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A new Wendroff-type weakly singular integral inequality and applications to partial fractional differential equations

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Abstract. In this paper, we establish a novel weakly singular integral inequality of Wendroff type. Using a combination of analytical and fractional calculus techniques, we derive sufficient conditions for the validity of this inequality. As a key application, we investigate the existence, uniqueness, and Ulam-Hyers stability of solutions to a class of nonlinear partial fractional differential equations. Our approach not only generalizes previous results but also provides a unified framework for analyzing fractional-order systems with singular kernels. An illustrative example is presented to demonstrate the applicability and effectiveness of the proposed method.

1. Introduction

Integral inequalities are fundamental to the qualitative analysis of differential and integral equations, especially in determining bounds, uniqueness, stability, and solution existence. Among classical inequalities, Wendroff-type inequalities have proven particularly powerful in solving problems with weakly singular integral operators [1–5]. Over time, these inequalities have been generalized and refined to meet the demands of diverse areas in mathematical analysis and applied sciences [6–8]. These extensions include the development of nonlinear versions to handle more complex nonlinearities, the incorporation of multiple integrals and delays for systems with memory effects, and the derivation of inequalities specifically tailored for the weakly singular kernels inherent to fractional calculus. A canonical form for a function $u(x, y)$ is given by $u(x, y) \leq a(x, y) + \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1} b(s, t)u(s, t)dtds$, which leads to a bound via a Mittag-Leffler function [4]. A significant recent generalization is the powered Wendroff-type inequality [7], $u(x, y) \leq a(x, y) + \sum_{i=1}^n \left[\int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} f_i(x, y, t, s)w_i(u(t, s))dsdt\right]^{p_i}$, which introduces powers $p_i \geq 1$ on the integral

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terms. This work contributes to this lineage by establishing, via a new technique, an inequality that handles two distinct singular kernels simultaneously. Specifically, we bound a function $\mathscr{U}(\xi,\omega)$ satisfying

$$\mathscr{U}(\xi,\omega) \leq \mathscr{A}(\xi,\omega) + \mathscr{B}(\xi,\omega)\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathscr{U}(\tau,\varsigma)d\varsigma d\tau$$

$$+C(\xi,\omega)\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathscr{U}(\tau,\varsigma)d\varsigma d\tau,$$

a structure essential for analyzing the complex multi-term fractional equations considered in this paper.

Recent years have witnessed increasing interest in fractional differential equations (FDEs) due to their unique ability to model complex systems with memory and hereditary effects [9, 10]. Unlike classical calculus, fractional-order derivatives and integrals provide a more accurate framework for describing phenomena in physics, engineering, biology, and finance [11–14]. However, a key challenge in analyzing these systems arises from the inherent nonlocality and singular kernels of fractional operators, which often hinder the direct application of traditional analytical methods [16–18].

We present a new Wendroff-type integral inequality designed for weakly singular kernels, extending and generalizing known results. Derived using techniques from fractional calculus and functional analysis, this inequality serves as a robust tool for probing the qualitative behavior of solutions to partial fractional differential equations (PFDEs).

The central application of our work is the analysis of the following nonlinear PFDE system:

$$^{C}D_{0^+}^{\mu}\left(\mathscr{U}(\xi,\omega) - I_{a^+}^{\varrho}\mathcal{F}_2(\xi,\omega,\mathscr{U}(\xi,\omega))\right) = \mathcal{F}_1(\xi,\omega,\mathscr{U}(\xi,\omega)),(\xi,\omega)\in J=[0,T_1]\times[0,T_2], \quad (1)$$

$$\begin{aligned}&\mathscr{U}(\xi,0)=\varphi(\xi),\ \xi\in[0,T_1],\\&\mathscr{U}(0,\omega)=\psi(\omega),\ \omega\in[0,T_2],\\&\varphi(0)=\psi(0),\end{aligned} \quad (2)$$

where $\mu=(\mu_1,\mu_2),\varrho=(\varrho_1,\varrho_2)\in(0,1)^2$, $^{C}D_{0^+}^{\mu}$ is the partial Caputo fractional derivative of order μ and $\mathcal{F}_1,\mathcal{F}_2 : J\times\mathbb{R}\to\mathbb{R}$, $\varphi:[0,T_1]\to\mathbb{R}$ and $\psi:[0,T_2]\to\mathbb{R}$ are given continuous functions.

Within this framework, we establish rigorous results on the existence, uniqueness, and Ulam-Hyers stability of solutions. To demonstrate the practical applicability of our theory, we conclude with a detailed example.

This paper is organized as follows: Section 2 presents fundamental concepts and preliminary results from fractional calculus essential for our analysis. In Section 3, we introduce and rigorously prove our novel Wendroff-type inequality. Building on these foundations, Section 4 develops the theoretical framework for applying this inequality to PFDEs, establishing key results on solution existence, uniqueness, and stability characteristics. Finally, Section 5 provides a demonstrative example that both validates and illustrates the practical implementation of our theoretical developments.

2. Preliminarily

Definition 1. *[15] The left-sided mixed Riemann–Liouville fractional integral of order σ of $\mathscr{U}(\xi,\omega)\in L^1(J)$ is defined by*

$$\left(I_{0^+}^{\sigma}\mathscr{U}\right)(\xi,\omega) = \frac{1}{\Gamma(\sigma_1)\Gamma(\sigma_2)}\int_0^{\xi}\int_0^{\omega}\frac{\mathscr{U}(\tau,\varsigma)}{(\xi-\tau)^{1-\sigma_1}(\omega-\varsigma)^{1-\sigma_2}}d\varsigma d\tau,$$

where $\sigma=(\sigma_1,\sigma_2)$ with σ_1, σ_2 are positive real numbers.

Definition 2. *[15] The mixed fractional Riemann Liouville derivative of order $\sigma = (\sigma_1, \sigma_2) \in (0,1)^2$ of $\mathscr{U}(\xi, \omega)$ is defined by*

$$\begin{aligned}\left(D_{0^+}^{\sigma}\mathscr{U}\right)(\xi,\omega) &= D_{\xi\omega}^{2}\left(I_{0^+}^{1-\sigma}\mathscr{U}\right)(\xi,\omega)\\ &= \frac{D_{\xi\omega}^{2}}{\Gamma(1-\sigma_1)\Gamma(1-\sigma_2)}\int_0^{\xi}\int_0^{\omega}\frac{\mathscr{U}(\tau,\varsigma)}{(\xi-\tau)^{\sigma_1}(\omega-\varsigma)^{\sigma_2}}d\varsigma d\tau,\end{aligned}$$

where, $D_{\xi\omega}^{2} = \frac{\partial^2}{\partial\xi\partial\omega}$.

Definition 3. *[15] The mixed fractional Caputo derivative of order $\sigma = (\sigma_1, \sigma_2) \in (0,1)^2$ of $\mathscr{U}(\xi, \omega)$ is defined as*

$$\left({}^{C}D_{0^+}^{\sigma}\mathscr{U}\right)(\xi,\omega) = \frac{D_{\xi\omega}^{2}}{\Gamma(1-\sigma_1)\Gamma(1-\sigma_2)}\int_0^{\xi}\int_0^{\omega}\frac{\mathscr{U}(\tau,\varsigma)-\mathscr{U}(\tau,0)-\mathscr{U}(0,\varsigma)+\mathscr{U}(0,0)}{(\xi-\tau)^{\sigma_1}(\omega-\varsigma)^{\sigma_2}}d\varsigma d\tau.$$

Lemma 1. *[4] Let $\sigma_1, \sigma_2 > 0$, and let $\mathscr{U}$ and $\mathscr{V}$ be two non-negative integrable functions on J. Consider a function $\mathcal{G} : J \to \mathbb{R}$ that is continuous, non-negative, and non-decreasing with respect to its variables. If*

$$\mathscr{U}(\xi,\omega) \leq \mathscr{V}(\xi,\omega) + \mathcal{G}(\xi,\omega)\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\sigma_1-1}(\omega-\varsigma)^{\sigma_2-1}\mathscr{U}(\tau,\varsigma)\,d\varsigma d\tau, \tag{3}$$

then,

$$\begin{aligned}\mathscr{U}(\xi,\omega) \leq\ & \mathscr{V}(\xi,\omega) + \int_0^{\xi}\int_0^{\omega}\sum_{k=1}^{\infty}\frac{(\mathcal{G}(\xi,\omega)\Gamma(\sigma_1)\Gamma(\sigma_2))^k}{\Gamma(k\sigma_1)\Gamma(k\sigma_2)}\\ &\times(\xi-\tau)^{k\sigma_1-1}(\omega-\varsigma)^{k\sigma_2-1}\mathscr{V}(\tau,\varsigma)\,d\varsigma d\tau.\end{aligned} \tag{4}$$

Moreover, if $\mathscr{V}$ is non-decreasing with respect to its variables, then

$$\mathscr{U}(\xi,\omega) \leq \mathscr{V}(\xi,\omega)\,\mathbb{E}\Big((\sigma_1,1),(\sigma_2,1);\mathcal{G}(\xi,\omega)\Gamma(\sigma_1)\Gamma(\sigma_2)\big(\xi-a_1\big)^{\sigma_1}\big(\omega-a_2\big)^{\sigma_2}\Big). \tag{5}$$

where

$$\mathbb{E}\left((\alpha_1,\beta_2),(\alpha_2,\beta_2);(z)\right) = \sum_{\kappa=0}^{+\infty}\frac{z^{\kappa}}{\Gamma(\kappa\alpha_1+\beta_1)\Gamma(\kappa\alpha_2+\beta_2)}.$$

for all $\alpha_1, \beta_1, \alpha_2, \beta_2, z \in \mathbb{C}$, with $\Re(\alpha_j), \Re(\beta_j) > 0$, $j = 1, 2$.

3. Integral inequalities

Lemma 2. *Let $\mu_1, \mu_2 \in (0,1)$, and let $\mathcal{P}$ be a positive real constant with $\mathcal{P} > 1$ and $\mathcal{P}(\min\{\mu_1,\mu_2\}) - \mathcal{P} + 1 > 0$. Then,*

$$\left[\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mathcal{P}(\mu_1-1)}(\omega-\varsigma)^{\mathcal{P}(\mu_2-1)}\exp\left(\mathcal{P}(\tau+\varsigma)\right)d\varsigma d\tau\right]^{\frac{1}{\mathcal{P}}} \leq \lambda_{\{\mu_1,\mu_2,\mathcal{P}\}}\exp(\xi+\omega), \tag{6}$$

where $\lambda_{\{\mu_1,\mu_2,\mathcal{P}\}} = \left(\frac{\Gamma\left(\mathcal{P}\mu_1-\mathcal{P}+1\right)\Gamma\left(\mathcal{P}\mu_2-\mathcal{P}+1\right)}{\mathcal{P}^{\mathcal{P}(\mu_1+\mu_2)-2\mathcal{P}+2}}\right)^{\frac{1}{\mathcal{P}}}$.

Proof. We have

$$\begin{aligned}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mathcal{P}(\mu_1-1)}(\omega-\varsigma)^{\mathcal{P}(\mu_2-1)}\exp\left(\mathcal{P}(\tau+\varsigma)\right)d\varsigma d\tau \\ =\int_0^{\xi}(\xi-\tau)^{\mathcal{P}(\mu_1-1)}\exp\left(\mathcal{P}\tau\right)d\tau\int_0^{\omega}(\omega-\varsigma)^{\mathcal{P}(\mu_2-1)}\exp\left(\mathcal{P}\varsigma\right)d\varsigma.\end{aligned}\tag{7}$$

Now, we proceed to calculate the following integral

$$\int_0^{\xi}(\xi-\tau)^{\mathcal{P}(\mu_1-1)}\exp\left(\mathcal{P}\tau\right)d\tau.$$

For $\chi=\xi-\tau$, we get

$$\begin{aligned}\int_0^{\xi}(\xi-\tau)^{\mathcal{P}(\mu_1-1)}\exp\left(\mathcal{P}\tau\right)d\tau &= \int_0^{\xi}\chi^{\mathcal{P}(\mu_1-1)}\exp\left(\mathcal{P}\left(\xi-\chi\right)\right)d\chi \\ &= \exp\left(\mathcal{P}\xi\right)\int_0^{\xi}\chi^{\left(\mathcal{P}\mu_1-\mathcal{P}+1\right)-1}\exp\left(-\mathcal{P}\chi\right)d\chi.\end{aligned}$$

For $r=\mathcal{P}\chi$, we obtain

$$\begin{aligned}\int_0^{\xi}(\xi-\tau)^{\mathcal{P}(\mu_1-1)}\exp\left(\mathcal{P}\tau\right)d\tau &= \frac{\exp\left(\mathcal{P}\xi\right)}{\mathcal{P}^{\mathcal{P}\mu_1-\mathcal{P}+1}}\int_0^{\mathcal{P}\xi}r^{\left(\mathcal{P}\mu_1-\mathcal{P}+1\right)-1}\exp\left(-r\right)dr \\ &\le \frac{\exp\left(\mathcal{P}\xi\right)}{\mathcal{P}^{\mathcal{P}\mu_1-\mathcal{P}+1}}\Gamma\left(\mathcal{P}\mu_1-\mathcal{P}+1\right).\end{aligned}\tag{8}$$

From (7) and 8 and after some simplifications, we arrive at (6). □

Theorem 1. *Let* $(\mu_1,\mu_2),(\varrho_1,\varrho_2)\in(0,1)^2$*, and let* $\mathcal{U}(\xi,\omega),\mathcal{A}(\xi,\omega),\mathcal{B}(\xi,\omega),\mathcal{C}(\xi,\omega)\in C\left([0,T_1]\times[0,T_2],\mathbb{R}_+\right)$ *with* $\mathcal{A}(\xi,\omega),\mathcal{B}(\xi,\omega)$ *and* $\mathcal{C}(\xi,\omega)$ *are nondecreasing in each of its variables. Assume that,*

$$\begin{aligned}\mathcal{U}(\xi,\omega) \le\ & \mathcal{A}(\xi,\omega)+\mathcal{B}(\xi,\omega)\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{U}(\tau,\varsigma)d\varsigma d\tau \\ &+\mathcal{C}(\xi,\omega)\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{U}(\tau,\varsigma)d\varsigma d\tau.\end{aligned}\tag{9}$$

Then,

$$\mathcal{U}(\xi,\omega)\le\mathcal{N}(\xi,\omega)\mathcal{A}(\xi,\omega)\exp\left(\xi+\omega\right),\qquad\forall\left(\xi,\omega\right)\in[0,T_1]\times[0,T_2],\tag{10}$$

where

$$\begin{aligned}\mathcal{N}(\xi,\omega)=\ & 2^{\frac{\mathcal{Q}_2-1}{\mathcal{Q}_2}}3^{\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\exp\left(\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right) \\ &\times\exp\left(\frac{2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\lambda^{\mathcal{Q}_2}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\xi\omega}{\mathcal{Q}_2}\right),\end{aligned}$$

with the expressions of $\lambda_{\{\mu_1,\mu_2,\mathcal{P}_1\}}$ *and* $\lambda_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}$ *are given as in Lemma 2.*
Moreover if $\mathcal{A}\equiv 0$*, then*

$$\mathcal{U}(\xi,\omega)\equiv 0,\qquad\forall\left(\xi,\omega\right)\in[0,T_1]\times[0,T_2].$$

Proof. Applying Hölder's inequality, we obtain

$$\begin{aligned}\mathcal{U}(\xi,\omega) \leq\;& \mathcal{A}(\xi,\omega)+\mathcal{B}(\xi,\omega)\left[\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mathcal{P}_1(\mu_1-1)}(\omega-\varsigma)^{\mathcal{P}_1(\mu_2-1)}\exp\left(\mathcal{P}_1(\tau+\varsigma)\right)d\varsigma d\tau\right]^{\frac{1}{\mathcal{P}_1}}\\
&\times\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_1(\tau+\varsigma)\right)\mathcal{U}^{Q_1}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{1}{Q_1}}\\
&+C(\xi,\omega)\left[\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mathcal{P}_2(\varrho_1-1)}(\omega-\varsigma)^{\mathcal{P}_2(\varrho_2-1)}\exp\left(\mathcal{P}_2(\tau+\varsigma)\right)d\varsigma d\tau\right]^{\frac{1}{\mathcal{P}_2}}\\
&\times\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2(\tau+\varsigma)\right)\mathcal{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{1}{Q_2}},\end{aligned}$$

where $\mathcal{P}_1, \mathcal{P}_2, Q_1, Q_2$ are real numbers strictly greater than one, and satisfy $\frac{1}{\mathcal{P}_1}+\frac{1}{Q_1}=1$ and $\frac{1}{\mathcal{P}_2}+\frac{1}{Q_2}=1$. It follows from Lamma 2 that

$$\begin{aligned}\mathcal{U}(\xi,\omega) \leq\;& \mathcal{A}(\xi,\omega)+\lambda_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}(\xi,\omega)\exp(\xi+\omega)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_1(\tau+\varsigma)\right)\mathcal{U}^{Q_1}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{1}{Q_1}}\\
&+\lambda_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}C(\xi,\omega)\exp(\xi+\omega)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2(\tau+\varsigma)\right)\mathcal{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{1}{Q_2}}.\end{aligned}$$

It implies that,

$$\begin{aligned}\mathcal{U}^{Q_1}(\xi,\omega) \leq\;& 3^{Q_1-1}\mathcal{A}^{Q_1}(\xi,\omega)\\
&+3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\exp\left(Q_1(\xi+\omega)\right)\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_1(\tau+\varsigma)\right)\mathcal{U}^{Q_1}(\tau,\varsigma)d\varsigma d\tau\\
&+3^{Q_1-1}\lambda^{Q_1}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}C^{Q_1}(\xi,\omega)\exp\left(Q_1(\xi+\omega)\right)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2(\tau+\varsigma)\right)\mathcal{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{Q_1}{Q_2}},\end{aligned}$$

and consequently,

$$\begin{aligned}&\exp\left(-Q_1(\xi+\omega)\right)\mathcal{U}^{Q_1}(\xi,\omega)\\
&\quad\leq 3^{Q_1-1}\mathcal{A}^{Q_1}(\xi,\omega)\exp\left(-Q_1(\xi+\omega)\right)\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_1(\tau+\varsigma)\right)\mathcal{U}^{Q_1}(\tau,\varsigma)d\varsigma d\tau\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}C^{Q_1}(\xi,\omega)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2(\tau+\varsigma)\right)\mathcal{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{Q_1}{Q_2}}\\
&\quad\leq 3^{Q_1-1}\mathcal{A}^{Q_1}(\xi,\omega)\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_1(\tau+\varsigma)\right)\mathcal{U}^{Q_1}(\tau,\varsigma)d\varsigma d\tau\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}C^{Q_1}(\xi,\omega)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2(\tau+\varsigma)\right)\mathcal{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{Q_1}{Q_2}}.\end{aligned}$$

Using Lemma 1 for $\sigma_1 = \sigma_2 = 1$, we obtain

$$\begin{aligned}
&\exp\left(-Q_1\left(\xi+\omega\right)\right)\mathscr{U}^{Q_1}(\xi,\omega)\\
&\quad\leq \mathbb{E}\Big((1,1),(1,1),3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega\Big)\Big[3^{Q_1-1}\mathcal{A}^{Q_1}(\xi,\omega)\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{Q_1}(\xi,\omega)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2\left(\tau+\varsigma\right)\right)\mathscr{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{Q_1}{Q_2}}\Big]\\
&\quad\leq \exp\Big(3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega\Big)\Big[3^{Q_1-1}\mathcal{A}^{Q_1}(\xi,\omega)\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{Q_1}(\xi,\omega)\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2\left(\tau+\varsigma\right)\right)\mathscr{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{Q_1}{Q_2}}\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\exp\left(-Q_1\left(\xi+\omega\right)\right)\mathscr{U}^{Q_1}(\xi,\omega)\\
&\quad\leq 3^{Q_1-1}\mathcal{A}^{Q_1}(\xi,\omega)\exp\Big(3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega\Big)\\
&\qquad+3^{Q_1-1}\lambda^{Q_1}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{Q_1}(\xi,\omega)\exp\Big(3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega\Big)\\
&\qquad\times\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2\left(\tau+\varsigma\right)\right)\mathscr{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{Q_1}{Q_2}}.
\end{aligned}$$

Since $Q_1 > 1$, we obtain

$$\begin{aligned}
&\exp\left(-\left(\xi+\omega\right)\right)\mathscr{U}(\xi,\omega)\\
&\quad\leq 3^{\frac{Q_1-1}{Q_1}}\mathcal{A}(\xi,\omega)\exp\left(\frac{3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega}{Q_1}\right)\\
&\qquad+3^{\frac{Q_1-1}{Q_1}}\lambda_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}(\xi,\omega)\exp\left(\frac{3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega}{Q_1}\right)\\
&\qquad\times\left[\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2\left(\tau+\varsigma\right)\right)\mathscr{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau\right]^{\frac{1}{Q_2}}.
\end{aligned}$$

It implies that

$$\begin{aligned}
&\exp\left(-Q_2\left(\xi+\omega\right)\right)\mathscr{U}^{Q_2}(\xi,\omega)\\
&\quad\leq 2^{Q_2-1}3^{Q_2\frac{Q_1-1}{Q_1}}\mathcal{A}^{Q_2}(\xi,\omega)\exp\left(Q_2\frac{3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega}{Q_1}\right)\\
&\qquad+2^{Q_2-1}3^{Q_2\frac{Q_1-1}{Q_1}}\lambda^{Q_2}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{Q_2}(\xi,\omega)\exp\left(Q_2\frac{3^{Q_1-1}\lambda^{Q_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{Q_1}(\xi,\omega)\xi\omega}{Q_1}\right)\\
&\qquad\times\int_0^{\xi}\int_0^{\omega}\exp\left(-Q_2\left(\tau+\varsigma\right)\right)\mathscr{U}^{Q_2}(\tau,\varsigma)d\varsigma d\tau.
\end{aligned}$$

Using Lemma 1 for $\sigma_1 = \sigma_2 = 1$, we obtain

$$\begin{aligned}
&\exp\left(-\mathcal{Q}_2(\xi+\omega)\right)\mathcal{U}^{\mathcal{Q}_2}(\xi,\omega)\\
&\le 2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\mathcal{A}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\\
&\quad\times\mathbb{E}\left((1,1),(1,1),\left(2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\lambda^{\mathcal{Q}_2}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\xi\omega\right)\right)\\
&\le 2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\mathcal{A}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\\
&\quad\times\exp\left(2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\lambda^{\mathcal{Q}_2}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\xi\omega\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{U}^{\mathcal{Q}_2}(\xi,\omega)\le\ & 2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\mathcal{A}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\\
&\times\exp\left(2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\lambda^{\mathcal{Q}_2}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\xi\omega\right)\\
&\times\exp\left(\mathcal{Q}_2(\xi+\omega)\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathcal{U}(\xi,\omega)\le\ & 2^{\frac{\mathcal{Q}_2-1}{\mathcal{Q}_2}}3^{\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\exp\left(\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\\
&\times\exp\left(\frac{2^{\mathcal{Q}_2-1}3^{\mathcal{Q}_2\frac{\mathcal{Q}_1-1}{\mathcal{Q}_1}}\lambda^{\mathcal{Q}_2}_{\{\varrho_1,\varrho_2,\mathcal{P}_2\}}\mathcal{C}^{\mathcal{Q}_2}(\xi,\omega)\exp\left(\mathcal{Q}_2\frac{3^{\mathcal{Q}_1-1}\lambda^{\mathcal{Q}_1}_{\{\mu_1,\mu_2,\mathcal{P}_1\}}\mathcal{B}^{\mathcal{Q}_1}(\xi,\omega)\xi\omega}{\mathcal{Q}_1}\right)\xi\omega}{\mathcal{Q}_2}\right)\\
&\times\mathcal{A}(\xi,\omega)\exp(\xi+\omega),
\end{aligned}$$

which is equivalent to:

$$\mathcal{U}(\xi,\omega)\le\mathcal{N}(\xi,\omega)\mathcal{A}(\xi,\omega)\exp(\xi+\omega).$$

□

4. Applications

4.1. Existence and uniqueness

We shall consider the following assumptions:

(H_1) There exist $\mathcal{R}_1,\mathcal{R}_2,\mathcal{T}_1,\mathcal{T}_2\in C\big(J,\mathbb{R}_+\big)$ such that, for all $(\xi,\omega)\in J$, and for all $\mathcal{U}\in\mathbb{R}$:

$$\begin{cases}|\mathcal{F}_1(\xi,\omega,\mathcal{U})|\le\mathcal{R}_1(\xi,\omega)|\mathcal{U}|+\mathcal{T}_1(\xi,\omega),\\|\mathcal{F}_2(\xi,\omega,\mathcal{U})|\le\mathcal{R}_2(\xi,\omega)|\mathcal{U}|+\mathcal{T}_2(\xi,\omega).\end{cases}$$

(H_2) There exist $\mathcal{H}_1,\mathcal{H}_2\in C(J,\mathbb{R}_+)$ such that, for all $(\xi,\omega)\in J$, and for all $\mathcal{U},\mathcal{V}\in\mathbb{R}$:

$$\begin{cases}|\mathcal{F}_1(\xi,\omega,\mathcal{U})-\mathcal{F}_1(\xi,\omega,\mathcal{V})|\le\mathcal{H}_1(\xi,\omega)|\mathcal{U}-\mathcal{V}|,\\|\mathcal{F}_2(\xi,\omega,\mathcal{U})-\mathcal{F}_2(\xi,\omega,\mathcal{V})|\le\mathcal{H}_2(\xi,\omega)|\mathcal{U}-\mathcal{V}|.\end{cases}$$

Theorem 2. *Suppose that* $(H1)$*is satisfied. Then there exists at least one solution to (1)–(2). Furthermore, if* $(H2)$ *is satisfied, then (1)–(2) has a unique solution on J.*

Proof. Let us consider the operator $\mathscr{B} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$\begin{aligned}(\mathscr{B}\mathscr{U})(\xi,\omega) &= \mathcal{T}_1(\xi,\omega) + \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{F}_1(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\,d\varsigma d\tau \\ &\quad + \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{F}_2(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\,d\varsigma d\tau,\end{aligned}$$

where $\mathcal{T}_1(\xi,\omega) = \varphi(\xi) + \psi(\omega) - \varphi(0)$. It follows from $(H1)$ and the continuity of $\mathcal{F}_1, \mathcal{F}_2, \varphi$ and ψ that the operator $\mathscr{B}$ is continuous and completely continuous. It remains to establish that the set

$$\mathscr{S} = \{\mathscr{U} \in C(J,\mathbb{R}),\ \mathscr{U} = (\theta\mathscr{B})(\mathscr{U}),\ \text{for } \theta \in (0,1)\}$$

is bounded, Let $\mathscr{U} \in \mathscr{S}$, then for $\theta \in (0,1)$ and $(\xi,\omega) \in J$, we have

$$\begin{aligned}\mathscr{U}(\xi,\omega) &= \theta\Big(\mathcal{T}_1(\xi,\omega) + \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{F}_1(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\,d\varsigma d\tau \\ &\quad + \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{F}_2(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\,d\varsigma d\tau\Big).\end{aligned}$$

So,

$$\begin{aligned}|\mathscr{U}(\xi,\omega)| &= \|\mathcal{T}_1\|_\infty + \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{T}_1(\tau,\varsigma)\,d\varsigma d\tau \\ &\quad + \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{R}_1(\tau,\varsigma)|\mathscr{U}(\tau,\varsigma)|\,d\varsigma d\tau \\ &\quad + \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{T}_2(\tau,\varsigma)\,d\varsigma d\tau \\ &\quad + \frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{R}_2(\tau,\varsigma)|\mathscr{U}(\tau,\varsigma)|\,d\varsigma d\tau \\ &\le \|\mathcal{T}_1\|_\infty + \frac{\|\mathcal{T}_1\|_\infty T_1^{\mu_1}T_2^{\mu_2}}{\Gamma(\mu_1+1)\Gamma(\mu_2+1)} + \frac{\|\mathcal{T}_2\|_\infty T_1^{\varrho_1}T_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)} \\ &\quad + \frac{\|\mathcal{R}_1\|_\infty}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}|\mathscr{U}(\tau,\varsigma)|\,d\varsigma d\tau \\ &\quad + \frac{\|\mathcal{R}_2\|_\infty}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}|\mathscr{U}(\tau,\varsigma)|\,d\varsigma d\tau.\end{aligned}$$

Now, applying Theorem 1, we get, for all $(\xi,\omega) \in J$:

$$\begin{aligned}|\mathscr{U}(\xi,\omega)| &\le \mathcal{A}\mathcal{N}(\xi,\omega)\exp(\xi+\omega), \\ &\le \mathcal{A}\mathcal{N}(T_1,T_2)\exp(T_1+T_2),\end{aligned}$$

where $\mathcal{N}$ is defined as in Theorem 1 with $\mathcal{B} = \frac{\|\mathcal{R}_1\|_\infty}{\Gamma(\mu_1)\Gamma(\mu_2)}$ and $\mathcal{C} = \frac{\|\mathcal{R}_2\|_\infty}{\Gamma(\varrho_1)\Gamma(\varrho_2)}$, and $\mathcal{A} = \|\varsigma_1\|_\infty + \frac{\|\mathcal{T}_1\|_\infty T_1^{\mu_1}T_2^{\mu_2}}{\Gamma(\mu_1+1)\Gamma(\mu_2+1)} + \frac{\|\mathcal{T}_2\|_\infty T_1^{\varrho_1}T_2^{\varrho_2}}{\Gamma(\varrho_1+1)\Gamma(\varrho_2+1)}$. From the Schaefer Fixed Point Theorem, it follows that the operator $\mathscr{B}$ has at least one fixed point in $C(J,\mathbb{R})$, which corresponds to a solution of (1)–(2).
Now, we suppose that (H_2) is satisfied, and let us assume that $\mathscr{U}_1(\xi,\omega), \mathscr{U}_2(\xi,\omega)$ are two solutions of (1)-(2).

Then,

$$
\begin{aligned}
&|\mathscr{U}_1(\xi,\omega)-\mathscr{U}_2(\xi,\omega)| \\
&\leq \frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathscr{H}_1(\tau,\varsigma)\left|\mathscr{U}_1(\tau,\varsigma)-\mathscr{U}_2(\tau,\varsigma)\right|d\varsigma d\tau \\
&+\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathscr{H}_2(\tau,\varsigma)\left|\mathscr{U}_1(\tau,\varsigma)-\mathscr{U}_2(\tau,\varsigma)\right|d\varsigma d\tau \\
&\leq \frac{\|\mathscr{H}_1\|_\infty}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\left|\mathscr{U}_1(\tau,\varsigma)-\mathscr{U}_2(\tau,\varsigma)\right|d\varsigma d\tau \\
&+\frac{\|\mathscr{H}_2\|_\infty}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\left|\mathscr{U}_1(\tau,\varsigma)-\mathscr{U}_2(\tau,\varsigma)\right|d\varsigma d\tau.
\end{aligned}
$$

Using Theorem 1, we obtain $\mathscr{U}_1=\mathscr{U}_2$. □

4.2. Ulam stability

We consider the following inequality for all $(\xi,\omega)\in J$ and for all $\epsilon>0$:

$$
\left|{}^{C}D^{\mu}_{a^+}\left(\mathscr{V}(\xi,\omega)-I^{\varrho}_{a^+}\mathscr{F}_2(\xi,\omega,\mathscr{V}(\xi,\omega))\right)-\mathscr{F}_1(\xi,\omega,\mathscr{V}(\xi,\omega))\right|\leq\epsilon. \tag{11}
$$

Definition 4. *Equation* (1) *is said to be Ulam–Hyers stable if there exist* $\mathcal{K}>0$, *such that for every* $\epsilon>0$, *and for every solution* $\mathscr{V}(\xi,\omega)$ *of (11) there exists a solution* $\mathscr{U}(\xi,\omega)$ *of equation (1) such that*

$$
|\mathscr{U}(\xi,\omega)-\mathscr{V}(\xi,\omega)|\leq\epsilon\mathcal{K}.
$$

Remark 1. *If* $\mathscr{V}(\xi,\omega)$ *is a solution of (11) then* $\mathscr{V}(\xi,\omega)$ *is a solution of*

$$
\begin{aligned}
&\left|\mathscr{V}(\xi,\omega)-\mathcal{T}_2(\xi,\omega)-\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathscr{F}_1(\tau,\varsigma,\mathscr{V}(\tau,\varsigma))d\varsigma d\tau\right. \\
&\left.-\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathscr{F}_2(\tau,\varsigma,\mathscr{V}(\tau,\varsigma))d\varsigma d\tau\right| \\
&\leq\frac{\epsilon T_1^{\mu_1}T_2^{\mu_2}}{\Gamma(\mu_1+1)\Gamma(\mu_2+1)},
\end{aligned}
$$

where $\mathcal{T}_2(\xi,\omega)=\mathscr{V}_2(\xi,0)+\mathscr{V}_2(0,\omega)-\mathscr{V}_2(0,0)$.

Theorem 3. *Suppose that* (H_2) *is satisfied. Therefore, Equation (1) is Ulam-Hyers stable.*

Proof. Let $\mathscr{V}(\xi,\omega)$ be a solution of (11) and $\mathscr{U}(\xi,\omega)$ the unique solution of the following problem

$$
\begin{cases}
{}^{C}D^{\mu}_{a^+}\left(\mathscr{U}(\xi,\omega)-I^{\varrho}_{a^+}\mathscr{F}_2(\xi,\omega,\mathscr{U}(\xi,\omega))\right)=\mathscr{F}_1(\xi,\omega,\mathscr{U}(\xi,\omega)),(\xi,\omega)\in J,\\
\mathscr{U}(\xi,0)=\mathscr{V}(\xi,0),\ \xi\in[0,T_1],\\
\mathscr{U}(0,\omega)=\mathscr{V}(0,\omega),\ \omega\in[0,T_2].
\end{cases} \tag{12}
$$

Then

$$
\begin{aligned}
\mathscr{U}(\xi,\omega) &= \mathcal{T}_2(\xi,\omega)+\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathscr{F}_1(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))d\varsigma d\tau \\
&+\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathscr{F}_2(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))d\varsigma d\tau
\end{aligned}
$$

where $\mathcal{T}_2(\xi,\omega)$ is defined as in Remark 1.
Using Remark 1 and ($H2$), we get

$$\begin{aligned}
&|\mathscr{V}(\xi,\omega)-\mathscr{U}(\xi,\omega)|\\
&=\Bigg|\mathscr{V}(\xi,\omega)-\mathcal{T}_2(\xi,\omega)-\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{F}_1(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\,d\varsigma d\tau\\
&\quad-\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{F}_2(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\,d\varsigma d\tau\Bigg|\\
&\le\Bigg|\mathscr{V}(\xi,\omega)-\mathcal{T}_2(\xi,\omega)-\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\mathcal{F}_1(\tau,\varsigma,\mathscr{V}(\tau,\varsigma))\,d\varsigma d\tau\\
&\quad-\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\mathcal{F}_2(\tau,\varsigma,\mathscr{V}(\tau,\varsigma))\,d\varsigma d\tau\Bigg|\\
&\quad+\frac{1}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\Big|\mathcal{F}_1(\tau,\varsigma,\mathscr{V}(\tau,\varsigma))-\mathcal{F}_1(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\Big|d\varsigma d\tau\\
&\quad+\frac{1}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\Big|\mathcal{F}_2(\tau,\varsigma,\mathscr{V}(\tau,\varsigma))-\mathcal{F}_2(\tau,\varsigma,\mathscr{U}(\tau,\varsigma))\Big|d\varsigma d\tau\\
&\le\frac{\epsilon T_1^{\mu_1}T_2^{\mu_2}}{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}+\frac{\|\mathcal{H}_1\|_\infty}{\Gamma(\mu_1)\Gamma(\mu_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\mu_1-1}(\omega-\varsigma)^{\mu_2-1}\Big|\mathscr{V}(\tau,\varsigma)-\mathscr{U}(\tau,\varsigma)\Big|d\varsigma d\tau\\
&\quad+\frac{\|\mathcal{H}_2\|_\infty}{\Gamma(\varrho_1)\Gamma(\varrho_2)}\int_0^{\xi}\int_0^{\omega}(\xi-\tau)^{\varrho_1-1}(\omega-\varsigma)^{\varrho_2-1}\Big|\mathscr{V}(\tau,\varsigma)-\mathscr{U}(\tau,\varsigma)\Big|d\varsigma d\tau.
\end{aligned}$$

Now, applying Theorem 1, we get, for all $(\xi,\omega)\in J$:

$$\begin{aligned}
|\mathscr{V}(\xi,\omega)-\mathscr{U}(\xi,\omega)| &\le \epsilon\mathcal{A}\mathcal{N}(\xi,\omega)\exp(\xi+\omega),\\
&\le \epsilon\mathcal{A}\mathcal{N}(T_1,T_2)\exp(T_1+T_2),
\end{aligned}$$

where $\mathcal{A}=\dfrac{T_1^{\mu_1}T_2^{\mu_2}}{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}$ and $\mathcal{N}$ is defined as in Theorem 1 with $\mathcal{B}=\frac{\|\mathcal{H}_1\|_\infty}{\Gamma(\mu_1)\Gamma(\mu_2)}$ and $\mathcal{C}=\frac{\|\mathcal{H}_2\|_\infty}{\Gamma(\varrho_1)\Gamma(\varrho_2)}$.
Thus, Equation (1) is Ulam-Hyers stable. □

5. Example

Let's consider the following problem

$$\begin{cases}
D_{0^+}^{\mu}\left(\mathscr{U}(\xi,\omega)-I_{a^+}^{\varrho}\left(\xi+\omega+\dfrac{1}{\xi+\omega+1}\sin(\mathscr{U}(\xi,\omega))\right)\right)\\
\qquad\qquad=\exp(\xi+\omega)+\arctan(\mathscr{U}(\xi,\omega)),\ (\xi,\omega)\in[0,1]\times[0,1],\\
\mathscr{U}(\xi,0)=\sin(\xi),\ \xi\in[0,1],\\
\mathscr{U}(0,\omega)=1-\cos(\omega),\ \omega\in[0,1],
\end{cases}\tag{13}$$

where $\mu=(\mu_1,\mu_2),\varrho=(\varrho_1,\varrho_2)\in(0,1)^2$.
We have

$$\begin{cases}
\mathcal{F}_1(\xi,\omega,\mathscr{U})=\xi+\omega+\dfrac{1}{\xi+\omega+1}\sin(\mathscr{U}),\\
\mathcal{F}_2(\xi,\omega,\mathscr{U})=\exp(\xi+\omega)+\arctan(\mathscr{U}).
\end{cases}$$

Moreover, for all $(\xi,\omega)\in[0,1]\times[0,1]$ and for all $\mathscr{U},\mathscr{V}\in\mathbb{R}$:

$$\begin{cases}
|\mathcal{F}_1(\xi,\omega,\mathscr{U})|\le\xi+\omega+\dfrac{1}{\xi+\omega+1}|\mathscr{U}|,\\
|\mathcal{F}_2(\xi,\omega,\mathscr{U})|\le\exp(\xi+\omega)+|\mathscr{U}|,
\end{cases}$$

and

$$\begin{cases} |\mathcal{F}_1(\xi,\omega,\mathscr{U}) - \mathcal{F}_1(\xi,\omega,\mathscr{V})| \leq \dfrac{1}{\xi+\omega+1}|\mathscr{U}-\mathscr{V}|, \\ |\mathcal{F}_2(\xi,\omega,\mathscr{U}) - \mathcal{F}_2(\xi,\omega,\mathscr{V})| \leq |\mathscr{U}-\mathscr{V}|. \end{cases}$$

Hence, the assumptions (H_1) and (H_2) are satisfied. It follows from Theorem 2 and Theorem 3 that the system (13) has a unique solution on $[0,1]\times[0,1]$, and this solution is Ulam-Hyers stable.

6. Conclusion

This paper presents a novel Wendroff-type integral inequality for weakly singular kernels, developed through an innovative combination of analytical methods and fractional calculus techniques. As a significant application, we establish a complete theoretical framework for investigating nonlinear partial fractional differential equations, deriving verifiable conditions that ensure solution existence, uniqueness, and Ulam-Hyers stability. The developed inequality not only advances theoretical understanding but also offers practical utility for analyzing complex systems with singular behaviors, as demonstrated through our rigorous analysis.

Our findings significantly advance the theoretical foundations for analyzing fractional-order systems with singular kernels while opening new research directions in their qualitative analysis. The illustrative example provided confirms the practical relevance and effectiveness of our approach. Looking ahead, this work naturally extends to several promising directions, including generalization to inequalities with variable-order fractional operators, applications to systems with time-varying delays, and extension to impulsive fractional dynamical systems.

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