Filomat 40:2 (2026), 515–535
https://doi.org/10.2298/FIL2602515B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some new Boole's inequalities with respect to the middle ending point using k-conformable fractional integral operators via different classes of differentiable functions

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Abstract. The objective of this study is to formulate a new definition of k-conformable fractional operators and to elucidate their boundedness and semigroup characteristics. Furthermore, we introduce an innovative identity for differentiable h-convex functions that employs k-conformable fractional operators. By employing this identity, we have formulated novel inequalities applicable to bounded functions as well as Lipschitzian functions. In the subsequent section, we utilized h-convexity in conjunction with the power-mean inequality and Hölder's inequality, respectively.

1. Introduction and preliminaries

Fractional calculus has recently received a lot of interest because of its numerous applications in various scientific domains. Because of their importance, several fractional integral operators have been introduced and explored. The Riemann-Liouville fractional operator is one of the most current fractional operators used in fractional calculus, and it has gotten a lot of attention recently as a result of its research in numerous scientific works (for example, see [11–16, 21, 22, 24, 27, 29–31]). Let f be a function in the space $L_1[a_1, a_2]$ (the set of all integrable functions defined on $[a_1, a_2]$). The left- and right-sided Riemann-Liouville fractional operators of order $\alpha > 0$ are expressed as follows:

$$\mathfrak{I}^{\alpha}_{a_1^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a_1}^{x} (x-\zeta)^{\alpha-1} f(\zeta) d\zeta, \quad a_1 < x \le a_2,$$

2020 *Mathematics Subject Classification*. Primary 26D07; Secondary 26D10, 26D15, 26A33, 26A51.
Keywords. k-Comformable fractional operators; h-Convex function; Middle ending point; Boole inequality; Hölder's inequality; Power-mean inequality; Lipschitzian functions; Bounded function.
Received: 23 August 2025; Accepted: 22 November 2025
Communicated by Dragan S. Djordjević
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$$\Im_{a_2^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{a_2} (\zeta - x)^{\alpha-1} f(\zeta) d\zeta, \quad a_1 \leq x < a_2,$$

where $\Gamma(\cdot)$ is defined for any values of $\alpha > 0$ as follows:

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \text{ with the property: } \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha).$$

Remark 1.1.

1. *The classical Riemann-Liouville fractional integrals are denoted by* $\Im_{a_1^+}^{\alpha} f(a_2)$ *and* $\Im_{a_2^-}^{\alpha} f(a_1)$.
2. *The middle ending point Riemann-Liouville fractional integrals are denoted by* $\Im_{a_1^+}^{\alpha} f\left(\frac{a_1+a_2}{2}\right)$ *and* $\Im_{a_2^-}^{\alpha} f\left(\frac{a_1+a_2}{2}\right)$.
3. *The middle starting point Riemann-Liouville fractional integrals are denoted by* $\Im_{\left(\frac{a_1+a_2}{2}\right)^+}^{\alpha} f(a_2)$ *and* $\Im_{\left(\frac{a_1+a_2}{2}\right)^-}^{\alpha} f(a_1)$.

In [25], the authors demonstrate the following fractional identity related to the third Simpson's rule, widely referred to as Simpson's 2/45 rule or Boole's rule.

Lemma 1.2. *Let $\alpha > 0$ and $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1([a_1, a_2])$, then the following identity holds:*

$$\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right] - \frac{\Gamma(\alpha+1)}{2(a_2-a_1)^{\alpha}}\left[\Im_{a_1^+}^{\alpha} f(a_2) + \Im_{a_2^-}^{\alpha} f(a_1)\right]$$

$$= \frac{(a_2-a_1)}{4} \int_0^1 G_{\alpha}(\zeta)\left[f'((1-\zeta)a_1 + \zeta a_2) - f'(\zeta a_1 + (1-\zeta)a_2)\right] d\zeta,$$

where

$$G_{\alpha}(\zeta) := \begin{cases} \zeta^{\alpha} - \frac{7}{90}, & if\ 0 \leq \zeta \leq \frac{1}{4}; \\ \zeta^{\alpha} - \frac{39}{90}, & if\ \frac{1}{4} \leq \zeta \leq \frac{1}{2}; \\ \zeta^{\alpha} - \frac{51}{90}, & if\ \frac{1}{2} \leq \zeta \leq \frac{3}{4}; \\ \zeta^{\alpha} - \frac{83}{90}, & if\ \frac{3}{4} \leq \zeta \leq 1. \end{cases}$$

The Boole inequality is determined for functions with convex absolute values of their first derivatives [25, Corollary 1]:

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\zeta) d\zeta\right|$$
$$\leq \frac{239(a_2-a_1)\left[\left|f'(a_1)\right| + \left|f'(a_2)\right|\right]}{6480}. \tag{1}$$

In recent work [6, 7], the authors introduce a novel class of functions termed B-functions, defined as follows:

Definition 1.3. *Let $H : [0, \infty) \to \mathbb{R}$ be a non-negative function. The function H is called a B-function if*

$$H(x-a_1) + H(a_2-x) \leq 2H\left(\frac{a_1+a_2}{2}\right), \tag{2}$$

where $a_1 \leq x \leq a_2$ *with* $a_1, a_2 \in [0, \infty)$.

If the inequality (2) is reversed, H is called A-function, or that H belongs to the class $A(a_1, a_2)$.

If we have equality in (2), H is called AB-function, or that H belongs to the class $AB(a_1, a_2)$.

In particular, taking $H = h$, $a_1 = 0$ and $a_2 = 1$ in (2), we obtain the inequality

$$h(x) + h(1 - x) \leq 2\,h\left(\frac{1}{2}\right). \tag{3}$$

Examples of such a function h satisfying the inequality (3) can be provided by $h_1(x) = 1$, $h_2(x) = x$ and $h_3(x) = x^s$ with $s \in (0, 1]$.

The application of convexity in functional analysis and optimization theory has rendered it a prominent subject of research. In [32], the author develops a new class of functions known as h-convex functions.

Definition 1.4. *Let* $h : [0, 1] \to \mathbb{R}$ *be a positive function and* $f : I \subset \mathbb{R} \to \mathbb{R}$. *For all* $x, y \in I$ *and* $\zeta \in [0, 1]$, *we define* f *as an h-convex function if*

$$f(\zeta x + (1 - \zeta)y) \leq h(\zeta)f(x) + h(1 - \zeta)f(y). \tag{4}$$

When the inequality (4) is reversed, f is referred to as an h-concave function.

Through setting up:

- $h(\zeta) = \zeta$, the idea of h-convexity evolves into the definition of a convex function.
- $h(\zeta) = 1$, the notion of h-convexity simplifies to that of P-functions [18, 26].
- $h(\zeta) = \zeta^s$, where $s \in (0, 1]$, the notion of h-convexity is streamlined to s-convex functions in the second sense [17].

The results concerning integral inequality fundamentally rely on two essential inequalities: Hölder inequality and the power-mean integral inequality.

Theorem 1.5 (Hölder inequality). *Let* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* Ψ *and* Φ *are real functions defined on* $[\lambda_1, \lambda_2]$ *and if* $|\Psi|^p$, $|\Phi|^q$ *are integrable functions on* $[\lambda_1, \lambda_2]$, *then*

$$\int_{\lambda_1}^{\lambda_2} |\Psi(\zeta)\Phi(\zeta)|\,d\zeta \leq \left(\int_{\lambda_1}^{\lambda_2} |\Psi(\zeta)|^p\,d\zeta\right)^{\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |\Phi(\zeta)|^q\,d\zeta\right)^{\frac{1}{q}}.$$

The power-mean integral inequality, derived from the Hölder inequality, can be expressed as follows:

Theorem 1.6 (Power-mean integral inequality). *Let* $p \geq 1$ *and* W, Φ *be two real functions defined on* $[\lambda_1, \lambda_2]$. *If* $|W|$, $|W||\Phi|^p$ *are integrable functions on* $[\lambda_1, \lambda_2]$, *then*

$$\int_{\lambda_1}^{\lambda_2} |W(\zeta)\Phi(\zeta)|\,d\zeta \leq \left(\int_{\lambda_1}^{\lambda_2} |W(\zeta)|\,d\zeta\right)^{1-\frac{1}{p}} \left(\int_{\lambda_1}^{\lambda_2} |W(\zeta)|\,|\Phi(\zeta)|^p\,d\zeta\right)^{\frac{1}{p}}.$$

For additional details and elaboration of the power-mean integral inequality, see references [8] and [19].

In another context, conformable fractional operators represent the latest category of fractional operators employed in fractional calculus, which has garnered significant attention recently due to its examination in numerous scientific publications (for example, see [1–3, 5, 9, 10, 20, 23, 33]).

2. k-conformable fractional integral operators

This section desires to define the left- and right-sided k-conformable fractional operators [28], stipulating the requisite condition that $\beta > 0$. Furthermore, we define the correct space within which these operators are bounded (for more detail and clarification, see [12]). Let $[a_1, a_2] \subseteq (0, +\infty)$, where $a_1 < a_2$.

Definition 2.1. *Let $k > 0$ and $\beta > 0$ be two real numbers, and let f be a function in the space $L[a_1, a_2]$. The left- and right-sided k-conformable fractional integral operators of order $\alpha > 0$ are expressed as follows:*

$$ {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f(x) := \frac{1}{k\,\Gamma_k(\alpha)} \int_{a_1}^{x} \left(\frac{(x-a_1)^{\beta} - (\zeta - a_1)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (\zeta - a_1)^{\beta-1} f(\zeta) d\zeta, \quad a_1 < x \le a_2 \tag{5} $$

and

$$ {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f(x) := \frac{1}{k\,\Gamma_k(\alpha)} \int_{x}^{a_2} \left(\frac{(a_2-x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1} f(\zeta) d\zeta, \quad a_1 \le x < a_2, \tag{6} $$

where $\Gamma_k(\cdot)$ and $\beta_k(\cdot,\cdot)$ are defined for any values of $k, x, y > 0$ as follows:

$$ \Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt, $$

and

$$ \beta_k(x; y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt, $$

verified the following properties:

$$ \Gamma_k(\alpha + k) = \alpha\,\Gamma_k(\alpha) \text{ and } \beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. $$

For $k = 1$, the two preceding functions $\Gamma_k(\cdot)$ and $\beta_k(\cdot,\cdot)$ reduce to the well-known Gamma and Beta Euler functions.
Let $1 \le p < +\infty$. The space $L_p^w[a_1, a_2]$ of all real-valued Lebesgue measurable functions f on $[a_1, a_2]$ with norm condition [4] :

$$ \| f \|_p^w = \left(\int_{a_1}^{a_2} | f(x) |^p \, w(x) dx \right)^{\frac{1}{p}} < \infty, $$

is known as weighted Lebesgue space, where w denotes a weight function.

1. Put $w \equiv 1$, the space $L_p^w[a_1, a_2]$ reduces to the classical Lebesgue space $L_p[a_1, a_2]$.
2. Choose $w(x) = (x - a_1)^{\beta-1}$, we get

$$ L_{X^p_{a_1}}[a_1, a_2] := \left\{ f : \ \|f\|_{X^p_{a_1}} = \left(\int_{a_1}^{a_2} \left| f(x) \right|^p (x - a_1)^{\beta-1} \, dx \right)^{\frac{1}{p}} < \infty \right\}. \tag{7} $$

3. Choose $w(x) = (a_2 - x)^{\beta-1}$, we get

$$ L_{X^p_{a_2}}[a_1, a_2] := \left\{ f : \ \|f\|_{X^p_{a_2}} = \left(\int_{a_1}^{a_2} \left| f(x) \right|^p (a_2 - x)^{\beta-1} \, dx \right)^{\frac{1}{p}} < \infty \right\}. \tag{8} $$

4. We denote

$$L_{X^p}[a_1,a_2] := L_{X^p_{a_1}}[a_1,a_2] \cap L_{X^p_{a_2}}[a_1,a_2]. \tag{9}$$

For example, consider the function $f(x) = x^{\frac{1}{p}}$, then $f \in L_{X^p_{a_1}}[a_1,a_2]$ and $f \in L_{X^p_{a_2}}[a_1,a_2]$. We conclude that

$$L_{X^p}[a_1,a_2] \neq \emptyset.$$

Remark 2.2. *Putting $\beta = 1$ gives the next identity:*

$$L_{X^p}[a_1,a_2] = L_{X^p_{a_1}}[a_1,a_2] = L_{X^p_{a_2}}[a_1,a_2] = L_p[a_1,a_2].$$

The boundedness of the k-conformable fractional operators is demonstrated in the following theorem.

Theorem 2.3. *Let $1 \le p < +\infty$. For any functions $f \in L_{X^p}[a_1,a_2]$, the k-conformable fractional operators (5) and (6) are well-defined, and we have*

$$ {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f(x) \in L_{X^p_{a_1}}[a_1,a_2] \text{ and } {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f(x) \in L_{X^p_{a_2}}[a_1,a_2]. \tag{10}$$

Moreover

$$\left\| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f \right\|_{X^p_{a_1}} \le C \left\| f \right\|_{X^p_{a_1}}, \tag{11}$$

and

$$\left\| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f \right\|_{X^p_{a_2}} \le C \left\| f \right\|_{X^p_{a_2}}, \tag{12}$$

where

$$C := \frac{1}{\Gamma_k^p(\alpha+k)} \left(\frac{(a_2-a_1)^{\beta}}{\beta} \right)^{\frac{p\alpha}{k}}. \tag{13}$$

Proof. 1. Let $p = 1$ and $f \in L_X[a_1,a_2]$. By applying Fubini's theorem, we obtain

$$\left\| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f \right\|_{X^p_{a_1}} = \int_{a_1}^{a_2} \left| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f(x) \right| (x-a_1)^{\beta-1} dx$$

$$= \int_{a_1}^{a_2} \left| \frac{1}{k\,\Gamma_k(\alpha)} \int_{a_1}^{x} \left(\frac{(x-a_1)^{\beta} - (\zeta-a_1)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (\zeta-a_1)^{\beta-1} f(\zeta) d\zeta \right| (x-a_1)^{\beta-1} dx$$

$$\le \frac{1}{k\,\Gamma_k(\alpha)} \int_{a_1}^{a_2} \int_{a_1}^{x} \left| f(\zeta) \right| \left(\frac{(x-a_1)^{\beta} - (\zeta-a_1)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (\zeta-a_1)^{\beta-1} (x-a_1)^{\beta-1} d\zeta\, dx$$

$$= \frac{1}{k\,\Gamma_k(\alpha)} \int_{a_1}^{a_2} \left| f(\zeta) \right| (\zeta-a_1)^{\beta-1} \left(\int_{\zeta}^{a_2} \left(\frac{(x-a_1)^{\beta} - (\zeta-a_1)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (x-a_1)^{\beta-1}\, dx \right) d\zeta$$

$$\le \frac{1}{\alpha\,\Gamma_k(\alpha)} \left(\frac{(a_2-a_1)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}} \int_{a_1}^{a_2} \left| f(\zeta) \right| (\zeta-a_1)^{\beta-1}\, d\zeta$$

$$= \frac{1}{\Gamma_k(\alpha+k)} \left(\frac{(a_2-a_1)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}} \left\| f \right\|_{X_{a_1}}.$$

This yields the required inequality (11). The same as when we demonstrate the first inequality (12).

2. Let $1 < p < +\infty$ where $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L_{X^p}[a_1, a_2]$, we have

$$\left\| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f \right\|^p_{X_{a_2}} = \int_{a_1}^{a_2} \left| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f(x) \right|^p (a_2 - x)^{\beta-1} dx$$

$$= \int_{a_1}^{a_2} \left| \frac{1}{k\,\Gamma_k(\alpha)} \int_x^{a_2} \left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1} f(\zeta) d\zeta \right|^p (a_2 - x)^{\beta-1} dx$$

$$= \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_{a_1}^{a_2} \left| \int_x^{a_2} f(\zeta) \left[\left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1} \right]^{\frac{1}{p}+\frac{1}{p'}} d\zeta \right|^p (a_2 - x)^{\beta-1} dx.$$

Using Hölder inequality for $p \geq 1$, we get

$$\left\| {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f \right\|^p_{X^p_{a_2}} \leq \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_{a_1}^{a_2} \left| \int_x^{a_2} f^p(\zeta) \left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1}\, d\zeta \right|$$

$$\times \left| \int_x^{a_2} \left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1}\, d\zeta \right|^{p-1} (a_2 - x)^{\beta-1} dx$$

$$\leq \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_{a_1}^{a_2} \left(\int_x^{a_2} \left| f(\zeta) \right|^p \left(\frac{(a_2 - x1)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1}\, d\zeta \right)$$

$$\times \left(\int_x^{a_2} \left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1}\, d\zeta \right)^{p-1} (a_2 - x)^{\beta-1} dx$$

$$= \frac{k^{p-1}}{\alpha^{p-1}} \frac{1}{k^p\,\Gamma_k^p(\alpha)} \int_{a_1}^{a_2} \int_x^{a_2} \left| f^p(\zeta) \right| \left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1}$$

$$\times \left(\frac{(a_2 - x)^{\beta}}{\beta} \right)^{\frac{(p-1)\alpha}{k}} (a_2 - x)^{\beta-1}\, d\zeta\, dx$$

$$\leq \frac{\alpha}{k\,\Gamma_k^p(\alpha + k)} \left(\frac{(a_2 - a_1)^{\beta}}{\beta} \right)^{\frac{(p-1)\alpha}{k}}$$

$$\times \int_{a_1}^{a_2} \int_x^{a_2} \left| f^p(\zeta) \right| \left(\frac{(a_2 - x)^{\beta} - (a_2 - \zeta)^{\beta}}{\beta} \right)^{\frac{\alpha}{k}-1} (a_2 - \zeta)^{\beta-1} (a_2 - x)^{\beta-1}\, d\zeta\, dx.$$

Applying Fubini's theorem, we deduce

$$\left\|{}_{k}^{\beta}\mathfrak{I}_{a_2^-}^{\alpha}f\right\|_{X_{a_2}^p}^p \leq \frac{\alpha}{k\,\Gamma_k^p(\alpha+k)}\left(\frac{(a_2-a_1)^\beta}{\beta}\right)^{\frac{(p-1)\alpha}{k}}$$

$$\times\int_{a_1}^{a_2}\left|f^p(\zeta)\right|(a_2-\zeta)^{\beta-1}\left(\int_{a_1}^{\zeta}\left(\frac{(a_2-x)^\beta-(a_2-\zeta)^\beta}{\beta}\right)^{\frac{\alpha}{k}-1}(a_2-x)^{\beta-1}\,dx\right)d\zeta$$

$$\leq \frac{1}{\Gamma_k^p(\alpha+k)}\left(\frac{(a_2-a_1)^\beta}{\beta}\right)^{\frac{(p-1)\alpha}{k}}\int_{a_1}^{a_2}\left|f^p(\zeta)\right|(a_2-\zeta)^{\beta-1}\left(\frac{(a_2-a_1)^\beta}{\beta}\right)^{\frac{\alpha}{k}}d\zeta$$

$$= \frac{1}{\Gamma_k^p(\alpha+k)}\left(\frac{(a_2-a_1)^\beta}{\beta}\right)^{\frac{p\alpha}{k}}\int_{a_1}^{a_2}\left|f(\zeta)\right|^p(a_2-\zeta)^{\beta-1}d\zeta$$

$$= \frac{1}{\Gamma_k^p(\alpha+k)}\left(\frac{(a_2-a_1)^\beta}{\beta}\right)^{\frac{p\alpha}{k}}\left\|f\right\|_{X_{a_2}^p}^p .$$

This gives the desired inequality (12). In a similar manner, we prove the second inequality (11). □

Now, we demonstrate the commutativity and the semigroup properties of the k-conformable fractional operators.

Theorem 2.4. *Let $\alpha,\beta>0$, then*

$${}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\alpha}\left({}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\beta}f(x)\right) = {}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\alpha+\beta}f(x) = {}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\beta}\left({}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\alpha}f(x)\right). \tag{14}$$

$${}_{k}^{\beta}\mathfrak{I}_{a_2^-}^{\alpha}\left({}_{k}^{\beta}\mathfrak{I}_{a_2^-}^{\beta}f(x)\right) = {}_{k}^{\beta}\mathfrak{I}_{a_2^-}^{\alpha+\beta}f(x) = {}_{k}^{\beta}\mathfrak{I}_{a_2^-}^{\beta}\left({}_{k}^{\beta}\mathfrak{I}_{a_2^-}^{\alpha}f(x)\right). \tag{15}$$

Equations (14) and (15) are satisfied in almost every point for $f\in L_{X^p}[a_1,a_2]$.

Proof. Using Fubini's theorem, we get

$$\left[k^2\,\Gamma_k(\alpha)\,\Gamma_k(\beta)\right]\ {}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\alpha}\left({}_{k}^{\beta}\mathfrak{I}_{a_1^+}^{\beta}f(x)\right)$$

$$= \int_{a_1}^{x}\left(\frac{(x-a_1)^\beta-(\zeta-a_1)^\beta}{\beta}\right)^{\frac{\alpha}{k}-1}(\zeta-a_1)^{\beta-1}\left(\int_{a_1}^{\zeta}\left(\frac{(\zeta-a_1)^\beta-(s-a_1)^\beta}{\beta}\right)^{\frac{\beta}{k}-1}(s-a_1)^{\beta-1}f(s)ds\right)d\zeta$$

$$= \int_{a_1}^{x}(s-a_1)^{\beta-1}f(s)\left(\int_{s}^{x}\left(\frac{(x-a_1)^\beta-(\zeta-a_1)^\beta}{\beta}\right)^{\frac{\alpha}{k}-1}\left(\frac{(\zeta-a_1)^\beta-(s-a_1)^\beta}{\beta}\right)^{\frac{\beta}{k}-1}(\zeta-a_1)^{\beta-1}d\zeta\right)ds. \tag{16}$$

By the change of variable $(\zeta - a_1)^\beta = y\left((x-a_1)^\beta - (s-a_1)^\beta\right) + (s-a_1)^\beta$, we have

$$\int_s^x \left(\frac{(x-a_1)^\beta - (\zeta-a_1)^\beta}{\beta}\right)^{\frac{\alpha}{k}-1} \left(\frac{(\zeta-a_1)^\beta - (s-a_1)^\beta}{\beta}\right)^{\frac{\beta}{k}-1} (\zeta-a_1)^{\beta-1} d\zeta$$

$$= \int_0^1 \left(\frac{\left((x-a_1)^\beta - (s-a_1)^\beta\right)(1-y)}{\beta}\right)^{\frac{\alpha}{k}-1} \left(\frac{(x-a_1)^\beta - (s-a_1)^\beta}{\beta} \cdot y\right)^{\frac{\beta}{k}-1} \left(\frac{(x-a_1)^\beta - (s-a_1)^\beta}{\beta}\right) dy$$

$$= \left(\frac{(x-a_1)^\beta - (s-a_1)^\beta}{\beta}\right)^{\frac{\alpha+\beta}{k}-1} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} (y)^{\frac{\beta}{k}-1} dy$$

$$= \left(\frac{(x-a_1)^\beta - (s-a_1)^\beta}{\beta}\right)^{\frac{\alpha+\beta}{k}-1} k\beta_k(\alpha,\beta). \tag{17}$$

Using k-beta property and (17) in (16), we deduce that

$${}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+}\left({}^{\beta}_{k}\mathfrak{I}^{\beta}_{a_1^+} f(x)\right) = \frac{1}{k\Gamma_k(\alpha+\beta)} \int_{a_1}^x \left(\frac{(x-a_1)^\beta - (s-a_1)^\beta}{\beta}\right)^{\frac{\alpha+\beta}{k}-1} (s-a_1)^{\beta-1} f(s) ds.$$

Hence, we get the desired equality (14), similarly to the equality (15). □

The aim of the remaining results is to derive a new Boole's identity and using it to derive several Boole's inequalities incorporates with k-conformable fractional operators, and employing a middle-ending point approach that accounts for h-convex functions. Furthermore, by employing this identity, we will obtain some inequalities applicable to bounded functions as well as Lipschitzian functions.

3. Basic Identity

Lemma 3.1. *Let k, $\alpha, \beta > 0$ and $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1([a_1, a_2])$. Then, the following identity holds:*

$$\begin{aligned} &\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right] \\ &- \frac{\beta^{\frac{\alpha}{k}} 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1+a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1+a_2}{2}\right)\right] \\ &= \frac{(a_2-a_1)}{4}\int_0^1 D_{k\beta,\alpha}(\zeta)\left[f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right] d\zeta, \end{aligned} \tag{18}$$

where

$$D_{k,\beta,\alpha}(\zeta) := \begin{cases} \left(1-(1-\zeta)^\beta\right)^{\frac{\alpha}{k}} - \frac{6}{45}, & if\ 0 \le \zeta \le \frac{1}{2}; \\ \left(1-(1-\zeta)^\beta\right)^{\frac{\alpha}{k}} - \frac{38}{45}, & if\ \frac{1}{2} \le \zeta \le 1. \end{cases} \tag{19}$$

Proof. By using the integration by parts, we deduce

$$I_1 = \int_0^{\frac{1}{2}} \left(\left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{6}{45}\right)\left[f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta$$

$$= \left(\frac{2}{a_2-a_1}\right)\left(\left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{6}{45}\right)\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]\Bigg|_0^{\frac{1}{2}}$$

$$-\left(\frac{2\alpha\beta}{k(a_2-a_1)}\right)\int_0^{\frac{1}{2}} \left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1}$$

$$\times\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta$$

$$= \frac{2}{a_2-a_1}\left\{\left(\left(1-\left(\frac{1}{2}\right)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{6}{45}\right)\left(f\left(\frac{a_1+3a_2}{4}\right) + f\left(\frac{3a_1+a_2}{4}\right)\right) + \frac{6}{45}\left(2f\left(\frac{a_1+a_2}{2}\right)\right)\right\}$$

$$-\left(\frac{2\alpha\beta}{k(a_2-a_1)}\right)\int_0^{\frac{1}{2}} \left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1}$$

$$\times\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta.$$

In a similar manner, we obtain

$$I_2 = \int_{\frac{1}{2}}^1 \left(\left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{38}{45}\right)\left[f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta$$

$$= \left(\frac{2}{a_2-a_1}\right)\left(\left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{38}{45}\right)\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]\Bigg|_{\frac{1}{2}}^{1}$$

$$-\left(\frac{2\alpha\beta}{k(a_2-a_1)}\right)\int_{\frac{1}{2}}^1 \left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1}$$

$$\times\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta,$$

then

$$I_2 = \frac{2}{a_2-a_1}\left\{\frac{7}{45}\left(f(a_2)+f(a_1)\right) - \left(\left(1-\left(\frac{1}{2}\right)^{\rho}\right)^{\frac{\alpha}{k}} - \frac{38}{45}\right)\left(f\left(\frac{a_1+3a_2}{4}\right) + f\left(\frac{3a_1+a_2}{4}\right)\right)\right\}$$

$$-\left(\frac{2\alpha\beta}{k(a_2-a_1)}\right)\int_{\frac{1}{2}}^1 \left(1-(1-\zeta)^{\beta}\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1}\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta.$$

As a result, we get

$$I_1 + I_2 = \frac{4}{a_2 - a_1}\left\{\frac{7}{90}\left(f(a_1) + f(a_2)\right) + \frac{32}{90}\left(f\left(\frac{3a_1 + a_2}{4}\right) + f\left(\frac{a_1 + 3a_2}{4}\right)\right) + \frac{12}{90}\left(f\left(\frac{a_1 + a_2}{2}\right)\right)\right\}$$

$$-\left(\frac{2\alpha\beta}{k(a_2 - a_1)}\right)\int_0^1 \left(1 - (1-\zeta)^\beta\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1}$$

$$\times\left[f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) + f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta.$$

Since

$$\int_0^1 \left(1 - (1-\zeta)^\beta\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1} f\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)d\zeta$$

$$= \left(\frac{2}{a_2 - a_1}\right)^{\frac{\beta\alpha}{k}} \int_{\frac{a_1+a_2}{2}}^{a_2}\left(\left(\frac{a_2 - a_1}{2}\right)^\beta - (a_2 - \tau)^\beta\right)^{\frac{\alpha}{k}-1}(a_2 - \tau)^{\beta-1} f(\tau)\,d\tau$$

$$= \left(\frac{2}{a_2 - a_1}\right)^{\frac{\beta\alpha}{k}} \beta^{\frac{\alpha}{k}-1}\,\Gamma_k(\alpha)\ {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right),$$

we deduce

$$\int_0^1 \left(1 - (1-\zeta)^\beta\right)^{\frac{\alpha}{k}-1}(1-\zeta)^{\beta-1} f\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)d\zeta = \left(\frac{2}{a_2 - a_1}\right)^{\frac{\beta\alpha}{k}} \beta^{\frac{\alpha}{k}-1}\,\Gamma_k(\alpha)\ {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right).$$

Then, we obtain

$$\left[\frac{7}{90}\left(f(a_1) + f(a_2)\right) + \frac{32}{90}\left(f\left(\frac{3a_1 + a_2}{4}\right) + f\left(\frac{a_1 + 3a_2}{4}\right)\right) + \frac{12}{90}\left(f\left(\frac{a_1 + a_2}{2}\right)\right)\right]$$

$$= \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right] + \frac{a_2 - a_1}{4}\left(I_1 + I_2\right).$$

This gives the required result. □

Putting $k = \beta = 1$ in the Lemma 3.1 gives the next corollary.

Corollary 3.2. *Let $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function such that $f' \in L_1([a_1, a_2])$. Then, the following identity holds:*

$$\begin{aligned}
&\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right] \\
&- \frac{\Gamma(\alpha + 1)}{2^{1-\alpha}(a_2 - a_1)^\alpha}\left[\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + \mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right] \\
&= \frac{(a_2 - a_1)}{4}\int_0^1 G_\alpha(\zeta)\left[f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right]d\zeta,
\end{aligned} \tag{20}$$

where

$$G_\alpha(\zeta) := \begin{cases} \zeta^\alpha - \frac{6}{45}, & if\ 0 \le \zeta \le \frac{1}{2}; \\ \zeta^\alpha - \frac{38}{45}, & if\ \frac{1}{2} \le \zeta \le 1\,. \end{cases} \tag{21}$$

4. Boole inequalities in the set of bounded functions

Theorem 4.1. *Let $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) such that $f' \in L_1([a_1, a_2])$. If there exist constants $-\infty < m < M < +\infty$ such that $m \le f'(x) \le M$ for all $x \in [a_1, a_2]$, then the following inequality holds:*

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right]\right.$$

$$\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right]\right|$$

$$\le \frac{(a_2 - a_1)(M - m)}{4} \int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta.$$

Proof. Through the Lemma 3.1, we have

$$\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right]$$

$$- \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right]$$

$$= \frac{(a_2 - a_1)}{4} \int_0^1 D_{k,\beta,\alpha}(\zeta)\left[\left(f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - \frac{M+m}{2}\right)\right.$$

$$\left. - \left(f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right) - \frac{M+m}{2}\right)\right] d\zeta.$$

Applying the absolute value to the previously equality, we obtain

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right]\right.$$

$$\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right]\right|$$

$$\le \frac{(a_2 - a_1)}{4} \int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left[\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - \frac{M+m}{2}\right|\right.$$

$$\left. + \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right) - \frac{M+m}{2}\right|\right] d\zeta. \tag{22}$$

Given that $m \leq f'(x) \leq M$ for all $x \in [a_1, a_2]$, we get

$$\left| f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right) - \frac{M+m}{2}\right| \leq \frac{M-m}{2} \tag{23}$$

and

$$\left| f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right) - \frac{M+m}{2}\right| \leq \frac{M-m}{2}. \tag{24}$$

Adding inequalities (23) and (24) to (22) yields

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right]\right.$$

$$\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[\, {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1+a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq \frac{(a_2-a_1)(M-m)}{4}\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta.$$

□

Take $k = \beta = \alpha = 1$ in Theorem 4.1, we get the following result via classical Riemann integral.

Corollary 4.2. *Let $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) such that $f' \in L_1([a_1, a_2])$. If there exist constants $-\infty < m < M < +\infty$ such that $m \leq f'(x) \leq M$ for all $x \in [a_1, a_2]$, then the following inequality holds:*

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right]\right.$$

$$\left. -\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\zeta)\, d\zeta\right| \leq \frac{239(a_2-a_1)(M-m)}{6480}.$$

5. Boole inequalities in the set of Lipschitzian functions

Theorem 5.1. *Let $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) such that $f' \in L_1([a_1, a_2])$. If f' is an L-Lipschitzian function on $[a_1, a_2]$, then*

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right]\right.$$

$$\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[\, {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1+a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq \frac{L(a_2-a_1)^2}{4}\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta.$$

Proof. According to Lemma 3.1, we have

$$\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]$$

$$-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]$$

$$=\frac{(a_2-a_1)}{4}\int_0^1 D_{k,\beta,\alpha}(\zeta)\left[\left(f'\left(\left(\frac{1-\zeta}{2}\right)a_1+\left(\frac{1+\zeta}{2}\right)a_2\right)-f'(a_1)\right)\right.$$

$$\left.-\left(f'\left(\left(\frac{1+\zeta}{2}\right)a_1+\left(\frac{1-\zeta}{2}\right)a_2\right)-f'(a_1)\right)\right]d\zeta.$$

Using the absolute value to the preceding equality, we derive

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]\right.$$

$$\left.-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq\frac{(a_2-a_1)}{8}\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|\left[\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1+\left(\frac{1+\zeta}{2}\right)a_2\right)-f'(a_1)\right|\right.$$

$$\left.+\left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1+\left(\frac{1-\zeta}{2}\right)a_2\right)-f'(a_1)\right|\right]d\zeta.$$

Given that f' is a L-Lipschitzian function on $[a_1,a_2]$, we deduce

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]\right.$$

$$\left.-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq\frac{(a_2-a_1)}{4}\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|\left[L\left(\frac{1+\zeta}{2}\right)(a_2-a_1)+L\left(\frac{1-\zeta}{2}\right)(a_2-a_1)\right]d\zeta$$

$$=\frac{L(a_2-a_1)^2}{4}\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|d\zeta.$$

□

Take $k=\beta=\alpha=1$ in Theorem 5.1, we get the following result via classical Riemann integral.

Corollary 5.2. *Let $f : [a_1, a_2] \to \mathbb{R}$ be a differentiable function on (a_1, a_2) such that $f' \in L_1([a_1, a_2])$. If f' is an L-Lipschitzian function on $[a_1, a_2]$, then*

$$\left| \frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right] - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(\zeta)\, d\zeta \right| \leq \frac{239\, L(a_2 - a_1)^2}{6480}.$$

6. Boole inequality via power-mean integral inequality

Theorem 6.1. *Let $p \geq 1$, h be a B-function on $(0, 1)$ and assume that the assumptions of Lemma 3.1 hold. If $|f'|^p$ is an h-convex function on $[a_1, a_2]$, then the following Boole inequality holds:*

$$\begin{aligned} &\left| \frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right] \right. \\ &\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k} - 1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right] \right| \\ &\leq \frac{(a_2 - a_1)}{2}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} D_{k,\beta,\alpha}\left(\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right)^{\frac{1}{p}}, \end{aligned} \tag{25}$$

where

$$D_{k,\beta,\alpha} := \int_0^{\frac{1}{2}} \left| \left(1 - (1 - \zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{6}{45} \right| d\zeta + \int_{\frac{1}{2}}^{1} \left| \left(1 - (1 - \zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{38}{45} \right| d\zeta. \tag{26}$$

Proof. The following inequality is required to establish the next results. Let $C, B \geq 0$ and $\lambda > 0$:

$$C^{\lambda} + B^{\lambda} \leq \max(1, 2^{1-\lambda})(C + B)^{\lambda}. \tag{27}$$

By applying the absolute value of identity (18), we conclude

$$\begin{aligned} &\left| \frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right] \right. \\ &\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k} - 1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right] \right| \\ &\leq \frac{(a_2 - a_1)}{4}\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left[\left| f'\left(\left(\frac{1 - \zeta}{2}\right)a_1 + \left(\frac{1 + \zeta}{2}\right)a_2\right)\right| + \left| f'\left(\left(\frac{1 + \zeta}{2}\right)a_1 + \left(\frac{1 - \zeta}{2}\right)a_2\right)\right| \right] d\zeta. \end{aligned} \tag{28}$$

For $p \geq 1$, the application of the power-mean integral inequality produces the following results:

$$\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left[\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right| + \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right|\right] d\zeta$$

$$\leq \left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta\right)^{1-\frac{1}{p}} \left[\left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right|^p d\zeta\right)^{\frac{1}{p}}\right.$$

$$\left. + \left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right|^p d\zeta\right)^{\frac{1}{p}}\right].$$

The inequality (27) yields $C^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(C+B)^{\frac{1}{p}}$, thus

$$\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left[\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right| + \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right|\right] d\zeta$$

$$\leq 2^{1-\frac{1}{p}} \left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta\right)^{1-\frac{1}{p}}$$

$$\times \left[\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left(\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right|^p + \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right|^p\right) d\zeta\right]^{\frac{1}{p}}.$$

Considering that $|f'|^p$ is an h-convex function, the utilization of inequality (3) with $x = \frac{\zeta}{2}$ produces the following result:

$$\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right|^p \leq h\left(\frac{1-\zeta}{2}\right)\left|f'(a_1)\right|^p + h\left(\frac{1+\zeta}{2}\right)\left|f'(a_2)\right|^p.$$

Following this, we get

$$\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right|^p + \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right|^p$$

$$\leq \left[h\left(1-\frac{\zeta}{2}\right) + h\left(\frac{1+\zeta}{2}\right)\right]\left(\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right)$$

$$\leq 2\,h\left(\frac{1}{2}\right)\left(\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right).$$

Hence,

$$\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left[\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1 + \left(\frac{1+\zeta}{2}\right)a_2\right)\right| + \left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1 + \left(\frac{1-\zeta}{2}\right)a_2\right)\right|\right] d\zeta$$

$$\leq \left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta\right)^{1-\frac{1}{p}} 2^{1-\frac{1}{p}} \left[\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| 2\,h\left(\frac{1}{2}\right)\left(\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right) d\zeta\right]^{\frac{1}{p}}.$$

Therefore,

$$\begin{aligned}
&\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| \left[\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1+\left(\frac{1+\zeta}{2}\right)a_2\right)\right|+\left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1+\left(\frac{1-\zeta}{2}\right)a_2\right)\right|\right] d\zeta \\
&\leq \left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta\right) 2\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left(\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{29}$$

Combine the inequalities (28) and (29), we obtain

$$\begin{aligned}
&\Bigg|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right] \\
&-\frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[\,{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+\,{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]\Bigg| \\
&\leq \frac{(a_2-a_1)}{2}\left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right| d\zeta\right)\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

This accomplishes the inequality in (25). □

Putting $k=\beta=\alpha=1$ in the Theorem 6.1 gets the next corollary.

Corollary 6.2. *Let $p\geq 1$, h be a B-function on $(0,1)$ and assume that the assumptions of Corollary 3.2 hold. If $|f'|^p$ is an h-convex function on $[a_1,a_2]$, then the following Boole inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\zeta)\,d\zeta\right| \\
&\leq \frac{239\,(a_2-a_1)}{3240}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left(\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{30}$$

Here

$$\int_0^{\frac{1}{2}}\left|\zeta-\frac{6}{45}\right| d\zeta=\frac{137}{1800} \quad \text{and} \quad \int_{\frac{1}{2}}^1\left|\zeta-\frac{38}{45}\right| d\zeta=\frac{1157}{16200}.$$

Replacing $p=1$ in Theorem 6.1 and Corollary 6.2 yields the next results.

Corollary 6.3. *Let h be a B-function on $(0,1)$ and assume that the assumptions of Lemma 3.1 hold. If $|f'|$ is an h-convex function on $[a_1,a_2]$, then the following Boole inequality for Riemann-Liouville fractional operators holds:*

$$\begin{aligned}
&\Bigg|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right] \\
&-\frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[\,{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+\,{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]\Bigg| \\
&\leq \frac{(a_2-a_1)}{2}\,h\left(\frac{1}{2}\right) D_{k,\beta,\alpha}\left(\left|f'(a_1)\right|+\left|f'(a_2)\right|\right),
\end{aligned} \tag{31}$$

where $D_{k,\beta,\alpha}$ is defined by (26). For $k=\beta=\alpha=1$, we have

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(\zeta)\,d\zeta\right| \leq \frac{239\,(a_2-a_1)}{3240}h\left(\frac{1}{2}\right)\left(\left|f'(a_1)\right|+\left|f'(a_2)\right|\right). \tag{32}$$

We will now examine certain cases of h-convexity.

1. Put $h(\zeta)=\zeta^s$ with $s\in(0,1]$ in Theorem 6.1 and Corollary 6.2, we get the following results.

   Corollary 6.4. *Assume k,β,α,p and f are defined according to Theorem 6.1. If $|f'|^p$ is an s-convex function on $[a_1,a_2]$, then we get*

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]\right.$$
$$\left.-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}_{a_2^-}^{\alpha}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}_{a_1^+}^{\alpha}f\left(\frac{a_1+a_2}{2}\right)\right]\right| \tag{33}$$
$$\leq\frac{(a_2-a_1)}{2}\left(\frac{1}{2}\right)^{\frac{s}{p}}D_{k,\beta,\alpha}\left(\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right)^{\frac{1}{p}},$$

   where, $D_{k,\beta,\alpha}$ is defined by (26) . For $k=\beta=\alpha=1$, we get

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}f(\zeta)\,d\zeta\right| \leq\frac{239\,(a_2-a_1)}{3240}\left(\frac{1}{2}\right)^{\frac{s}{p}}\left(\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right)^{\frac{1}{p}}. \tag{34}$$

   By setting $s=p=1$, the inequality (34) is a novel generalization of the Boole inequality through the Riemann integral for convex functions.

2. Considering that $h(\zeta)=1$ in Theorem 6.1 and Corollary 6.2 produces a novel result for the class of P-functions. This is also analogous to the scenarios $s\to0^+$ in inequalities (33) and (34).

   Corollary 6.5. *Assume k, β, α, p and f are defined according to Theorem 6.1. If $|f'|^p$ is a P-function on $[a_1,a_2]$, then*

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]\right. \tag{35}$$
$$\left.-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}_{a_2^-}^{\alpha}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}_{a_1^+}^{\alpha}f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$
$$\leq\frac{(a_2-a_1)}{2}D_{k,\beta,\alpha}\left(\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right)^{\frac{1}{p}}.$$

Taking $k = \beta = \alpha = 1$ gives

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right] - \frac{1}{a_2 - a_1}\int_{a_1}^{a_2} f(\zeta)\,d\zeta\right|$$
$$\leq \frac{239\,(a_2 - a_1)}{3240}\left(\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right)^{\frac{1}{p}}. \tag{36}$$

7. Boole inequality via Hölder inequality

Theorem 7.1. *Let h be a B-function on $(0, 1)$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume that α, f are defined as in Lemma 3.1. If $\left|f'\right|^p$ is an h-convex function on $[a_1, a_2]$, then the following Boole inequality holds:*

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right]\right.$$
$$\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k} - 1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right]\right| \tag{37}$$
$$\leq \frac{(a_2 - a_1)}{2}\left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}}\left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}}\left[\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right]^{\frac{1}{p}},$$

where

$$\left|D_{k,\beta,\alpha}(\zeta)\right|^q := \begin{cases} \left|\left(1 - (1 - \zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{6}{45}\right|^q, & if\ 0 \leq \zeta \leq \frac{1}{2}; \\ \left|\left(1 - (1 - \zeta)^{\beta}\right)^{\frac{\alpha}{k}} - \frac{38}{45}\right|^q, & if\ \frac{1}{2} \leq \zeta \leq 1. \end{cases} \tag{38}$$

Proof. By employing the absolute value of the identity (18), we obtain

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1 + a_2}{4}\right) + 12f\left(\frac{a_1 + a_2}{2}\right) + 32f\left(\frac{a_1 + 3a_2}{4}\right) + 7f(a_2)\right]\right.$$
$$\left. - \frac{\beta^{\frac{\alpha}{k}}\, 2^{\frac{\beta\alpha}{k} - 1}\Gamma_k(\alpha + k)}{(a_2 - a_1)^{\frac{\beta\alpha}{k}}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1 + a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1 + a_2}{2}\right)\right]\right|$$
$$\leq \frac{(a_2 - a_1)}{4}\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right|\left|f'\left(\left(\frac{1 - \zeta}{2}\right)a_1 + \left(\frac{1 + \zeta}{2}\right)a_2\right)\right| d\zeta$$
$$+ \frac{(a_2 - a_1)}{4}\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right|\left|f'\left(\left(\frac{1 + \zeta}{2}\right)a_1 + \left(\frac{1 - \zeta}{2}\right)a_2\right)\right| d\zeta.$$

By using Hölder's inequality and $C^{\frac{1}{p}} + B^{\frac{1}{p}} \leq 2^{1 - \frac{1}{p}}(C + B)^{\frac{1}{p}}$, it yields

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]\right.$$

$$\left.-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq\frac{(a_2-a_1)}{4}\left(\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}}\left(\int_0^1\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1+\left(\frac{1+\zeta}{2}\right)a_2\right)\right|^p d\zeta\right)^{\frac{1}{p}}$$

$$+\frac{(a_2-a_1)}{4}\left(\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}}\left(\int_0^1\left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1+\left(\frac{1-\zeta}{2}\right)a_2\right)\right|^p d\zeta\right)^{\frac{1}{p}}$$

$$\leq\frac{(a_2-a_1)}{4}\left(\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}}\;2^{1-\frac{1}{p}}\left[\int_0^1\left|f'\left(\left(\frac{1-\zeta}{2}\right)a_1+\left(\frac{1+\zeta}{2}\right)a_2\right)\right|^p d\zeta\right.$$

$$\left.+\int_0^1\left|f'\left(\left(\frac{1+\zeta}{2}\right)a_1+\left(\frac{1-\zeta}{2}\right)a_2\right)\right|^p d\zeta\right]^{\frac{1}{p}}.$$

Given that $|f'|^p$ is an h-convex function, we get

$$\left|\frac{1}{90}\left[7f(a_1)+32f\left(\frac{3a_1+a_2}{4}\right)+12f\left(\frac{a_1+a_2}{2}\right)+32f\left(\frac{a_1+3a_2}{4}\right)+7f(a_2)\right]\right.$$

$$\left.-\frac{\beta^{\frac{\alpha}{k}}\,2^{\frac{\beta\alpha}{k}-1}\Gamma_k(\alpha+k)}{(a_2-a_1)^{\frac{\beta\alpha}{k}}}\left[{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_2^-}f\left(\frac{a_1+a_2}{2}\right)+{}_k^{\beta}\mathfrak{I}^{\alpha}_{a_1^+}f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq\frac{(a_2-a_1)}{4}\left(\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}}\;2^{\frac{1}{q}}\left[\int_0^1\left(h\left(\frac{1-\zeta}{2}\right)\left|f'(a_1)\right|^p+h\left(\frac{1+\zeta}{2}\right)\left|f'(a_2)\right|^p\right)d\zeta\right.$$

$$\left.+\int_0^1\left(h\left(\frac{1+\zeta}{2}\right)\left|f'(a_1)\right|^p+h\left(\frac{1-\zeta}{2}\right)\left|f'(a_2)\right|^p\right)d\zeta\right]^{\frac{1}{p}}$$

$$\leq\frac{(a_2-a_1)}{4}\left(\int_0^1\left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}}\;2^{\frac{1}{q}}\left(\int_0^1\left[h\left(\frac{1-\zeta}{2}\right)+h\left(\frac{1+\zeta}{2}\right)\right]d\zeta\right)^{\frac{1}{p}}\left[\left|f'(a_1)\right|^p+\left|f'(a_2)\right|^p\right]^{\frac{1}{p}}.$$

Assuming inequality (3) for $x = \frac{\zeta}{2}$, we obtain

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right]\right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a_2-a_1)^{\alpha}}\left[{}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_2^-} f\left(\frac{a_1+a_2}{2}\right) + {}^{\beta}_{k}\mathfrak{I}^{\alpha}_{a_1^+} f\left(\frac{a_1+a_2}{2}\right)\right]\right|$$

$$\leq \frac{(a_2-a_1)}{2}\left(\int_0^1 \left|D_{k,\beta,\alpha}(\zeta)\right|^q d\zeta\right)^{\frac{1}{q}} \left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left[\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right]^{\frac{1}{p}}.$$

This completes the proof of the inequality (37).

□

Considering $k = \beta = \alpha = 1$ in Theorem 7.1, the following Boole inequality via classical Riemann integral for h-convex functions holds.

Corollary 7.2. *Let h be a B-function on $(0,1)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and assume that the assumptions of Lemma 3.1 hold. If $\left|f'\right|^p$ is a h-convex function on $[a_1, a_2]$, then*

$$\left|\frac{1}{90}\left[7f(a_1) + 32f\left(\frac{3a_1+a_2}{4}\right) + 12f\left(\frac{a_1+a_2}{2}\right) + 32f\left(\frac{a_1+3a_2}{4}\right) + 7f(a_2)\right] - \frac{1}{a_2-a_1}\int_{a_1}^{a_2} f(\zeta)\,d\zeta\right|$$

$$\leq \frac{a_2-a_1}{2}\left(\int_0^1 |F_1(\zeta)|^q d\zeta\right)^{\frac{1}{q}} \left(h\left(\frac{1}{2}\right)\right)^{\frac{1}{p}} \left[\left|f'(a_1)\right|^p + \left|f'(a_2)\right|^p\right]^{\frac{1}{p}},$$

where

$$|F_1(\zeta)|^q := \begin{cases} \left|\zeta - \frac{6}{45}\right|^q, & if\ 0 \leq \zeta \leq \frac{1}{2}; \\ \left|\zeta - \frac{38}{45}\right|^q, & if\ \frac{1}{2} \leq \zeta \leq 1. \end{cases}$$

8. Conclusion

In this paper, we formulated a new definition of k-conformable fractional operators and elucidated their boundedness and semigroup characteristics. Furthermore, we introduced an innovative identity for differentiable h-convex functions that employs k-conformable fractional operators. By employing this identity, we have formulated novel inequalities applicable to bounded functions as well as Lipschitzian functions. Moreover, we utilized h-convexity in conjunction with the power-mean inequality and Hölder's inequality, respectively to derive some new results. We believe that the interested researcher will use these results for new and fascinating directions in fractional, quantum, post-quantum, multiplicative calculus, and time scales, and its applications to different types of convexity.

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