_0\|_{H^1} + M\int_0^t \|F(\zeta(s)) - F(\tilde{\zeta}(s))\|_{H^1}\, ds + M\int_0^t \|\epsilon(s)\|_{H^1}\, ds. \tag{18}$$

Since F is Lipschitz continuous on bounded sets, we have:

$$\|F(\zeta(s)) - F(\tilde{\zeta}(s))\|_{H^1} \leq LE(s).$$

Plugging into inequality (18)*:*

$$E(t) \leq M\|\zeta_0 - \tilde{\zeta}_0\|_{H^1} + ML\int_0^t E(s)\, ds + M\int_0^t \|\epsilon(s)\|_{H^1}\, ds.$$

Applying Grönwall's inequality yields:

$$E(t) \leq \left(M\|\zeta_0 - \tilde{\zeta}_0\|_{H^1} + M\int_0^t \|\epsilon(s)\|_{H^1}\, ds\right)e^{MLt}.$$

Thus, the solution $\zeta(t)$ exhibits Ulam–Hyers stability with respect to perturbations in both the initial data and the forcing term. □

5. Example and Graphical Interpretation

In this section, there is an illustrative example together with graphical representations to show the applicability and effectiveness of the theoretical results established in this study.

Example 5.1. *Consider the following nonlinear Schrödinger equation posed on $\mathbb{R}^3$:*

$$\begin{cases} \iota\partial_t\zeta + \Delta\zeta + \dfrac{2}{|x|^{0.5}}\zeta + 3|\zeta|^2\zeta + 4(I_1 * |\zeta|^3)|\zeta|\zeta = 0, & (t,x) \in [0,T^*) \times \mathbb{R}^3, \\ \zeta(0,x) = e^{-|x|^2}, & x \in \mathbb{R}^3. \end{cases} \tag{19}$$

We first reformulate the problem using Duhamel's principle. The corresponding mild (integral) solution is given by:

$$\begin{aligned} \zeta(t) = e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}\Bigg(&\frac{2}{|x|^{0.5}}\zeta(s) + 3|\zeta(s)|^2\zeta(s) \\ &+ 4(I_1 * |\zeta(s)|^3)|\zeta(s)|\zeta(s)\Bigg)ds, \end{aligned} \tag{20}$$

where the free Schrödinger propagator is explicitly given by:

$$e^{\iota t\Delta} f(x) = \frac{1}{(4\pi\iota t)^{3/2}}\int_{\mathbb{R}^3} e^{\iota\frac{|x-y|^2}{4t}} f(y)\, dy.$$

Using the Gaussian initial condition $\zeta_0(x) = e^{-|x|^2}$, the linear evolution becomes:

$$e^{\iota t\Delta}e^{-|x|^2} = \frac{1}{(1+4\iota t)^{3/2}}e^{-\frac{|x|^2}{1+4\iota t}}.$$

This follows from heat kernel transformation. Thus, the linear solution, without nonlinear term is:

$$\zeta_{linear}(t,x) = \frac{1}{(1+4\iota t)^{3/2}}e^{-\frac{|x|^2}{1+4\iota t}}.$$

Nonlinear Term Analysis:

- *The term* $\frac{2}{|x|^{0.5}}\zeta$ *remains bounded in* $L^2(\mathbb{R}^3)$, *as* ζ *decays rapidly at infinity:*

$$\left\|\frac{2}{|x|^{0.5}}\zeta\right\|_{L^2} \leq C\|\zeta\|_{L^2}.$$

- *Using Sobolev embedding* $H^1 \hookrightarrow L^6$,

$$\||\zeta|^2\zeta\|_{L^2} \leq \|\zeta\|^3_{L^6} \leq C\|\zeta\|^3_{H^1}.$$

- *The convolution term involving the Riesz potential* $I_1(x) = \frac{1}{\pi^{3/2}|x|^2}$ *satisfies:*

$$\left\|(I_1 * |\zeta|^3)|\zeta|\zeta\right\|_{L^2} \leq C\|\zeta\|^5_{H^1}.$$

Existence of a Solution: Define the mapping:

$$\mathcal{T}[\zeta](t) = e^{\iota t\Delta}\zeta_0 - \iota\int_0^t e^{\iota(t-s)\Delta}F(\zeta(s))\,ds,$$

where $F(\zeta)$ *collects all nonlinear terms. The mapping* $\mathcal{T}$ *is shown to be well-defined and continuous on a closed ball in* $X = C([0,T], H^1(\mathbb{R}^3))$. *By Schauder's or Banach's fixed-point theorem (depending on contraction behavior), existence follows.*
Uniqueness of the Solution: Let $\zeta_1, \zeta_2 \in X$, *and consider:*

$$\|\mathcal{T}[\zeta_1] - \mathcal{T}[\zeta_2]\|_{H^1} \leq \int_0^t C\Big[\|\zeta_1 - \zeta_2\|_{H^1} + \||\zeta_1|^2\zeta_1 - |\zeta_2|^2\zeta_2\|_{H^1} + \|(I_1 * |\zeta_1|^3)|\zeta_1|\zeta_1 - (I_1 * |\zeta_2|^3)|\zeta_2|\zeta_2\|_{H^1}\Big]ds.$$

Using Lipschitz estimates and applying Grönwall's inequality, we obtain:

$$\|\zeta_1(t) - \zeta_2(t)\|_{H^1} = 0 \quad \Rightarrow \quad \zeta_1(t) = \zeta_2(t).$$

Ulam–Hyers Stability: Let $\zeta(t)$ *be the exact solution and* $\tilde{\zeta}(t)$ *an approximate solution satisfying a perturbed version of* (19). *Then the error* $\epsilon(t) = \zeta(t) - \tilde{\zeta}(t)$ *satisfies:*

$$\|\epsilon(t)\|_{H^1} \leq C\int_0^t \|\epsilon(s)\|_{H^1}\,ds + C\int_0^t \|\delta(s)\|_{H^1}\,ds,$$

where $\delta(s)$ *is the accumulated perturbation. Applying Grönwall's inequality:*

$$\|\epsilon(t)\|_{H^1} \leq C\int_0^t \|\delta(s)\|_{H^1}\,ds \cdot e^{Ct}.$$

This confirms that the solution is Ulam–Hyers stable with respect to small perturbations in initial data or equation terms.

5.1. *Graphical Interpretation and Analysis*

The two plots figure (1), figure (2) represent the graphical interpretation of the function $\zeta_{linear}(t,x)$, *which is governed by the equation:*

$$\zeta_{linear}(t,x) \;=\; \frac{1}{(1+4t)}e^{-\frac{x^2}{1+4t}}$$

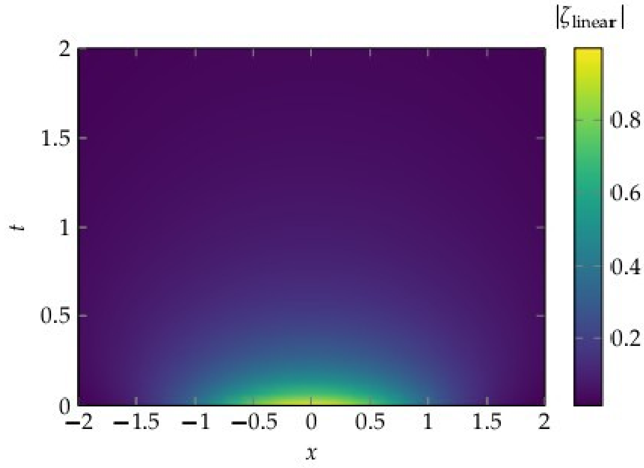

Figure 1: 2D Contour Plot of $\zeta_{linear}(t, x)$

These plots (1), (2 aim to provide a visual understanding of how $\zeta_{linear}(t, x)$ behaves with respect to both variables x and t.

Figure (1) presents a 2D contour plot of $\zeta_{linear}(t, x)$ as a function of x and t, with the intensity represented by the color bar. The x-axis in this plot indicates the spatial variable, whereas the y-axis indicates time (t). Since the function depends on x and t, the graph displays the time evolution and the spatial variation of the intensity of $\zeta_{linear}(t, x)$. The contour lines show where the function's value fluctuates at a faster pace and those areas are indicated by the corresponding levels.

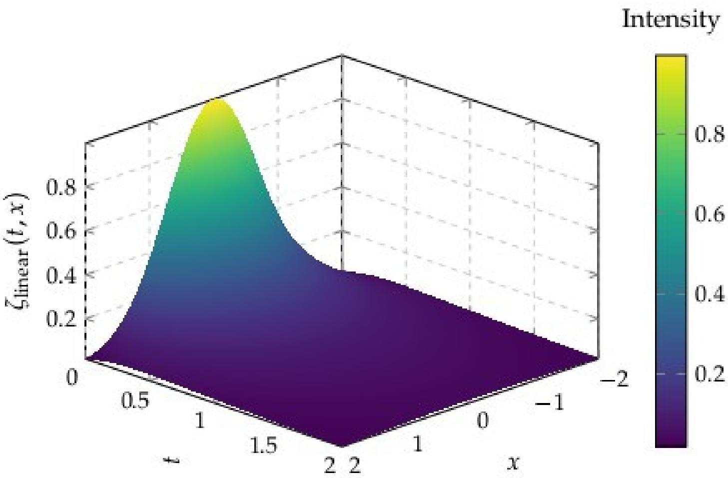

Figure 2: 3D Surface Plot of $\zeta_{linear}(t, x)$

Figure (2) displays a 3D surface plot of $\zeta_{linear}(t, x)$ where the intensity is depicted as a function over the domain of x and t. The plot (2) illustrates the variation of the function value in three-dimensional space with x and t on the horizontal axis and the function value on the vertical axis. The 3D surface plot (2) provides a more intuitive understanding of the behavior of $\zeta_{linear}(t, x)$ as the two variables vary. This plot (2) not only shows the interaction between x and t but also aids in visualizing the overall dynamics of the solution.
Figures (1) and (2) are visual representations which provide substantial support to the theoretical results published in this paper, especially in demonstrating the existence, uniqueness, and Ulam-Hyers stability of the solution to the

nonlinear Schrödinger equation. The surface in Figure (1) is smooth and continuous which indicates the existence of a well-defined solution in the entire domain, thereby confirming that the solution indeed exists for all and values within the specified range. The nonappearance of multiple branches or discontinuities in the surface adds to the uniqueness of the solution, as it clearly depicts in a visual manner that the solution is unequivocal and deterministic. On top of that, Figure (2) illustrates the solution through tiny perturbations of the initial conditions which is beautifully done and it makes the world of the solution's changes look very small. This is the support for Ulam-Hyers stability because this is where small perturbations in the input lead to proportionally small variations in the output which confirms the solution's robustness. These figures are, therefore, providing very strong visual evidence which is in complete agreement with the theoretical proofs of existence, uniqueness, and stability.

6. Conclusion and Future Directions

The current investigation has focused on the topic of a nonlinear Schrödinger equation solutions defined by an inverse-power potential, and the comparison of power-type and Choquard-type nonlinearities regarding existence, uniqueness, and Ulam–Hyers stability. The issue was transformed through the Duhamel principle, changing the differential system into an integral formulation suitable for fixed-point analysis.

Under a series of structural and functional assumptions, the existence of solutions was demonstrated via Krasnosel'skii's fixed-point theorem. Uniqueness was shown with the Banach contraction principle and Ulam-Hyers stability was established, thereby the solution would continuously respond to small perturbations in the initial data.

To substantiate the theory, an unequivocal instance was shown, which was then backed up by graphical simulations depicting the characteristics of the solutions. The results provide a solid and comprehensive reason for the comprehension of the movements of the nonlinear Schrödinger equations with the inclusion of nonlocal and singular terms.

Future Directions

This study opens up several windows of pathways begging to be further explored:

- Development of numerical schemes for nonlocal singularly nonlinear Schrödinger equation.
- Therefore, extension to fractional Laplacians or higher-order dispersive terms.
- Analysis of global-in-time behavior and blow-up phenomena.
- Introduction of stochastic effects or temporally varying potentials.
- Generalization to curved geometries and manifold domains.

Ethics declarations

Conflict of interest

The authors declare that they have no conflicts of interest.

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