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Uninorms with non-trivial unit element on the bounded lattices

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Abstract. In this paper, we focus on uninorms with non-trivial unit on bounded lattices. Based on the existence a t-norm T acting on $[0, e]$ as well as a t-conorm S acting on $[e, 1]$, we present some construction methods for uninorms on a bounded lattice, where some additional conditions on the bounded lattice and the neutral element e are required. The role of these additional conditions is emphasized. Finally, we present some results and some illustrative examples derived from these methods.

1. Introduction

There has recently been a rapidly growing interest in uninorms. They play an important role in theory and practice, some of their uses are decision-making process, fuzzy control systems, fuzzy inference systems, fuzzy logic and fuzzy systems modeling ([5, 17, 19–21]). Uninorms on unit interval, which is a bounded lattice, has been introduced by Yager and Rybalov in ([22]). Some structures and properties of uninorms on unit interval have been studied in ([8]). The notation of uninorms has been expanded on bounded lattices in the work of Karaçal and Mesiar ([14]). They have also presented methods for constructing uninorms as well as some properties of uninorms on bounded lattices. In some other works, researchers have focused on various methods to generate aggregation functions such as t-norm, uninorm, uni-nullnorm on bounded lattices see ([1, 2, 6, 9–13, 15]) .

Uninorms on bounded lattices are functions that satisfies commutative, associative, increasing with respect to the both variables and has a neutral element e. In this study, we concentrate on the construction methods of uninorm on bounded lattices. It is clear that $U(e, e) = e$ if U is a uninorm on the bounded lattice L with the neutral element e. Thus, we interested in the existence of uninorms satisfying $U(x, y) = e$ for some $(x, y) \in (L\backslash\{e\})^2$. More precisely, in this paper, we introduce some construction methods which satisfy $U(x, y) = e$ for some $(x, y) \in I_e \times I_e$. The proposed methods are significant because they address a gap in the literature: uninorms with non-unit elements have not yet been studied.

The structure of the article is arranged as follows. In Section 2, we list some definitions, theorems and notations concerning lattices and aggregation functions which will be use in the paper. Section 3 offers some construction methods to produce on bounded lattices, and gives some explanations derived from these methods. Also, in this section additional examples are provided. Conclusions are discussed in Section 4.

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2. Preliminaries

A set L is called a partially ordered set denoted by $(L, \leq)$ if it is equipped with a binary relation $\leq$ satisfying reflexivity ($a \leq a$ for all $a \in L$), antisymmetry ($a = b$ whenever $a \leq b$ and $b \leq a$ for $a, b \in L$) and transitivity ($a \leq c$ whenever $a \leq b$ and $b \leq c$ for $a, b, c \in L$). $a \in L$ ($b \in L$) is called the smallest element (the greatest element), denoted by 0 (1), if $a \leq x$ ($x \leq b$) for all $x \in L$.

Let $A \subseteq L$. $a \in L$ ($b \in L$) is called a lower bound of A (a upper bound of A) if $a \leq x$ ($x \leq b$) for $x \in A$. The set of the lower bounds of A (the set of the upper bounds of A) is represented by $\underline{A}$ ($\overline{A}$). $1_{\underline{A}}$ ($0_{\overline{A}}$) is called infumum (supremum) if $\underline{A}$ ($\overline{A}$) has the smallest element $1_{\underline{A}}$ ($0_{\overline{A}}$). Specifically, when $A = \{a, b\}$, the supremum (infumum) of A is denoted by $a \vee b$ ($a \wedge b$).

L is called a lattice if there exists $a \vee b$ and $a \wedge b$ for all $a, b \in L$. Also, it is called bounded lattice, denoted by $(L, \leq, 0, 1)$ if the smallest element $0 \in L$ and the greatest element1 $1 \in L$.

Let $(L, \leq, 0, 1)$ be a bounded lattice and $a, b \in L$ with $a \leq b$. Sub-intervals of L are defined as:

$[a, b] = \{x \in L \mid a \leq x \leq b\}$, $(a, b] = \{x \in L \mid a < x \leq b\}$,

$[a, b) = \{x \in L \mid a \leq x < b\}$, $(a, b) = \{x \in L \mid a < x < b\}$.

The elements x and y are called comparable if $x \leq y$ or $y \leq x$, denoted by $x \nparallel y$. Otherwise, x and y are called incomparable, denoted by $x \parallel y$. I_a denotes the set of incomparable elements with $a \in L$, that is $I_a = \{x \in L \mid x \parallel a\}$. Readers can refer to [3] for more information.

We denote the cardinality of a set A by $|A|$.

2.1. Some definitions and theorems concerning uninorms on bounded lattices L

This section presents essential concepts of triangular norms, triangular conorms and uninorms on bounded lattices and some results related to alternating associativity that will be used through the paper.

Definition 2.1. *([11]) Let S be a nonempty set, A, B, C be subsets of S and U be a binary operation on S. H is called alternating associative on (A, B, C) if for any $[X, Y, Z]$ of (A, B, C) it satisfies*

$$H(H(x, y), z) = H(x, H(y, z)) \text{ for all } x \in X,\ y \in Y,\ z \in Z.$$

Theorem 2.2. *([11]). Let S be a nonempty set and A, B, C be subsets of S and H be a commutative binary operation on S.*

(i) If $H(H(x, y), z) = H(x, H(y, z)) = H(H(x, z), y)$ for all $x \in A$, $y \in B$ and $z \in C$, then H is alternating associative on (A, B, C).

(ii) If $H(H(x, y), z) = H(x, H(y, z))$ for all $x, y \in A$, $z \in B$, then H is alternating associative on (A, A, B).

(iii) If $H(H(x, y), z) = H(x, H(y, z))$ for all $x \in A$, $y, z \in B$, then H is alternating associative on (A, B, B).

Theorem 2.3. *([11]) Let $A_1, A_2, \cdots, A_n$ be nonempty,*

$$S = \bigcup_{i=1}^{n} A_i$$

and H be commutative binary operation on S. Then H associative on S if the following statements hold:

(i) H is alternating associative on (A_i, A_j, A_k) for every combination $\{i, j, k\}$ of size 3 chosen $\{1, 2, \cdots, n\}$.

(ii) H is alternating associative on (A_i, A_i, A_j) for every combination $\{i, j\}$ of size 2 chosen $\{1, 2, \cdots, n\}$.

(iii) H is alternating associative on (A_i, A_j, A_j) for every combination $\{i, j\}$ of size 2 chosen $\{1, 2, \cdots, n\}$.

(iv) H is alternating associative on (A_i, A_i, A_i) for every $i \in \{1, 2, \cdots, n\}$.

Remark 2.4. *It can be easily seen that if (iii) holds, then (iv) also holds in Theorem 2.3.*

Definition 2.5. *([16]) Let $(L, \leq, 0, 1)$ be a bounded lattice. A triangular (co)norm T (briefly t-(co)norm) is a binary operation on L which is commutative, associative, monotone and has neutral element* 1 (0).

We will denote by $\mathcal{T}^a$ and $\mathcal{S}_a$ the set of all t-norms on sublattice $[0, a]$ and the set of all t-conorms on sublattice $[a, 1]$ for $a \in L$, respectively.

Example 2.6. *[16] Let $(L, \leq, 0, 1)$ be a bounded lattice. Two basic t-norms T_W and $T_\wedge$ on a bounded lattice L are respectively given by*

$$T_W(x, y) = \begin{cases} y & \text{if } x = 1, \\ x & \text{if } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$T_\wedge(x, y) = x \wedge y.$$

Two basic t-conorms S_D and $S_\vee$ on a bounded lattice L are respectively given as follows:

$$S_D(x, y) = \begin{cases} y & \text{if } x = 0, \\ x & \text{if } y = 0, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$S_\vee(x, y) = x \vee y.$$

Let T_1 (S_1) and T_2 (S_2) be two t-norms (t-conorms) on L. T_1 (S_1) is called smaller than T_2 (S_2) if for any elements $x, y \in L$, $T_1(x, y) \leq T_2(x, y)$ ($S_1(x, y) \leq S_2(x, y)$). The smallest and greatest t-norms (t-conorms) on a bounded lattice L are given respectively by T_W ($S_\vee$) and $T_\wedge$ (S_D).

Definition 2.7. *([13, 14]) Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \longrightarrow L$ is called a uninorm on L (shortly a uninorm, if L is fixed) if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.*

Proposition 2.8. *([14]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then the following results hold:*

i) $T_e = U|_{[0,e]^2} : [0, e]^2 \to [0, e]$ *is a t-norm on* $[0, e]^2$.

ii) $S_e = U|_{[e,1]^2} : [e, 1]^2 \to [e, 1]$ *is a t-conorm on* $[e, 1]^2$.

Proposition 2.9. *([14]) Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ and U be a uninorm on L with the neutral element e. Then the following properties hold:*

i) $x \wedge y \leq U(x, y) \leq x \vee y$ *for* $(x, y) \in [0, e] \times [e, 1] \cup [e, 1] \times [0, e]$.

ii) $U(x, y) \leq x$ *for* $(x, y) \in L \times [0, e]$.

iii) $U(x, y) \leq y$ *for* $(x, y) \in [0, e] \times L$.

iv) $U(x, y) \leq x \wedge y$ *for* $(x, y) \in [0, e] \times [0, e]$.

v) $x \leq U(x, y)$ *for* $(x, y) \in L \times [e, 1]$.

vi) $y \leq U(x, y)$ *for* $(x, y) \in [e, 1] \times L$.

vii) $x \vee y \leq U(x, y)$ *for* $(x, y) \in [e, 1] \times [e, 1]$.

Definition 2.10. *([4]) Let* $(L, \leq, 0, 1)$ *be a bounded lattice. An operation* $U : L^2 \longrightarrow L$ *is called locally internal if it satisfies* $U(x, y) \in \{x, x \wedge y, x \vee y, y\}$.

Proposition 2.11. *([4]) Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $e \in L \backslash \{0, 1\}$ *and* U *be a uninorm on* L *with the neutral element* e. *If* U *is locally internal, then* U *is an idempotent uninorm.*

3. Eight uninorm construction methods on bounded lattices

Since our aim is to investigate the existence and construction methods of uninorms with the neutral element e on the bounded lattices, which satisfy $U(x, y) = e$ for some $x, y \in L \backslash \{e\}$ in addition to the trivial case $U(e, e) = e$. Before doing so, we introduce the notation of unit element that will serve for the construction and analysis of such uninorms.

Definition 3.1. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $x \in L$, $e \in L \backslash \{0, 1\}$ *and* $U : L^2 \rightarrow L$ *be a uninorm with the neutral element* e. x *is called a unit element of* L *if there exist an element* y *of* L *such that* $U(x, y) = e$. *From* $U(e, e) = e$ *for the uninorm* U, e *is a trivial unit element. If* $x \in L \backslash \{e\}$ *is a unit element, then it is called a non-trivial unit element. In addition, if* U *has non-trivial unit elements, it is called a uninorm with non-trivial unit element on* L.

In the following proposition, we find that for which cardinality of I_e the condition $U(x, y) = e$ holds for all $(x, y) \in I_e \times I_e$.

Proposition 3.2. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $e \in L \backslash \{0, 1\}$ *and* $U : L^2 \rightarrow L$ *be a uninorm with the neutral element* e. *If* $U(x, y) = e$ *for all* $(x, y) \in (I_e)^2$, *then* $|I_e| \leq 1$.

Proof. Let us suppose that $I_e \neq \emptyset$ and $x, y \in I_e$. We have that $U(x, U(y, y)) = U(x, e) = x$ and $U(U(x, y), y) = U(e, y) = y$. Then, the associativity of U implies that $x = y$. Consequently, $|I_e| \leq 1$. □

Proposition 3.3. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $e \in L \backslash \{0, 1\}$ *and* $U : L^2 \rightarrow L$ *be a uninorm with the neutral element* e *such that* $U(x, x) = e$ *for all* $x \in I_e$. *If* $a \neq b$ *for* $a, b \in I_e$, *then* $a \parallel b$.

Proof. Let $a, b \in I_e$ with $a \neq b$. Suppose that $a < b$. It is obtained that $b = U(e, b) = U(U(a, a), b) = U(a, U(a, b)) \leq U(b, U(a, b)) = U(U(b, b), a) = U(e, a) = a$ by the properties of U, which contradicts $a < b$. Similarly, it can be shown that $b \nless a$. Hence, it follows that $a \parallel b$. □

Proposition 3.4. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice and* $e \in L \backslash \{0, 1\}$. *There is no uninorm* $U : L^2 \rightarrow L$ *with the neutral element* e *such that* $U(x, y) = e$ *for all* $(x, y) \in [0, e) \times (e, 1]$.

Proof. Assume that $U : L^2 \rightarrow L$ is a uninorm with the neutral element e satisfying $U(x, y) = e$ for all $(x, y) \in [0, e) \times (e, 1]$. From $y \in (e, 1]$, it follows that $e < y = U(e, y) \leq U(y, y)$. Hence $U(y, y) \in (e, 1]$. We have that $U(x, U(y, y)) = e$ and $U(U(x, y), y) = U(e, y) = y$ for $(x, y) \in [0, e) \times (e, 1]$. It is obtained that $y = e$ from the associativity of U, which is a contradiction. □

Proposition 3.5. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $e \in L \backslash \{0, 1\}$ *and* $I_e = \{a, b, c\}$ *such that* $a \parallel b \parallel c$. *Then, there is no uninorm with the neutral element* e *satisfying the conditions* $U(a, a) = U(b, c) = e$, $U(b, b) = c$ *and* $U(c, c) = b$.

Proof. Assume that there exists a uninorm that satisfies the conditions above. If $U(a, b) \leq e$, then $U(a, U(a, b)) \leq U(a, e) = a$. Also, we have $U(a, U(a, b)) = U(U(a, a), b) = b$ by the associativity of U. Consequently, it is obtained that $b \leq a$, which contradicts $a \parallel b$. It follows $U(a, b) \nleq e$. Similarly, it is seen that $U(a, b) \ngeq e$. Finally, let us suppose that $U(a, b) \in I_e$. If $U(a, b) = a$, then $U(a, U(a, b)) = U(a, a) = e$. We also know $U(U(a, a), b) = U(e, b) = b$. Thus, the associativity of U implies $e = b$, which contradicts $b \in I_e$. Similarly, the contradiction is obtained when $U(a, b) \in \{b, c\}$. Consequently there doesn't exist a uninorm that satisfies the conditions above. □

Definition 3.6. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$ and $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$. Denote the class of all uninorms U on L with the neutral element e satisfying the conditions $U(a, a) = U(b, b) = U(c, c) = e$ by $\mathcal{U}^e_{abc}$.*

The following proposition characterizes some properties of U, where $U \in \mathcal{U}^e_{abc}$.

Proposition 3.7. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$ and $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$. If $U \in \mathcal{U}^e_{abc}$, then the following statements holds:*
i) $U(a, b) = c$,
ii) $U(b, c) = a$,
iii) $U(a, c) = b$.

Proof. We only prove that statement (i) holds. First, we show that $U(a, b) \in I_e$. Suppose that $U(a, b) \leq e$ or $U(a, b) > e$. Then, from the monotonicity and associativity of U it follows that $b \leq a$ or $a \leq b$, which is a contradiction. Hence, $U(a, b) \in I_e$, that is, $U(a, b) = a$ or $U(a, b) = b$ or $U(a, b) = c$. We have that $b = U(b, e) = U(b, U(a, a)) = U(a, U(a, b)) = U(a, a) = e$ by the associativity of U when $U(a, b) = a$, which is contradiction. Hence, it is obtained that $U(a, b) \neq a$. In a similar way, we can observe that $U(a, b) \neq b$. Consequently, it must hold that $U(a, b) = c$. □

Theorem 3.8. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$. Define the binary operation $U^{TS}_{abc} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U^{TS}_{abc}(x, y) = \begin{cases} T(x, y) & (x, y) \in [0, e)^2, \\ e & (x, y) \in \{(a, a), (b, b), (c, c)\}, \\ c & (x, y) \in \{(a, b), (b, a)\}, \\ a & (x, y) \in \{(b, c), (c, b)\}, \\ b & (x, y) \in \{(a, c), (c, a)\}, \\ y & (x, y) \in I_e \times ([0, e) \cup (e, 1]) \cup (e, 1] \times [0, e) \cup \{e\} \times L, \\ x & (x, y) \in ([0, e) \cup (e, 1]) \times I_e \cup [0, e) \times (e, 1] \cup L \times \{e\}, \\ S(x, y) & (x, y) \in (e, 1]^2. \end{cases} \tag{1}$$

Then, U^{TS}_{abc} is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Proof. Necessity. Let $x \in [0, e)$, $y \in I_e$. By the monotonicity of U^{TS}_{abc}, we have that $x = U^{TS}_{abc}(x, y) \leq U^{TS}_{abc}(e, y) = y$. By $x \in (e, 1]$, $y \in I_e$, it is trivial that $x \neq y$. Similarly, it can be seen that $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Sufficiency. It is clear that U^{TS}_{abc} is commutative and e is a neutral element of U^{TS}_{abc}. Thus, we demonstrate only the monotonicity and the associativity of U^{TS}_{abc}.

• Monotonicity: We prove that if $y \leq z$ for $y, z \in L$, then $U^{TS}_{abc}(x, y) \leq U^{TS}_{abc}(x, z)$ for all $x \in L$. If $x = e$ or $(y, z) = (e, e)$ or $(y, z) \in [0, e)^2 \cup (e, 1]^2$ or $(y, z) \in I_e \times I_e$ with $y = z$, then the proof is immediate. Thus, let us show that it is valid for all remaining cases.

1. Suppose that $(x, y) \in [0, e)^2$ and $z \in [e, 1] \cup I_e$.

$$U^{TS}_{abc}(x, y) = T(x, y) \leq x = U^{TS}_{abc}(x, z).$$

2. Suppose that $(x, y) \in \{(a, a), (b, b), (c, c)\}$ and $z \in (e, 1]$.

$$U^{TS}_{abc}(x, y) = e < z = U^{TS}_{abc}(x, z).$$

3. Suppose that $(x, y) \in \{(a, b), (b, a)\}$ or $(x, y) \in \{(b, c), (c, b)\}$ or $(x, y) \in \{(a, c), (c, a)\}$, and $z \in (e, 1]$. Take $(x, y) \in \{(a, b), (b, a)\}$ without losing generality.

$$U^{TS}_{abc}(x, y) = c < z = U^{TS}_{abc}(x, z).$$

4. Suppose that $(x, y) \in \{a, b, c\} \times [0, e)$. Take $x = a$ without losing generality.

4.1. If $z = e$, then

$$U^{TS}_{abc}(x, y) = y < a = U^{TS}_{abc}(x, z).$$

4.2. If $z \in (e, 1]$, then

$$U^{TS}_{abc}(x, y) = y < z = U^{TS}_{abc}(x, z).$$

4.3. If $z = a$, then

$$U^{TS}_{abc}(x, y) = y < e = U^{TS}_{abc}(x, z).$$

4.4. If $z = b$, then

$$U^{TS}_{abc}(x, y) = y < c = U^{TS}_{abc}(x, z).$$

4.5. If $z = c$, then

$$U^{TS}_{abc}(x, y) = y < b = U^{TS}_{abc}(x, z).$$

5. Suppose that $(x, y) \in [0, e) \times (I_e \cup \{e\})$ and $z \in (e, 1]$.

$$U^{TS}_{abc}(x, y) = x = U^{TS}_{abc}(x, z).$$

6. Suppose that $(x, y) \in (e, 1] \times (I_e \cup \{e\})$ and $z \in (e, 1]$.

$$U^{TS}_{abc}(x, y) = x \leq S(x, z) = U^{TS}_{abc}(x, z).$$

7. Suppose that $(x, y) \in (e, 1] \times [0, e)$.

7.1. If $z \in (e, 1]$, then

$$U^{TS}_{abc}(x, y) = y < x \leq S(x, z) = U^{TS}_{abc}(x, z).$$

7.2. If $z \in I_e \cup \{e\}$, then

$$U^{TS}_{abc}(x, y) = y < x = U^{TS}_{abc}(x, z).$$

8. Suppose that $(x, y) \in I_e \times \{e\}$ and $z \in (e, 1]$.

$$U^{TS}_{abc}(x, y) = x < z = U^{TS}_{abc}(x, z).$$

• Associativity: We prove that $U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(x, U^{TS}_{abc}(y, z))$ for all $x, y, z \in L$. If $e \in \{x, y, z\}$, the proof is immediate. By Theorems 2.2 and 2.3 considering $L \backslash \{e\} = \{a\} \cup \{b\} \cup \{c\} \cup [0, e) \cup (e, 1]$ we show the associativity of U^{TS}_{abc}. If $x = y = z = a$ or $x = y = z = b$ or $x = y = z = c$ or $x, y, z \in [0, e)^2$ or $x, y, z \in (e, 1]^2$, then the proof is clear. Thus, we only check the following cases:

1. Let $x = a$, $y = b$ and $z = c$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(U^{TS}_{abc}(a, b), c) = U^{TS}_{abc}(c, c) = e,$$

$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(a, U^{TS}_{abc}(b, c)) = U^{TS}_{abc}(a, a) = e,$$

$$U^{TS}_{abc}(U^{TS}_{abc}(x, z), y) = U^{TS}_{abc}(U^{TS}_{abc}(a, c), b) = U^{TS}_{abc}(b, b) = e.$$

2. Let $x = a$, $y = b$ or $y = c$, and $z \in [0, e) \cup (e, 1]$. Take $y = b$ without losing generality.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(U^{TS}_{abc}(a, b), z) = U^{TS}_{abc}(c, z) = z,$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(a, U^{TS}_{abc}(b, z)) = U^{TS}_{abc}(a, z) = z,$$
$$U^{TS}_{abc}(U^{TS}_{abc}(x, z), y) = U^{TS}_{abc}(U^{TS}_{abc}(a, z), b) = U^{TS}_{abc}(z, b) = z.$$

3. Let $x = b$, $y = c$, and $z \in [0, e) \cup (e, 1]$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(U^{TS}_{abc}(b, c), z) = U^{TS}_{abc}(a, z) = z,$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(b, U^{TS}_{abc}(c, z)) = U^{TS}_{abc}(b, z) = z,$$
$$U^{TS}_{abc}(U^{TS}_{abc}(x, z), y) = U^{TS}_{abc}(U^{TS}_{abc}(b, z), c) = U^{TS}_{abc}(z, c) = z.$$

4. Let $x \in I_e$, $y \in (e, 1]$, $z \in [0, e)$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(y, z) = z,$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(x, z) = z,$$
$$U^{TS}_{abc}(U^{TS}_{abc}(x, z), y) = U^{TS}_{abc}(z, y) = z.$$

5. Let $x = y = a$ and $z = b$ or $z = c$. Take $z = b$ without losing generality.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(U^{TS}_{abc}(a, a), b) = U^{TS}_{abc}(e, b) = b,$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(a, U^{TS}_{abc}(a, b)) = U^{TS}_{abc}(a, c) = b.$$

6. Let $x = y = b$ and $z = a$ or $z = c$. Similar to item (V).

7. Let $x = y = c$ and $z = a$ or $z = b$. Similar to item (V).

8. Let $x = y \in I_e$ and $z \in [0, e) \cup (e, 1]$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(e, z) = z,$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(x, z) = z.$$

9. Let $x, y \in [0, e)$ and $z \in I_e \cup (e, 1]$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(T(x, y), z) = T(x, y),$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(x, y) = T(x, y).$$

10. Let $x, y \in (e, 1]$ and $z \in I_e$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(S(x, y), z) = S(x, y),$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(x, y) = S(x, y).$$

11. Let $x, y \in (e, 1]$ and $z \in [0, e)$.

$$U^{TS}_{abc}(U^{TS}_{abc}(x, y), z) = U^{TS}_{abc}(S(x, y), z) = z,$$
$$U^{TS}_{abc}(x, U^{TS}_{abc}(y, z)) = U^{TS}_{abc}(x, z) = z.$$

□

In particularly, if $T = T_W$ and $S = S_\vee$ in Theorem 3.8, then the following corollary can be obtained.

Corollary 3.9. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$. Define the binary operation $U_{abc}^{T_W S_\vee} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_{abc}^{T_W S_\vee}(x,y) = \begin{cases} 0 & (x,y) \in [0,e)^2, \\ e & (x,y) \in \{(a,a),(b,b),(c,c)\}, \\ c & (x,y) \in \{(a,b),(b,a)\}, \\ a & (x,y) \in \{(b,c),(c,b)\}, \\ b & (x,y) \in \{(a,c),(c,a)\}, \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup (e,1] \times [0,e) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup [0,e) \times (e,1] \cup L \times \{e\}, \\ x \vee y & (x,y) \in (e,1]^2. \end{cases} \tag{2}$$

Then, $U_{abc}^{T_W S_\vee}$ is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

If we consider the construction method in Theorem 3.8 , the t-norm $T \in \mathcal{T}^e$ given by $T = T_W$ and the t-conorm $S \in \mathcal{S}_e$ given by $S = S_\vee$, then we obtain the following result that yields the smallest element of the set of all uninorms on L which possesses the following properties.

Proposition 3.10. *$(L, \leq, 0, 1)$ be a bounded lattice, $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$, $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$. If $L\backslash I_e$ is a chain, then the smallest element of $\mathcal{U}_{abc}^e$ is $U_{abc}^{T_W S_\vee}$.*

Proof. Let $U \in \mathcal{U}_{abc}^e$ be an arbitrary uninorm on L. We show that $U_{abc}^{T_W S_\vee}(x, y) \leq U(x, y)$ for all $(x, y) \in L \times L$. When $y = e$, the proof is clear. Then, considering the commutative property of U and $U_{abc}^{T_W S_\vee}$, we only examine the following cases.

- Let $(x, y) \in [0, e)^2$. Then, $U_{abc}^{T_W S_\vee}(x, y) = 0 \leq U(x, y)$.
- Let $(x, y) \in (e, 1]^2$. Then, $U_{abc}^{T_W S_\vee}(x, y) = x \vee y$. Also, we know that $x \vee y \leq U(x, y)$ by Proposition 2.9 (vii). Hence, $U_{abc}^{T_W S_\vee}(x, y) = x \vee y \leq U(x, y)$.
- Let $(x, y) \in [0, e) \times (e, 1]$. Then, $U_{abc}^{T_W S_\vee}(x, y) = x \leq U(x, y)$ by Proposition 2.9 (vii).
- Let $(x, y) \in I_e \times I_e$. Due to the definition of $U_{abc}^{T_W S_\vee}$ and Proposition 3.7, it follows that $U_{abc}^{T_W S_\vee}(x, y) = U(x, y)$.
- Let $(x, y) \in I_e \times ([0, e) \cup (e, 1])$. Take $x = a$ without losing generality and $y < e$. We obtain that $U(a, y) \leq U(a, e) = a$ from the monotonocity of U. If $U(a, y) = a$, then $y = e$, which is a contradiction. So, it is obtained that $U(a, y) < a$. Hence, we have $U(a, y) \nparallel y$ from the fact that $L\backslash I_e$ is a chain. Let us suppose that $U(a, y) \neq y$. It follows that $U(a, y) < y$ or $U(a, y) > y$. If $U(a, y) < y$, then from the monotonocity and the associativity of U, we obtain that $y = U(e, y) = U(U(a, a), y) = U(a, U(a, y)) \leq U(a, y)$, which is a contradiction. Similarly, it can be seen that that $U(a, y) \nless y$. Hence, we have that $U(a, y) = y$.

In summary, it is obtained that $U_{abc}^{T_W S_\vee}(a, y) = y = U(a, y)$ when $x = a$ and $y < e$. Analogously, we can show that $U_{abc}^{T_W S_\vee}(a, y) = y = U(a, y)$ for $y > e$. Consequently, $U_{abc}^{T_W S_\vee}(x, y) \leq U(x, y)$ if $(x, y) \in I_e \times ([0, e) \cup (e, 1])$. □

The next theorem gives a new construction method for uninorm which is an element of the set $\mathcal{U}_{abc}^e$.

Theorem 3.11. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$. Define the binary operation $U_{abc}^{1TS} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_{abc}^{1TS}(x,y) = \begin{cases} T(x,y) & (x,y) \in [0,e)^2, \\ e & (x,y) \in \{(a,a),(b,b),(c,c)\}, \\ c & (x,y) \in \{(a,b),(b,a)\}, \\ a & (x,y) \in \{(b,c),(c,b)\}, \\ b & (x,y) \in \{(a,c),(c,a)\}, \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x,y) \in (e,1] \times [0,e) \cup [0,e) \times (e,1], \\ S(x,y) & (x,y) \in (e,1]^2. \end{cases} \tag{3}$$

Then, U_{abc}^{1TS} is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

In particularly, if $T = T_\wedge$ and $S = S_D$ in Theorem 3.11, then the following corollary can be obtained.

Corollary 3.12. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$. Define the binary operation $U_{abc}^{1T_\wedge S_D} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_{abc}^{1T_\wedge S_D}(x, y) = \begin{cases} x \wedge y & (x, y) \in [0, e)^2, \\ e & (x, y) \in \{(a, a), (b, b), (c, c)\}, \\ c & (x, y) \in \{(a, b), (b, a)\}, \\ a & (x, y) \in \{(b, c), (c, b)\}, \\ b & (x, y) \in \{(a, c), (c, a)\}, \\ y & (x, y) \in I_e \times ([0, e) \cup (e, 1]) \cup \{e\} \times L, \\ x & (x, y) \in ([0, e) \cup (e, 1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x, y) \in (e, 1] \times [0, e) \cup [0, e) \times (e, 1], \\ 1 & (x, y) \in (e, 1]^2. \end{cases} \tag{4}$$

Then, $U_{abc}^{1T_\wedge S_D}$ is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

The following proposition presents the greatest element of $\mathcal{U}_{abc}^e$ on certain bounded lattices.

Proposition 3.13. *$(L, \leq, 0, 1)$ be a bounded lattice, $I_e = \{a, b, c\}$ such that $a \parallel b \parallel c$, $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$. If $L\backslash I_e$ is a chain, then the greatest element of $\mathcal{U}_{abc}^e$ is $U_{abc}^{1T_\wedge S_D}$.*

In the following example, we first present a lattice L_1 with a t-norm $T = T_\wedge$ and a t-conorm $S = S_\vee$. Next if we take in Theorem 3.8 with $a = x_3$, $b = x_4$, $c = x_5$, the uninorm $U_{x_3x_4x_5}^{T_\wedge S_\vee}$ can be obtained as in Table 1.

Example 3.14. *Consider the bounded lattice $(L_1 = \{0, x_1, x_2, x_3, x_4, x_5, e, x_6, x_7, x_8, x_9, x_{10}, x_{11}, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 1.*

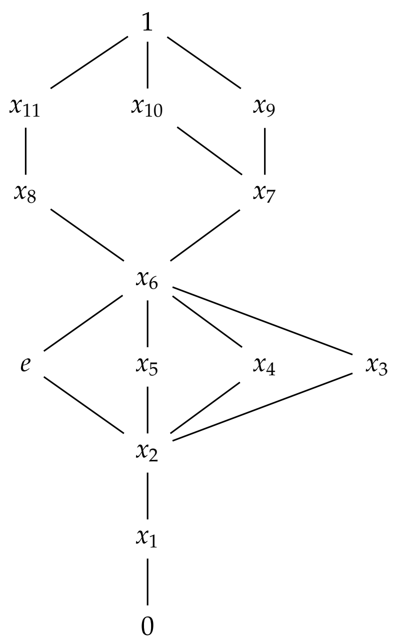

Figure 1: Lattice diagram of L_1.

If we apply the formula (1) in Theorem 3.8 with $a = x_3$, $b = x_4$, $c = x_5$ and take that $T = T_\wedge$, $S = S_\vee$, the corresponding uninorm $U^{T_\wedge S_\vee}_{x_3x_4x_5}$ is given in Table 1.

$U^{T_\wedge S_\vee}_{x_3x_4x_5}$	0	x_1	x_2	x_3	x_4	x_5	e	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1	0	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	0	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
x_3	0	x_1	x_2	e	x_5	x_4	x_3	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_4	0	x_1	x_2	x_5	e	x_3	x_4	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_5	0	x_1	x_2	x_4	x_3	e	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
e	0	x_1	x_2	x_3	x_4	x_5	e	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_6	0	x_1	x_2	x_6	x_6	x_6	x_6	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_7	0	x_1	x_2	x_7	x_7	x_7	x_7	x_7	x_7	1	x_9	x_{10}	1	1
x_8	0	x_1	x_2	x_8	x_8	x_8	x_8	x_8	1	x_8	1	1	x_{11}	1
x_9	0	x_1	x_2	x_9	x_9	x_9	x_9	x_9	x_9	1	x_9	1	1	1
x_{10}	0	x_1	x_2	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	1	1	x_{10}	1	1
x_{11}	0	x_1	x_2	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	1	x_{11}	1	1	x_{11}	1
1	0	x_1	x_2	1	1	1	1	1	1	1	1	1	1	1

Table 1: The uninorm $U^{T_\wedge S_\vee}_{x_3x_4x_5}$ on L_1.

By applying the formula (3) in Theorem 3.11 with $a = x_3$, $b = x_4$, $c = x_5$ and $T = T_\wedge$, $S = S_\vee$, the uninorm $U^{1T_\wedge S_\vee}_{x_3x_4x_5}$ can be obtained as in Table 2.

$U^{1T_\wedge S_\vee}_{x_3x_4x_5}$	0	x_1	x_2	x_3	x_4	x_5	e	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
0	0	0	0	0	0	0	0	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_1	0	x_1	x_1	x_1	x_1	x_1	x_1	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_2	0	x_1	x_2	x_2	x_2	x_2	x_2	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_3	0	x_1	x_2	e	x_5	x_4	x_3	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_4	0	x_1	x_2	x_5	e	x_3	x_4	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_5	0	x_1	x_2	x_4	x_3	e	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
e	0	x_1	x_2	x_3	x_4	x_5	e	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_6	x_6	x_6	x_6	x_6	x_6	x_6	x_6	x_6	x_7	x_8	x_9	x_{10}	x_{11}	1
x_7	x_7	x_7	x_7	x_7	x_7	x_7	x_7	x_7	x_7	1	x_9	x_{10}	1	1
x_8	x_8	x_8	x_8	x_8	x_8	x_8	x_8	x_8	1	x_8	1	1	x_{11}	1
x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	1	x_9	1	1	1
x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	1	1	x_{10}	1	1
x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	1	x_{11}	1	1	x_{11}	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 2: The uninorm $U^{1T_\wedge S_\vee}_{x_3x_4x_5}$ on L_1.

.

Remark 3.15. *It is easy to check that uninorms U^{TS}_{abc} and U^{1TS}_{abc} given in Tables 1 and 2 are different from each other.*

Definition 3.16. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$ and $I_e = \{a, b\}$ such that $a \parallel b$. Denote the class of all uninorms U on L with the neutral element e satisfying the condition $U(a, b) = e$ by $\mathcal{U}^e_{ab}$.*

The following proposition characterizes some properties of U, where $U \in \mathcal{U}^e_{ab}$. Also, it will be used in the proof of Proposition 3.22.

Proposition 3.17. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$ and $I_e = \{a, b\}$ such that $a \parallel b$. If $U \in \mathcal{U}^e_{ab}$, then the following statements holds:*
i) $U(a, a) = b$,
ii) $U(b, b) = a$.

Proof. i) First, we prove that $U(a, a) \parallel e$. Let us suppose that $U(a, a) \nparallel e$. Then, either $U(a, a) \leq e$ or $U(a, a) > e$. If $U(a, a) \leq e$, then $U(U(a, a), b) \leq U(e, b) = b$ from the monotonicity of U. Also, it is easy to obtain that $U(U(a, a), b) = a$ by the associativity of U. Hence, $a \leq b$, contradicting $a \parallel b$. Then, $U(a, a) \nleq e$. Similarly, verify $U(a, a) \ngtr e$. Then, $U(a, a) \parallel e$. Suppose that $U(a, a) = a$. It follows that $a = U(e, a) = U(U(b, a), a) = U(b, U(a, a)) = U(b, a) = e$ by the properties of U. Consequently, we obtain $a = e$, which is a contradiction. Then, this implies that $U(a, a) = b$.

ii) It can be proven easily in a smilar way to (i). □

In the following, by Theorem 3.18 and Theorem 3.23, we investigate the existence of uninorms with non-trivial unit element under which some conditions when $| I_e |= 2$.

Theorem 3.18. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a, b\}$ such that $a \parallel b$. Define the binary operation $U^{TS}_{ab} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U^{TS}_{ab}(x, y) = \begin{cases} T(x, y) & (x, y) \in [0, e)^2, \\ e & (x, y) = (a, b) \text{ or } (x, y) = (b, a), \\ b & (x, y) = (a, a), \\ a & (x, y) = (b, b), \\ y & (x, y) \in I_e \times ([0, e) \cup (e, 1]) \cup (e, 1] \times [0, e) \cup \{e\} \times L, \\ x & (x, y) \in ([0, e) \cup (e, 1]) \times I_e \cup [0, e) \times (e, 1] \cup L \times \{e\}, \\ S(x, y) & (x, y) \in (e, 1]^2. \end{cases} \tag{5}$$

Then, U^{TS}_{ab} is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Proof. Necessity. Let $x \in [0, e)$, $y \in I_e$. By the monotonicity of U^{TS}_{ab}, we have that $x = U^{TS}_{ab}(x, y) \leq U^{TS}_{ab}(e, y) = y$. It is trivial that $x \neq y$ from $x \in (e, 1]$, $y \in I_e$. Hence, $x < y$ holds. Dually it can be easily obtained other case.

Sufficiency. We can see that U^{TS}_{ab} is commutative and e is a neutral element of U^{TS}_{ab}. Hence, we show only the monotonicity and the associativity of U^{TS}_{ab}.

• Monotonicity: We prove that if $y \leq z$ for $y, z \in L$, then $U^{TS}_{ab}(x, y) \leq U^{TS}_{ab}(x, z)$ for all $x \in L$. If $x = e$ or $(y, z) = (e, e)$ or $(y, z) \in [0, e)^2 \cup (e, 1]^2$ or $(y, z) \in I_e \times I_e$ with $y = z$, then the proof is immediate. Hence, we consider only remain the following cases.

1. Suppose that $(x, y) \in [0, e)^2$ and $z \in [e, 1] \cup I_e$.

$$U^{TS}_{ab}(x, y) = T(x, y) \leq x = U^{TS}_{ab}(x, z).$$

2. Suppose that $(x, y) = (a, b)$ or $(x, y) = (b, a)$, and $z \in (e, 1]$.

$$U^{TS}_{ab}(x, y) = e < z = U^{TS}_{ab}(x, z).$$

3. Suppose that $(x, y) = (a, a)$ or $(x, y) = (b, b)$, and $z \in (e, 1]$. Take $(x, y) = (a, a)$ without losing generality.

$$U^{TS}_{ab}(x, y) = b < z = U^{TS}_{ab}(x, z).$$

4. Suppose that $(x, y) \in \{a\} \times [0, e)$ or $(x, y) \in \{b\} \times [0, e)$. Take $(x, y) \in \{a\} \times [0, e)$ without losing generality.

   4.1. If $z = e$, then

$$U^{TS}_{ab}(x, y) = y < x = U^{TS}_{ab}(x, z).$$

4.2. If $z \in (e, 1]$, then

$$U_{ab}^{TS}(x, y) = y < z = U_{ab}^{TS}(x, z).$$

4.3. If $z = a$, then

$$U_{ab}^{TS}(x, y) = y < b = U_{ab}^{TS}(x, z).$$

4.4. If $z = b$, then

$$U_{ab}^{TS}(x, y) = y < e = U_{ab}^{TS}(x, z).$$

5. Suppose that $(x, y) \in (e, 1] \times [0, e)$.

5.1. If $z \in (e, 1]$, then

$$U_{ab}^{TS}(x, y) = y < x \leq S(x, z) = U_{ab}^{TS}(x, z).$$

5.2. If $z \in I_e \cup \{e\}$, then

$$U_{ab}^{TS}(x, y) = y < x = U_{ab}^{TS}(x, z).$$

6. Suppose that $(x, y) \in [0, e) \times (I_e \cup \{e\})$ and $z \in (e, 1]$.

$$U_{ab}^{TS}(x, y) = x = U_{ab}^{TS}(x, z).$$

7. Suppose that $(x, y) \in (e, 1] \times (I_e \cup \{e\})$ and $z \in (e, 1]$.

$$U_{ab}^{TS}(x, y) = x \leq S(x, z) = U_{ab}^{TS}(x, z).$$

8. Suppose that $(x, y) \in I_e \times \{e\}$ and $z \in (e, 1]$.

$$U_{ab}^{TS}(x, y) = x < z = U_{ab}^{TS}(x, z).$$

• Associativity: We prove that $U_{ab}^{TS}(U_{ab}^{TS}(x, y), z) = U_{ab}^{TS}(x, U_{ab}^{TS}(y, z))$ for all $x, y, z \in L$. The proof is clear when $e \in \{x, y, z\}$ since e is the neutral element of U_{ab}^{TS}. Let us consider Theorems 2.2 and 2.3 and $L \backslash \{e\} = \{a\} \cup \{b\} \cup [0, e) \cup (e, 1]$. If $x = y = z = a$ or $x = y = z = b$ or $x, y, z \in [0, e)^2$ or $x, y, z \in (e, 1]^2$, then the proof is clear. Hence, we only check the remain following cases:

1. Let $x = a$, $y = b$ and $z \in [0, e) \cup (e, 1]$.

$$U_{ab}^{TS}(U_{ab}^{TS}(x, y), z) = U_{ab}^{TS}(U_{ab}^{TS}(a, b), z) = U_{ab}^{TS}(e, z) = z,$$

$$U_{ab}^{TS}(x, U_{ab}^{TS}(y, z)) = U_{ab}^{TS}(a, U_{ab}^{TS}(b, z)) = U_{ab}^{TS}(a, z) = z,$$

$$U_{ab}^{TS}(U_{ab}^{TS}(x, z), y) = U_{ab}^{TS}(U_{ab}^{TS}(a, z), b) = U_{ab}^{TS}(z, b) = z.$$

2. Let $x \in I_e$, $y \in (e, 1]$ and $z \in [0, e)$.

$$U_{ab}^{TS}(U_{ab}^{TS}(x, y), z) = U_{ab}^{TS}(y, z) = z,$$

$$U_{ab}^{TS}(x, U_{ab}^{TS}(y, z)) = U_{ab}^{TS}(x, z) = z,$$

$$U_{ab}^{TS}(U_{ab}^{TS}(x, z), y) = U_{ab}^{TS}(z, y) = z.$$

3. Let $x = y = a$ and $z = b$, or $x = y = b$ and $z = a$. Take $x = y = a$ and $z = b$ without losing generality.

$$U_{ab}^{TS}(U_{ab}^{TS}(x,y),z) = U_{ab}^{TS}(U_{ab}^{TS}(a,a),b) = U_{ab}^{TS}(b,b) = a,$$

$$U_{ab}^{TS}(x,U_{ab}^{TS}(y,z)) = U_{ab}^{TS}(a,U_{ab}^{TS}(a,b)) = U_{ab}^{TS}(a,e) = a.$$

4. Let $x = y = a$ or $x = y = b$, and $z \in [0,e) \cup (e,1]$. Take $x = y = a$ without losing generality.

$$U_{ab}^{TS}(U_{ab}^{TS}(x,y),z) = U_{ab}^{TS}(U_{ab}^{TS}(a,a),z) = U_{ab}^{TS}(b,z) = z,$$

$$U_{ab}^{TS}(x,U_{ab}^{TS}(y,z))U_{ab}^{TS}(a,U_{ab}^{TS}(a,z)) = U_{ab}^{TS}(a,z) = z.$$

5. Let $x,y \in [0,e)$ and $z \in I_e \cup (e,1]$.

$$U_{ab}^{TS}(U_{ab}^{TS}(x,y),z) = U_{ab}^{TS}(T(x,y),z) = T(x,y),$$

$$U_{ab}^{TS}(x,U_{ab}^{TS}(y,z)) = U_{ab}^{TS}(x,y) = T(x,y).$$

6. Let $x,y \in (e,1]$ and $z \in I_e$.

$$U_{ab}^{TS}(U_{ab}^{TS}(x,y),z) = U_{ab}^{TS}(S(x,y),z) = S(x,y),$$

$$U_{ab}^{TS}(x,U_{ab}^{TS}(y,z)) = U_{ab}^{TS}(x,y) = S(x,y).$$

7. Let $x,y \in (e,1]$ and $z \in [0,e)$.

$$U_{ab}^{TS}(U_{ab}^{TS}(x,y),z) = U_{ab}^{TS}(S(x,y),z) = z,$$

$$U_{ab}^{TS}(x,U_{ab}^{TS}(y,z)) = U_{ab}^{TS}(x,z) = z.$$

□

Remark 3.19. *In Theorem 3.18, in general, the constraint $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$ cannot be omitted. In the following, we provide an example of a lattice that does not satisfy this condition on which the function U_{ab}^{TS} defined in Theorem 3.18 is not a uninorm. Further we demonstrate that $\mathcal{U}_{ab}^{e} = \emptyset$.*

Example 3.20. *Consider the bounded lattice $(L_2 = \{0,e,x,a,b,1\},\leq,0,1)$ characterized by the Hasse diagram in Figure 2. There is no uninorm $U : L^2 \to L$ with the neutral element e such that $U(a,b) = e$. Suppose that there is such a uninorm U . Then, it is obtained $U(0,b) \leq U(e,b) = b$ by the monotonicity of U. Observe that in this case then either $U(0,b) = 0$ or $U(0,b) = b$. Also, it follows $U(x,a) = 0$ or $U(x,a) = a$ from $U(x,a) \leq U(e,a) = a$. Let $U(x,a) = 0$. We have $U(x,U(a,b)) = U(x,e) = x$ and $U(U(x,a),b) = U(0,b)$. It is obtained $x = U(0,b)$, which is a contradiction. Similarly, when $U(x,a) = a$, we obtain a contradiction. Hence, $\mathcal{U}_{ab}^{e} = \emptyset$.*

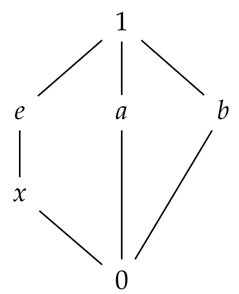

Figure 2: Lattice diagram of L_2.

In particularly, if $T = T_W$ and $S = S_\vee$ in Theorem 3.18, then the following corollary can be obtained.

Corollary 3.21. *Let* $(L, \leq, 0, 1)$ *be a bounded lattice,* $e \in L\backslash\{0,1\}$ *and* $I_e = \{a, b\}$ *such that* $a \parallel b$. *Define the binary operation* $U_{ab}^{T_W S_\vee} : L^2 \to L$ *given, for all* $x, y \in L$, *as*

$$U_{ab}^{T_W S_\vee}(x,y) = \begin{cases} 0 & (x,y) \in [0,e)^2, \\ e & (x,y) = (a,b) \text{ or } (x,y) = (b,a), \\ b & (x,y) = (a,a), \\ a & (x,y) = (b,b), \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup (e,1] \times [0,e) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup [0,e) \times (e,1] \cup L \times \{e\}, \\ x \vee y & (x,y) \in (e,1]^2. \end{cases} \quad (6)$$

Then, $U_{ab}^{T_W S_\vee}$ *is a uninorm if and only if* $x < y$ *for all* $x \in [0,e)$, $y \in I_e$ *and* $x > y$ *for all* $x \in (e,1]$, $y \in I_e$.

The following proposition presents the smallest element of $\mathcal{U}_{ab}^e$ on certain bounded lattices.

Proposition 3.22. $(L, \leq, 0, 1)$ *be a bounded lattice,* $I_e = \{a, b\}$ *such that* $a \parallel b$, $x < y$ *for all* $x \in [0,e)$, $y \in I_e$ *and* $x > y$ *for all* $x \in (e,1]$, $y \in I_e$. *If* $L\backslash I_e$ *is a chain, then the smallest element of* $\mathcal{U}_{ab}^e$ *is* $U_{ab}^{T_W S_\vee}$.

Proof. Let $U \in \mathcal{U}_{ab}^e$ be an arbitrary uninorm on L. We show that $U_{ab}^{T_W S_\vee}(x,y) \leq U(x,y)$ for all $(x,y) \in L \times L$. When $y = e$, the proof is clear. Also, it is proved in a manner similar to Proposition (3.8) when $(x,y) \in [0,e)^2 \cup (e,1]^2 \cup [0,e) \times (e,1]$. Then, considering the commutative property of U and $U_{ab}^{T_W S_\vee}$, we only examine the following cases.

• Let $(x,y) \in I_e \times I_e$. Due to the definition of $U_{ab}^{T_W S_\vee}$ and Proposition 3.17, it follows that $U_{ab}^{T_W S_\vee}(x,y) = U(x,y)$.

• Let $(x,y) \in I_e \times ([0,e) \times (e,1])$. Take $x = a$ without losing generality and $y < e$. We obtain that $U(a,y) \leq U(a,e) = a$ from the monotonocity of U. If $U(a,y) = a$, then $y = e$, which is a contradiction. So, it must be that $U(a,y) < a$. Hence, we have $U(a,y) \nparallel y$ from the fact that $L\backslash I_e$ is a chain.

• When $U(a,y) \leq y$. From the monotonocity and the associativity of U, it follows $U(b,y) = U(U(a,a),y) = U(a,U(a,y)) \leq U(a,y)$. Then, $U(b,y) \leq U(a,y)$. Furthermore, we know that $U(b,y) \leq U(b,e) = b$ from the monotonocity of U. If $U(b,y) = b$, then $y = e$, which is a contradiction. Then, it must be that $U(b,y) < b$. Hence, we have $U(b,y) \nparallel y$ from the fact that $L\backslash I_e$ is a chain.

- Suppose that $U(b,y) \geq y$. It follows $U(b,y) \geq U(a,y)$ that from $U(a,y) \leq y$. We have that $U(a,y) = U(b,y)$ by $U(a,y) \leq U(b,y)$ and $U(b,y) \leq U(a,y)$.

- Suppose that $U(b,y) < y$. It is obtained that $U(a,y) = U(U(b,b),y) = U(b,U(b,y)) \leq U(b,y)$ by the monotonocity and associativity of U. We again have that $U(a,y) = U(b,y)$ by $U(a,y) \leq U(b,y)$ and $U(b,y) \leq U(a,y)$. In summary, we obtain $U(a,y) = U(b,y)$ when $U(b,y) \nparallel y$. Finally, it follows the following equations:

$$\begin{aligned} U(a,U(a,y)) &= U(a,U(b,y)) \\ U(U(a,a),y) &= U(U(a,b),y) \\ U(b,y) &= U(e,y) = y. \end{aligned}$$

Verify that $U(b,y) = y$ implies $U(a,y) = y$. $U_{ab}^{T_W S_\vee}(a,y) = y = U(a,y)$.

• When $U(a,y) > y$. $U_{ab}^{T_W S_\vee}(a,y) = y < U(a,y)$.

Hence, it is obtained that $U_{ab}^{T_W S_\vee}(a,y) = y \leq U(a,y)$ when $x = a$ and $y < e$. In an analogous way, we can show that $U_{ab}^{T_W S_\vee}(a,y) \leq U(a,y)$ for $y > e$. Consequently, $U_{ab}^{T_W S_\vee}(x,y) \leq U(x,y)$ if $(x,y) \in I_e \times ([0,e) \cup (e,1])$.
□

In the following, we will introduce another uninorm construction method on the bounded lattices satisfying the constrains as in Theorem 3.18.

Theorem 3.23. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a, b\}$ such that $a \parallel b$. Define the binary operation $U_{ab}^{1TS} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_{ab}^{1TS}(x,y) = \begin{cases} T(x,y) & (x,y) \in [0,e)^2, \\ e & (x,y) = (a,b) \text{ or } (x,y) = (b,a), \\ b & (x,y) = (a,a), \\ a & (x,y) = (b,b), \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x,y) \in [0,e) \times (e,1] \cup (e,1] \times [0,e), \\ S(x,y) & (x,y) \in (e,1]^2. \end{cases} \tag{7}$$

Then, U_{ab}^{1TS} is a uninorm if and only if $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$.

Proof. The proof is similar to the proof of Theorem 3.18. □

In particularly, if $T = T_\wedge$ and $S = S_D$ in Theorem 3.23, then the following corollary can be obtained.

Corollary 3.24. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$ and $I_e = \{a, b\}$ such that $a \parallel b$. Define the binary operation $U_{ab}^{1T_\wedge S_D} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_{ab}^{1T_\wedge S_D}(x,y) = \begin{cases} x \wedge y & (x,y) \in [0,e)^2, \\ e & (x,y) = (a,b) \text{ or } (x,y) = (b,a), \\ b & (x,y) = (a,a), \\ a & (x,y) = (b,b), \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x,y) \in [0,e) \times (e,1] \cup (e,1] \times [0,e), \\ 1 & (x,y) \in (e,1]^2. \end{cases} \tag{8}$$

Then, $U_{ab}^{1T_\wedge S_D}$ is a uninorm if and only if $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$.

The following proposition is proved in a way dual to Proposition 3.22.

Proposition 3.25. *$(L, \leq, 0, 1)$ be a bounded lattice, $I_e = \{a, b\}$ such that $a \parallel b$, $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$. If $L\backslash I_e$ is a chain, then the greatest element of $\mathcal{U}_{ab}^e$ is $U_{ab}^{1T_\wedge S_D}$.*

The following example demonstrates that the methods described in Theorems 3 and 4 are different on the same bounded lattice.

Example 3.26. *Consider the bounded lattice $(L_3 = \{0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, e, x_9, x_{10}, x_{11}, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 3, $T = T_\wedge$ and $S = S_\vee$. Besides, the lattice L_3 can be seen to easily satisfy the constraints of Theorem 3.18 and Theorem 3.23.*

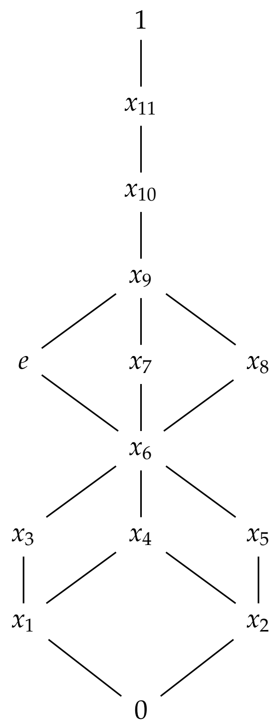

Figure 3: Lattice diagram of L_3.

Take $a = x_7$, $b = x_8$. From the construction method in Theorem 3.18, we obtain the uninorm defined in Table 3.

$U_{x_7x_8}^{T_\wedge S_\vee}$	0	x_1	x_2	x_3	x_4	x_5	x_6	e	x_7	x_8	x_9	x_{10}	x_{11}	1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
x_1	0	x_1	0	x_1	x_1	0	x_1	x_1	x_1	x_1	x_1	x_1	x_1	x_1
x_2	0	0	x_2	0	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
x_3	0	x_1	0	x_3	x_1	0	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
x_4	0	x_1	x_2	x_1	x_4	x_2	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4
x_5	0	0	x_2	0	x_2	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5
x_6	0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_6	x_6	x_6	x_6	x_6	x_6
e	0	x_1	x_2	x_3	x_4	x_5	x_6	e	x_7	x_8	x_9	x_{10}	x_{11}	1
x_7	0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	e	x_9	x_{10}	x_{11}	1
x_8	0	x_1	x_2	x_3	x_4	x_5	x_6	x_8	e	x_7	x_9	x_{10}	x_{11}	1
x_9	0	x_1	x_2	x_3	x_4	x_5	x_6	x_9	x_9	x_9	x_9	x_{10}	x_{11}	1
x_{10}	0	x_1	x_2	x_3	x_4	x_5	x_6	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{11}	1
x_{11}	0	x_1	x_2	x_3	x_4	x_5	x_6	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	1
1	0	x_1	x_2	x_3	x_4	x_5	x_6	1	1	1	1	1	1	1

Table 3: The uninorm$U_{x_7x_8}^{T_\wedge S_\vee}$ on L_3.

By applying the formula (7) in Theorem 3.23, where $a = x_7, b = x_8$, the uninorm $U_{x_7x_8}^{1T_\wedge S_\vee}$ can be obtained as in Table 4.

$U^{1T_\wedge S_\vee}_{x_7x_8}$	0	x_1	x_2	x_3	x_4	x_5	x_6	e	x_7	x_8	x_9	x_{10}	x_{11}	1
0	0	0	0	0	0	0	0	0	0	0	x_9	x_{10}	x_{11}	1
x_1	0	x_1	0	x_1	x_1	0	x_1	x_1	x_1	x_1	x_9	x_{10}	x_{11}	1
x_2	0	0	x_2	0	x_2	x_2	x_2	x_2	x_2	x_2	x_9	x_{10}	x_{11}	1
x_3	0	x_1	0	x_3	x_1	0	x_3	x_3	x_3	x_3	x_9	x_{10}	x_{11}	1
x_4	0	x_1	x_2	x_1	x_4	x_2	x_4	x_4	x_4	x_4	x_9	x_{10}	x_{11}	1
x_5	0	0	x_2	0	x_2	x_5	x_5	x_5	x_5	x_5	x_9	x_{10}	x_{11}	1
x_6	0	x_1	x_2	x_3	x_4	x_5	x_6	x_6	x_6	x_6	x_9	x_{10}	x_{11}	1
e	0	x_1	x_2	x_3	x_4	x_5	x_6	e	x_7	x_8	x_9	x_{10}	x_{11}	1
x_7	0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	e	x_9	x_{10}	x_{11}	1
x_8	0	x_1	x_2	x_3	x_4	x_5	x_6	x_8	e	x_7	x_9	x_{10}	x_{11}	1
x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_9	x_{10}	x_{11}	1
x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{10}	x_{11}	1
x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	x_{11}	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 4: The uninorm $U^{1T_\wedge S_\vee}_{x_7x_8}$ on L_3.

Therefore, it is worth nothing that the uninorms U^{TS}_{ab} and U^{1TS}_{ab} may not be same in general.

Proposition 3.27. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$ and $I_e = \{a, b\}$ such that $a \parallel b$. Then, there is no uninorm $U : L^2 \to L$ with the neutral element e such that $U(a, a) = U(b, b) = e$.*

Proof. Suppose that $U : L^2 \to L$ with the neutral element e such that $U(a, a) = U(b, b) = e$. We claim that $U(a, b) \parallel e$. Suppose that $U(a, b) \leq e$. From the monotonicity, associativity and commutativity of U, it follows that $b = U(b, e) = U(b, U(a, a)) = U(a, U(a, b)) \leq U(a, e) = a$, which contradicts $a \parallel b$. Then, it holds that $U(a, b) \nleq e$. Analogously, we can observe that $U(a, b) \ngeq e$. Hence, it must hold that $U(a, b) \in I_e$. We obtain $U(a, b) = a$ or $U(a, b) = b$ from $U(a, b) \in I_e$. Let us suppose that $U(a, b) = a$. It follows that $U(a, U(a, b)) = U(a, a) = e$ and $b = U(e, b) = U(U(a, a), b)$. By the associativity of U we have that $b = e$, which contradicts $b \in I_e$. Thus, we obtain that $U(a, b) \neq a$. It is similarly observed that $U(a, b) \neq b$. Hence, there is no uninorm $U : L^2 \to L$ with the neutral element e such that $U(a, a) = U(b, b) = e$. □

Definition 3.28. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$ and $I_e = \{a\}$. Denote the class of all uninorms U on L with the neutral element e satisfying the condition $U(a, a) = e$ by $\mathcal{U}^e_a$.*

Next, we introduce four new methods to produce uninorms on bounded lattices, where $| I_e |= 1$ and $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$. We omit their proofs since they can be proven in similar fashion as done in Theorem 3.18.

Theorem 3.29. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a\}$. Define the binary operation $U^{TS}_a : L^2 \to L$ given, for all $x, y \in L$, as*

$$U^{TS}_a(x, y) = \begin{cases} T(x, y) & (x, y) \in [0, e)^2, \\ e & (x, y) \in I_e \times I_e, \\ y & (x, y) \in I_e \times ([0, e) \cup (e, 1]) \cup (e, 1] \times [0, e) \cup \{e\} \times L, \\ x & (x, y) \in ([0, e) \cup (e, 1]) \times I_e \cup [0, e) \times (e, 1] \cup L \times \{e\}, \\ S(x, y) & (x, y) \in (e, 1]^2. \end{cases} \tag{9}$$

Then, U^{TS}_a is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Example 3.30. *Consider the bounded lattice $(L_4 = \{0, x_1, x_2, e, x_3, x_4, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 4, $T = T_\wedge$ and $S = S_\vee$.*

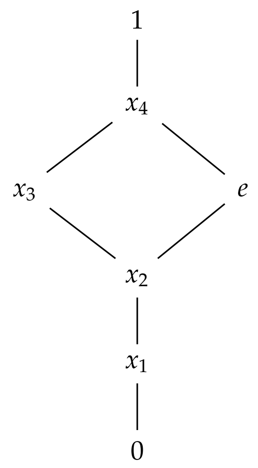

Figure 4: Lattice diagram of L_4.

By applying the formula (9) in Theorem 3.29 with $a = x_3$, the uninorm $U_{x_3}^{T_\wedge S_\vee}$ can be obtained as in Table 5.

$U_{x_3}^{T_\wedge S_\vee}$	0	x_1	x_2	e	x_3	x_4	1
0	0	0	0	0	0	0	0
x_1	0	x_1	x_1	x_1	x_1	x_1	x_1
x_2	0	x_1	x_2	x_2	x_2	x_2	x_2
e	0	x_1	x_2	e	x_3	x_4	1
x_3	0	x_1	x_2	x_3	e	x_4	1
x_4	0	x_1	x_2	x_4	x_4	x_4	1
1	0	x_1	x_2	1	1	1	1

Table 5: The uninorm $U_{x_3}^{T_\wedge S_\vee}$ on L_4.

In particularly, if $T = T_W$ and $S = S_\vee$ in Theorem 3.29, then the following corollary can be obtained.

Corollary 3.31. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a\}$. Define the binary operation $U_a^{T_W S_\vee} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_a^{T_W S_\vee}(x, y) = \begin{cases} 0 & (x, y) \in [0, e)^2, \\ e & (x, y) \in I_e \times I_e, \\ y & (x, y) \in I_e \times ([0, e) \cup (e, 1]) \cup (e, 1] \times [0, e) \cup \{e\} \times L, \\ x & (x, y) \in ([0, e) \cup (e, 1]) \times I_e \cup [0, e) \times (e, 1] \cup L \times \{e\}, \\ x \vee y & (x, y) \in (e, 1]^2. \end{cases} \tag{10}$$

Then, $U_a^{T_W S_\vee}$ is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Proposition 3.32. *$(L, \leq, 0, 1)$ be a bounded lattice, $I_e = \{a\}$, $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$. If $L\backslash I_e$ is a chain, then the smallest element of $\mathcal{U}_a^e$ is $U_a^{T_W S_\vee}$.*

Proof. The proof can be shown in a manner similar to that of Proposition 3.22. □

Theorem 3.33. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a\}$. Define the binary operation $U_a^{1TS} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_a^{1TS}(x,y) = \begin{cases} T(x,y) & (x,y) \in [0,e)^2, \\ e & (x,y) \in I_e \times I_e, \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x,y) \in (e,1] \times [0,e) \cup [0,e) \times (e,1], \\ S(x,y) & (x,y) \in (e,1]^2. \end{cases} \quad (11)$$

Then, U_a^{1TS} is a uninorm if and only if $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$.

In particularly, if $T = T_\wedge$ and $S = S_D$ in Theorem 3.33, then the following corollary can be obtained.

Corollary 3.34. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a\}$. Define the binary operation $U_a^{1T_\wedge S_D} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_a^{1T_\wedge S_D}(x,y) = \begin{cases} x \wedge y & (x,y) \in [0,e)^2, \\ e & (x,y) \in I_e \times I_e, \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x,y) \in (e,1] \times [0,e) \cup [0,e) \times (e,1], \\ 1 & (x,y) \in (e,1]^2. \end{cases} \quad (12)$$

Then, $U_a^{1T_\wedge S_D}$ is a uninorm if and only if $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$.

Proposition 3.35. *$(L, \leq, 0, 1)$ be a bounded lattice, $I_e = \{a\}$, $x < y$ for all $x \in [0,e)$, $y \in I_e$ and $x > y$ for all $x \in (e,1]$, $y \in I_e$. If $L\backslash I_e$ is a chain, then the greatest element of $\mathcal{U}_a^e$ is $U_a^{1T_\wedge S_D}$.*

Proof. The proof can be shown in a manner similar to that of Proposition 3.22. □

Example 3.36. *Consider the bounded lattice $(L_4 = \{0, x_1, x_2, e, x_3, x_4, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 4. By applying the formula (11) in Theorem 3.33 with $a = x_3$, the uninorm $U_{x_3}^{1T_\wedge S_\vee}$ can be obtained as in Table 6.*

$U_{x_3}^{1T_\wedge S_\vee}$	0	x_1	x_2	e	x_3	x_4	1
0	0	0	0	0	0	x_4	1
x_1	0	x_1	x_1	x_1	x_1	x_4	1
x_2	0	x_1	x_2	x_2	x_2	x_4	1
e	0	x_1	x_2	e	x_3	x_4	1
x_3	0	x_1	x_2	x_3	e	x_4	1
x_4	x_4	x_4	x_4	x_4	x_4	x_4	1
1	1	1	1	1	1	1	1

Table 6: The uninorm $U_{x_3}^{1T_\wedge S_\vee}$ on L_4.

Definition 3.37. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0,1\}$ and $I_e = \{a\}$. Denote the class of all uninorms U on L with the neutral element e satisfying the condition $U(a,a) = a$ by $\mathcal{U}_a^a$.*

In the following, we get the following Theorems 3.38 and 3.41 satisfying this condition.

Theorem 3.38. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a\}$. Define the binary operation $U_a^{2TS} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_a^{2TS}(x,y) = \begin{cases} T(x,y) & (x,y) \in [0,e)^2, \\ a & (x,y) \in I_e \times I_e, \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup (e,1] \times [0,e) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup [0,e) \times (e,1] \cup L \times \{e\}, \\ S(x,y) & (x,y) \in (e,1]^2. \end{cases} \quad (13)$$

Then, U_a^{2TS} is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Example 3.39. *Consider the bounded lattice $(L_4 = \{0, x_1, x_2, e, x_3, x_4, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 4, $T = T_\wedge$ and $S = S_\vee$. By using the construction approach in Theorem 3.38 with $a = x_3$, we find the uninorm $U_{x_3}^{2T_\wedge S_\vee}$ defined in Table 7.*

$U_{x_3}^{2T_\wedge S_\vee}$	0	x_1	x_2	e	x_3	x_4	1
0	0	0	0	0	0	0	0
x_1	0	x_1	x_1	x_1	x_1	x_1	x_1
x_2	0	x_1	x_2	x_2	x_2	x_2	x_2
e	0	x_1	x_2	e	x_3	x_4	1
x_3	0	x_1	x_2	x_3	x_3	x_4	1
x_4	0	x_1	x_2	x_4	x_4	x_4	1
1	0	x_1	x_2	1	1	1	1

Table 7: The uninorm $U_{x_3}^{2T_\wedge S_\vee}$ on L_4.

Remark 3.40. *Due to the Tables 5, 6 and 7, it is easy to check that the uninorms U_a^{TS}, U_a^{1TS} and U_a^{2TS} are different from each other.*

Theorem 3.41. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$, $T \in \mathcal{T}^e$, $S \in \mathcal{S}_e$ and $I_e = \{a\}$. Define the binary operation $U_a^{3TS} : L^2 \to L$ given, for all $x, y \in L$, as*

$$U_a^{3TS}(x,y) = \begin{cases} T(x,y) & (x,y) \in [0,e)^2, \\ a & (x,y) \in I_e \times I_e, \\ y & (x,y) \in I_e \times ([0,e) \cup (e,1]) \cup \{e\} \times L, \\ x & (x,y) \in ([0,e) \cup (e,1]) \times I_e \cup L \times \{e\}, \\ x \vee y & (x,y) \in (e,1] \times [0,e) \cup [0,e) \times (e,1], \\ S(x,y) & (x,y) \in (e,1]^2. \end{cases} \quad (14)$$

Then, U_a^{3TS} is a uninorm if and only if $x < y$ for all $x \in [0, e)$, $y \in I_e$ and $x > y$ for all $x \in (e, 1]$, $y \in I_e$.

Example 3.42. *Consider the bounded lattice $(L_4 = \{0, x_1, x_2, e, x_3, x_4, 1\}, \leq, 0, 1)$ characterized by the Hasse diagram in Figure 4. We exploit the construction method in Theorem 3.41 with $a = x_3$ to construct a uninorm such that $U(a, a) = a$ and we choose $T = T_\wedge$ and $S = S_\vee$. The uninorm $U_{x_3}^{3T_\wedge S_\vee}$ is listed in Table 8.*

$U_{x_3}^{3T_\wedge S_\vee}$	0	x_1	x_2	e	x_3	x_4	1
0	0	0	0	0	0	x_4	1
x_1	0	x_1	x_1	x_1	x_1	x_4	1
x_2	0	x_1	x_2	x_2	x_2	x_4	1
e	0	x_1	x_2	e	x_3	x_4	1
x_3	0	x_1	x_2	x_3	x_3	x_4	1
x_4	x_4	x_4	x_4	x_4	x_4	x_4	1
1	1	1	1	1	1	1	1

Table 8: The uninorm $U_{x_3}^{3T_\wedge S_\vee}$ on L_4.

Remark 3.43. *Two uninorms U_a^{2TS} and U_a^{3TS} given by the Examples 3.39 and 3.42 differ from each other on the bounded lattice L_4.*

Remark 3.44. *(i) It is easy to see that the uninorms derived from Theorems 3.8, 3.11, 3.18, 3.23, 3.29 and 3.33 are not idempotent. Therefore, we conclude that these uninorms are not locally internal.*
(ii) If we take $T = T_\wedge$ and $S = S_\vee$ in Theorems 3.38 and 3.41, then the uninorms derived from them are locally internal.

4. Conclusions

In Proposition 3.2, it has been demonstrated that if U is a uninorm on L with neutral element e and satisfies $U(x, y) = e$ for all $(x, y) \in (I_e)^2$, then $|I_e| = 1$. Additionally, we have proposed construction methods that reveal the existence of the uninorms discussed in Proposition 3.2. Motivated by this proposition 3.2, we have revealed some construction methods for uninorms that satisfy the condition $U(x, y) = e$ for some $(x, y) \in I_e \times I_e$, particularly when $|I_e| \neq 1$. We have also discussed the relationships among the presented methods. Furthermore, these methods have been introduced within the new uninorm classes: $U_{abc}^{TS}, U_{abc}^{1TS} \in \mathcal{U}_{abc}^e$, $U_{ab}^{TS}, U_{ab}^{1TS} \in \mathcal{U}_{ab}^e$, $U_a^{TS}, U_a^{1TS} \in \mathcal{U}_a^e$, $U_a^{2TS}, U_a^{3TS} \in \mathcal{U}_a^a$. Moreover, we have found that the greatest and smallest elements of these classes on some special lattices.

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