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Majorization and convolution properties for analytic functions involving the Dziok-Srivastava linear operator associated with telephone numbers

Şahsene Altınkaya[a,*], Asena Çetinkaya[b]

[a]*Department of Mathematics and Statistics, University of Turku, 20014 Turku, Finland*
[b]*Department of Mathematics and Computer Science, Istanbul Kültür University, Istanbul, Türkiye*

Abstract. This paper introduces a novel subclass of Ma-Minda type starlike functions by incorporating the Dziok-Srivastava linear operator, which is associated with telephone numbers. The study provides an in-depth analysis of the majorization properties within this function class, highlighting their mathematical significance. Additionally, the convolution conditions for these functions are thoroughly investigated, offering new insights into their structural behavior. The results obtained not only extend the existing theory but also open up new avenues for further research in the field of Geometric Function Theory.

1. Introduction

Symbolize by $\mathcal{A}$ the family of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with $f(0) = 0 = f'(0) - 1$. When f is univalent, we denote the subfamily of $\mathcal{A}$ by $\mathcal{S}$. For a detailed survey, we refer to the recent study by Thomas et al. [27].

For analytic functions f_1 and f_2 in $\mathbb{D}$, f_1 is subordinate to f_2, symbolized by $f_1 \prec f_2$, if a Schwarz function $\vartheta(z)$ exists, where $\vartheta(z) = \sum_{n=1}^{\infty} \varsigma_n z^n$ is analytic in $\mathbb{D}$ with $\vartheta(0) = 0$ and $|\vartheta(z)| < 1$, such that $f_1(z) = f_2(\vartheta(z))$. For the function $\kappa(z) = z + b_2 z^2 + \cdots = z + \sum_{n=2}^{\infty} b_n z^n$, the convolution $f * \kappa$ is expressed by $(f * \kappa)(z) = z + \sum\limits_{n=2}^{\infty} a_n b_n z^n$. For multiple subfamilies, the convolution properties have addressed in [4, 11].

The concept of majorization was proposed by MacGregor [16]. Assume f_1 and f_2 are analytic in $\mathbb{D}$. We can show that f_1 is majorized by f_2,

$$f_1(z) \ll f_2(z),$$

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* Corresponding author: Şahsene Altınkaya
Email addresses: sahsenealtinkaya@gmail.com (Şahsene Altınkaya), asnfigen@hotmail.com (Asena Çetinkaya)
ORCID iDs: https://orcid.org/0000-0002-7950-8450 (Şahsene Altınkaya), https://orcid.org/0000-0002-8815-5642 (Asena Çetinkaya)

if an analytic function φ in $\mathbb{D}$ with

$$|\varphi(z)| \leq 1 \ \text{ and } \ f_1(z) = \varphi(z) f_2(z) \tag{1}$$

exists.

The family $\mathcal{S}^*(\Phi)$ was established by

$$\mathcal{S}^*(\Phi) = \left\{ f \in \mathcal{A} : \frac{z f'(z)}{f(z)} \prec \Phi(z) \right\},$$

where Φ is analytic and univalent in $\mathbb{D}$ with $\Phi(\mathbb{D})$ being convex, $\Phi(0) = 1$, and $\Re(\Phi(z)) > 0$ [15]. The family $\mathcal{S}^*(\Pi, \Sigma)$ of Janowski starlike functions, the specific case $\Phi(z) = (1 + \Pi z)/(1 + \Sigma z)$ $(-1 \leq \Sigma < \Pi \leq 1)$, was introduced in [12]. The family $\mathcal{S}^*(\delta)$ of starlike functions of order δ comes from $\Pi = 1 - 2\delta$ $(0 \leq \delta < 1)$ and $\Sigma = -1$. In particular, for $\delta = 0$, we deal with the family $\mathcal{S}^*$ of starlike functions.

The study of analytic and univalent functions within the unit disk is a significant area of research in Geometric Function Theory (GFT). By exploring the properties of such functions, mathematicians can uncover fundamental aspects of complex analysis and its applications.

The classical telephone numbers can be represented using a recurrence

$$T(\eta) = T(\eta - 1) + (\eta - 1) T(\eta - 2) \quad (\eta \geq 2)$$

with $T(0) = T(1) = 1$. Heinrich August Rothe, in 1800, made an observation that established the connection of the numbers with symmetric groups. He noted that $T(\eta)$ represents the count of involutions (permutations that are self-inverse) within the symmetric group [6, 14].

While Włoch and Wołowiec-Musiał [29] defined $T(\beta, \eta)$, the generalized telephone numbers, for integers $\eta \geq 0$ and $\beta \geq 1$ by

$$T(\beta, \eta) = \beta T(\beta, \eta - 1) + (\eta - 1) T(\beta, \eta - 2)$$

with $T(\beta, 0) = 1$, $T(\beta, 1) = \beta$, Bednarz and Wołowiec-Musiał [2] established the generalization

$$T_\beta(\eta) = T_\beta(\eta - 1) + \beta(\eta - 1) T_\beta(\eta - 2)$$

with $T_\beta(0) = T_\beta(1) = 1$ $(\eta \geq 2, \ \beta \geq 1)$. In those papers, the generating function, direct formula and matrix generators were also emphasized. On the other hand, they explored various interpretations and demonstrated several properties of these numbers in relation to congruences. The summation formula

$$e^{x + \beta \frac{x^2}{2}} = \sum_{\eta=0}^{\infty} T_\beta(\eta) \frac{x^\eta}{\eta!}, \quad (\beta \geq 1)$$

were mentioned. Here, we arrive the classical telephone numbers $T(\eta)$ for parameter $\beta = 1$.

In [7], Deniz considered the function

$$\Phi(z) = e^{z + \beta \frac{z^2}{2}} = 1 + z + \frac{z^2}{2} + \frac{(1+\beta) z^3}{6} + \frac{(1 + 3\beta) z^4}{24} + \cdots \tag{2}$$

within its domain in $\mathbb{D}$ (see [17, 28]).

The generalized hypergeometric function, traditionally has the notation ${}_m\mathcal{F}_s$ for parameters $a_j \in \mathbb{C}$ $(j = 1, ..., m)$ and $b_j \in \mathbb{C}$ $(b_j \neq 0, -1, -2, \ldots; j = 1, \ldots, s)$, is expressed by

$${}_m\mathcal{F}_s(a_1, \ldots, a_i, \ldots, a_m; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_m)_n}{(b_1)_n \ldots (b_s)_n} \frac{z^n}{n!}$$

$$(m \leq s + 1, \ \ m, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \ z \in \mathbb{D})$$

Here $(.)_n$ is the Pochhammer symbol, written via the Gamma function Γ, by

$$(\xi)_n = \frac{\Gamma(\xi+n)}{\Gamma(\xi)} = \xi(\xi+1)\ldots(\xi+n-1), \ \ ((\xi)_0 = 1, n \in \mathbb{N}).$$

By making use of the generalized hypergeometric function ${}_m\mathcal{F}_s$, Dziok and Srivastava [9, 10] introduced and discussed the Dziok-Srivastava linear operator

$$\mathcal{H}_s^m(a_1,\ldots,a_m;b_1,\ldots,b_s) : \mathcal{A} \to \mathcal{A}$$

expressed by

$$\begin{aligned}\mathcal{H}_s^m(a_1,\ldots,a_m;b_1,\ldots,b_s)f(z) &= z\,{}_m\mathcal{F}_s(a_1,\ldots,a_m;b_1,\ldots,b_s;z) * f(z)\\ &= z + \sum_{n=2}^{\infty}\Gamma_n a_n z^n, \quad (z \in \mathbb{D})\end{aligned}$$

where

$$\Gamma_n = \frac{(a_1)_{n-1}\ldots(a_m)_{n-1}}{(b_1)_{n-1}\ldots(b_s)_{n-1}}\frac{1}{(n-1)!}.$$

For convenience, if we formulated

$$\mathcal{H}_s^m[a_1]f(z) = \mathcal{H}_s^m(a_1,\ldots,a_m;b_1,\ldots,b_s)f(z),$$

then

$$z(\mathcal{H}_s^m[a_1]f(z))' = a_1\mathcal{H}_s^m[a_1+1]f(z) - (a_1-1)\mathcal{H}_s^m[a_1]f(z). \tag{3}$$

The literature on Geometric Function Theory also includes investigations of analytic function classes associated with a further generalization of the Dziok–Srivastava linear operator, known as the Srivastava–Wright linear convolution operator, which is defined by means of the Fox–Wright generalized hypergeometric function [13], [23], and the references therein. On the other hand, for some particular cases of the Dziok–Srivastava linear operator, we arrive Carlson-Shaffer [5], Ruscheweyh derivative [20] and generalized Bernardi-Libera-Livingston [3] operators.

This paper aims to contribute to this field by introducing a novel subclass $\mathcal{S}_s^m[a_1]$ of Ma-Minda type starlike functions, which incorporates the Dziok-Srivastava linear operator. This novel approach provides a fresh perspective on the analysis of these functions, particularly in terms of their majorization properties and convolution conditions.

Definition 1.1. *If a function $f \in \mathcal{A}$ fulfills*

$$\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} \prec \Phi(z) := e^{z+\beta\frac{z^2}{2}}, \tag{4}$$

it is regarded as a member of the class $\mathcal{S}_s^m[a_1]$.

We note that for $m = 2, s = 1, a_1 = 1$, we get $\mathcal{S}_s^m[a_1] = \mathcal{S}^*(\beta)$, (see [7]).

2. Majorization Properties

Recent studies have advanced the understanding of majorization problems for various classes of analytic univalent functions [1], [21], [22], [24], [25]. Moreover, the concept of majorization plays a crucial role in understanding the hierarchical structure of functions within this new subclass [26]. By examining how one function can dominate another, we gain deeper insights into their comparative behavior and intrinsic properties. Additionally, the convolution conditions explored in this paper offer a new length through which to view the interactions between different functions within the subclass $\mathcal{S}_s^m[a_1]$. These conditions not only enhance our theoretical understanding but also have practical implications for various applications in mathematical analysis. Hence, this section is devoted to find majorization properties..

Lemma 2.1. *The function $\Phi(z) = e^{z+\beta\frac{z^2}{2}}$ holds*

$$\min_{|z|=r} \Re(\Phi(z)) = \min_{|z|=r} |\Phi(z)| = \Phi(-r)$$

and

$$\max_{|z|=r} \Re(\Phi(z)) = \max_{|z|=r} |\Phi(z)| = \Phi(r),$$

where $r \in (0, 1)$.

Proof. The function $\Re(\Phi(re^{i\theta})) = e^{r\cos\theta+\beta\frac{r^2\cos 2\theta}{2}}(\cos(r\sin\theta)\cos(r^2\sin 2\theta))$, $(\theta \in [0, 2\pi))$ reaches its minimum at $\theta = \pi$, maximum at $\theta = 0$. Hence

$$\min_{|z|=r} \Re(\Phi(z)) = \min_{|z|=r} |\Phi(z)| = e^{-r+\beta\frac{r^2}{2}}$$

and

$$\max_{|z|=r} \Re(\Phi(z)) = \max_{|z|=r} |\Phi(z)| = e^{r+\beta\frac{r^2}{2}}.$$

□

Theorem 2.2. *Assume $f \in \mathcal{A}$, and let $g \in \mathcal{S}_s^m[a_1]$ with $\mathcal{H}_s^m[a_1]f(z) \ll \mathcal{H}_s^m[a_1]g(z)$. Then for $|z| \leq r_1$, we arrive*

$$|\mathcal{H}_s^m[a_1 + 1]f(z)| \leq |\mathcal{H}_s^m[a_1 + 1]g(z)|,$$

where r_1 represents the smallest positive solution of

$$(1 - r^2)\left(|a_1| + e^{-r+\beta\frac{r^2}{2}} - 1\right) - 2r = 0.$$

Proof. Since $g \in \mathcal{S}_s^m[a_1]$, then from (4) we write

$$\frac{z(\mathcal{H}_s^m[a_1]g(z))'}{\mathcal{H}_s^m[a_1]g(z)} = e^{\vartheta(z)+\beta\frac{(\vartheta(z))^2}{2}}. \tag{5}$$

Now using (3) in (5) and making simple calculations we get

$$\mathcal{H}_s^m[a_1]g(z) = \frac{a_1}{a_1 - 1 + e^{\vartheta(z)+\beta\frac{(\vartheta(z))^2}{2}}}\mathcal{H}_s^m[a_1 + 1]g(z). \tag{6}$$

Considering the conditions of the Schwarz function ϑ, we derive

$$|\mathcal{H}_s^m[a_1]g(z)| \leq \frac{|a_1|}{|a_1| + e^{|z|+\beta\frac{|z|^2}{2}} - 1}|\mathcal{H}_s^m[a_1 + 1]g(z)|. \tag{7}$$

Since $\mathcal{H}_s^m[a_1]f$ is majorized by $\mathcal{H}_s^m[a_1]g$, thus, from (1), we record

$$\mathcal{H}_s^m[a_1]f(z) = \varphi(z)\mathcal{H}_s^m[a_1]g(z). \tag{8}$$

Then taking the derivative of both sides of (8) with respect to z, utilizing (3), yields

$$\mathcal{H}_s^m[a_1 + 1]f(z) = \frac{1}{a_1}z\varphi'(z)\mathcal{H}_s^m[a_1]g(z) + \varphi(z)\mathcal{H}_s^m[a_1 + 1]g(z). \tag{9}$$

Due to Nehari [18, p.168], we know that Schwarz function satisfies

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}. \tag{10}$$

Hence, substituting (7) and (10) into (9), we get

$$|\mathcal{H}_s^m[a_1+1]f(z)| \leq \left[\frac{|z|(1-|\varphi(z)|^2)}{(1-|z|^2)}\frac{1}{|a_1|+e^{|z|+\beta\frac{|z|^2}{2}}-1}+|\varphi(z)|\right]|\mathcal{H}_s^m[a_1+1]g(z)|. \tag{11}$$

Taking $|z| = r$ and $|\varphi(z)| = \rho$ $(0 \leq \rho \leq 1)$, and in view of Lemma 2.1, the last inequality may be expressed by

$$|\mathcal{H}_s^m[a_1+1]f(z)| \leq \Theta(r,\rho)|\mathcal{H}_s^m[a_1+1]g(z)|,$$

where

$$\Theta(r,\rho) = \frac{r(1-\rho^2)}{(1-r^2)}\frac{1}{|a_1|+e^{-r+\beta\frac{r^2}{2}}-1}+\rho.$$

In order to determine r_1, we choose

$$\begin{aligned} r_1 &= \max\{r \in (0,1) : \Theta(r,\rho) \leq 1, \forall \rho \in [0,1]\} \\ &= \max\{r \in (0,1) : \nu(r,\rho) \geq 0, \forall \rho \in [0,1]\}, \end{aligned}$$

where

$$\nu(r,\rho) = (1-r^2)(|a_1|+e^{-r+\beta\frac{r^2}{2}}-1)-r(1+\rho).$$

Given that $\frac{\partial}{\partial\rho}\nu(r,\rho) = -r < 0$, it follows that $\nu(r,\rho)$ attains its minimum at $\rho = 1$. Specifically,

$$\min\{\nu(r,\rho) \geq 0, \rho \in [0,1]\} = \nu(r,1) := \nu(r),$$

where

$$\nu(r) = (1-r^2)(|a_1|+e^{-r+\beta\frac{r^2}{2}}-1)-2r.$$

Furthermore, given that

$$\nu(0) = |a_1| \geq 0, \text{ and } \nu(1) = -2 < 0,$$

we conclude that there exists a value r_1 such that $\nu(r) \geq 0$ $(r \in [0, r_1]]$, where r_1 is the smallest positive solution □

For $m = 2, s = 1, a_1 = 1$, we arrive:

Corollary 2.3. *Suppose $f \in \mathcal{A}$ and $g \in \mathcal{S}^*(\beta)$ such that $f(z) \ll g(z)$. Then, for $|z| \leq r_1$, it follows that*

$$|f'(z)| \leq |g'(z)|,$$

where r_1 is the smallest positive solution of $(1-r^2)e^{-r+\beta\frac{r^2}{2}}-2r = 0$.

3. Convolution properties

Theorem 3.1. *$f \in \mathcal{S}_s^m[a_1]$ iff*

$$\frac{1}{z}\left[\mathcal{H}_s^m[a_1]f(z) * \frac{z-\Delta z^2}{(1-z)^2}\right] \neq 0 \tag{12}$$

for all $\Delta = \frac{e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}$, *where $\theta \in [0,2\pi]$ and also $\Delta = 1$.*

Proof. Assume $f \in \mathcal{S}_s^m[a_1]$, then

$$\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} \prec e^{z+\beta\frac{z^2}{2}} \tag{13}$$

Considering the function $\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)}$ being analytic in $\mathbb{D}$, it implies

$$\mathcal{H}_s^m[a_1]f(z) \neq 0$$

for $z \in \mathbb{D}^* = \mathbb{D}\backslash\{0\}$; in other words, $(1/z)\mathcal{H}_s^m[a_1]f(z) \neq 0$, which is equivalent to the assertion that (12) holds for $\Delta = 1$. Given the relation (13), we deduce

$$\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} = e^{\vartheta(z)+\beta\frac{(\vartheta(z))^2}{2}}, \tag{14}$$

where ϑ is a Schwarz function, or equivalently

$$\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} \neq e^{i\theta+\beta\frac{(i\theta)^2}{2}}, \tag{15}$$

so that

$$\frac{1}{z}\Big[z(\mathcal{H}_s^m[a_1]f(z))' - \Big(e^{i\theta+\beta\frac{(i\theta)^2}{2}}\Big)\mathcal{H}_s^m[a_1]f(z)\Big] \neq 0. \tag{16}$$

For functions $f \in \mathcal{S}_s^m[a_1]$, we observe that

$$\mathcal{H}_s^m[a_1]f(z) * \frac{z}{1-z} = \mathcal{H}_s^m[a_1]f(z)$$

and

$$\mathcal{H}_s^m[a_1]f(z) * \frac{z}{(1-z)^2} = z(\mathcal{H}_s^m[a_1]f(z))',$$

then expression (16) is written as

$$\frac{1}{z}\Big[\mathcal{H}_s^m[a_1]f(z) * \Big(\frac{z}{(1-z)^2} - \frac{(e^{i\theta+\beta\frac{(i\theta)^2}{2}})z}{(1-z)}\Big)\Big] \neq 0.$$

Therefore, we have

$$\frac{1-e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{z}\left[\mathcal{H}_s^m[a_1]f(z) * \frac{z - \frac{e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}z^2}{(1-z)^2}\right] \neq 0. \tag{17}$$

Conversely, given the assumption (12) holds for $\Delta = 1$, it implies

$$(1/z)\mathcal{H}_s^m[a_1]f(z) \neq 0,$$

thus $\psi(z) = \frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)}$ is analytic in $\mathbb{D}$, and regular at $z = 0$ with $\psi(0) = 1$. As demonstrated in the initial part of the proof, assumption (12) is equivalent to (15), thereby we arrive

$$\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} \neq e^{i\theta+\beta\frac{(i\theta)^2}{2}} \tag{18}$$

and if we denote

$$\Phi(z) = e^{z+\beta\frac{z^2}{2}},$$

this relationship demonstrates that the simply connected domain $\psi(\mathbb{D})$ lies within a connected component of $\mathbb{C}\backslash\Phi(\partial\mathbb{D})$. Leveraging the fact $\psi(0) = \Phi(0)$ alongside the univalence of the function Φ, implies $\psi \prec \Phi$, which corresponds to (13). Therefore, $f \in \mathcal{S}_s^m[a_1]$. □

Corollary 3.2. *For the function f to be in the class $\mathcal{S}_s^m[a_1]$, it is necessary and sufficient that*

$$1-\sum_{n=2}^{\infty}\frac{n-e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}\Gamma_n a_n z^{n-1}\neq 0. \tag{19}$$

Proof. The function $f\in\mathcal{S}_s^m[a_1]$ iff

$$\frac{1}{z}\left[\mathcal{H}_s^m[a_1]f(z)*\frac{z-\Delta z^2}{(1-z)^2}\right]\neq 0 \tag{20}$$

for all $\Delta=\frac{e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}$ and also $\Delta=1$. From (20), we arrive

$$\begin{aligned}&\frac{1}{z}\left[\mathcal{H}_s^m[a_1]f(z)*\left(\frac{z}{(1-z)^2}-\frac{\Delta z^2}{(1-z)^2}\right)\right]\\&=\frac{1}{z}\left\{z(\mathcal{H}_s^m[a_1]f(z))'-\Delta(z(\mathcal{H}_s^m[a_1]f(z))'-\mathcal{H}_s^m[a_1]f(z))\right\}\\&=1-\sum_{n=2}^{\infty}(n(\Delta-1)-\Delta)\Gamma_n a_n z^{n-1}\\&=1-\sum_{n=2}^{\infty}\frac{n-e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}\Gamma_n a_n z^{n-1},\end{aligned}$$

which completes the proof. □

Next, we establish coefficient estimates theorem.

Theorem 3.3. *Assume $f\in\mathcal{A}$. If*

$$\sum_{n=2}^{\infty}\left|\frac{n-e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}\right|\Gamma_n|a_n|<1, \tag{21}$$

then $f\in\mathcal{S}_s^m[a_1]$.

Proof. Performing a straightforward calculation based on expression (19), we find

$$\begin{aligned}&\left|1-\sum_{n=2}^{\infty}(n(\Delta-1)-\Delta)\Gamma_n a_n z^{n-1}\right|\\&\geq 1-\sum_{n=2}^{\infty}\left|(n(\Delta-1)-\Delta)\Gamma_n a_n z^{n-1}\right|\\&>1-\sum_{n=2}^{\infty}|(n(\Delta-1)-\Delta)|\Gamma_n|a_n|\\&=1-\sum_{n=2}^{\infty}\left|\frac{n-e^{i\theta+\beta\frac{(i\theta)^2}{2}}}{e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1}\right|\Gamma_n|a_n|>0,\end{aligned}$$

if the inequality (21) holds. □

Corollary 3.4. *Assume $f\in\mathcal{A}$. If*

$$|a_n|<\frac{|e^{i\theta+\beta\frac{(i\theta)^2}{2}}-1|}{|n-e^{i\theta+\beta\frac{(i\theta)^2}{2}}|\Gamma_n},\ (n\geq 2)$$

then $f\in\mathcal{S}_s^m[a_1]$.

Next, we compute radius of starlikeness for functions $f \in \mathcal{S}_s^m[a_1]$.

Theorem 3.5. *Let $f \in \mathcal{S}_s^m[a_1]$, then $f \in \mathcal{S}^*(\delta)$ $(0 \leq \delta < 1)$ in $|z| = r < r^*$, where*

$$r^* = \inf_{n\geq 2}\left\{\frac{|n - e^{i\theta+\beta\frac{(i\theta)^2}{2}}|}{|e^{i\theta+\beta\frac{(i\theta)^2}{2}} - 1|} \times \left(\frac{1-\delta}{n-\delta}\right)\right\}^{1/n-1}. \tag{22}$$

Proof. To establish $f \in \mathcal{S}^*(\delta)$, it is sufficient to demonstrate that

$$\left|\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} - 1\right| < 1 - \delta, \quad (|z| < r^*). \tag{23}$$

Indeed we have

$$\left|\frac{z(\mathcal{H}_s^m[a_1]f(z))'}{\mathcal{H}_s^m[a_1]f(z)} - 1\right| < \frac{\sum_{n=2}^{\infty}(n-1)\Gamma_n|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}\Gamma_n|a_n||z|^{n-1}}.$$

Thus, the inequality (23) is true if

$$\sum_{n=2}^{\infty}\frac{n-\delta}{1-\delta}\Gamma_n|a_n|r^{n-1} < 1,$$

where we deduce that $f \in \mathcal{S}^*(\delta)$ if

$$|a_n| < \frac{1-\delta}{(n-\delta)\Gamma_n r^{n-1}}.$$

Using Corollary 3.4, we arrive at

$$\frac{|e^{i\theta+\beta\frac{(i\theta)^2}{2}} - 1|}{|n - e^{i\theta+\beta\frac{(i\theta)^2}{2}}|\Gamma_n} < \frac{1-\delta}{(n-\delta)\Gamma_n r^{n-1}}$$

or equivalently

$$r < \left\{\frac{|n - e^{i\theta+\beta\frac{(i\theta)^2}{2}}|}{|e^{i\theta+\beta\frac{(i\theta)^2}{2}} - 1|} \times \left(\frac{1-\delta}{n-\delta}\right)\right\}^{1/n-1},$$

which conclude the proof. □

4. Conclusion

In this paper, we have introduced a novel subclass of Ma-Minda type starlike functions, incorporating the Dziok-Srivastava linear operator associated with telephone numbers. This subclass, denoted as $\mathcal{S}_s^m[a_1]$, has been rigorously analyzed in terms of its majorization properties and convolution conditions.

We began by defining the new subclass and establishing its foundational properties. Through Lemma 2.1, we characterized the extremal behavior of the exponential function $\Phi(z) = e^{z+\beta\frac{z^2}{2}}$, which plays a central role in our analysis. This lemma was crucial in proving Theorem 2.2, where we derived majorization properties for the subclass $\mathcal{S}_s^m[a_1]$. Specifically, we demonstrated that for any function $f \in \mathcal{A}$ and $g \in \mathcal{S}_s^m[a_1]$ such that $\mathcal{H}_s^m[a_1]f(z) \ll \mathcal{H}_s^m[a_1]g(z)$, the inequality $|\mathcal{H}_s^m[a_1+1]f(z)| \leq |\mathcal{H}_s^m[a_1+1]g(z)|$ holds within a specified radius.

We also extended our results to a more general setting by establishing convolution properties for functions in $\mathcal{S}_s^m[a_1]$. Theorem 3.1 presented a necessary and sufficient condition involving convolution with a specific kernel, thus broadening the understanding of convolution operations within this function class.

Our results highlight the interplay between analytic function theory and combinatorial constructs such as telephone numbers. The integration of the Dziok-Srivastava linear operator and the generalized hypergeometric function provided a rich framework for exploring new function classes with robust geometric and analytic properties.

Future work can explore further generalizations of these function classes, potentially incorporating other types of linear operators or different combinatorial sequences. Additionally, extending the analysis to include more complex domains or higher-dimensional analogs could provide new insights and applications. The connections between these analytic properties and their combinatorial counterparts also offer a fertile ground for interdisciplinary research.

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