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Perturbations of Weyl spectra for 2×2 relation matrices

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Abstract. This paper solves the completion problem for 2×2 block operator matrices whose entries are linear relations. We provide necessary and sufficient conditions for the completed matrix M_X to be Weyl, right Weyl, or left Weyl, whether the unknown entry X is a single- valued bounded operator or a multi-valued bounded relation. Consequently, we characterize the perturbations of the associated Weyl spectra. Our results generalize known theorems for operator matrices and provide tools for analyzing systems beyond the scope of standard operator theory.

1. Introduction

A linear relation is a generalization of the concept of an operator in the multi-valued case. The linear relation will naturally appear in considering the adjoint of non densely defined operators and the inverse of certain operators, and it is shown to be very useful in various research fields such as nonlinear analysis, control theory and differential equations (cf.[4, 12, 16, 18] and references therein). Partial relation matrices are relation matrices the entries of which are specified only on a subset of its positions, while a completion of a partial relation matrix is the conventional relation matrix resulting from filling in its unspecified entries. Usually, one concerns the conditions under which a partial relation matrix has completions with some given properties. The completion problem was shown to be very useful in various pure and applied mathematical fields, e.g., in relation theory, numerical analysis, optimal theory, systems theory and engineering problems (cf.[13] and references therein).

1.1. *Basic Definitions*

A linear relation $T : \mathcal{H} \to \mathcal{K}$ is a mapping such that $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all nonzero scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ and $x_1, x_2 \in \mathrm{dom}T$, where $\mathrm{dom}T \subseteq \mathcal{H}$ is the domain of T, and $T(x_1)$, $T(x_2) \subseteq \mathcal{K}$ are nonempty. The set $\mathcal{LR}(\mathcal{H}, \mathcal{K})$ denotes the class of linear relations with $\mathrm{dom}T = \mathcal{H}$ and $T(\mathrm{dom}T) \subseteq \mathcal{K}$. The set $\mathcal{LR}(\mathcal{H}, \mathcal{K})$ denotes the class of linear relations with $\mathrm{dom}T = \mathcal{H}$ and $T(\mathrm{dom}T) \subseteq \mathcal{K}$, and write $\mathcal{LR}(\mathcal{H}) := \mathcal{LR}(\mathcal{H}, \mathcal{H})$. If $T(0) = \{0\}$, then T is called an operator. The class of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ is denoted by $\mathcal{B}(\mathcal{H}, \mathcal{K})$. The graph $G(T)$ of $T \in \mathcal{LR}(\mathcal{H}, \mathcal{K})$ is

$$G(T) = \{(u, v) \in \mathcal{H} \oplus \mathcal{K} : u \in \mathrm{dom}T, v \in T(u)\}.$$

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For a subspace $\mathcal{M} \subset \mathrm{dom}T$, then $T\mid_{\mathcal{M}}$ is defined by

$$G(T\mid_{\mathcal{M}}) = \{(u,v) \in \mathcal{H} \oplus \mathcal{K} : u \in \mathcal{M}, v \in T(u)\},$$

and $P_{\mathcal{M}}$ for the orthogonal projection onto $\mathcal{M}$ along $\mathcal{M}^{\perp}$ when $\mathcal{M}$ is closed. The inverse of T is the relation T^{-1} given by

$$G(T^{-1}) = \{(v,u) \in \mathcal{K} \oplus \mathcal{H} : (u,v) \in G(T)\},$$

and the closure $\overline{T}$ of T is the linear relation defined by $G(\overline{T}) = \overline{G(T)}$. If $G(T) \subseteq \mathcal{H} \oplus \mathcal{K}$ is a closed subspace, then T is said to be closed. The class of such relations is denoted by $\mathcal{CR}(\mathcal{H},\mathcal{K})$. As usual, write $\mathrm{ker}T = \{x \in \mathcal{H} : (x,0) \in G(T)\}$ for the kernel of T, and $\mathrm{ran}T := T(\mathrm{dom}T)$ for the range of T; write $n(T) := \dim \mathrm{ker}T$ and $d(T) := \mathrm{ran}T^{\perp}$. The adjoint $T^* \in \mathcal{LR}(\mathcal{K},\mathcal{H})$ of $T \in \mathcal{LR}(\mathcal{H},\mathcal{K})$ is defined by

$$G(T^*) = \{(v,v') \in \mathcal{K} \oplus \mathcal{H} : \langle u',v\rangle = \langle u,v'\rangle \text{ for all } \langle u,u'\rangle \in G(T)\}.$$

For $T \in \mathcal{LR}(\mathcal{H},\mathcal{K})$,

$$\mathrm{ker}T^* = \mathrm{ran}T^{\perp}, T^*(0) = \mathrm{dom}T^{\perp}, \mathrm{ker}\overline{T} = \mathrm{ran}(T^*)^{\perp}, \overline{T}(0) = \mathrm{dom}(T^*)^{\perp}.$$

The quotient map from $\mathcal{K}$ to $\mathcal{K}/\overline{T(0)}$ is denoted by Q_T. Clearly Q_TT is an operator, so that we can define $\|Tx\| = \|Q_TTx\|$ for $x \in \mathrm{dom}T$ and $\|T\| = \|Q_TT\|$. Notice that for $u \in \mathrm{dom}T$, $Q_TTu = Q_Tv$. Indeed, since $v \in Tu$ if and only if $T(u) = v + T(0)$, then $Q_TTu = Q_Tv$. If $T \in \mathcal{LR}(\mathcal{H},\mathcal{K})$ such that $\|T\| < \infty$, then T is said to be bounded. By $\mathcal{BR}(\mathcal{H},\mathcal{K})$ denote the subset of $\mathcal{LR}(\mathcal{H},\mathcal{K})$, whose elements are bounded. Write $\mathcal{BCR}(\mathcal{H},\mathcal{K}) := \mathcal{CR}(\mathcal{H},\mathcal{K}) \cap \mathcal{BR}(\mathcal{H},\mathcal{K})$. As abbreviations, $\mathcal{BR}(\mathcal{H}) := \mathcal{BR}(\mathcal{H},\mathcal{H})$, $\mathcal{CR}(\mathcal{H}) := \mathcal{CR}(\mathcal{H},\mathcal{H})$ and $\mathcal{BCR}(\mathcal{H}) := \mathcal{BCR}(\mathcal{H},\mathcal{H})$.

1.2. *Fredholm and Weyl Relations*

Let $T \in \mathcal{BCR}(\mathcal{H},\mathcal{K})$ with closed range $\mathrm{ran}T$. Then the relation T is said to be right Fredholm, if $d(T) < \infty$; while if $n(T) < \infty$, we say T is left Fredholm. If T is both right and left Fredholm, then it is called Fredholm. Write $\mathrm{ind}T := n(T) - d(T)$ for the index of T. Then T is called right Weyl if it is right Fredholm with $\mathrm{ind}T \geq 0$, left Weyl if left Fredholm with $\mathrm{ind}\,T \leq 0$, and Weyl if Fredholm with $\mathrm{ind}\,T = 0$. Obviously, T is Weyl if and only if T is both right and left Weyl. We denote the collections of right Fredholm, left Fredholm, Fredholm, right Weyl, left Weyl, and Weyl relations as: $\Phi_-(\mathcal{H},\mathcal{K}), \Phi_+(\mathcal{H},\mathcal{K}), \Phi(\mathcal{H},\mathcal{K}), \Phi_-^+(\mathcal{H},\mathcal{K}), \Phi_+^-(\mathcal{H},\mathcal{K})$ and $\Phi_0(\mathcal{H},\mathcal{K})$. Again, we have the abbreviations $\Phi_-(\mathcal{H}), \Phi_+(\mathcal{H}), \Phi(\mathcal{H}), \Phi_-^+(\mathcal{H}), \Phi_+^-(\mathcal{H})$ and $\Phi_0(\mathcal{H})$ of the above classes of relations like $\mathcal{BR}(\mathcal{H})$.

For $T \in \mathcal{BR}(\mathcal{H})$, the sets

$$\begin{aligned}
\sigma_{re}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not right Fredholm}\},\\
\sigma_{le}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left Fredholm}\},\\
\sigma_{e}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},\\
\sigma_{rw}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not right Weyl}\},\\
\sigma_{lw}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not left Weyl}\},\\
\sigma_{w}(T) &= \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}
\end{aligned}$$

are called the right essential spectrum, left essential spectrum, essential spectrum, right Weyl spectrum, left Weyl spectrum and Weyl spectrum, respectively.

Recall that an operator T^+ is the maximal Tseng inverse of $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ if and only if

$$\mathcal{D}(T^{\dagger}) = \mathrm{ran}T \oplus \mathrm{ran}T^{\perp}, \quad T^{\dagger}T = P_{\mathrm{Ker}T^{\perp}}, \quad TT^{\dagger} = P_{\overline{\mathrm{ran}T}}|_{\mathrm{ran}T\oplus\mathrm{ran}T^{\perp}}.$$

It is clear that $u = TT^{\dagger}u$ for any $u \in \mathrm{ran}T$.

1.3. *The Matrix* M_X

When the relations $A \in \mathcal{BR}(\mathcal{H}), B \in \mathcal{BR}(\mathcal{K})$ and $C \in \mathcal{BR}(\mathcal{K}, \mathcal{H})$ are given, we define

$$M_X := \begin{bmatrix} A & C \\ X & B \end{bmatrix} \tag{1}$$

with an unknown relation $X \in \mathcal{BR}(\mathcal{H}, \mathcal{K})$. Hereafter, the symbol M_X is reserved for the relation matrix with the form as in (1). In particular, if $C = 0$, then M_X admit an upper triangular relation matrix form, which has been studied by some authors [2, 3, 5, 10, 11, 19]. Note that if A, B, C and X are operators, then then Fredholm-type and Weyl-type properties of the general operator matrix M_X have been studied by different authors [14, 15, 21, 22]. The purpose of this paper is to extend Weyl-type properties of matrix operators developed in [6, 7, 22] to linear relations. In [3, 5, 11], the authors studied the completion problem of partial upper relation matrices in which the known element is multi-valued, but the unknown element is single-valued. In fact, the multivalued part of unknown element is also worth considering. This paper is concerned with the completion problem of partial relation matrices under the condition of single-valued and multi-valued unknown element respectively.

The spectral theory of multivalued linear operators, which provides the foundation for our work, has been comprehensively treated in [20]. This paper builds upon this foundation to address a specific and fundamental problem: the completion problem for 2×2 relation matrices. To further clarify the scope and novelty of our work, we pursue the following three objectives, which extend beyond the conventional focus on single-valued operators: (1) Resolve the completion problem for 2×2 block matrices with linear relation entries, focusing on Weyl, right Weyl, and left Weyl properties of M_X; (2) Unify analysis by allowing X to be either a single-valued operator or a multi-valued linear relation, thus bridging a critical gap between these two settings; (3) Characterize perturbations of Weyl spectra induced by varying X. These objectives fill a gap in existing theory, which typically restricts entries to single-valued operators.

In this paper, we establish a necessary and sufficient condition under which $M_X \in \Phi_0(\mathcal{H} \oplus \mathcal{K})$ ($M_X \in \Phi_-^+(\mathcal{H} \oplus \mathcal{K})$, or $M_X \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$) holds for some bounded entry $X : \mathcal{H} \to \mathcal{K}$, which can be either a single-valued operator or a multi-valued linear relation. Moreover, we characterize the perturbation of $\sigma_w(M_X)$, $\sigma_{rw}(M_X)$ and $\sigma_{lw}(M_X)$, when the X runs over the set $\mathcal{B}(\mathcal{H}, \mathcal{K})$ (or $\mathcal{BR}(\mathcal{H}, \mathcal{K})$). As a byproduct, we also obtain a necessary and sufficient condition is given for

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \Phi_0(\mathcal{H} \oplus \mathcal{K}), \ \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \Phi_-^+(\mathcal{H} \oplus \mathcal{K}), \ \text{or} \ \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K}))$$

some bounded entry $X : \mathcal{K} \to \mathcal{H}$, which can be either a single-valued operator or a multi-valued linear relation.

2. Some auxiliary results

We begin with some basic lemmas, which are useful for the proofs of the main results of this paper.

Lemma 2.1 (see [17, Lemma 6]). *Let $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then for any $\epsilon > 0$ there exist the orthogonal decompositions $\mathcal{K} = \mathcal{K}_\epsilon \oplus \mathcal{K}^\epsilon$ and $\mathcal{H} = \mathcal{H}_\epsilon \oplus \mathcal{H}^\epsilon$ such that*

$$C(\mathcal{K}_\epsilon) \subset \mathcal{H}_\epsilon, \ \|Cx\| \le \epsilon\|x\| \text{ for all } x \in \mathcal{K}_\epsilon,$$
$$C(\mathcal{K}^\epsilon) \subset \mathcal{H}^\epsilon, \ \|Cx\| \ge \epsilon\|x\| \text{ for all } x \in \mathcal{K}^\epsilon.$$

Lemma 2.2 (see [1, Remark 1.54]). *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be a right (left) Fredholm operator. Then there exists $\epsilon := \epsilon(T) > 0$ such that $S \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\|S\| < \epsilon$ implies that $T + S$ is also a right (left) Fredholm operator. Moreover,*

$$n(T + S) \le n(T), \ d(T + S) \le d(T), \ \text{ind}(T + S) = \text{ind}\, T.$$

Lemma 2.3 (see [1, Remark 1.54]). *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, and let $S \in \mathcal{B}(\mathcal{H},\mathcal{K})$ be a compact operator. Then*
(i) $T \in \Phi_+(\mathcal{H},\mathcal{K})$ *if and only if* $T + S \in \Phi_+(\mathcal{H},\mathcal{K})$ *with* $\mathrm{ind}(T + S) = \mathrm{ind}T$.
(ii) $T \in \Phi_-(\mathcal{H},\mathcal{K})$ *if and only if* $T + S \in \Phi_-(\mathcal{H},\mathcal{K})$ *with* $\mathrm{ind}(T + S) = \mathrm{ind}T$.

Lemma 2.4 (see [1, Theorem 1.53]). *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Then*
(i) $T \in \Phi_+(\mathcal{H},\mathcal{K})$ *if and only if there exist* $U \in \mathcal{B}(\mathcal{K},\mathcal{H})$ *and* $K_1 \in \mathcal{B}(\mathcal{H})$ *such that* $UT = I_{\mathcal{H}} + K_1$, *where* $\dim K_1 < \infty$.
(ii) $T \in \Phi_-(\mathcal{H},\mathcal{K})$ *if and only if there exist* $V \in \mathcal{B}(\mathcal{K},\mathcal{H})$ *and* $K_2 \in \mathcal{B}(\mathcal{K})$ *such that* $TV = I_{\mathcal{K}} + K_2$, *where* $\dim K_2 < \infty$.

Lemma 2.5 (see [8, Lemma 5.8]). *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$, then T is a compact operator if and only if* $\mathrm{ran}T$ *contains no closed infinite dimensional subspaces.*

For relations $A \in \mathcal{BR}(\mathcal{H})$ and $C \in \mathcal{BR}(\mathcal{K},\mathcal{H})$, write

$$\mathcal{N}(A \mid C) := \{G \in \mathcal{B}(\mathcal{K},\mathcal{H}) : \mathrm{ran}AG + C(0) \subseteq \mathrm{ran}C + A(0)\},$$

and $[A\ C] : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H}$ denotes the row relation.

Lemma 2.6 (see [4, Proposition II.5.3]). *Let $T \in \mathcal{LR}(\mathcal{H},\mathcal{K})$, then T is closed if and only if $Q_T T$ is closed, and $T(0)$ is a closed subspace.*

Lemma 2.7 (see [4, Proposition III.1.2, Corollary III.1.13]). *Let $T \in \mathcal{LR}(\mathcal{H},\mathcal{K})$, then T^* is closed and $\|T^*\| \leq \|T\|$.*

Lemma 2.8 (see [2, Lemma 4.2]). *Let $T \in \mathcal{BCR}(\mathcal{H})$, Then*
(i) $T \in \Phi_+(\mathcal{H})$ *if and only if* $Q_T T \in \Phi_+(\mathcal{H},\mathcal{H}/T(0))$. *In such case,* $\mathrm{ind}T = \mathrm{ind}(Q_T T)$.
(ii) $T \in \Phi_-(\mathcal{H})$ *if and only if* $Q_T T \in \Phi_-(\mathcal{H},\mathcal{H}/T(0))$. *In such case,* $\mathrm{ind}T = \mathrm{ind}(Q_T T)$.

Lemma 2.9 (see [4, Theorem III.1.10]). *Let $\mathcal{M} \subseteq \mathcal{H}$, and let $J_{\mathcal{M}}$ denote the natural injection map of $\mathcal{M}$ into $\mathcal{H}$, i.e.,* $\mathrm{dom}J_{\mathcal{M}} = \mathcal{M}$ *and $J_{\mathcal{M}}x = x$ for $x \in \mathcal{M}$. Then, $(Q_{\mathcal{M}})^* = J_{\mathcal{M}^\perp}$ and $(J_{\mathcal{M}})^* = Q_{\mathcal{M}^\perp}$.*

Lemma 2.10 (see [4, Theorem III.1.6]). *Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be separable Hilbert spaces, and let $T \in \mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2,\mathcal{H}_3)$, then $G(T^*S^*) \subseteq G((ST)^*)$. Furthermore, if one of the following statements is fulfilled:*
(i) $\mathrm{ran}T^* = \mathcal{H}_1$ *and* $\mathrm{dom}S \subseteq \mathrm{ran}T$;
(ii) $\mathrm{ran}S^* = \mathcal{H}_3$ *and* $\mathrm{ran}T \subseteq \mathrm{dom}S$,
then $(ST)^* = T^*S^*$.

Lemma 2.11. *Let $A \in \mathcal{LR}(\mathcal{H}), B \in \mathcal{LR}(\mathcal{K}), C \in \mathcal{LR}(\mathcal{K},\mathcal{H})$ and $X \in \mathcal{LR}(\mathcal{H},\mathcal{K})$, then*

$$Q_{M_X} M_X = \begin{bmatrix} Q_{[A\ C]}A & Q_{[A\ C]}C \\ Q_{[X\ B]}X & Q_{[X\ B]}B \end{bmatrix}.$$

Proof. Let $(\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]) \in G(M_X)$. Then there exist $u_1 \in Ax$, $u_2 \in Cy$, $v_1 \in Xx$ and $v_2 \in By$ such that $u = u_1 + u_2$ and $v = v_1 + v_2$. This implies that

$$Q_{M_X} M_X \begin{bmatrix} x \\ y \end{bmatrix} = M_X \begin{bmatrix} u \\ v \end{bmatrix}.$$

Notice that $\left[\begin{smallmatrix} u' \\ v' \end{smallmatrix}\right] \in Q_{M_X} \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ if and only if $\left[\begin{smallmatrix} u' \\ v' \end{smallmatrix}\right] - \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in \overline{M_X(0)}$, i.e.,

$$\begin{cases} u' - u \in \overline{A(0) + C(0)}, \\ v' - v \in \overline{X(0) + B(0)}. \end{cases}$$

That is equivalent to

$$\begin{cases} u' \in Q_{[A\ C]}u = Q_{[A\ C]}(u_1 + u_2) = Q_{[A\ C]}u_1 + Q_{[A\ C]}u_2, \\ v' \in Q_{[X\ B]}v = Q_{[X\ B]}(v_1 + v_2) = Q_{[X\ B]}v_1 + Q_{[X\ B]}v_2. \end{cases}$$

This shows that

$$Q_{M_X}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} Q_{[A\ C]}u_1 & Q_{[A\ C]}u_2 \\ Q_{[X\ B]}v_1 & Q_{[X\ B]}v_2 \end{bmatrix}.$$

Since $u_1 \in Ax$, $u_2 \in Cy$, $v_1 \in Xx$ and $v_2 \in By$, it follows that

$$Q_{[A\ C]}u_1 = Q_{[A\ C]}Ax, Q_{[A\ C]}u_2 = Q_{[A\ C]}Cy,$$

$$Q_{[X\ B]}v_1 = Q_{[X\ B]}Xx, Q_{[X\ B]}v_2 = Q_{[X\ B]}By.$$

Therefore,

$$Q_{M_X}M_X\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Q_{[A\ C]}Ax & Q_{[A\ C]}Cy \\ Q_{[X\ B]}Xx & Q_{[X\ B]}By \end{bmatrix} = \begin{bmatrix} Q_{[A\ C]}A & Q_{[A\ C]}C \\ Q_{[X\ B]}X & Q_{[X\ B]}B \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

This completed the proof.
□

Lemma 2.12. *Let $A \in \mathcal{BR}(\mathcal{H}), B \in \mathcal{BR}(\mathcal{K})$, $C \in \mathcal{BR}(\mathcal{K}, \mathcal{H})$ and $X \in \mathcal{BR}(\mathcal{H}, \mathcal{K})$. Then M_X is closed if and only if $A(0) + C(0)$ and $X(0) + B(0)$ are closed.*

Proof. Let $A(0) + C(0)$ and $X(0) + B(0)$ are closed, then $M_X(0)$ is closed. By Lemma 2.6, we need only show that $Q_{M_X}M_X$ is closed in order to prove the desired result. Since $A \in \mathcal{BR}(\mathcal{H})$ and $\|Q_{[A\ C]}Ax\| \le \|Q_A Ax\| \le \|A\|\|x\|$, $x \in \mathcal{H}$, it follows that $Q_{[A\ C]}A \in \mathcal{B}(\mathcal{H}, \mathcal{H}/(A(0) + C(0))$. Similarly, we have

$$Q_{[A\ C]}C \in \mathcal{B}(\mathcal{K}, \mathcal{H}/(A(0) + C(0)), \ \ Q_{[X\ B]}X \in \mathcal{B}(\mathcal{H}, \mathcal{K}/(X(0) + B(0))$$

and $Q_{[X\ B]}B \in \mathcal{B}(\mathcal{K}, \mathcal{K}/(X(0) + B(0))$. Therefore,

$$Q_{M_X}M_X \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}, \mathcal{H} \oplus \mathcal{K}/M_X(0)),$$

and hence $Q_{M_X}M_X$ is closed.

Conversely, if M_X is closed, then $M_X(0)$ is closed, and hence $A(0) + C(0)$ and $X(0) + B(0)$ are closed.
□

Lemma 2.13. *Let $A \in \mathcal{BCR}(\mathcal{H}), B \in \mathcal{BCR}(\mathcal{K})$, $C \in \mathcal{BCR}(\mathcal{K}, \mathcal{H})$ and $X \in \mathcal{BCR}(\mathcal{H}, \mathcal{K})$ with $A(0) + C(0)$ and $X(0) + B(0)$ are closed. Then the following statements are hold:*

(i) *The adjoint M_X^* is an operator;*

(ii) *The explicit form of M_X^* is given by*

$$M_X^* = \begin{bmatrix} A^* & X^* \\ C^* & B^* \end{bmatrix} : \begin{bmatrix} (A(0) + C(0))^\perp \\ (X(0) + B(0))^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}.$$

Proof. By Lemma 2.11, $\|Q_{M_X}M_X\| < \infty$ and hence $\|M_X\| < \infty$. It follows from the closedness of M_X^* that $\mathrm{dom}M_X^*$ is closed according to Lemma 2.7. Lemma 2.12 ensures that M_X is closed, which together with the closedness of relations A, C, X and B, we obtain that

$$\mathrm{dom}M_X^* = M_X(0)^\perp = (A(0) + C(0))^\perp \oplus (X(0) + B(0))^\perp$$

and

$$\operatorname{dom}\begin{bmatrix} A^* & X^* \\ C^* & B^* \end{bmatrix} = (\operatorname{dom}A^* \cap \operatorname{dom}C^*) \oplus (\operatorname{dom}X^* \cap \operatorname{dom}B^*)$$
$$= (A(0)^{\perp} \cap C(0)^{\perp}) \oplus (X(0)^{\perp} \cap B(0)^{\perp}).$$

This together with

$$A(0)^{\perp} \cap C(0)^{\perp} = (A(0) + C(0))^{\perp}, X(0)^{\perp} \cap B(0)^{\perp} = (X(0) + B(0))^{\perp}$$

implies that

$$\operatorname{dom}M_X^* = \operatorname{dom}\begin{bmatrix} A^* & X^* \\ C^* & B^* \end{bmatrix}.$$

Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{H} \oplus \mathcal{K}$ and $\left[\begin{smallmatrix} x^* \\ y^* \end{smallmatrix}\right] \in \operatorname{dom}M_X^* = \operatorname{dom}\left[\begin{smallmatrix} A^* & X^* \\ C^* & B^* \end{smallmatrix}\right]$. Then

$$\begin{aligned}\langle M_X^* \begin{bmatrix} x^* \\ y^* \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\rangle &= \langle \begin{bmatrix} x^* \\ y^* \end{bmatrix}, \begin{bmatrix} Ax + Cy \\ Xx + By \end{bmatrix}\rangle \\ &= \langle x^*, Ax + Cy\rangle + \langle y^*, Xx + By\rangle \\ &= \langle A^*x^*, x\rangle + \langle C^*x^*, y\rangle + \langle X^*y^*, x\rangle + \langle B^*y^*, y\rangle \\ &= \langle A^*x^* + X^*y^*, x\rangle + \langle C^*x^* + B^*y^*, y\rangle \\ &= \langle \begin{bmatrix} A^* & X^* \\ C^* & B^* \end{bmatrix}\begin{bmatrix} x^* \\ y^* \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}\rangle,\end{aligned}$$

and hence

$$M_X^* = \begin{bmatrix} A^* & X^* \\ C^* & B^* \end{bmatrix}.$$

Finally, we emphasize the key conclusion that M_X^* is is an operator. Since $A \in \mathcal{BCR}(\mathcal{H})$, $\operatorname{dom}A$ is dense in $\mathcal{H}$, so $A^*(0) = \operatorname{dom}A^{\perp} = \{0\}$. Similarly, we have that $C^*(0) = \{0\}$, $X^*(0) = \{0\}$ and $B^*(0) = \{0\}$, which means that $M_X^*(0) = \{0\}$, i.e. M_X^* is an operator. □

Theorem 2.14. *Let $A \in \mathcal{B}(\mathcal{H})$, $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and $m \in Z^+$. If $[A\ C] \in \Phi_-(\mathcal{H} \oplus \mathcal{K}, \mathcal{K})$, C is non-compact and $\mathcal{N}(A \mid C)$ contains a non-compact operator, then there exists $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $A + CX \in \Phi(\mathcal{H})$ and* $\operatorname{ind}(A + CX) = m$.

Proof. The core idea is to transform the problem of finding $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $A + CX$ is Fredholm with $\operatorname{ind}(A + CX) = m$ into an equivalent problem involving the block operator matrix

$$M_Y := \begin{bmatrix} A & C \\ Y & I \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$$

where $Y = -X$. Using the matrix identity (2), we show this equivalence preserves Fredholmness and index. We then employ a decomposition technique (Lemma 2.1) to separate C into components with different norm properties. The proof proceeds by constructing suitable operators Z and Y in two cases (depending on whether a certain component C_ϵ is compact or not) to make M_Y Fredholm with $\operatorname{ind}(A + CX) = m$, which yields the desired $X = -Y$.

Since

$$\begin{bmatrix} I & -C \\ 0 & I \end{bmatrix}\begin{bmatrix} A & C \\ -X & I \end{bmatrix}\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} A + CX & 0 \\ 0 & I \end{bmatrix}, \tag{2}$$

it follows that $A + CX \in \Phi(\mathcal{H})$ with $\mathrm{ind}(A + CX) = m$ if and only if $\left[\begin{smallmatrix} A & C \\ -X & I \end{smallmatrix}\right] \in \Phi(\mathcal{H} \oplus \mathcal{K})$ with $\mathrm{ind}\left[\begin{smallmatrix} A & C \\ -X & I \end{smallmatrix}\right] = m$. Write

$$M_Y := \begin{bmatrix} A & C \\ Y & I \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}).$$

By Lemma 2.1, for any $\epsilon > 0$, there exist the orthogonal decompositions $\mathcal{K} = \mathcal{K}_\epsilon \oplus \mathcal{K}^\epsilon$ and $\mathcal{H} = \mathcal{H}_\epsilon \oplus \mathcal{H}^\epsilon$ such that

$$\begin{aligned} &C(\mathcal{K}_\epsilon) \subset \mathcal{H}_\epsilon, \ \ \|Cx\| \leq \epsilon\|x\| \text{ for all } x \in \mathcal{K}_\epsilon, \\ &C(\mathcal{K}^\epsilon) \subset \mathcal{H}^\epsilon, \ \ \|Cx\| \geq \epsilon\|x\| \text{ for all } x \in \mathcal{K}^\epsilon. \end{aligned}$$

Actually, we have

$$C = \begin{bmatrix} C|_{\mathcal{K}_\epsilon} & 0 \\ 0 & C|_{\mathcal{K}^\epsilon} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_\epsilon \\ \mathcal{K}^\epsilon \end{bmatrix} \to \begin{bmatrix} \mathcal{H}_\epsilon \\ \mathcal{H}^\epsilon \end{bmatrix}.$$

It is clear that $C|_{\mathcal{K}^\epsilon}$ is left invertible for $0 < \epsilon < \|C\|$ and $[A\ C]$, as an operator from $\mathcal{H} \oplus \mathcal{K}^\epsilon \oplus \mathcal{K}_\epsilon$ to $\mathrm{ran}(C|_{\mathcal{K}^\epsilon}) \oplus (\mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \oplus \mathcal{H}_\epsilon)$, has the block representation

$$[A\ C] = \begin{bmatrix} A_1(\epsilon) & C^\epsilon & 0 \\ A_2(\epsilon) & 0 & C_\epsilon \end{bmatrix},$$

where

$$C|_{\mathcal{K}^\epsilon} = \begin{bmatrix} C^\epsilon \\ 0 \end{bmatrix} : \mathcal{K}^\epsilon \to \begin{bmatrix} \mathrm{ran}(C|_{\mathcal{K}^\epsilon}) \\ \mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \end{bmatrix}, \ \ C_\epsilon = \begin{bmatrix} 0 \\ C|_{\mathcal{K}_\epsilon} \end{bmatrix} : \mathcal{K}_\epsilon \to \begin{bmatrix} \mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \\ \mathcal{H}_\epsilon \end{bmatrix}$$

with $C^\epsilon : \mathcal{K}^\epsilon \to \mathrm{ran}(C|_{\mathcal{K}^\epsilon})$ invertible and $\|C_\epsilon\| \leq \epsilon$. Then M_Y admits the following new representation

$$M_Y = \begin{bmatrix} A_1(\epsilon) & C^\epsilon & 0 \\ A_2(\epsilon) & 0 & C_\epsilon \\ Y & I_1(\epsilon) & I_2(\epsilon) \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K}^\epsilon \\ \mathcal{K}_\epsilon \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(C|_{\mathcal{K}^\epsilon}) \\ \mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \oplus \mathcal{H}_\epsilon \\ \mathcal{K} \end{bmatrix},$$

and hence the invertible operators $U \in \mathcal{B}(\mathrm{ran}(C|_{\mathcal{K}^\epsilon}) \oplus (\mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \oplus \mathcal{H}_\epsilon) \oplus \mathcal{K})$ and $V \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K}^\epsilon \oplus \mathcal{K}_\epsilon)$ given by

$$U := \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -I_1(\epsilon)(C^\epsilon)^{-1} & 0 & I \end{bmatrix}, \quad V := \begin{bmatrix} I & 0 & 0 \\ -(C^\epsilon)^{-1}A_1(\epsilon) & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

are such that

$$UM_YV = \begin{bmatrix} 0 & C^\epsilon & 0 \\ A_2(\epsilon) & 0 & C_\epsilon \\ Y - I_1(\epsilon)(C^\epsilon)^{-1}A_1(\epsilon) & 0 & I_2(\epsilon) \end{bmatrix}. \tag{3}$$

Since

$$\left\| \begin{bmatrix} A_1(\epsilon) & C^\epsilon & 0 \\ A_2(\epsilon) & 0 & C_\epsilon \end{bmatrix} - \begin{bmatrix} A_1(\epsilon) & C^\epsilon & 0 \\ A_2(\epsilon) & 0 & 0 \end{bmatrix} \right\| = \|C_\epsilon\| \leq \epsilon,$$

by Lemma 2.2, the right Fredholmness of $[A\ \ C]$ implies that

$$[A\ C|_{\mathcal{K}^\epsilon}] = \begin{bmatrix} A_1(\epsilon) & C^\epsilon \\ A_2(\epsilon) & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K}^\epsilon \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(C|_{\mathcal{K}^\epsilon}) \\ \mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \oplus \mathcal{H}_\epsilon \end{bmatrix}$$

is right Fredholm for sufficiently small $\epsilon > 0$. For such $\epsilon > 0$, $A_2(\epsilon)$ is then a right Fredholm operator. Let $\mathrm{Ker}A_2(\epsilon) = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are closed subspaces of $\mathrm{Ker}A_2(\epsilon)$, $\dim\mathcal{M}_1 = \infty$ and $\dim\mathcal{M}_2 = m + d(A_2(\epsilon))$.

If C_ϵ is compact, then $n(A_2(\epsilon)) = d(I_2(\epsilon)) = \infty$ from $\mathcal{N}(A \mid C)$ contains a non-compact operator and C is a non-compact operator. Define

$$Z = \begin{bmatrix} Z_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \\ \mathrm{Ker}(A_2(\epsilon))^\perp \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(I_2(\epsilon))^\perp \\ \mathrm{ran}(I_2(\epsilon)) \end{bmatrix},$$

where $Z_1 : \mathcal{M}_1 \to \mathcal{R}(I_2(\epsilon))^\perp$ is a unitary operator. Then $\begin{bmatrix} A_2(\epsilon) & 0 \\ Z & I_2(\epsilon) \end{bmatrix}$ is a Fredholm operator with $\mathrm{ind}\begin{bmatrix} A_2(\epsilon) & 0 \\ Z & I_2(\epsilon) \end{bmatrix} = m$ and so is

$$\begin{bmatrix} A_2(\epsilon) & C_\epsilon \\ Z & I_2(\epsilon) \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K}_\epsilon \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \oplus \mathcal{H}_\epsilon \\ \mathcal{K} \end{bmatrix}. \tag{4}$$

Define $Y := Z + B_1(\epsilon)(C^\epsilon)^{-1}A_1(\epsilon)$ and we have from (3) that $M_Y \in \Phi(\mathcal{H} \oplus \mathcal{K})$ with $\mathrm{ind}M_Y = m$. Define $X = -Y$. Then we get the desired result from (2).

In the following, we suppose that C_ϵ is a non-compact operator. Since $\mathcal{K}_\epsilon = \mathrm{Ker}(I_2(\epsilon))^\perp$ and $\mathrm{ran}(C|_{\mathcal{K}^\epsilon})^\perp \oplus \mathcal{H}_\epsilon = \mathrm{ran}(A_2(\epsilon)) \oplus \mathrm{ran}(A_2(\epsilon))^\perp$, the operator matrix defined as in (4) can be written as the following new representation

$$\begin{bmatrix} A_{21}(\epsilon) & C_\epsilon^1 & C_\epsilon^2 \\ 0 & C_\epsilon^3 & C_\epsilon^4 \\ Z & I_{21}(\epsilon) & I_{22}(\epsilon) \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_2(\epsilon)) \\ \mathrm{ran}(A_2(\epsilon))^\perp \\ \mathcal{K} \end{bmatrix}, \tag{5}$$

where $A_{21}(\epsilon)$ is right invertible and $I_{21}(\epsilon)$ is left invertible. Obviously, C_ϵ^1 is non-compact, and $C_\epsilon^2, C_\epsilon^3$, C_ϵ^4 and $I_{22}(\epsilon)$ are of finite rank. Since $A_{21}(\epsilon)$ is right invertible and $I_{21}(\epsilon) : \mathcal{M}_1 \to \mathcal{K}$ is left invertible and C_ϵ^1 is non-compact, $\mathcal{N}(A_{21}(\epsilon) \mid C_\epsilon^1)$ and $\mathcal{N}(I_{21}^*(\epsilon) \mid C_\epsilon^{1*})$ clearly contain non-compact operators. Using [21, Theorem 1], we can choose a suitable $Z \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that

$$\begin{bmatrix} A_{21}(\epsilon) & C_\epsilon^1 \\ Z & I_{21}(\epsilon) \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{M}_1 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_2(\epsilon)) \\ \mathcal{K} \end{bmatrix}$$

is invertible. This together with $d(A_2(\epsilon)) = 0$ and $\dim\mathcal{M}_2 = m + d(A_2(\epsilon))$ deduces that

$$\begin{bmatrix} A_{21}(\epsilon) & C_\epsilon^1 & 0 \\ 0 & 0 & 0 \\ Z & I_{21}(\epsilon) & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(A_2(\epsilon)) \\ \mathrm{ran}(A_2(\epsilon))^\perp \\ \mathcal{K} \end{bmatrix}$$

is a Fredholm operator and

$$\mathrm{ind}\begin{bmatrix} A_{21}(\epsilon) & C_\epsilon^1 & 0 \\ 0 & 0 & 0 \\ Z & I_{21}(\epsilon) & 0 \end{bmatrix} = m.$$

From (5), Lemma 2.3 and (3), it follows that $M_Y \in \Phi(\mathcal{H} \oplus \mathcal{K})$ with $\mathrm{ind}M_Y = m$ for $Y = Z + I_1(\epsilon)(C^\epsilon)^{-1}A_1(\epsilon)$. Define $X = -Y$. Then we immediately have the desired result from (2). □

3. Main Results

In this section, we present proofs of the main results of this paper, i.e., Theorems 3.1, 3.3, 3.5, 3.8, 3.10 and 3.12. As their corollaries, some related properties are also mentioned.

Theorem 3.1. *Let* $A \in \mathcal{BCR}(\mathcal{H})$, $B \in \mathcal{BCR}(\mathcal{K})$ *and* $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$ *with* $A(0)+C(0)$ *is closed. Then* $M_X \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$ *for some* $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *if and only if* $[A\ C] \in \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$, $[B^*\ C^*\ |_{(A(0)+C(0))^\perp}] \in \Phi_-(B(0)^\perp\oplus(A(0)+C(0))^\perp,\mathcal{K})$, *and one of the following statements is fulfilled:*

(i) $\mathcal{N}(A \mid C)$ *and* $\mathcal{N}(B^* \mid (C^*\ |_{(A(0)+C(0))^\perp}))$ *contain non-compact operators;*

(ii) $M_0 = \begin{bmatrix} A & C \\ 0 & B\end{bmatrix} \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$.

Proof. The sufficiency proof is constructive. The core idea is to use the assumptions on $[A\ C]$ to build an invertible operator L that block-diagonalizes the quotient map $M_X Q_{M_X}$.This reduces the problem to finding an X such that a specific compression becomes Fredholm with a prescribed index.

Sufficiency. Let $\mathcal{H}' = \text{ran}A + \text{ran}C$. If assertion (ii) holds, then the result is trival. Now assume that assertion (i) holds. It is clear that $n([A\ C]) = \infty$. Then there exists a left invertible operator $\begin{bmatrix} E \\ F\end{bmatrix} : \mathcal{K} \to \mathcal{H}\oplus\mathcal{K}$ such that $\text{ran}\begin{bmatrix} E \\ F\end{bmatrix} = \ker[A\ C]$. Since $Q_{[A\ C]}[A\ \ C] = [Q_{[A\ C]}A\ \ Q_{[A\ C]}C]$ and $A(0) + C(0)$ is closed, it follows that

$$\ker[A\ \ C] = \ker Q_{[A\ C]}[A\ \ C] = \ker[Q_{[A\ C]}A\ \ Q_{[A\ C]}C],$$

and hence

$$(Q_{[A\ C]}A)E + (Q_{[A\ C]}C)F = 0. \tag{6}$$

In view of $[A\ \ C] \in \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$, we obtain that

$$[Q_{[A\ C]}A\ \ Q_{[A\ C]}C] \in \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H}/(A(0) + C(0))).$$

Then there exists an invertible operator

$$\begin{bmatrix} Y \\ Z\end{bmatrix} : \mathcal{H}'/(A(0) + C(0)) \to \ker[Q_{[A\ C]}A\ \ Q_{[A\ C]}C]^\perp$$

such that

$$[Q_{[A\ C]}A\ \ Q_{[A\ C]}C]\begin{bmatrix} Y \\ Z\end{bmatrix} = Q_{[A\ C]}AY + Q_{[A\ C]}CZ = I_{\mathcal{H}'/(A(0)+C(0))}. \tag{7}$$

Put

$$L = \begin{bmatrix} Y & E \\ Z & F\end{bmatrix} : \begin{bmatrix} \mathcal{H}'/(A(0) + C(0)) \\ \mathcal{K}\end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K}\end{bmatrix}.$$

Then, $L : \mathcal{H}'/(A(0) + C(0)) \oplus \mathcal{K} \to \mathcal{H}\oplus\mathcal{K}$ is an invertible operator. Indeed, since $\begin{bmatrix} E \\ F\end{bmatrix} : \mathcal{K} \to \mathcal{H}\oplus\mathcal{K}$ is a left invertible operator, then there exists a right invertible operator $[Q\ \ R] : \mathcal{H}\oplus\mathcal{K} \to \mathcal{K}$ such that $QE + RF = I_{\mathcal{K}}$. From the relation

$$\begin{bmatrix} Q_{[A\ C]}A & Q_{[A\ C]}C \\ Q & R\end{bmatrix}\begin{bmatrix} Y & E \\ Z & F\end{bmatrix} = \begin{bmatrix} I_{\mathcal{H}'/(A(0)+C(0))} & 0 \\ QY + RZ & I_{\mathcal{K}}\end{bmatrix},$$

we derive that L is left invertible. By (7), we have that

$$\text{ran}\begin{bmatrix} Y \\ Z\end{bmatrix} + \ker[Q_{[A\ C]}A\ \ Q_{[A\ C]}C] = \mathcal{H}\oplus\mathcal{K},$$

and hence

$$\text{ran}L = \text{ran}\begin{bmatrix} Y \\ Z\end{bmatrix} + \text{ran}\begin{bmatrix} E \\ F\end{bmatrix} = \mathcal{H}\oplus\mathcal{K}.$$

This proves the invertibility of L. It is clear that

$$Q_{M_X}M_XL=\begin{bmatrix} I_{\mathcal{H}'/(A(0)+C(0))} & 0 \\ (Q_BX)Y+(Q_BB)Z & (Q_BX)E+(Q_BB)F \end{bmatrix} \tag{8}$$

from Lemma 2.11. Since $\mathcal{N}(A \mid C)$ contains a non-compact operator, it follows that there exists a closed infinite dimensional subspace $\mathcal{M}\subseteq\mathcal{H}$ such that

$$\mathrm{ran}A\mid_{\mathcal{M}}+C(0)\subseteq\mathrm{ran}C+A(0),$$

which means that $\mathcal{M}\subseteq\mathrm{ran}E$, and hence E is a non-compact operator from Lemma 2.5. By Lemmas 2.9 and 2.10, we obtain that $(Q_{[A\ \ C]}C)^*=C^*J_{(A(0)+C(0))^\perp}$, which together with $[B^*\ C^*\mid_{(A(0)+C(0))^\perp}]\in\Phi_-(B(0)^\perp\oplus(A(0)+C(0))^\perp,\mathcal{K})$ and $\mathrm{ran}B^*=\mathrm{ran}(Q_BB)^*$ implies that

$$[(Q_{[A\ \ C]}C)^*\ \ (Q_BB)^*]\in\Phi_-((A(0)+C(0))^\perp\oplus B(0)^\perp,\mathcal{K}).$$

Then

$$\begin{bmatrix} Q_{[A\ \ C]}C \\ Q_BB \end{bmatrix}\in\Phi_+(\mathcal{K},\mathcal{H}/(A(0)+C(0))\oplus\mathcal{K}/B(0)),$$

it follows that there exists $[G_1\ \ L_1]:\mathcal{H}\oplus\mathcal{K}\to\mathcal{K}$ such that

$$G_1Q_{[A\ \ C]}C+L_1Q_BB=I_{\mathcal{K}}+K, \tag{9}$$

where $K\in\mathcal{B}(\mathcal{K})$ is a finite rank operator. Also since $\left[\begin{smallmatrix} E \\ F\end{smallmatrix}\right]:\mathcal{K}\to\mathcal{H}\oplus\mathcal{K}$ is left invertible, then there exists $[G_2\ \ L_2]:\mathcal{H}/(A(0)+C(0))\oplus\mathcal{K}/B(0)\to\mathcal{K}$ such that

$$G_2E+L_2F=I_{\mathcal{K}}. \tag{10}$$

By (9), we have $G_1(Q_{[A\ \ C]}C)F+L_1(Q_BB)F=(I_{\mathcal{K}}+K)F=F+FK$, which together with (10) and (6) implies that

$$(G_2-L_2G_1(Q_{[A\ \ C]}A))E+L_2L_1(Q_BB)F=I_{\mathcal{K}}+FK.$$

By Lemma 2.4, we obtain that

$$\begin{bmatrix} E \\ (Q_BB)F \end{bmatrix}\in\Phi_+(\mathcal{K},\mathcal{H}\oplus\mathcal{K}/B(0)),$$

and hence $[((Q_BB)F)^*\ \ E^*]\in\Phi_-(B(0)^\perp\oplus\mathcal{H},\mathcal{K})$. Since

$$G\in\mathcal{N}(B^*\mid(C^*\mid_{(A(0)+C(0))^\perp})),$$

then there exists $L_3\in\mathcal{B}(\mathcal{H})$ such that $B^*G=C^*\mid_{(A(0)+C(0))^\perp}L_3$, which together with $\mathrm{dom}B^*=B(0)^\perp$ implies that $B^*J_{B(0)^\perp}G=C^*J_{(A(0)+C(0))^\perp}L_3$, and therefore $(Q_BB)^*G=(Q_{[A\ \ C]}C)^*L_3$. From

$$((Q_BB)F)^*G=F^*(Q_BB)^*G=F^*(Q_{[A\ \ C]}C)^*L_3=-E^*(Q_{[A\ \ C]}A)^*L_3,$$

we have $G\in\mathcal{N}(((Q_BB)F)^*\mid E^*)$. Notice that, G is a non-compact operator. Then, by Lemma 2.14, there exists $X_1\in\mathcal{B}(\mathcal{H},\mathcal{K}/B(0))$ such that $(Q_BB)F+X_1E$ is Fredholm and

$$\mathrm{ind}((Q_BB)F+X_1E)=\dim\mathcal{H}'^\perp. \tag{11}$$

Define $X\in\mathcal{B}(\mathcal{H},\mathcal{K})$ by

$$Xx=P_{B(0)^\perp}y,\ \ x\in\mathcal{H},$$

where $[y] = X_1x$. It is clear that $X_1 = Q_BX$. By (8), we have $\mathrm{ran}Q_{M_X}M_XL$ is closed, $n(Q_{M_X}M_XL) = n((Q_BB)F + X_1E)$ and

$$d(Q_{M_X}M_XL) = d((Q_BB)F + X_1E) + \dim\mathcal{H}'^{\perp}.$$

This together with (11) demonstrates that $Q_{M_X}M_XL$ is Weyl. By Lemma 2.8, we have $M_X \in \Phi_0(\mathcal{H} \oplus \mathcal{K})$.

Necessity. Assume that $M_X \in \Phi_0(\mathcal{H} \oplus \mathcal{K})$ for some $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Since B is a closed relation, it follows that $B(0)$ is closed. Notice that $A(0) + C(0)$ is closed. Then M_X admits the following representation

$$M_X = \begin{bmatrix} A_1 & C_1 \\ A_2 & C_2 \\ X_1 & B_1 \\ X_2 & B - B \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} (A(0) + C(0))^{\perp} \\ A(0) + C(0) \\ B(0)^{\perp} \\ B(0) \end{bmatrix}. \tag{12}$$

Clearly, we have

$$M'_{X_1} = \begin{bmatrix} A_1 & C_1 \\ X_1 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} (A(0) + C(0))^{\perp} \\ B(0)^{\perp} \end{bmatrix} \tag{13}$$

is a Weyl operator, it follows that there exist

$$\begin{bmatrix} E & S \\ H & T \end{bmatrix} \in \mathcal{B}((A(0) + C(0))^{\perp} \oplus B(0)^{\perp}, \mathcal{H} \oplus \mathcal{K})$$

and finite rank operators $F_{11} \in \mathcal{B}((A(0) + C(0))^{\perp}, \mathcal{H})$, $F_{12} \in \mathcal{B}(B(0)^{\perp}, \mathcal{H})$ $F_{21} \in \mathcal{B}((A(0) + C(0))^{\perp}, \mathcal{K})$, $F_{22} \in \mathcal{B}(B(0)^{\perp}, \mathcal{K})$ such that

$$\begin{bmatrix} A_1 & C_1 \\ X_1 & B_1 \end{bmatrix} \begin{bmatrix} E & S \\ H & T \end{bmatrix} = \begin{bmatrix} I_{(A(0)+C(0))^{\perp}} & 0 \\ 0 & I_{B(0)^{\perp}} \end{bmatrix} + \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}.$$

Therefore

$$A_1E + C_1H = I_{(A(0)+C(0))^{\perp}} + F_{11}, X_1S + B_1T = I_{B(0)^{\perp}} + F_{22}, A_1S + C_1T = F_{12}.$$

It follows from $A_1E + C_1H = I_{(A(0)+C(0))^{\perp}} + F_{11}$ that $[A_1\ C_1] \in \Phi_-(\mathcal{H} \oplus \mathcal{K}, (A(0) + C(0))^{\perp})$. This together with the closedness of $A(0) + C(0)$ implies that $[A\ C] \in \Phi_-(\mathcal{H} \oplus \mathcal{K}, \mathcal{H})$. By the Weylness of M_X and Lemma 2.13 implies that

$$M_X^* = \begin{bmatrix} A^* & X^* \\ C^* & B^* \end{bmatrix} : \begin{bmatrix} (A(0) + C(0))^{\perp} \\ B(0)^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$$

is Weyl. Therefore, $[B^*\ C^* |_{(A(0)+C(0))^{\perp}}] \in \Phi_-(B(0)^{\perp} \oplus (A(0) + C(0))^{\perp}, \mathcal{K})$.

Assume that $\mathcal{N}(A \mid C)$ contains only compact operators. Let $\left[\begin{smallmatrix} E \\ F \end{smallmatrix}\right] : \mathcal{K}_1 \to \mathcal{H} \oplus \mathcal{K}$ is a left invertible operator and $\mathrm{ran}\left[\begin{smallmatrix} E \\ F \end{smallmatrix}\right] = \ker[A\ C]$, where $\mathcal{K}_1$ is a Hilbert space with $\dim\mathcal{K}_1 = \dim\ker[A\ C]$. Similar to the proof of the sufficiency, there exists an invertible operator

$$\begin{bmatrix} Y \\ Z \end{bmatrix} : \mathcal{H}'/(A(0) + C(0)) \to \ker[Q_{[A\ C]}A\ \ Q_{[A\ C]}C]^{\perp}$$

such that

$$L = \begin{bmatrix} Y & E \\ Z & F \end{bmatrix} : \begin{bmatrix} \mathcal{H}'/(A(0) + C(0)) \\ \mathcal{K}_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}$$

is an invertible operator. Then we still have (8). If $\dim\mathcal{K}/B(0) = \infty$, then the Weylness of M_X and (8) implies that $\dim\mathcal{K}_1 = \infty$. From

$$\operatorname{ran}\begin{bmatrix} E \\ F \end{bmatrix} = \ker[Q_{[A\ C]}A\ \ Q_{[A\ C]}C] = \ker[A\ C],$$

we get that there exists a unitary operator $V : \mathcal{K} \to \mathcal{K}_1$ such that $EV \in \mathcal{N}(A \mid C)$. Since $\mathcal{N}(A \mid C)$ contains only compact operators, then E is a compact operator. If $\dim\mathcal{K}/B(0) < \infty$, then $\dim\mathcal{K}_1 < \infty$, which implies that $E : \mathcal{K}_1 \to \mathcal{H}$ is a compact operator. From (8), the Weylness of M_X implies that $(Q_B B)F + XE$ is Fredholm and $\operatorname{ind}((Q_B B)F + XE) = \dim\mathcal{H}'^{\perp}$. Notice that $E : \mathcal{K}_1 \to \mathcal{H}$ is a compact operator, it follows from Lemma 2.3 that $(Q_B B)F$ is Fredholm and

$$\operatorname{ind}((Q_B B)F) = \dim\mathcal{H}'^{\perp}. \tag{14}$$

Take $X = 0$ in (8). Then it follows from (8), the Fredholmness of $(Q_B B)F$ and (14) that $Q_{M_0}M_0 L$ is Weyl. By Lemma 2.8, we obtain that $M_0 \in \Phi_0(\mathcal{H} \oplus \mathcal{K})$.

Now assume that $\mathcal{N}(B^* \mid (C^* \mid_{(A(0)+C(0))^{\perp}}))$ contains only compact operators. Since M_X^* is Weyl, then we get M_0^* is Weyl in the similar way as the proof of the above, and hence M_0 is Weyl. □

Corollary 3.2. *Let $A \in \mathcal{BCR}(\mathcal{H}), B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$ with $A(0) + C(0)$ is closed. Then*

$$\begin{aligned}\bigcap_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_w(M_X) = &\{\lambda \in \mathbb{C} : [A - \lambda I\ \ C] \notin \Phi_-(\mathcal{H} \oplus \mathcal{K}, \mathcal{H})\} \\ &\cup\{\lambda \in \mathbb{C} : [B^* - \overline{\lambda} I\ \ C^* \mid_{(A(0)+C(0))^{\perp}}] \notin \Phi_-(\mathcal{K} \oplus \mathcal{H}, \mathcal{K})\} \\ &\cup\{\lambda \in \sigma_w(\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]) : \text{either } \mathcal{N}(A - \lambda I \mid C) \text{ or} \\ &\quad \mathcal{N}(B^* - \overline{\lambda} I \mid (C^* \mid_{(A(0)+C(0))^{\perp}})) \text{ contains only compact operators}\}.\end{aligned}$$

Theorem 3.3. *Let $A \in \mathcal{BR}(\mathcal{H}), B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then $M_X \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ if and only if $[A\ C] \in \Phi_-(\mathcal{H} \oplus \mathcal{K}, \mathcal{H})$, and one of the following statements is fulfilled:*

(i) $\mathcal{N}(A \mid C)$ *contains a non-compact operator;*

(ii) $M_0 = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right] \in \Phi_-^+(\mathcal{H} \oplus \mathcal{K})$.

Proof. Sufficiency. If assertion (ii) holds, then sufficiency is clear. Now assume that $\mathcal{N}(A \mid C)$ contains a non-compact operator. By Lemma 2.5, we have that there exists a closed a closed infinite dimensional subspace $\mathcal{M} \subseteq \mathcal{H}$ such that

$$\operatorname{ran}A \mid_{\mathcal{M}} + C(0) \subseteq \operatorname{ran}C + A(0),$$

and hence

$$\operatorname{ran}P_{(A(0)+C(0))^{\perp}}AP_{\mathcal{M}} \subseteq \operatorname{ran}P_{(A(0)+C(0))^{\perp}}C \subseteq \operatorname{dom}(P_{(A(0)+C(0))^{\perp}}C)^{+}.$$

Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are closed infinite-dimensional subspaces of $\mathcal{M}$. We take a right invertible operator $S \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with $\operatorname{Ker}S^{\perp} = \mathcal{M}_1$. Since $\operatorname{ran}P_{(A(0)+C(0))^{\perp}}AP_{\mathcal{M}_1} \subseteq \operatorname{ran}P_{(A(0)+C(0))^{\perp}}C$, it follows that

$$(P_{(A(0)+C(0))^{\perp}}C)^{\dagger}P_{(A(0)+C(0))^{\perp}}AP_{\mathcal{M}_1} \in \mathcal{B}(\mathcal{H},\mathcal{K}).$$

Define $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ by

$$X := S + P_{B(0)^{\perp}}B(P_{(A(0)+C(0))^{\perp}}C)^{\dagger}P_{(A(0)+C(0))^{\perp}}AP_{\mathcal{M}_1}.$$

Then $M_X \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$. Indeed, let $\mathcal{H}' = \operatorname{ran}A + \operatorname{ran}C$. We clearly have $\operatorname{ran}M_X \subseteq \mathcal{H}' \oplus \mathcal{K}$. Let $\left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in \mathcal{H}' \oplus \mathcal{K}$. Since $\operatorname{ran}A \mid_{\mathcal{M}} + C(0) \subseteq \operatorname{ran}C + A(0)$, there exist $x_0 \in \mathcal{M}^{\perp}$ and $y_0 \in \mathcal{K}$ such that $u_1 \in Ax_0 + Cy_0$. From the definition of S, it follows that $u_2 \in S\hat{x}_0 + By_0$ for some $\hat{x}_0 \in \mathcal{M}_1$. If we choose $x_1 := x_0 + \hat{x}_0$ and $y_1 := y_0 - (P_{(A(0)+C(0))^{\perp}}C)^{+}P_{(A(0)+C(0))^{\perp}}A\hat{x}_0$, then we get

$$
\begin{aligned}
M_X\begin{bmatrix}x_1\\y_1\end{bmatrix} &= \begin{bmatrix} Ax_0+Cy_0+A\hat{x}_0- \\ (P_{(A(0)+C(0))^\perp}C+P_{A(0)+C(0)}C)(P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}A\hat{x}_0 \\ \\ S\hat{x}_0+By_0+P_{B(0)^\perp}B(P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}AP_{\mathcal{M}_1}\hat{x}_0- \\ (P_{B(0)^\perp}B+B-B)(P_{(A(0)+C(0))^\perp}C)^+P_{(A(0)+C(0))^\perp}AP_{\mathcal{M}_1}\hat{x}_0 \end{bmatrix} \\
&= \begin{bmatrix} Ax_0+Cy_0+A\hat{x}_0-P_{(A(0)+C(0))^\perp}A\hat{x}_0 \\ S\hat{x}_0+By_0 \end{bmatrix} \\
&= \begin{bmatrix} Ax_0+Cy_0+A\hat{x}_0-P_{(A(0)+C(0))^\perp}A\hat{x}_0-P_{(A(0)+C(0))}A\hat{x}_0 \\ S\hat{x}_0+By_0 \end{bmatrix} \\
&= \begin{bmatrix} Ax_0+Cy_0 \\ S\hat{x}_0+By_0 \end{bmatrix},
\end{aligned}
$$

which means that

$$
\begin{bmatrix}u_1\\u_2\end{bmatrix} \in M_X\begin{bmatrix}x_1\\y_1\end{bmatrix},
$$

and hence $\mathcal{H}'\oplus\mathcal{K}\subseteq \operatorname{ran}M_X$. This together with $[A\ C]\in\Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$ demonstrates that $M_X\in\Phi_-(\mathcal{H}\oplus\mathcal{K})$. It remains to prove $n(M_X)\geq d(M_X)$. Indeed, there exists $y_0'\in\mathcal{K}$ such that $0\in Ax_0'+Cy_0'$ for all $x_0'\in\mathcal{M}_2$, since $\operatorname{ran}A\mid_{\mathcal{M}}+C(0)\subseteq\operatorname{ran}C+A(0)$. The right invertibility of S further implies $0\in S\hat{x}_0'+By_0'$ for some $\hat{x}_0'\in\mathcal{M}_1$. Define $x_1:=x_0'+\hat{x}_0'$ and $y_2:=y_0'-C_1^+A_1\hat{x}_0'$, then

$$
\begin{bmatrix}0\\0\end{bmatrix} \in M_X\begin{bmatrix}x_1\\y_2\end{bmatrix},
$$

i.e., $\left[\begin{smallmatrix}x_1\\y_2\end{smallmatrix}\right]\in\ker M_X$. The arbitrariness of $x_0'\in\mathcal{M}_2$ results in $n(M_X)=\infty>d(M_X)$.

Necessity. Assume that $M_X\in\Phi_-^+(\mathcal{H}\oplus\mathcal{K})$ for some $X\in\mathcal{B}(\mathcal{H},\mathcal{K})$. Since B is a closed relation, it follows that $B(0)$ is closed. Notice that $A(0)+C(0)$ is closed. Then M_X has the representation (12). Clearly, we have M'_{X_1} defined as in (13) is a right Weyl operator, it follows that $[A_1\ C_1]$ is right Fredholm. This together with the closedness of $A(0)+C(0)$ implies that $[A\ C]\in\Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$. Assume that $\mathcal{N}(A\mid C)$ contains only compact operators. Let $\left[\begin{smallmatrix}E\\F\end{smallmatrix}\right]:\mathcal{K}_1\to\mathcal{H}\oplus\mathcal{K}$ is a left invertible operator and $\operatorname{ran}\left[\begin{smallmatrix}E\\F\end{smallmatrix}\right]=\ker[A\ C]$, where $\mathcal{K}_1$ is a Hilbert space with $\dim\mathcal{K}_1=\dim\ker[A\ C]$. Similar to the proof of the necessity of Theorem 3.1, we obtain that $M_0=\left[\begin{smallmatrix}A&C\\0&B\end{smallmatrix}\right]\in\Phi_-^+(\mathcal{H}\oplus\mathcal{K})$.
□

Corollary 3.4. *Let $A\in\mathcal{BR}(\mathcal{H}), B\in\mathcal{BR}(\mathcal{K})$ and $C\in\mathcal{BR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then*

$$
\begin{aligned}
\bigcap_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})}\sigma_{rw}(M_X) = \{\lambda\in\mathbb{C}:\ [A-\lambda I\ \ C]\notin\Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})\} \\
\cup\{\lambda\in\sigma_{rw}(\left[\begin{smallmatrix}A&C\\0&B\end{smallmatrix}\right]):\mathcal{N}(A-\lambda I\mid C)\text{ contains only compact operators}\}.
\end{aligned}
$$

Theorem 3.5. *Let $A\in\mathcal{BCR}(\mathcal{H}), B\in\mathcal{BCR}(\mathcal{K})$ and $C\in\mathcal{BCR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then $M_X\in\Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for some $X\in\mathcal{B}(\mathcal{H},\mathcal{K})$ if and only if $[B^*\ C^*\mid_{(A(0)+C(0))^\perp}]\in\Phi_-(B(0)^\perp\oplus(A(0)+C(0))^\perp,\mathcal{K})$, and one of the following statements is fulfilled:*

(i) $\mathcal{N}(B^*\ \mid(C^*\mid_{(A(0)+C(0))^\perp}))$ *contains a non-compact operator;*

(ii) $M_0=\left[\begin{smallmatrix}A&C\\0&B\end{smallmatrix}\right]\in\Phi_+^-(\mathcal{H}\oplus\mathcal{K})$.

Proof. Since B is a closed relation, it follows that $B(0)$ is closed, which together with the closedness of $A(0)+C(0)$ implies that M_X is a closed relation follows from Lemma 2.12. Notice that M_X is left Weyl if and only if

$$M_X^* = \begin{bmatrix} B^* & C^* \\ X^* & A^* \end{bmatrix} : \begin{bmatrix} B(0)^\perp \\ (A(0)+C(0))^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{H} \end{bmatrix}$$

is right Weyl. According to Theorem 3.3, we immediately obtain the desired result. □

Corollary 3.6. *Let $A \in \mathcal{BCR}(\mathcal{H})$, $B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then*

$$\bigcap_{X\in\mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_{lw}(M_X) = \{\lambda \in \mathbb{C} : [B^* - \overline{\lambda} I \;\; C^* |_{(A(0)+C(0))^\perp}] \notin \Phi_-(\mathcal{K}\oplus\mathcal{H},\mathcal{K})\}$$
$$\cup\{\lambda \in \sigma_{lw}(\left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right]) : \mathcal{N}(B^* - \overline{\lambda} I \mid (C^* |_{(A(0)+C(0))^\perp})) \text{ contains } \text{only compact operators}\}.$$

Remark 3.7. Theorems 3.1, 3.3 and 3.5 extended the Theorems 3.9, 3.2 and 3.5 of [22] to the general case of linear relations.

The Weylness and right (left) Weylness of relation matrices $M_X = \left[\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right]$ with $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is presented above. In the sequel, we turn our attention to the analogues for which the unknown element $X : \mathcal{H} \to \mathcal{K}$ is taken as a bounded linear relation.

Theorem 3.8. *Let $A \in \mathcal{BCR}(\mathcal{H})$, $B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$. If $A(0)+C(0)$ is closed, then $M_X \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ with $X(0)+B(0)$ is closed, if and only if there exists a constant relation $S \in \mathcal{BR}(\mathcal{K})$ such that $B(0)+S(0)$ is closed,*

$$[B^* |_{(B(0)+S(0))^\perp} \;\; C^* |_{(A(0)+C(0))^\perp}] \in \Phi_-((B(0)+S(0))^\perp \oplus (A(0)+C(0))^\perp, \mathcal{K}),$$

$[A \;\; C] \in \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$, and one of the following statements is fulfilled:
(i) *$\mathcal{N}(A \mid C)$ and $\mathcal{N}((B^* |_{(B(0)+S(0))^\perp}) \mid (C^* |_{(A(0)+C(0))^\perp}))$ contain non-compact operators;*
(ii) *$M_0^S = \left[\begin{smallmatrix} A & C \\ 0 & B+S \end{smallmatrix}\right] \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$.*

Proof. Sufficiency. Write $M_X^S = \left[\begin{smallmatrix} A & C \\ X & B+S \end{smallmatrix}\right] \in \mathcal{BR}(\mathcal{H}\oplus\mathcal{K})$. Assume that there exists a constant relation $S \in \mathcal{BR}(\mathcal{K})$ such that $[B^* |_{(B(0)+S(0))^\perp} \;\; C^* |_{(A(0)+C(0))^\perp}] \in \Phi_-((B(0)+S(0))^\perp \oplus (A(0)+C(0))^\perp, \mathcal{K})$, $[A \;\; C] \in \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$ and one of the assertions (i) or (ii) hold. Notice that

$$(B+S)^* |_{(B(0)+S(0))^\perp} = B^* |_{(B(0)+S(0))^\perp} \tag{15}$$

and

$$(M_0^S)^* = \begin{bmatrix} B^* & C^* \\ 0 & A^* \end{bmatrix} : \begin{bmatrix} (B(0)+S(0))^\perp \\ (A(0)+C(0))^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{H} \end{bmatrix}. \tag{16}$$

By Theorem 3.1, we obtain that

$$M_{X_1}^S = \begin{bmatrix} A & C \\ X_1 & B+S \end{bmatrix} \in \Phi_0(\mathcal{H}\oplus\mathcal{K}) \tag{17}$$

for some $X_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Define $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ by

$$X = X_1 + X - X,$$

where $(X - X)x = S(0)$ for $x \in \mathcal{H}$. It is clear that $M_X \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$.

Necessity. Let $M_X \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ with $X(0)+B(0)$ is closed. Define $S \in \mathcal{BR}(\mathcal{K})$ by $Sx = X(0)$ for $x \in \mathcal{K}$, then $B(0)+S(0)$ is closed. This together with $M_X \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$ implies that

$$M_X^S = \begin{bmatrix} A & C \\ X & B+S \end{bmatrix} \in \Phi_0(\mathcal{H}\oplus\mathcal{K}),$$

and hence

$$(M_X^S)^* = \begin{bmatrix} B^* & C^* \\ X^* & A^* \end{bmatrix} : \begin{bmatrix} (B(0)+S(0))^{\perp} \\ (A(0)+C(0))^{\perp} \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{H} \end{bmatrix}$$

is Weyl. From Theorem 3.1, the desired result follows right away. □

Corollary 3.9. *Let $A \in \mathcal{BCR}(\mathcal{H}), B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then,*

$$\begin{aligned}
&\bigcap_{\substack{X\in\mathcal{BR}(\mathcal{H},\mathcal{K})\\ \overline{X(0)+B(0)}=X(0)+B(0)}} \sigma_w(M_X)\\
&= \{\lambda \in \mathbb{C} : [A-\lambda I \;\; C] \notin \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})\}\\
&\quad \cup\{\lambda \in \mathbb{C} : [B^* - \overline{\lambda} I\,|_{(B(0)+S(0))^{\perp}} \;\; C^*\,|_{(A(0)+C(0))^{\perp}}]\\
&\quad \notin \Phi_-((B(0)+S(0))^{\perp}\oplus(A(0)+C(0))^{\perp},\mathcal{K})\\
&\quad \text{for any constant relation } S\in\mathcal{BR}(\mathcal{K}) \text{ with } \overline{S(0)+B(0)} = S(0)+B(0)\}\\
&\quad \cup\{\lambda \in \sigma_w(\left[\begin{smallmatrix} A & C \\ 0 & B+S \end{smallmatrix}\right]) : \textit{either } \mathcal{N}(A-\lambda I \mid C) \text{ or}\\
&\quad \mathcal{N}((B^*-\overline{\lambda} I\,|_{(B(0)+S(0))^{\perp}}) \mid (C^*\,|_{(A(0)+C(0))^{\perp}}))\\
&\quad \text{contains only compact operators for any constant relation}\\
&\quad S\in\mathcal{BR}(\mathcal{K}) \text{ with } \overline{S(0)+B(0)} = S(0)+B(0)\}.
\end{aligned}$$

Theorem 3.10. *Let $A \in \mathcal{BR}(\mathcal{H}), B \in \mathcal{BR}(\mathcal{K})$ and $C \in \mathcal{BR}(\mathcal{K},\mathcal{H})$. If $A(0)+C(0)$ is closed, then $M_X \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ with $X(0)+B(0)$ is closed, if and only if $[A\ C] \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$.*

Proof. Let $[A\ C] \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$. Define $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ by $Xx = \mathcal{K}$, for $x \in \mathcal{H}$. It is clear that $X(0)+B(0) = \mathcal{K}$ is closed, which together with the closedness of $A(0)+C(0)$ implies that M_X is a closed relation follows from Lemma 2.12. On the other hand, since $\mathrm{ran}M_X = \mathrm{ran}[A\ C]\oplus\mathcal{K}$ and $\ker M_X = \ker[A\ C]$, it follows that $n(M_X) = n([A\ C])$ and $d(M_X) = d([A\ C])$. This proves that $M_X \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$.

Conversely, Let $A(0)+C(0)$ is closed. Assume that $M_X \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ with $X(0)+B(0)$ is closed. Then M_X has the following relation matrix representation:

$$M_X = \begin{bmatrix} A_1 & C_1 \\ A_2 & C_2 \\ X_1 & B_1 \\ X_2 & B_2 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} (A(0)+C(0))^{\perp} \\ A(0)+C(0) \\ (X(0)+B(0))^{\perp} \\ X(0)+B(0) \end{bmatrix}.$$

Clearly, we have

$$M'_{X_1} = \begin{bmatrix} A_1 & C_1 \\ X_1 & B_1 \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \to \begin{bmatrix} (A(0)+C(0))^{\perp} \\ (X(0)+B(0))^{\perp} \end{bmatrix}$$

is right Weyl. Similar to the proof of Theorem 3.3, we obtain that $[A\ C] \in \Phi_-(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$. Since $d([A\ C]) \le d(M_X)$ and $n(M_X) \le n([A\ C])$, which together with $d(M_X) \le n(M_X)$ implies that $d([A\ C]) \le n([A\ C])$. This prove that $[A\ C] \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K},\mathcal{H})$.

□

Corollary 3.11. *Let $A \in \mathcal{BR}(\mathcal{H}), B \in \mathcal{BR}(\mathcal{K})$ and $C \in \mathcal{BR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then*

$$\bigcap_{\substack{X\in\mathcal{BR}(\mathcal{H},\mathcal{K})\\ \overline{X(0)+B(0)}=X(0)+B(0)}} \sigma_{rw}(M_X) = \{\lambda \in \mathbb{C} : [A-\lambda I \;\; C] \notin \Phi_-^+(\mathcal{H}\oplus\mathcal{K},\mathcal{H})\}.$$

Theorem 3.12. *Let $A \in \mathcal{BCR}(\mathcal{H}), B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$. If $A(0)+C(0)$ is closed, then $M_X \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ with $X(0)+B(0)$ is closed, if and only if there exists a constant relation $S \in \mathcal{BR}(\mathcal{K})$ such that $B(0)+S(0)$ is closed,*

$$[B^* |_{(B(0)+S(0))^\perp} \;\; C^* |_{(A(0)+C(0))^\perp}] \in \Phi_-((B(0)+S(0))^\perp \oplus (A(0)+C(0))^\perp, \mathcal{K}),$$

and one of the following statements is fulfilled:
(i) $\mathcal{N}((B^* |_{(B(0)+S(0))^\perp}) \mid (C^* |_{(A(0)+C(0))^\perp}))$ *contains a non-compact operator;*
(ii) $M_0^S = \left[\begin{smallmatrix} A & C \\ 0 & B+S \end{smallmatrix}\right] \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$.

Proof. Sufficiency. Write $M_X^S = \left[\begin{smallmatrix} A & C \\ X & B+S \end{smallmatrix}\right] \in \mathcal{BR}(\mathcal{H}\oplus\mathcal{K})$. Assume that there exists a constant relation $S \in \mathcal{BR}(\mathcal{K})$ such that $[B^* |_{(B(0)+S(0))^\perp} \;\; C^* |_{(A(0)+C(0))^\perp}] \in \Phi_-((B(0)+S(0))^\perp \oplus (A(0)+C(0))^\perp, \mathcal{K})$, and one of the assertions (i) or (ii) hold. Notice that (15) and (16). By Theorem 3.5, we obtain that $M_{X_1}^S$ defined as in (17) is a left Weyl relation for some $X_1 \in \mathcal{B}(\mathcal{H},\mathcal{K})$. Define $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ by

$$X = X_1 + X - X,$$

where $(X-X)x = S(0)$ for $x \in \mathcal{H}$. It is clear that $M_X \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$.

Necessity. Let $M_X \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{H},\mathcal{K})$ with $X(0)+B(0)$ is closed. Define $S \in \mathcal{BR}(\mathcal{K})$ by $Sx = X(0)$ for $x \in \mathcal{K}$, then $B(0)+S(0)$ is closed. This together with $M_X \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$, implies that $M_X^S \in \Phi_+^-(\mathcal{H}\oplus\mathcal{K})$, and hence

$$(M_X^S)^* = \begin{bmatrix} B^* & C^* \\ X^* & A^* \end{bmatrix} : \begin{bmatrix} (B(0)+S(0))^\perp \\ (A(0)+C(0))^\perp \end{bmatrix} \to \begin{bmatrix} \mathcal{K} \\ \mathcal{H} \end{bmatrix}$$

is right Weyl. From Theorem 3.3, the desired result follows right away. □

Corollary 3.13. *Let $A \in \mathcal{BCR}(\mathcal{H}), B \in \mathcal{BCR}(\mathcal{K})$ and $C \in \mathcal{BCR}(\mathcal{K},\mathcal{H})$ with $A(0)+C(0)$ is closed. Then,*

$$\begin{aligned}
&\bigcap_{\substack{X\in\mathcal{BR}(\mathcal{H},\mathcal{K}) \\ \overline{X(0)+B(0)}=X(0)+B(0)}} \sigma_{lw}(M_X) \\
&= \{\lambda \in \mathbb{C} : [B^* - \overline{\lambda} I |_{(B(0)+S(0))^\perp} \;\; C^* |_{(A(0)+C(0))^\perp}] \\
&\quad \notin \Phi_-((B(0)+S(0))^\perp \oplus (A(0)+C(0))^\perp, \mathcal{K}) \\
&\quad \text{for any constant relation } S \in \mathcal{BR}(\mathcal{K}) \text{ with } \overline{S(0)+B(0)} = S(0)+B(0)\} \\
&\quad \cup \{\lambda \in \sigma_{lw}(\left[\begin{smallmatrix} A & C \\ 0 & B+S \end{smallmatrix}\right]) : \mathcal{N}((B^* - \overline{\lambda} I |_{(B(0)+S(0))^\perp}) \mid (C^* |_{(A(0)+C(0))^\perp})) \\
&\quad \text{contains only compact operators for any constant relation} \\
&\quad S \in \mathcal{BR}(\mathcal{K}) \text{ with } \overline{S(0)+B(0)} = S(0)+B(0)\}.
\end{aligned}$$

Now, we see the Weylness and right (left) Weylness of upper triangular relation matrix

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathcal{BR}(\mathcal{H}\oplus\mathcal{K}).$$

The following three Corollaries are direct consequences of Theorems 3.1, 3.3 and 3.5.

Corollary 3.14 (see [11, Corollary 3.2]). *Let $A \in \mathcal{BCR}(\mathcal{H})$ and $B \in \mathcal{BCR}(\mathcal{K})$. Then $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if $A \in \Phi_+(\mathcal{H})$, $B \in \Phi_-(\mathcal{K})$, and one of the following statements is fulfilled:*
(i) $d(A) = n(B) = \infty$;
(ii) $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] \in \Phi_0(\mathcal{H}\oplus\mathcal{K})$.

Corollary 3.15. *Let $A \in \mathcal{BR}(\mathcal{H})$ and $B \in \mathcal{BR}(\mathcal{K})$ with $A(0)$ is closed. Then $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$ for some $X \in \mathcal{B}(\mathcal{K},\mathcal{H})$ if and only if $B \in \Phi_-(\mathcal{K})$, and one of the following statements is fulfilled:*
(i) $n(B) = \infty$;
(ii) $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] \in \Phi_-^+(\mathcal{H}\oplus\mathcal{K})$.

Remark 3.16. Corollary 3.15 is an extension of Corollary 3.1 of [11], since A is a closed relation, it follows that $A(0)$ is closed.

Corollary 3.17. *Let $A \in \mathcal{BR}(\mathcal{H})$ and $B \in \mathcal{BR}(\mathcal{K})$ with $B(0)$ is closed. Then $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $A \in \Phi_+(\mathcal{H})$, and one of the following statements is fulfilled:*
(i) $d(A) = \infty$;
(ii) $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right] \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$.

Remark 3.18. Theorem 3.1 of [11] follows from Corollary 3.17 immediately, since B is a closed relation, it follows that $B(0)$ is closed.

Corollary 3.19. *Let $A \in \mathcal{BCR}(\mathcal{H})$ and $B \in \mathcal{BCR}(\mathcal{K})$. Then $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] \in \Phi_0(\mathcal{H} \oplus \mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{K}, \mathcal{H})$ with $A(0) + X(0)$ is closed, if and only if there exists a constant relation $S \in \mathcal{BR}(\mathcal{H})$ such that $A(0) + S(0)$ is closed, $A^* \mid_{(A(0)+S(0))^\perp} \in \Phi_-((A(0) + S(0))^\perp, \mathcal{H})$, $B \in \Phi_-(\mathcal{K})$, and one of the following statements is fulfilled:*
(i) $n(A^* \mid_{(A(0)+S(0))^\perp}) = n(B) = \infty$;
(ii) $M_0^S = \left[\begin{smallmatrix} A+S & 0 \\ 0 & B \end{smallmatrix}\right] \in \Phi_0(\mathcal{H} \oplus \mathcal{K})$.

Proof. By Theorem 3.8 and Lemma 2.5, we easily obtain the desired result. □

Corollary 3.20. *Let $A \in \mathcal{BR}(\mathcal{H})$ and $B \in \mathcal{BR}(\mathcal{K})$. Then $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] \in \Phi_-^+(\mathcal{H} \oplus \mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{K}, \mathcal{H})$ with $A(0) + X(0)$ is closed, if and only if $B \in \Phi_-(\mathcal{K})$.*

Proof. Notice that $[B \ \ 0] \in \Phi_-^+(\mathcal{K} \oplus \mathcal{H}, \mathcal{K})$ if and only if $B \in \Phi_-(\mathcal{K})$. From Theorem 3.10, we have the result. □

Corollary 3.21. *Let $A \in \mathcal{BCR}(\mathcal{H})$ and $B \in \mathcal{BCR}(\mathcal{K})$. Then $\left[\begin{smallmatrix} A & X \\ 0 & B \end{smallmatrix}\right] \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$ for some $X \in \mathcal{BR}(\mathcal{K}, \mathcal{H})$ with $A(0) + X(0)$ is closed, if and only if there exists a constant relation $S \in \mathcal{BR}(\mathcal{H})$ such that $A(0) + S(0)$ is closed, $A^* \mid_{(A(0)+S(0))^\perp} \in \Phi_-((A(0) + S(0))^\perp, \mathcal{H})$, and one of the following statements is fulfilled:*
(i) $n(A^* \mid_{(A(0)+S(0))^\perp}) = \infty$;
(ii) $M_0^S = \left[\begin{smallmatrix} A+S & 0 \\ 0 & B \end{smallmatrix}\right] \in \Phi_+^-(\mathcal{H} \oplus \mathcal{K})$.

Proof. By Theorem 3.8 and Lemma 2.5, we easily obtain the desired result. □

We conclude this section with two illustrating examples of the previous results.

Example 3.22. *Let $\mathcal{H} = \mathcal{K} = \ell^2$, and let $A, B, C \in \mathcal{BR}(\ell^2)$ be defined by*

$$Ax = (0, 0, x_1, 0, x_5, 0, x_9, 0, x_{13}, \cdots),\ Bx = (x_1, x_2, \tfrac{x_3}{3}, x_4, \tfrac{x_5}{5}, x_6, \tfrac{x_7}{7}, \cdots),$$
$$Cx = (0, 0, x_1, 0, x_3, 0, x_5, 0, x_7, \cdots) + C(0)$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$, where

$$C(0) = \{(0, 0, 0, x_2, 0, x_4, 0, x_6, 0, x_8, \cdots) : (x_1, x_2, x_3, \cdots) \in \ell^2\}.$$

Then we claim that $M_X = \left[\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right] \in \Phi_0(\ell^2 \oplus \ell^2)$ for some $X \in \mathcal{B}(\ell^2)$.

Direct calculations show that $[A \ C] \in \Phi_-(\ell^2 \oplus \ell^2, \ell^2)$, $[B^* \ C^* \mid_{(A(0)+C(0))^\perp}] \in \Phi_-(B(0)^\perp \oplus (A(0) + C(0))^\perp, \mathcal{K})$, both $\mathcal{N}(A \mid C)$ and $\mathcal{N}(B^* \mid (C^* \mid_{(A(0)+C(0))^\perp}))$ contain non-compact operators. By Theorem 3.1, there exists $X \in \mathcal{B}(\ell^2)$ such that $M_X \in \Phi_0(\ell^2 \oplus \ell^2)$. Indeed, define $X \in \mathcal{B}(\ell^2)$ by

$$(x_3 + x_1, 0, x_4 + \frac{x_5}{3}, 0, x_5 + \frac{x_9}{5}, 0, x_6 + \frac{x_{13}}{7}, 0, \cdots)$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$. Then we can check that ranM_X is closed and $n(M_X) = d(M_X) = 2$, and hence $M_X \in \Phi_0(\ell^2 \oplus \ell^2)$.

Example 3.23. *Let $\mathcal{H} = \mathcal{K} = \ell^2$, and let $A, B, C \in \mathcal{BR}(\ell^2)$ be defined by*

$$Ax = (0, 0, x_3, 0, x_4, 0, x_5, 0, x_6, \cdots),\ Bx = (0, 0, x_2, x_3, x_4, x_5, \cdots),$$
$$Cx = (0, 0, 0, x_2, 0, x_3, 0, x_4, 0, x_5, \cdots)$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$. Then we claim that $M_X = \left[\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right] \notin \Phi_0(\ell^2 \oplus \ell^2)$ for any $X \in \mathcal{B}(\ell^2)$, but $M_X = \left[\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right] \in \Phi_0(\ell^2 \oplus \ell^2)$ for some $X \in \mathcal{BR}(\ell^2)$.

Direct calculations verify that $[A\ C] \in \Phi_-(\ell^2 \oplus \ell^2, \ell^2)$, but $M_0 = \left[\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right] \notin \Phi_0(\ell^2 \oplus \ell^2)$ and $\mathcal{N}(A \mid C)$ contains only compact operators. By Theorem 3.1, there does not exist any $X \in \mathcal{B}(\ell^2)$ such that $M_X \in \Phi_0(\ell^2 \oplus \ell^2)$. Define a constant relation $S \in \mathcal{BR}(\ell^2)$ by

$$Sx = \mathcal{M}$$

for $x = (x_1, x_2, x_3, \cdots) \in \ell^2$, where $\mathcal{M} = \{(x_1, 0, 0, 0, \cdots) : (x_1, x_2, x_3, \cdots) \in \ell^2\}$. It is easy to see that $B(0) + S(0)$ and $\operatorname{ran} M_0^S = \operatorname{ran} \left[\begin{smallmatrix} A & C \\ 0 & B+S \end{smallmatrix}\right]$ are closed and $n(M_0^S) = d(M_0^S) = 3$, then $M_0^S \in \Phi_0(\ell^2 \oplus \ell^2)$. By Theorem 3.8, there exists $X \in \mathcal{BR}(\ell^2)$ such that $M_X \in \Phi_0(\ell^2 \oplus \ell^2)$. In fact, define a relation $X \in \mathcal{BR}(\ell^2)$ by $X := S$. Therefore, $M_X = \left[\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right] \in \Phi_0(\ell^2 \oplus \ell^2)$.

4. Applications

In this section, we illustrate how Corollary 3.2 can be applied to address problems in the stabilization of singular systems in control theory. A typical problem in this area involves stabilizing a singular plant via state feedback. Consider a system described by the state equation:

$$\dot{x} = Ax + Cu,$$

(Note: The output equation $y = Bx$ is not required for the state feedback stabilization problem.) the goal is to find a state feedback operator X such that the control law $u = Xx$ stabilizes the system. This problem can be reformulated using the relation matrix:

$$M_X = \begin{bmatrix} A & C \\ X & I \end{bmatrix}.$$

A key stabilization objective is to ensure that M_X is is a Weyl relation, which corresponds to a well-posed closed-loop system with desirable spectral properties, particularly stability. A practical goal is to place the Weyl spectrum $\sigma_w(M_X)$ in the left half of the complex plane, ensuring exponential stability. In this context, Corollary 3.2 plays a fundamental role. It characterizes the spectral points that are unavoidable-those lying in the intersection

$$\bigcap_{X \in \mathcal{B}(\mathcal{H},\mathcal{K})} \sigma_w(M_X),$$

regardless of the choice of feedback operator X. This characterization leads to a systematic two-step procedure for assessing stabilizability:

Assess Intrinsic Stabilizability: The first step is to verify that all unavoidable spectral points (i.e., those in the intersection described above) lie in the left half-plane. If any unavoidable point lies in the closed right half-plane, the system cannot be stabilized by any feedback law X.

Design a Stabilizing Feedback: If the system is intrinsically stabilizable (all unavoidable points are stable), Corollary 3.2 provides a complete characterization of the feedback operators X that achieve stabilization. The corollary explicitly identifies the λ-dependent conditions that must be satisfied to exclude the remaining, avoidable spectral points from the closed right half-plane.

In summary, Corollary 3.2 transforms the stabilization problem for a broad class of singular systems into a concrete spectral analysis task. It clearly separates the fundamental limitations (unavoidable spectra) from the design freedom (tunable spectra), thereby providing control theorists with a powerful and precise tool for analyzing stabilizability and synthesizing stabilizing controllers. This application underscores the practical value of the abstract results presented in this work.

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