Filomat 40:8 (2026), 2847–2869
https://doi.org/10.2298/FIL2608847A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: `http://www.pmf.ni.ac.rs/filomat`

On lattice generalizations of bipolar soft rough approximations

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Abstract. In this paper, we introduce bipolar soft sets, defined over a complete atomic Boolean lattice (Ξ), and discuss some basic properties of bipolar soft sets (*BSs*) on Ξ. We introduce bipolar soft rough approximations (*BSRA*) based on a complete atomic Boolean lattice to generalize bipolar soft rough approximations. We study some of their properties and characterization of bipolar soft sets and support them with illustrative examples. Finally, an application of a bipolar soft rough set on Ξ in decision-making problems is discussed.

1. Introduction

In our daily existence, we frequently encounter issues characterized by uncertain and imprecise data. These issues are notably prevalent in numerous fields such as economics, engineering, environmental sciences, and medical diagnosis, where traditional mathematical approaches often fail to capture the inherent vagueness and granularity of real-world information.

Molodtsov [12] pioneered an innovative concept of soft sets as a flexible mathematical framework for handling uncertainty without the limitations of parameterization required in fuzzy set theory. Since then, soft set theory has attracted considerable attention and has been successfully applied in various disciplines, including function smoothness, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory [12, 13]. More recently, practical applications have expanded significantly, particularly in decision-making problems, such as multi-person decision-making strategies based on ranked soft sets [24].

Maji et al. [11] investigated decision-making problems using soft sets. Subsequently, Yang et al. [26] combined interval-valued fuzzy sets with soft sets, introducing interval-valued fuzzy soft sets.

Rough set theory, originally proposed by Pawlak [21], was developed to address vagueness and information granularity through lower and upper approximations. Rough sets have been widely applied in

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2020 *Mathematics Subject Classification*. Primary 03E72; Secondary 06D35.
Keywords. Bipolar soft sets, bipolar soft rough sets, complete atomic Boolean lattice, seft rough approximations, decision making.
Received: 24 August 2024; Revised: 03 January 2026; Accepted: 15 January 2026
Communicated by Ljubiša D. R. Kočinac

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machine learning, pattern recognition, knowledge discovery, image processing, signal analysis, and decision analysis [22]. In recent years, rough-set-based models have shown remarkable effectiveness in medical diagnosis and decision-making, including, for instance, heart failure diagnosis and COVID-19-related medical data analysis [3, 4].

Feng et al. [5] introduced the notion of soft rough approximations and demonstrated their applicability in multi-criteria group decision-making problems. Recently, improved soft rough approximation models based on soft neighborhoods have been proposed to enhance the flexibility and descriptive power of approximation operators [20].

The bipolar nature of information was incorporated into soft sets by Shabir and Naz [25], who defined bipolar soft sets and studied their fundamental operations and applications in decision-making. Further developments led to generalized bipolar-soft set frameworks and their associated topological structures, particularly in decision-making contexts [17].

Several paradigms of rough sets have also been investigated via containment neighborhoods and ideal-based approaches, providing alternative mechanisms for rough approximations [19]. In addition, metric-based approaches to attribute reduction in rough set theory have been explored using soft metrics [16]. Some bipolar soft ideal rough set models have further been applied to real-life problems, including medical decision-making scenarios [18].

To summarize the main differences between existing bipolar soft rough set models defined on classical universes and the proposed lattice-based framework, a comparative analysis is presented in Table 1.

As illustrated in Table 1, classical bipolar soft rough set models can be regarded as particular cases of the proposed approach when the underlying lattice reduces to a power set. This observation clearly highlights the structural limitations of classical universe-based models and motivates the adoption of a lattice-theoretic framework.

Motivation for lattice-based formulation. Given that many categories of information granules are naturally governed by lattice ordering [23], there has been a renewed interest in lattice-theoretic approaches [1]. Lattice-based models provide enhanced structural generality and finer granularity compared to classical universe-based frameworks.

In particular, complete atomic Boolean lattices offer a powerful algebraic structure in which elements can be decomposed into atomic components, enabling more expressive approximation mechanisms. Järvinen demonstrated that rough approximation operators can be effectively modeled within such lattices. Building upon this direction, H.I.Mustafa [14] introduced soft sets over complete atomic Boolean lattices and defined corresponding soft rough approximations.

The main purpose of this paper is to introduce bipolar soft sets defined over a complete atomic Boolean lattice. We investigate their fundamental properties and operations, including complement, extended and restricted unions and intersections. Furthermore, we propose bipolar soft rough approximations on a complete atomic Boolean lattice as a generalization of existing bipolar soft rough models and study their properties. Finally, we present an application of the proposed framework to decision-making problems, supported by illustrative examples.

2. Preliminaries

In this section, we recall the basic notions, notations, and results that will be used throughout the paper. The preliminaries mainly concern soft sets defined on complete atomic Boolean lattices and bipolar soft sets. Standard lattice-theoretic terminology and notation are assumed, as in [2, 6], and the presentation follows [10, 14].

Throughout this paper, $\Xi = (\Xi, \leq)$ denotes a complete atomic Boolean lattice with least and greatest elements 0 and 1, respectively, and ζ denotes a nonempty set of parameters.

Definition 2.1. ([14]) Let $\varsigma \subseteq \zeta$ be a nonempty set of parameters. A soft set over Ξ with support ς is a mapping
$\Upsilon_\varsigma : \varsigma \to \Xi$, where $\Upsilon_\varsigma(\varrho)$ represents the ϱ-approximate element of Ξ.

Table 1: Comparison between classical bipolar soft rough set models and the proposed lattice-based bipolar soft rough model

Aspect	**Classical Bipolar Soft Rough Set Models**	**Proposed Bipolar Soft Rough Model on a Complete Atomic Boolean Lattice**
Underlying universe	A classical crisp universe U	A complete atomic Boolean lattice Ξ
Mathematical framework	Set-theoretic framework	Algebraic lattice-theoretic framework
Information granularity	Fixed and relatively coarse	Multi-level and finer granularity
Basis of rough approximation	Equivalence relations or binary relations	Lattice order and atomic decomposition
Representation of bipolarity	Positive and negative parameter sets treated separately	Positive and negative information embedded within lattice elements
Expressive power	Limited to binary membership representation	Capable of representing hierarchical and graded information
Structural flexibility	Restricted by classical set structures	High flexibility due to lattice generality
Algebraic properties	Weak or implicit algebraic structure	Rich algebraic structure (completeness, atomicity, Boolean operations)
Stability of approximations	Highly sensitive to the choice of relations	More stable due to lattice completeness
Applicability	Mainly suitable for simple decision-making problems	Applicable to complex uncertainty modeling and multi-criteria decision making
Theoretical generality	A particular case of the proposed framework	A general framework encompassing classical models
Main limitation	Inability to model hierarchical uncertainty	Higher mathematical complexity

Definition 2.2. ([14])Let $\varsigma_1, \varsigma_2 \subseteq \zeta$ and let Υ_{ς_1} and Q_{ς_2} be two soft sets over Ξ.

i. Υ_{ς_1} is called a *soft subset* of Q_{ς_2}, denoted by $\Upsilon_{\varsigma_1} \subseteq Q_{\varsigma_2}$, if $\varsigma_1 \subseteq \varsigma_2$ and $\Upsilon(\varrho) \leq Q(\varrho)$ for all $\varrho \in \varsigma_1$.
ii. Υ_{ς_1} and Q_{ς_2} are said to be *soft equal* if $\Upsilon_{\varsigma_1} \subseteq Q_{\varsigma_2}$ and $Q_{\varsigma_2} \subseteq \Upsilon_{\varsigma_1}$.

Definition 2.3. ([14])Let $\varsigma \subseteq \zeta$ and Υ_ς be a soft set over Ξ.

i. Υ_ς is called the *null soft set*, denoted by 0_ς, if $\Upsilon(\varrho) = 0$ for all $\varrho \in \varsigma$.
ii. Υ_ς is called the *absolute soft set*, denoted by 1_ς, if $\Upsilon(\varrho) = 1$ for all $\varrho \in \varsigma$.

Definition 2.4. ([14])Let $\varsigma \subseteq \zeta$ and Υ_ς be a soft set over Ξ. The complement of Υ_ς is defined by

$$\Upsilon_\varsigma(\varrho)^c = \big(\Upsilon(\varrho)\big)^{\backprime}, \quad \forall \varrho \in \varsigma.$$

Definition 2.5. ([14])Let $\varsigma_1, \varsigma_2 \subseteq \zeta$ and let Υ_{ς_1} and Q_{ς_2} be two soft sets over Ξ.

i. The *intersection* of Υ_{ς_1} and Q_{ς_2} is the soft set $\Upsilon_{\varsigma_1} \sqcap Q_{\varsigma_2}$ defined on $\varsigma_1 \cap \varsigma_2$ by

$$(\Upsilon \sqcap Q)(\varrho) = \Upsilon(\varrho) \wedge Q(\varrho).$$

ii. The *union* of Υ_{ς_1} and Q_{ς_2} is the soft set $\Upsilon_{\varsigma_1} \sqcup Q_{\varsigma_2}$ defined on $\varsigma_1 \cup \varsigma_2$ by

$$(\Upsilon \sqcup Q)(\varrho) = \begin{cases} \Upsilon(\varrho), & \varrho \in \varsigma_1 \setminus \varsigma_2, \\ Q(\varrho), & \varrho \in \varsigma_2 \setminus \varsigma_1, \\ \Upsilon(\varrho) \vee Q(\varrho), & \varrho \in \varsigma_1 \cap \varsigma_2. \end{cases}$$

Proposition 2.6. ([14]) *Let $\varsigma_1, \varsigma_2, \varsigma_3 \subseteq \zeta$ with soft sets Υ_{ς_1}, Q_{ς_2}, and R_{ς_3} over Ξ. Then the following properties hold:*

i. $\Upsilon_{\varsigma_1} \sqcup \Upsilon_{\varsigma_1} = \Upsilon_{\varsigma_1}$.
ii. $\Upsilon_{\varsigma_1} \sqcup Q_{\varsigma_2} = Q_{\varsigma_2} \sqcup \Upsilon_{\varsigma_1}$.
iii. $(\Upsilon_{\varsigma_1} \sqcup Q_{\varsigma_2}) \sqcup R_{\varsigma_3} = \Upsilon_{\varsigma_1} \sqcup (Q_{\varsigma_2} \sqcup R_{\varsigma_3})$.

Definition 2.7. ([10])Let $\varsigma = \{\varrho_1, \varrho_2, \ldots, \varrho_n\}$ be a set of parameters. The *NOT set* of ς is defined as

$$\neg\varsigma = \{\neg\varrho_1, \neg\varrho_2, \ldots, \neg\varrho_n\}.$$

Definition 2.8. ([10])Let U be a universal set and $\varsigma \subseteq \zeta$. Let (Υ, ς) and $(\Omega, \neg\varsigma)$ be two soft sets over U such that

$$\Upsilon(\varrho) \cap \Omega(\neg\varrho) = \emptyset \quad \forall \varrho \in \varsigma.$$

Then the triple $(\Upsilon, \Omega, \varsigma)_U$ is called a *bipolar soft set* over U.

Proposition 2.9. ([14]) *Let Υ_ς be a soft set over a complete atomic Boolean lattice Ξ. If Υ_ς induces a partition of Ξ, then*

$$(\Xi^{\vee^+}, \geq) \cong (\Xi^{\wedge^+}, \leq).$$

Proposition 2.10. ([14]) *Let Υ_ς be a soft set over a complete atomic Boolean lattice Ξ. Then the following assertions hold:*

I. *If Υ_ς is full, then:*
  i. $z^{\vee^+} \leq z \leq z^{\wedge^+}$ *for all* $z \in \Xi$;
  ii. $1^{\vee^+} = 1^{\wedge^+} = 1$.
II. *If Υ_ς is supremum-preserving, then:*
  i. *for every $z \in \Xi$, there exists $\varrho \in \varsigma$ such that $z^{\vee^+} = \Upsilon(\varrho)$;*
  ii. *for every $z \in \Xi$, there exists $\varrho \in \varsigma$ such that $z^{\wedge^+} = \Upsilon(\varrho)$.*
III. *If Υ_ς is both full and supremum-preserving, then $z^{\wedge^+} = 1$ for every $z \in \Xi \setminus \{0\}$.*

For the convenience of the reader, Table 2 summarizes the main symbols and notations used throughout the paper.

Symbol	Description
$\Xi = (\Xi, \leq)$	A complete atomic Boolean lattice
ζ	Universal set of parameters
$\varsigma, \varsigma_1, \varsigma_2, \varsigma_3$	Parameter subsets of ζ
$(\Upsilon, \Omega)_\varsigma$	A bipolar soft set over Ξ with parameter set ς
$\Upsilon(\varrho)$	Positive approximation function of a bipolar soft set
$\Omega(\neg\varrho)$	Negative approximation function of a bipolar soft set
$(\Upsilon, \Omega)^c_\varsigma$	Complement of a bipolar soft set
$\vee$, $\wedge$	Join and meet operations in the Boolean lattice Ξ
$\widetilde{\sqcup}_B$	Extended union of bipolar soft sets
$\widetilde{\sqcap}_B$	Extended intersection of bipolar soft sets
$\sqcup^{\mathfrak{R}}_B$	Restricted union of bipolar soft sets
$\sqcap^{\mathfrak{R}}_B$	Restricted intersection of bipolar soft sets
$(0, 1)_\varsigma$	Null bipolar soft set over ς
$(1, 0)_\varsigma$	Absolute bipolar soft set over ς
$H(\varrho)$, $K(\neg\varrho)$	Resulting positive and negative mappings after bipolar soft operations

Table 2: List of symbols and notations used in the paper

3. Bipolar soft sets on a complete atomic Boolean lattice

In this section, we introduce the concept of a bipolar soft set over a complete atomic Boolean lattice (BSs) and provide definitions of complement, extended and restricted unions and intersections, as well as some fundamental properties.

Definition 3.1. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice, and let $\varsigma = \{\varrho_1, \varrho_2, \ldots, \varrho_i\}$ be a set of parameters. A triple $(\Upsilon, \Omega, \varsigma)$, denoted by $(\Upsilon, \Omega)_\varsigma$, is called a *bipolar soft set* over Ξ, where

$$\Upsilon : \varsigma \longrightarrow \Xi, \quad \Omega : \neg\varsigma \longrightarrow \Xi$$

are maps such that

$$\Upsilon(\varrho) \wedge \Omega(\neg\varrho) = 0, \quad \forall \varrho \in \varsigma.$$

Remark 3.2. If $\Xi = \mathcal{P}(U)$, then the above definition coincides with the classical bipolar soft set introduced in [10].

Definition 3.3. Let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. We say that $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ is a *bipolar soft subset* of $(\Upsilon_2, \Omega_2)_{\varsigma_2}$, denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \subseteq_B (\Upsilon_2, \Omega_2)_{\varsigma_2},$$

if:

1. $\varsigma_1 \subseteq \varsigma_2$;
2. $\Upsilon_1(\varrho) \leq \Upsilon_2(\varrho)$ and $\Omega_2(\neg\varrho) \leq \Omega_1(\neg\varrho)$ for every $\varrho \in \varsigma_1$.

Similarly, $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ is a *bipolar soft super-set* of $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ if $(\Upsilon_2, \Omega_2)_{\varsigma_2} \subseteq_B (\Upsilon_1, \Omega_1)_{\varsigma_1}$.

Definition 3.4. Two bipolar soft sets $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ over Ξ are said to be *equal* if

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \subseteq_B (\Upsilon_2, \Omega_2)_{\varsigma_2} \quad \text{and} \quad (\Upsilon_2, \Omega_2)_{\varsigma_2} \subseteq_B (\Upsilon_1, \Omega_1)_{\varsigma_1}.$$

Definition 3.5. Let $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ.

i. $(\Upsilon, \Omega)_\varsigma$ is called *null*, denoted by $(0, 1)_\varsigma$, if

$$\Upsilon(\varrho) = 0 \quad \text{and} \quad \Omega(\neg\varrho) = 1, \quad \forall \varrho \in \varsigma.$$

ii. $(\Upsilon, \Omega)_\varsigma$ is called *absolute*, denoted by $(1, 0)_\varsigma$, if

$$\Upsilon(\varrho) = 1 \quad \text{and} \quad \Omega(\neg\varrho) = 0, \quad \forall \varrho \in \varsigma.$$

Definition 3.6. The *complement* of a bipolar soft set $(\Upsilon, \Omega)_\varsigma$ over Ξ, denoted by $(\Upsilon, \Omega)^c_\varsigma$, is defined as

$$(\Upsilon, \Omega)^c_\varsigma = (\Upsilon^c, \Omega^c)_\varsigma,$$

where

$$\Upsilon(\varrho)^c = \Omega(\neg\varrho), \quad \Omega(\neg\varrho)^c = \Upsilon(\varrho), \quad \forall \varrho \in \varsigma.$$

Definition 3.7. Let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. The *bipolar soft set "and" operation* is denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \wedge (\Upsilon_2, \Omega_2)_{\varsigma_2}$$

and is defined as

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \wedge (\Upsilon_2, \Omega_2)_{\varsigma_2} = (M, L, \varsigma_1 \times \varsigma_2),$$

where, for all $(\varrho_1, \varrho_2) \in \varsigma_1 \times \varsigma_2$,

$$M(\varrho_1, \varrho_2) = \Upsilon_1(\varrho_1) \wedge \Upsilon_2(\varrho_2), \quad L(\neg\varrho_1, \neg\varrho_2) = \Omega_1(\neg\varrho_1) \vee \Omega_2(\neg\varrho_2).$$

Definition 3.8. Let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. The *bipolar soft set "or" operation* is denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \vee (\Upsilon_2, \Omega_2)_{\varsigma_2}$$

and is defined as

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \vee (\Upsilon_2, \Omega_2)_{\varsigma_2} = (M, L, \varsigma_1 \times \varsigma_2),$$

where, for all $(\varrho_1, \varrho_2) \in \varsigma_1 \times \varsigma_2$,

$$M(\varrho_1, \varrho_2) = \Upsilon_1(\varrho_1) \vee \Upsilon_2(\varrho_2), \quad L(\neg\varrho_1, \neg\varrho_2) = \Omega_1(\neg\varrho_1) \wedge \Omega_2(\neg\varrho_2).$$

Proposition 3.9. *Let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. Then the following properties hold:*

$$\Big[(\Upsilon_1, \Omega_1)_{\varsigma_1} \vee (\Upsilon_2, \Omega_2)_{\varsigma_2}\Big]^c = (\Upsilon_1, \Omega_1)^c_{\varsigma_1} \wedge (\Upsilon_2, \Omega_2)^c_{\varsigma_2};$$

$$\Big[(\Upsilon_1, \Omega_1)_{\varsigma_1} \wedge (\Upsilon_2, \Omega_2)_{\varsigma_2}\Big]^c = (\Upsilon_1, \Omega_1)^c_{\varsigma_1} \vee (\Upsilon_2, \Omega_2)^c_{\varsigma_2}.$$

Proof. Straightforward. □

Definition 3.10. Let $\varsigma_1, \varsigma_2 \subseteq \zeta$ and let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. The *extended union* of $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$, denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup_B} (\Upsilon_2, \Omega_2)_{\varsigma_2},$$

is defined as $(H, K)_{\varsigma}$, where $\varsigma = \varsigma_1 \cup \varsigma_2$ and for all $\varrho \in \varsigma$,

$$H(\varrho) = \begin{cases} \Upsilon_1(\varrho), & \varrho \in \varsigma_1 \setminus \varsigma_2, \\ \Upsilon_2(\varrho), & \varrho \in \varsigma_2 \setminus \varsigma_1, \\ \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho), & \varrho \in \varsigma_1 \cap \varsigma_2, \end{cases}$$

and

$$K(\neg\varrho) = \begin{cases} \Omega_1(\neg\varrho), & \neg\varrho \in \neg\varsigma_1 \setminus \neg\varsigma_2, \\ \Omega_2(\neg\varrho), & \neg\varrho \in \neg\varsigma_2 \setminus \neg\varsigma_1, \\ \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho), & \neg\varrho \in \neg\varsigma_1 \cap \neg\varsigma_2. \end{cases}$$

Definition 3.11. Let $\varsigma_1, \varsigma_2 \subseteq \zeta$ and let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. The *extended intersection* of $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$, denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcap_B} (\Upsilon_2, \Omega_2)_{\varsigma_2},$$

is defined as $(H, K)_{\varsigma}$, where $\varsigma = \varsigma_1 \cup \varsigma_2$ and for all $\varrho \in \varsigma$,

$$H(\varrho) = \begin{cases} \Upsilon_1(\varrho), & \text{if } \varrho \in \varsigma_1 \setminus \varsigma_2, \\ \Upsilon_2(\varrho), & \text{if } \varrho \in \varsigma_2 \setminus \varsigma_1, \\ \Upsilon_1(\varrho) \wedge \Upsilon_2(\varrho), & \text{if } \varrho \in \varsigma_1 \cap \varsigma_2, \end{cases}$$

and

$$K(\neg\varrho) = \begin{cases} \Omega_1(\neg\varrho), & \text{if } \neg\varrho \in \neg\varsigma_1 \setminus \neg\varsigma_2, \\ \Omega_2(\neg\varrho), & \text{if } \neg\varrho \in \neg\varsigma_2 \setminus \neg\varsigma_1, \\ \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho), & \text{if } \neg\varrho \in \neg\varsigma_1 \cap \neg\varsigma_2. \end{cases}$$

Definition 3.12. Let $\varsigma_1, \varsigma_2 \subseteq \zeta$ and let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ.The *restricted union* of $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$, denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcup_B^{\mathcal{R}} (\Upsilon_2, \Omega_2)_{\varsigma_2},$$

is defined as $(H, K)_{\varsigma}$, where $\varsigma = \varsigma_1 \cap \varsigma_2$ and for all $\varrho \in \varsigma$,

$$H(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho), \quad K(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho).$$

Definition 3.13. Let $\varsigma_1, \varsigma_2 \subseteq \zeta$ and let $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ. The *restricted intersection* of $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$, denoted by

$$(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap_B^{\mathfrak{R}} (\Upsilon_2, \Omega_2)_{\varsigma_2},$$

is defined as $(H, K)_{\varsigma}$, where $\varsigma = \varsigma_1 \cap \varsigma_2$ and for all $\varrho \in \varsigma$,

$$H(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_2(\varrho), \quad K(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho).$$

Remark 3.14. [Extended vs. Restricted Operations] We clarify the difference between the *extended* and *restricted* operations: extended operations include all parameters from both bipolar soft sets, preserving all available information, while restricted operations consider only common parameters, focusing on shared characteristics. Semantically, extended operations retain all contributions, whereas restricted operations emphasize agreement or conflict on shared parameters.

Proposition 3.15. *Let* $\varsigma_1, \varsigma_2, \varsigma_3 \subseteq \zeta$ *and* $(\Upsilon_1,\Omega_1)_{\varsigma_1}$, $(\Upsilon_2,\Omega_2)_{\varsigma_2}$, $(\Upsilon_3,\Omega_3)_{\varsigma_3}$ *be three bipolar soft sets over* Ξ*. Then:*

i. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_1,\Omega_1)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$.
ii. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_1,\Omega_1)_{\varsigma_1}$.
iii. $[(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2}]\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}[(\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3}]$.
iv. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2}$.
v. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(0,1)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$.
vi. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(1,0)_{\varsigma_1} = (1,0)_{\varsigma_1}$.

Proof. Parts (i), (ii), (v), and (vi) follow directly from the definition of the extended union. We prove (iii). Let $[(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2}]\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (S,T)_{\varsigma}$, $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}[(\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3}] = (L,Z)_{\varsigma}$.
$(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (I,C)_{\varsigma_1\cup\varsigma_2}$, $(\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (O,J)_{\varsigma_2\cup\varsigma_3}$
For any $\varrho \in \varsigma$ where $\varsigma = \varsigma_1 \cup \varsigma_2 \cup \varsigma_3$ it follows that $\varrho \in \varsigma_1$, *or* $\varrho \in \varsigma_2$, *or* $\varrho \in \varsigma_3$.

Case(1) ($\varrho \in \varsigma_3$)

(a) If $\varrho \notin \varsigma_1$ *and* $\varrho \notin \varsigma_2$ then $(S,T)_{\varsigma} = (\Upsilon_3,\Omega_3)_{\varsigma_3} = (O,J)_{\varsigma_2\cup\varsigma_3} = (L,Z)_{\varsigma}$.
(b) If $\varrho \notin \varsigma_1$ *and* $\varrho \in \varsigma_2$ then $(S,T)_{\varsigma} = (\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (L,Z)_{\varsigma}$, where $S(\varrho) = \Upsilon_2(\varrho) \vee \Upsilon_3(\varrho)$, $T(\neg\varrho) = \Omega_2(\neg\varrho) \wedge \Omega_3(\neg\varrho)$.
(c) If $\varrho \in \varsigma_1$ *and* $\varrho \notin \varsigma_2$ then $(S,T)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(O,J)_{\varsigma_2\cup\varsigma_3} = (L,Z)_{\varsigma}$, where $S(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_3(\varrho)$, $T(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_3(\neg\varrho)$, and $(\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (\Upsilon_3,\Omega_3)_{\varsigma_3} = (O,J)_{\varsigma_2\cup\varsigma_3}$, then $(L,Z)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (S,T)_{\varsigma}$.
(d) If $\varrho \in \varsigma_1$ *and* $\varrho \in \varsigma_2$ then $(S,T)_{\varsigma} = (I,C)_{\varsigma_1\cup\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (L,Z)_{\varsigma}$, where $S(\varrho) = I(\varrho) \vee \Upsilon_3 = \Upsilon_1 \vee \Upsilon_2 \vee \Upsilon_3$, $T(\neg\varrho) = C(\neg\varrho) \wedge \Omega_3(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho) \wedge \Omega_3(\neg\varrho)$, $\varrho \in \varsigma_1 \cap \varsigma_2 \cap \varsigma_3$, and $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(O,J)_{\varsigma_2\cup\varsigma_3} = (L,Z)_{\varsigma}$
Where $L(\varrho) = \Upsilon_1(\varrho) \vee O(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho) \vee \Upsilon_3(\varrho)$, $Z(\neg\varrho) = \Omega_1(\neg\varrho) \wedge J(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho) \wedge \Omega_3(\neg\varrho)$

Case(2) ($\varrho \notin \varsigma_3$)

(a) If $\varrho \notin \varsigma_1$ *and* $\varrho \in \varsigma_2$ then $(S,T)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_2,\Omega_2)_{\varsigma_2}$, and $(L,Z)_{\varsigma} = (\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (\Upsilon_2,\Omega_2)_{\varsigma_2}$ then $(S,T)_{\varsigma} = (L,Z)_{\varsigma}$.
(b) If $\varrho \in \varsigma_1$ *and* $\varrho \notin \varsigma_2$ then $(S,T)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$, and $(L,Z)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(0,1)_{\varsigma_2\cup\varsigma_3} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$. Where $\varsigma \notin \varsigma_2 \cup \varsigma_3$, *then* $(\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (0,1)_{\varsigma_2\cup\varsigma_3}$, $(S,T)_{\varsigma} = (L,Z)_{\varsigma}$.
(c) If $\varrho \in \varsigma_1$ *and* $\varrho \in \varsigma_2$, then $(S,T)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (I,C)_{\varsigma_1\cup\varsigma_2}$.
$I(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho)$ *and* $C(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho)$, $(L,Z)_{\varsigma} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(O,J)_{\varsigma_2\cup\varsigma_3} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2}$.
Then $L(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho)$, $Z(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho)$, $(S,T)_{\varsigma} = (L,Z)_{\varsigma}$.
Thus $[(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2}]\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3} = (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}[(\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcup_B}(\Upsilon_3,\Omega_3)_{\varsigma_3}]$

(4) If $\varrho \in \varsigma_1$ *and* $\varrho \in \varsigma_2$.
L.H.S $= (\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (S,T)_{\varsigma_1\cup\varsigma_2}$, where $S(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho)$, $T(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho)$.
R.H.S $= (\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2} = (L,Z)_{\varsigma_1\cap\varsigma_2}$, where $L(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho)$, $Z(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho)$ for all $\varrho \in \varsigma_1 \cap \varsigma_2$. Then $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2}$. □

Proposition 3.16. *Let* $\varsigma_1, \varsigma_2, \varsigma_3 \subseteq \zeta$ *and* $(\Upsilon_1,\Omega_1)_{\varsigma_1}$, $(\Upsilon_2,\Omega_2)_{\varsigma_2}$, $(\Upsilon_3,\Omega_3)_{\varsigma_3}$ *be three bipolar soft sets over* Ξ*. Then the following properties hold:*

i. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcap_B}(\Upsilon_1,\Omega_1)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$.
ii. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcap_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_2,\Omega_2)_{\varsigma_2}\widetilde{\sqcap_B}(\Upsilon_1,\Omega_1)_{\varsigma_1}$.
iii. $[(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3,\Omega_3)_{\varsigma_3} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B [(\Upsilon_2,\Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3,\Omega_3)_{\varsigma_3}]$.
iv. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcap_B}(\Upsilon_2,\Omega_2)_{\varsigma_2} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2}$.
v. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcap_B}(0,1)_{\varsigma_1} = (0,1)_{\varsigma_1}$.
vi. $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcap_B}(1,0)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$.

Proof. Parts (i), (ii), (v), and (vi) follow directly from the definition of the extended intersection. We prove (iii). Let $[(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3,\Omega_3)_{\varsigma_3} = (S,T)_{\varsigma}$, $(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B [(\Upsilon_2,\Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3,\Omega_3)_{\varsigma_3}] = (L,Z)_{\varsigma}$.
$(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2,\Omega_2)_{\varsigma_2} = (I,C)_{\varsigma_1\cap\varsigma_2}$, $(\Upsilon_2,\Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3,\Omega_3)_{\varsigma_3} = (O,J)_{\varsigma_2\cap\varsigma_3}$.

For any $\varrho \in \varsigma,$ *where* $\varsigma = \varsigma_1 \cap \varsigma_2 \cap \varsigma_3$, it follows that $\varrho \in \varsigma_1,\ \varrho \in \varsigma_2,\ \varrho \in \varsigma_3$.
Then $I(\varrho) = \Upsilon_1 \wedge \Upsilon_2,\ C(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho)$ and $(I, C)_{\varsigma_1\cap\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = (S, T)_{\varsigma}$, where $S(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho),\ T(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)$, and $O(\varrho) = \Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho),\ J(\neg\varrho) = \Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)$, where $(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (O, J)_{\varsigma_2\cap\varsigma_3} = (L, Z)_{\varsigma}, L(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho),\ Z(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)$.
Then $(S, T)_{\varsigma} = (L, Z)_{\varsigma}$.

(4) Let $(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcap}_B (\Upsilon_2, \Omega_2)_{\varsigma_2} = (S, T)_{\varsigma_1\cup\varsigma_2}$, where $S(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_2(\varrho),\ T(\neg\varsigma) = \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho)$, and put $(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2, \Omega_2)_{\varsigma_2} = (O, J)_{\varsigma_1\cap\varsigma_2}$, where $O(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_2(\varrho),\ J(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_2(\neg\varrho)$.
Then $(S, T)_{\varsigma_1\cup\varsigma_2} = (O, J)_{\varsigma_1\cap\varsigma_2}$. □

Proposition 3.17. *Let* $\varsigma_1, \varsigma_2, \varsigma_3 \subseteq \zeta$ *and* $(\Upsilon_1, \Omega_1)_{\varsigma_1}, (\Upsilon_2, \Omega_2)_{\varsigma_2}, (\Upsilon_3, \Omega_3)_{\varsigma_3}$ *be three bipolar soft sets over* Ξ*. Then the following properties hold:*

i. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \widetilde{\sqcup}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
ii. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \widetilde{\sqcup}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \sqcap^{\mathfrak{R}}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \widetilde{\sqcup}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
iii. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \sqcap^{\mathfrak{R}}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
iv. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \sqcup^{\mathfrak{R}}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
v. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \widetilde{\sqcup}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
vi. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \sqcap^{\mathfrak{R}}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
vii. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcap}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \widetilde{\sqcap}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcup^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
viii. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcup^{\mathfrak{R}}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \widetilde{\sqcap}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcap}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \sqcup^{\mathfrak{R}}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \widetilde{\sqcap}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.

Proof. (i) Let $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = (O, J)_{(\varsigma_1\cup\varsigma_2)\cap\varsigma_3}$,
$[(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \widetilde{\sqcup}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] = (L, Z)_{(\varsigma_1\cap\varsigma_3)\cup(\varsigma_2\cap\varsigma_3)}$
Obviously, $(\varsigma_1 \cup \varsigma_2) \cap \varsigma_3 = (\varsigma_1 \cap \varsigma_3) \cup (\varsigma_2 \cap \varsigma_3)$ *for every* $\varrho \in (\varsigma_1 \cup \varsigma_2) \cap \varsigma_3$.
We have the following three disjoint cases:
(a) If $\varrho \notin \varsigma_1 \cap \varsigma_3$ *and* $\varrho \in \varsigma_2 \cap \varsigma_3$, *then* $\varrho \notin \varsigma_1, \varrho \in \varsigma_2,$ *and* $\varrho \in \varsigma_3$.
So $((\Upsilon_1 \widetilde{\sqcup}_B \Upsilon_2) \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = \Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho),\ (\Omega_1 \widetilde{\sqcup}_B \Omega_2) \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) = \Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)$.
$(\Upsilon_1 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) \widetilde{\sqcup}_B (\Upsilon_2 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = 0 \vee (\Upsilon_2 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = \Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho)$.
$(\Omega_1 \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) \widetilde{\sqcup}_B (\Omega_2 \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) = 1 \wedge (\Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)) = \Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)$.
(b) If $\varrho \in \varsigma_1 \cap \varsigma_3$ *and* $\varrho \notin \varsigma_2 \cap \varsigma_3$, *then* $\varrho \in \varsigma_1, \varrho \notin \varsigma_2,$ *and* $\varrho \in \varsigma_3$.
So $((\Upsilon_1 \widetilde{\sqcup}_B \Upsilon_2) \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_3(\varrho)$.
$(\Omega_1 \widetilde{\sqcup}_B \Omega_2) \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_3(\neg\varrho)$,
$(\Upsilon_1 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) \widetilde{\sqcup}_B (\Upsilon_2 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = (\Upsilon_1(\varrho) \wedge \Upsilon_3(\varrho)) \vee 0 = \Upsilon_1(\varrho) \wedge \Upsilon_3(\varrho)$.
$(\Omega_1 \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) \widetilde{\sqcup}_B (\Omega_2 \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) = (\Omega_1(\neg\varrho) \vee \Omega_3(\neg\varrho)) \wedge 1 = \Omega_1(\neg\varrho) \vee \Omega_3(\neg\varrho)$.
(c) If $\varrho \in \varsigma_1 \cap \varsigma_3$ *and* $\varrho \in \varsigma_2 \cap \varsigma_3$, *then* $\varrho \in \varsigma_1, \varrho \in \varsigma_2,$ *and* $\varrho \in \varsigma_3$.
So $((\Upsilon_1 \widetilde{\sqcup}_B \Upsilon_2) \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho),\ (\Omega_1 \widetilde{\sqcup}_B \Omega_2) \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)$, and
$(\Upsilon_1 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) \widetilde{\sqcup}_B (\Upsilon_2 \sqcap^{\mathfrak{R}}_B \Upsilon_3)(\varrho) = (\Upsilon_1(\varrho) \wedge \Upsilon_3(\varrho)) \vee (\Upsilon_2(\varrho) \wedge \Upsilon_3(\varrho)) = (\Upsilon_1(\varrho) \vee \Upsilon_2(\varrho)) \wedge \Upsilon_3(\varrho)$.
$(\Omega_1 \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) \widetilde{\sqcup}_B (\Omega_2 \sqcap^{\mathfrak{R}}_B \Omega_3)(\neg\varrho) = (\Omega_1(\neg\varrho) \vee \Omega_3(\neg\varrho)) \wedge (\Omega_2(\neg\varrho) \vee \Omega_3(\neg\varrho)) = \Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho) \vee \Omega_1(\neg\varrho)$.
Thus $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}] \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3} = [(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}] \widetilde{\sqcup}_B [(\Upsilon_2, \Omega_2)_{\varsigma_2} \sqcap^{\mathfrak{R}}_B (\Upsilon_3, \Omega_3)_{\varsigma_3}]$.
Similarly, we can check for the remaining parts. □

Proposition 3.18. *Let* $\varsigma_1, \varsigma_2 \subseteq \zeta$*. Let* $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ *and* $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ *be two bipolar soft sets over* Ξ*. Then the following properties hold:*

i. $[(\Upsilon_1, \Omega_1)^c_{\varsigma_1}]^c = (\Upsilon_1, \Omega_1)_{\varsigma_1}$
ii. $(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_1, \Omega_1)^c_{\varsigma_1} = (1, 0)_{\varsigma_1}$
iii. $(\Upsilon_1, \Omega_1)_{\varsigma_1} \sqcap^{\mathfrak{R}}_B (\Upsilon_1, \Omega_1)^c_{\varsigma_1} = (0, 1)_{\varsigma_1}$
iv. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}]^c = (\Upsilon_1, \Omega_1)^c_{\varsigma_1} \widetilde{\sqcap}_B (\Upsilon_2, \Omega_2)^c_{\varsigma_2}$
v. $[(\Upsilon_1, \Omega_1)_{\varsigma_1} \widetilde{\sqcap}_B (\Upsilon_2, \Omega_2)_{\varsigma_2}]^c = (\Upsilon_1, \Omega_1)^c_{\varsigma_1} \widetilde{\sqcup}_B (\Upsilon_2, \Omega_2)^c_{\varsigma_2}$

vi. $[(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcup_B^{\Re} (\Upsilon_2,\Omega_2)_{\varsigma_2}]^c = (\Upsilon_1,\Omega_1)^c_{\varsigma_1} \sqcap_B^{\Re} (\Upsilon_2,\Omega_2)^c_{\varsigma_2}$

vii. $[(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap_B^{\Re} (\Upsilon_2,\Omega_2)_{\varsigma_2}]^c = (\Upsilon_1,\Omega_1)^{c\ c}_{\varsigma_1} \sqcup_B^{\Re} (\Upsilon_2,\Omega_2)^c_{\varsigma_2}$

Proof. We prove properties (i), (ii), (iii), and (iv) as follows.
(i) Let ${(\Upsilon_1,\Omega_1)_{\varsigma_1}}^c = (h,k)_{\varsigma_1}$, ${(h,k)_{\varsigma_1}}^c = (m,n)_{\varsigma_1}$.
Since ${(\Upsilon_1,\Omega_1)_{\varsigma_1}}^c = ({\Upsilon_1}^c,{\Omega_1}^c)_{\varsigma_1} = (h,k)_{\varsigma_1}$, for any $\varrho \in \varsigma_1$. Then $h(\varrho) = {\Upsilon_1}^c(\varrho) = \Omega_1(\neg\varrho)$, $k(\neg\varrho) = \Omega_1(\neg\varrho)^c = \Upsilon_1(\varrho)$. From ${(h,k)_{\varsigma_1}}^c = (m,n)_{\varsigma_1}$, then $h(\varrho)^c = \Upsilon_1(\varrho)$, $k(\neg\varrho)^c = \Omega_1(\neg\varrho)$ and $(h(\varrho)^c,k(\neg\varrho)^c)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1}$.
Thus $[(\Upsilon_1,\Omega_1)^c_{\varsigma_1}]^c = (\Upsilon_1,\Omega_1)_{\varsigma_1}$.

(ii) Put $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup}_B{(\Upsilon_1,\Omega_1)_{\varsigma_1}}^c = (h,k)_{\varsigma_1}$, for any $\varrho \in \varsigma_1$. Then $h(\varrho) = \Upsilon_1(\varrho) \vee \Upsilon_1(\varrho)^c = \Upsilon_1(\varrho) \vee \Omega_1(\neg\varrho) = 1$, $k(\neg\varrho) = \Omega_1(\neg\varrho) \wedge \Omega_1(\neg\varrho)^c = \Omega_1(\neg\varrho) \wedge \Upsilon_1(\varrho) = 0$. Hence $(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup}_B(\Upsilon_1,\Omega_1)^c_{\varsigma_1} = (1,0)_{\varsigma_1}$.

(iii) $(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap_B^{\Re} (\Upsilon_1,\Omega_1)^c_{\varsigma_1} = (h,k)_{\varsigma_1}$, for any $\varrho \in \varsigma_1$. Then $h(\varrho) = \Upsilon_1(\varrho) \wedge \Upsilon_1(\varrho)^c = \Upsilon_1(\varrho) \wedge \Omega_1(\neg\varrho) = 0$, $k(\neg\varrho) = \Omega_1(\neg\varrho) \vee \Omega_1(\neg\varrho)^c = \Omega_1(\neg\varrho) \vee \Upsilon_1(\varrho) = 1$. Hence $(\Upsilon_1,\Omega_1)_{\varsigma_1} \sqcap_B^{\Re} (\Upsilon_1,\Omega_1)^c_{\varsigma_1} = (0,1)_{\varsigma_1}$.

(iv) Let $\varrho \in \varsigma_1 \cup \varsigma_2$ are three cases :
(a) If $\varrho \in \varsigma_1 - \varsigma_2$, then
$(\Upsilon_1\widetilde{\sqcup}_B\Upsilon_2)^c(\varrho) = \Upsilon_1(\varrho)^c = \Omega_1(\neg\varrho)$ *and* $({\Upsilon_1}^c\widetilde{\sqcap}_B{\Upsilon_2}^c)(\varrho) = (\Omega_1\widetilde{\sqcap}_B\Upsilon_2)(\neg\varrho) = \Omega_1(\neg\varrho)$, $(\Omega_1\widetilde{\sqcup}_B\Omega_2)^c(\neg\varrho) = \Omega_1(\neg\varrho)^c = \Upsilon_1(\varrho)$ *and* $({\Omega_1}^c\widetilde{\sqcap}_B{\Omega_2}^c)(\neg\varrho) = (\Upsilon_1\widetilde{\sqcap}_B\Upsilon_2)(\varrho) = \Upsilon_1(\varrho)$.
(b) If $\varrho \in \varsigma_2 - \varsigma_1$, then
$(\Upsilon_1\widetilde{\sqcup}_B\Upsilon_2)^c(\varrho) = \Upsilon_2(\varrho)^c = \Omega_2(\neg\varrho) = ({\Upsilon_1}^c\widetilde{\sqcap}_B{\Upsilon_2}^c)(\varrho)$, $(\Omega_1\widetilde{\sqcup}_B\Omega_2)^c(\neg\varrho) = \Omega_2(\neg\varrho)^c = \Upsilon_2(\varrho) = ({\Omega_1}^c\widetilde{\sqcap}_B{\Omega_2}^c)(\neg\varrho)$.
(c) If $\varrho \in \varsigma_1 \cap \varsigma_2$, then
$(\Upsilon_1\widetilde{\sqcup}_B\Upsilon_2)^c(\varrho) = (\Upsilon_1(\varrho) \vee \Upsilon_2(\varrho))^c = \Upsilon_1(\varrho)^c \wedge \Upsilon_2(\varrho)^c$ and $({\Upsilon_1}^c\widetilde{\sqcap}_B{\Upsilon_2}^c)(\varrho) = \Upsilon_1(\varrho)^c \wedge \Upsilon_2(\varrho)^c$,
$(\Omega_1\widetilde{\sqcup}_B\Omega_2)^c(\neg\varrho) = (\Omega_1(\neg\varrho) \wedge \Omega_2(\neg\varrho))^c = \Omega_1(\neg\varrho)^c \vee \Omega_2(\neg\varrho)^c$ and $({\Omega_1}^c\widetilde{\sqcap}_B{\Omega_2}^c)(\neg\varrho) = \Omega_1(\neg\varrho)^c \vee \Omega_2(\neg\varrho)^c$.
Then from (a), (b), (c), we obtain $[(\Upsilon_1,\Omega_1)_{\varsigma_1}\widetilde{\sqcup}_B(\Upsilon_2,\Omega_2)_{\varsigma_2}]^c = (\Upsilon_1,\Omega_1)^c_{\varsigma_1}\ \widetilde{\sqcap}_B\ (\Upsilon_2,\Omega_2)^c_{\varsigma_2}$.
Conditions (v), (vi), and (vii) can be proved similarly. □

Definition 3.19. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice, and let $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ.

i $(\Upsilon,\Omega)_\varsigma$ is called *full* if $\bigvee_{\varrho\in\varsigma}\Upsilon(\varrho) = 1$ and $\bigvee_{\neg\varrho\in\neg\varsigma}\Omega(\neg\varrho) = 1$.
ii $(\Upsilon,\Omega)_\varsigma$ is *infimum-preserving* if for any $\varrho_1,\varrho_2 \in \varsigma$, there exists $\varrho_3 \in \varsigma$ such that $\Upsilon(\varrho_1) \wedge \Upsilon(\varrho_2) = \Upsilon(\varrho_3)$, $\Omega(\neg\varrho_1) \vee \Omega(\neg\varrho_2) = \Omega(\neg\varrho_3)$.
iii $(\Upsilon,\Omega)_\varsigma$ is *supremum-preserving* if for any $\varrho_1,\varrho_2 \in \varsigma$, there exists $\varrho_3 \in \varsigma$ such that$\Upsilon(\varrho_1) \vee \Upsilon(\varrho_2) = \Upsilon(\varrho_3)$, $\Omega(\neg\varrho_1) \wedge \Omega(\neg\varrho_2) = \Omega(\neg\varrho_3)$.
iv $(\Upsilon,\Omega)_\varsigma$ is called a *partition* of Ξ if:
  (a) $\bigvee\limits_{\varrho\in\varsigma}\Upsilon(\varrho) = 1$ and $\bigvee\limits_{\neg\varrho\in\neg\varsigma}\Omega(\neg\varrho) = 1$.
  (b) For every $\varrho \in \varsigma$, $\Upsilon(\varrho) \neq 0$ and $\Omega(\neg\varrho) \neq 0$.
  (c) For every $\varrho_1,\varrho_2 \in \varsigma$, either $\Upsilon(\varrho_1) = \Upsilon(\varrho_2)$ or $\Upsilon(\varrho_1) \wedge \Upsilon(\varrho_2) = 0$, and either $\Omega(\neg\varrho_1) = \Omega(\neg\varrho_2)$ or $\Omega(\neg\varrho_1) \wedge \Omega(\neg\varrho_2) = 0$.

Now, we give the following example of the resulting bipolar soft sets obtained by applying the operations mentioned above on $(\Upsilon_1,\Omega_1)_{\varsigma_1}$ and $(\Upsilon_2,\Omega_2)_{\varsigma_2}$.

Example 3.20. Let $\Xi = \{0,a,b,c,d,e,f,1\}$ and let the order $\leq$ be defined as in Figure(1). The set of atoms of a complete atomic Boolean lattice $\Xi = (\Xi,\leq)$ is $\{a,b,c\}$. Let $\varsigma_1,\varsigma_2 \subseteq \zeta$, where $\zeta = \{\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5,\varrho_6\}$ be a set of parameters, $\neg\zeta = \{\neg\varrho_1,\neg\varrho_2,\neg\varrho_3,\neg\varrho_4,\neg\varrho_5,\neg\varrho_6\}$. Suppose that $\varsigma_1 = \{\varrho_1,\varrho_2,\varrho_3,\varrho_6\}$ and $\varsigma_2 = \{\varrho_2,\varrho_3,\varrho_4,\varrho_5\}$. Let $(\Upsilon_1,\Omega_1)_{\varsigma_1}$ and $(\Upsilon_2,\Omega_2)_{\varsigma_2}$ be two bipolar soft sets over Ξ defined as follows:
$\Upsilon_1(\varrho_1) = e$, $\Upsilon_1(\varrho_2) = b$, $\Upsilon_1(\varrho_3) = c$, $\Upsilon_1(\varrho_6) = 0$.
$\Omega_1(\neg\varrho_1) = b$, $\Omega_1(\neg\varrho_2) = e$, $\Omega_1(\neg\varrho_3) = 0$, $\Omega_1(\neg\varrho_6) = c$.
and
$\Upsilon_2(\varrho_2) = d$, $\Upsilon_2(\varrho_3) = f$, $\Upsilon_2(\varrho_4) = a$, $\Upsilon_2(\varrho_5) = c$.

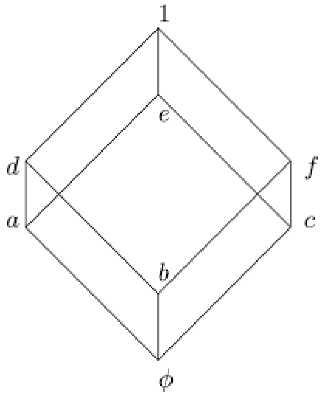

Figure 1:

$\Omega_2(\neg\varrho_2) = f$, $\Omega_2(\neg\varrho_3) = a$, $\Omega_2(\neg\varrho_4) = c$, $\Omega_2(\neg\varrho_5) = d$.

Let $(\Upsilon_1, \Omega_1, \varsigma_1)\widetilde{\sqcup}_B(\Upsilon_2, \Omega_2, \varsigma_2) = (H_1, K_1, \varrho_1 \cup \varrho_2)$.
Then $H_1(\varrho_1) = e \quad H_1(\varrho_2) = b \vee d = d \quad H_1(\varrho_3) = c \vee f = f \quad H_1(\varrho_4) = a \quad H_1(\varrho_5) = c \quad H_1(\varrho_6) = 0$,
and $K_1(\neg\varrho_1) = b \quad K_1(\neg\varrho_2) = e \wedge f = c \quad K_1(\neg\varrho_3) = 0 \wedge a = 0 \quad K_1(\neg\varrho_4) = c \quad K_1(\neg\varrho_5) = d \quad K_1(\neg\varrho_6) = c$.

Let $(\Upsilon_1, \Omega_1, \varsigma_1)\widetilde{\sqcap}_B(\Upsilon_2, \Omega_2, \varsigma_2) = (H_2, K_2, \varrho_1 \cup \varrho_2)$.
Then $H_2(\varrho_1) = e \quad H_2(\varrho_2) = b \wedge d = b \quad H_2(\varrho_3) = c \wedge f = c \quad H_2(\varrho_4) = a \quad H_2(\varrho_5) = c \quad H_2(\varrho_6) = 0$,
and $K_2(\neg\varrho_1) = b \quad K_2(\neg\varrho_2) = e \vee f = 1 \quad K_2(\neg\varrho_3) = 0 \vee a = a \quad K_2(\neg\varrho_4) = c \quad K_2(\neg\varrho_5) = d \quad K_2(\neg\varrho_6) = c$.

Let $(\Upsilon_1, \Omega_1, \varsigma_1)\sqcap_B{}^{\Re}(\Upsilon_2, \Omega_2, \varsigma_2) = (H_3, K_3, \varrho_1 \cap \varrho_2)$.
Then $H_3(\varrho_2) = b \wedge d \quad H_3(\varrho_3) = c \wedge f = c$,
and $K_3(\neg\varrho_2) = e \vee f = 1 \quad K_3(\neg\varrho_3) = 0 \vee a = a$.

Let $(\Upsilon_1, \Omega_1, \varsigma_1)\sqcup_B{}^{\Re}(\Upsilon_2, \Omega_2, \varsigma_2) = (H_4, K_4, \varrho_1 \cap \varrho_2)$.
Then $H_4(\varrho_2) = b \vee d = d \quad H_4(\varrho_3) = c \vee f = f$,
and $K_4(\neg\varrho_2) = e \wedge f = c \quad K_4(\neg\varrho_3) = 0 \wedge a = 0$.

Obviously, $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ and $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ are full since $\bigvee_{\varrho\in\varsigma_1} \Upsilon_1(\varrho) = e \vee b \vee c = 1$, $\bigvee_{\neg\varrho\in\neg\varsigma_1} \Omega_1(\neg\varrho) = b \vee e \vee c = 1$ and $\bigvee_{\varrho\in\varsigma_2} \Upsilon_2(\varrho) = d \vee f \vee a \vee c = 1$, $\bigvee_{\neg\varrho\in\neg\varsigma_2} \Omega_2(\neg\varrho) = f \vee a \vee c \vee d = 1$. Also $(\Upsilon_1, \Omega_1)_{\varsigma_1}$ is infimum-preserving but $(\Upsilon_2, \Omega_2)_{\varsigma_2}$ is not infimum-preserving.

Now, we introduce the lattice structure of bipolar soft sets over a complete atomic Boolean lattice Ξ. We denote:
$(S, T)_{(\Xi,\zeta)} = \{(\Upsilon, \Omega)_\zeta \mid (\Upsilon, \Omega)_\zeta$ is a bipolar soft set over $\Xi\}$,
$(S_1, T_1)_{(\Xi,\varsigma)} = \{(\Upsilon, \Omega)_\varsigma \mid \varsigma \subseteq \zeta$ and $(\Upsilon, \Omega)_\varsigma$ is a bipolar soft set over $\Xi\}$.

Theorem 3.21. *For any* $(\Upsilon_1, \Omega_1)_{\varsigma_1}, (\Upsilon_2, \Omega_2)_{\varsigma_2} \in (S, T)_{(\Xi,\zeta)}$, *define*
$(\Upsilon_1, \Omega_1)_{\varsigma_1} \leq (\Upsilon_2, \Omega_2)_{\varsigma_2} \iff (\Upsilon_1, \Omega_1)_{\varsigma_1} \subseteq (\Upsilon_2, \Omega_2)_{\varsigma_2}$,
and $(\Upsilon_1, \Omega_1)_{\varsigma_1} \vee (\Upsilon_2, \Omega_2)_{\varsigma_2} = (\Upsilon_1, \Omega_1)_{\varsigma_1} \cup (\Upsilon_2, \Omega_2)_{\varsigma_2}$, $\quad (\Upsilon_1, \Omega_1)_{\varsigma_1} \wedge (\Upsilon_2, \Omega_2)_{\varsigma_2} = (\Upsilon_1, \Omega_1)_{\varsigma_1} \cap (\Upsilon_2, \Omega_2)_{\varsigma_2}$.
Then $(S, T)_{(\Xi,\varsigma)}$ *forms a distributive lattice with the smallest element* $(0, 1)_\varsigma$ *and the greatest element* $(1, 0)_\varsigma$.

Proof. Obvious. □

Theorem 3.22. *For any* $(\Upsilon_1, \Omega_1)_{\varsigma_1}, (\Upsilon_2, \Omega_2)_{\varsigma_2} \in (S_1, T_1)_{(\Xi,\varsigma)}$, *define*
$(\Upsilon_1, \Omega_1)_{\varsigma_1} \leq (\Upsilon_2, \Omega_2)_{\varsigma_2} \iff (\Upsilon_1, \Omega_1)_{\varsigma_1} \subseteq (\Upsilon_2, \Omega_2)_{\varsigma_2}$,
$(\Upsilon_1, \Omega_1)_{\varsigma_1} \vee (\Upsilon_2, \Omega_2)_{\varsigma_2} = (\Upsilon_1, \Omega_1)_{\varsigma_1} \cup (\Upsilon_2, \Omega_2)_{\varsigma_2}$, $(\Upsilon_1, \Omega_1)_{\varsigma_1} \wedge (\Upsilon_2, \Omega_2)_{\varsigma_2} = (\Upsilon_1, \Omega_1)_{\varsigma_1} \cap (\Upsilon_2, \Omega_2)_{\varsigma_2}$.
Then $(S_1, T_1)_{(\Xi,\varsigma)}$ *forms a Boolean lattice.*

Proof. Denote $(W, Z) = (S_1, T_1)_{(\Xi,\varsigma)}$, it easily proved that $(S_1, T_1)_{(\Xi,\varsigma)}$ is a distributive lattice with least element $(0, 1)_{(W,Z)}$ and greatest element $(1, 0)_{(W,Z)}$. Let $(\Upsilon_1, \Omega_1)_{\varsigma_1} \in (W, Z)$, put $(H, K)_{\varsigma_1} = (\Upsilon_1, \Omega_1)_{\varsigma_1} \vee (\Upsilon_1, \Omega_1)^c_{\varsigma_1}$.

Since $(H,K)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \cup (\Upsilon_1,\Omega_1)^c_{\varsigma_1}$. Then for any $\varrho \in \varsigma_1, \varsigma_1 \subseteq \zeta$, $H(\varrho) = \Upsilon_1(\varrho) \vee (\Upsilon_1(\varrho))^c = \Upsilon_1(\varrho) \vee \Omega_1(\neg\varrho) = 1$, $K(\neg\varrho) = \Omega_1(\neg\varrho) \wedge (\Omega_1(\neg\varrho))^c = \Omega_1(\neg\varrho) \wedge \Upsilon_1(\varrho) = 0$. So $(H,K)_{\varsigma_1} = (1,0)_{\varsigma_1} = (1,0)_{(W,Z)}$. Then $(\Upsilon_1,\Omega_1)_\zeta \vee (\Upsilon_1,\Omega_1)^c_\zeta = (1,0)_{(W,Z)}$. We prove that $(\Upsilon_1,\Omega_1)_{\varsigma_1} \wedge (\Upsilon_1,\Omega_1)^c_{\varsigma_1} = (0,1)_{(W,Z)}$
Let $(H,K)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \wedge (\Upsilon_1,\Omega_1)^c_{\varsigma_1}$, since $(H,K)_{\varsigma_1} = (\Upsilon_1,\Omega_1)_{\varsigma_1} \cap {(\Upsilon_1,\Omega_1)_{\varsigma_1}}^c$. Then for any $\varrho \in \varsigma_1$, $H(\varrho) = \Upsilon_1(\varrho) \wedge (\Upsilon_1(\varrho))^c = \Upsilon_1(\varrho) \wedge \Omega_1(\neg\varrho) = 0$, $K(\neg\varrho) = \Omega_1(\neg\varrho) \vee (\Omega_1(\neg\varrho))^c = 1$.
So $(H,K)_{\varsigma_1} = (0,1)_{\varsigma_1} = (0,1)_{(W,Z)}$. Hence $(\Upsilon_1,\Omega_1)_\zeta \wedge {(\Upsilon_1,\Omega_1)_\zeta}^c = (0,1)_{(W,Z)}$. Then $(S_1,T_1)_{(\Xi,\varsigma)}$ is a Boolean lattice. □

4. Bipolar soft rough approximation operators on a complete atomic Boolean lattice

In this section, we introduce the concept of bipolar soft rough approximations within a complete atomic Boolean lattice (*BSRA*) and study their fundamental properties.

Definition 4.1. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice, and let $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ. For any $z \in \Xi$, we define the following approximation operators:
$z^{\vee^+}, z^{\vee^-}, z^{\wedge^+}, z^{\wedge^-} : \Xi \to \Xi$ as follows:

$$z^{\vee^+} = \bigvee \left\{\lambda \in \Lambda(\Xi) : \exists \varrho \in \varsigma \text{ s.t. } \lambda \leq \Upsilon(\varrho) \leq z\right\},$$
$$z^{\vee^-} = \bigvee \left\{\lambda \in \Lambda(\Xi) : \exists \neg\varrho \in \neg\varsigma \text{ s.t. } \lambda \leq \Omega(\neg\varrho),\ \Omega(\neg\varrho) \wedge z' \neq 0\right\},$$
$$z^{\wedge^+} = \bigvee \left\{\lambda \in \Lambda(\Xi) : \exists \varrho \in \varsigma \text{ s.t. } \lambda \leq \Upsilon(\varrho),\ \Upsilon(\varrho) \wedge z \neq 0\right\},$$
$$z^{\wedge^-} = \bigvee \left\{\lambda \in \Lambda(\Xi) : \exists \neg\varrho \in \neg\varsigma \text{ s.t. } \lambda \leq \Omega(\neg\varrho) \leq z'\right\},$$

where z' denotes the complement of z in Ξ.
The elements $z^{\vee^+}, z^{\vee^-}, z^{\wedge^+}$, and $z^{\wedge^-}$ are called, respectively, the *soft lower positive*, *soft lower negative*, *soft upper positive*, and *soft upper negative* approximations of z over Ξ.
Two elements $z_1, z_2 \in \Xi$ are said to be *bipolar soft equivalent* if
$z_1^{\vee^+} = z_2^{\vee^+}, \quad z_1^{\vee^-} = z_2^{\vee^-}, \quad z_1^{\wedge^+} = z_2^{\wedge^+}, \quad z_1^{\wedge^-} = z_2^{\wedge^-}$.

Definition 4.2. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice and $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ. For any $z \in \Xi$. Then

$$z_B^{\vee} = (z^{\vee^+}, z^{\vee^-}) \text{ and } \ z_B^{\wedge} = (z^{\wedge^+}, z^{\wedge^-})$$

are called bipolar soft rough approximations of z .

Definition 4.3. Let $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ. For any $w, z \in \Xi$.

i. ${z_B}^{\vee} \leq_B {w_B}^{\vee} \iff z^{\vee^+} \leq w^{\vee^+}$ and $w^{\vee^-} \leq z^{\vee^-}$;
ii. ${z_B}^{\wedge} \leq_B {w_B}^{\wedge} \iff z^{\wedge^+} \leq w^{\wedge^+}$ and $w^{\wedge^-} \leq z^{\wedge^-}$.

Proposition 4.4. *Let $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ. For any $z \in \Xi$. Then*

$$z_B^{\vee} \ \leq \ z_B^{\wedge}$$

Proof. The first part $z^{\vee^+} \leq z^{\wedge^+}$ was proved in [14]. We prove the other part $z^{\wedge^-} \leq z^{\vee^-}$. Let $\lambda \in \Lambda(\Xi)$, $\lambda \leq z^{\wedge^-}$ from Definition 4.1, then $\lambda \leq \Omega(\neg\varrho) \leq z'$ for some $\neg\varrho \in \neg\varsigma$. Thus, it follows that $\Omega(\neg\varrho) \wedge z' \neq 0$, and $\lambda \leq \Omega(\neg\varrho)$. Therefore, we get $\lambda \leq z^{\vee^-}$ then $z^{\wedge^-} \leq z^{\vee^-}$. □

Definition 4.5. Let $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ. For any $w, z \in \Xi$. The union of bipolar soft $B-$lower approximations and bipolar soft $B-$upper approximations of w, z over Ξ are defined respectively as,

$${z_B}^{\vee} \vee_B {w_B}^{\vee} = (z^{\vee^+} \vee w^{\vee^+}, z^{\vee^-} \wedge w^{\vee^-})$$
$${z_B}^{\wedge} \vee_B {w_B}^{\wedge} = (z^{\wedge^+} \vee w^{\wedge^+}, z^{\wedge^-} \wedge w^{\wedge^-})$$

Definition 4.6. Let $(\Upsilon,\Omega)_{\varsigma}$ be a bipolar soft set over Ξ. For any $w,z \in \Xi$. The intersection of bipolar soft $B-$lower approximations and bipolar soft $B-$ upper approximations of w,z over Ξ are defined respectively as,

$$z_B{}^{\vee} \wedge_B w_B{}^{\vee} = (z^{\vee^+} \wedge w^{\vee^+}, z^{\vee^-} \vee w^{\vee^-})$$
$$z_B{}^{\wedge} \wedge_B w_B{}^{\wedge} = (z^{\wedge^+} \wedge w^{\wedge^+}, z^{\wedge^-} \vee w^{\wedge^-})$$

Lemma 4.7. *Let $(\Upsilon,\Omega)_{\varsigma}$ be a bipolar soft set over Ξ. Then $\forall \lambda \in \Lambda(\Xi)$, $z \in \Xi$.*

i. $\lambda \leq z^{\vee^+} \iff \exists\ \varrho \in \varsigma$ *s.t* $\lambda \leq \Upsilon(\varrho)$ *and* $\Upsilon(\varrho) \leq z$;
ii. $\lambda \leq z^{\vee^-} \iff \exists\ \neg\varrho \in \neg\varsigma$ *s.t* $\lambda \leq \Omega(\neg\varrho)$ *and* $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$;
iii. $\lambda \leq z^{\wedge^+} \iff \exists\ \varrho \in \varsigma$ *s.t* $\lambda \leq \Upsilon(\varrho)$ *and* $\Upsilon(\varrho) \wedge z \neq 0$;
iv. $\lambda \leq z^{\wedge^-} \iff \exists\ \neg\varrho \in \neg\varsigma$ *s.t* $\lambda \leq \Omega(\neg\varrho)$ *and* $\Omega(\neg\varrho) \leq z^{\backprime}$.

Proof. Parts (i) and (iii) were proved in [14], we prove (ii).
($\Rightarrow$) Let $\lambda \leq z^{\vee^-} = \bigvee\{b \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma$ s.t $b \leq \Omega(\neg\varrho)$, $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\}$. Assume that $\forall\neg\varrho \in \neg\varsigma$ either $\lambda \nleq \Omega(\neg\varrho)$ or $\Omega(\neg\varrho) \wedge z^{\backprime} = 0$. If $\forall\neg\varrho \in \neg\varsigma$, $\Omega(\neg\varrho) \wedge z^{\backprime} = 0$ Then $\lambda \nleq z^{\vee^-}$, a contradiction. If $\forall\neg\varrho \in \neg\varsigma$, $\lambda \nleq \Omega(\neg\varrho)$, then $\lambda \wedge z^{\vee^-} = \lambda \wedge \bigvee\{b \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma$ s.t $b \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\}$ $= \bigvee\{\lambda \wedge b : b \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma$ s.t $b \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\}$.
Since $\lambda \nleq \Omega(\neg\varrho)$ then $\lambda \neq b$. So $\lambda \wedge b = 0$ because $\lambda, b \in \Lambda(\Xi)$. Hence $\lambda \wedge z^{\vee^-} = 0$. This implies that $\lambda \leq (z^{\vee^-})^{\backprime}$, which is a contradiction.
($\Leftarrow$) Let $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$. Then $\lambda \leq \bigvee\{b \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma$ s.t $b \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\} = z^{\vee^-} \Rightarrow \lambda \leq z^{\vee^-}$.
(iv) This can be demonstrated similarly. □

Proposition 4.8. *Let $(\Upsilon,\Omega)_{\varsigma}$ be a bipolar soft set over Ξ. Then for all $z \in \Xi$.*

i. $z^{\vee^+} = \bigvee\{\Upsilon(\varrho) : \varrho \in \varsigma$ *and* $\Upsilon(\varrho) \leq z\} \leq z$;
ii. $z^{\vee^-} = \bigvee\{\Omega(\neg\varrho) : \neg\varrho \in \neg\varsigma$ *and* $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\}$;
iii. $z^{\wedge^+} = \bigvee\{\Upsilon(\varrho) : \varrho \in \varsigma$ *and* $\Upsilon(\varrho) \wedge z \neq 0\}$;
iv. $z^{\wedge^-} = \bigvee\{\Omega(\varrho) : \neg\varrho \in \neg\varsigma$ *and* $\Omega(\neg\varrho) \leq z^{\backprime}\} \leq z^{\backprime}$.

Proof. Parts (i) and (iii) were proved in [14], we prove (ii).
Let $\lambda \in \Lambda(\Xi)$ s.t $\lambda \leq z^{\vee^-}, \exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$. So $\lambda \leq \bigvee\{\Omega(\neg\varrho) : \exists\ \neg\varrho \in \neg\varsigma$ *and* $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\} = z^{\vee^-}$. Conversely, let $\lambda \in \Lambda(\Xi)$ s.t $\lambda \leq \bigvee\{\Omega(\neg\varrho) : \exists\ \neg\varrho \in \neg\varsigma$ *and* $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\}$. Hence $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$. In fact, if $\neg\varrho \in \neg\varsigma$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0 \Rightarrow \lambda \nleq \Omega(\neg\varrho)$ and $\lambda \wedge (\Omega(\neg\varrho))^{\backprime} \neq 0$, therefore $\lambda \leq (\Omega(\neg\varrho))^{\backprime}$ s.t $\lambda \in \Lambda(\Xi)$. Thus $\lambda \leq \bigvee\{(\Omega(\neg\varrho))^{\backprime} : \neg\varrho \in \neg\varsigma$ *and* $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\}$ where $(\Omega(\neg\varrho))^{\backprime} = (\Omega(\neg\varrho))^c = \Upsilon(\varrho)$. So $\lambda \leq \bigvee\{\Omega(\neg\varrho) \wedge (\Omega(\neg\varrho))^{\backprime} : \neg\varrho \in \neg\varsigma$ *and* $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0\} = 0$, a contradiction. So $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$ and consequently $\lambda \leq z^{\vee^-}$.
(iv) This can be demonstrated similarly. □

Proposition 4.9. *Let $(\Upsilon,\Omega)_{\varsigma}$ be a bipolar soft set over Ξ. Then $\forall\ w,z \in \Xi$, the following holds:*

i. $0_B{}^{\vee} = 0_B{}^{\wedge} = (0\ ,\ \bigvee_{\neg\varrho\in\neg\varsigma}\Omega(\neg\varrho))$;
ii. $1_B{}^{\vee} = 1_B{}^{\wedge} = (\bigvee_{\varrho\in\varsigma}\Upsilon(\varrho)\ ,\ 0)$;
iii. $z_B \leq w_B\ \Rightarrow\ z_B^{\vee} \leq w_B^{\vee}$ *and* $z_B^{\wedge} \leq w_B^{\wedge}$.

Proof. (i) The first part $0^{\vee^+} = 0^{\wedge^+} = 0$ was proved in [14]. The other part $0^{\vee^-} = 0^{\wedge^-} = \bigvee_{\neg\varrho\in\neg\varsigma}\Omega(\neg\varrho)$ is obvious.
(ii) The first part $1^{\vee^+} = 1^{\wedge^+} = \bigvee_{\varrho\in\varsigma}\Upsilon(\varrho)$ was proved in [14]. The other part $1^{\vee^-} = 1^{\wedge^-} = 0$ is obvious.
(iii) The first part $z \leq w \Rightarrow z^{\vee^+} \leq w^{\vee^+}$ and $z^{\wedge^+} \leq w^{\wedge^+}$ was proved in [14]. The other part $z \leq w \Rightarrow w^{\vee^-} \leq z^{\vee^-}$ and $w^{\wedge^-} \leq z^{\wedge^-}$ can be demonstrated similarly. □

Definition 4.10. Let $(\Upsilon,\Omega)_{\varsigma}$ be a bipolar soft set over Ξ and $z \in \Xi$. The complement of $z_B{}^{\vee}$ *and* $z_B{}^{\wedge}$ are defined

$$z_B{}^{\vee^c} = (z^{\vee^-}\ ,\ z^{\vee^+});$$
$$z_B{}^{\wedge^c} = (z^{\wedge^-}\ ,\ z^{\wedge^+}).$$

Proposition 4.11. *Let* $(\Upsilon, \Omega)_\varsigma$ *be a bipolar soft set over* Ξ*.Then* $\forall w, z \in \Xi$*, the following holds:*

i. $(z_B{}^{\vee^c})^c = z_B{}^{\vee}$;
ii. $(z_B{}^{\wedge^c})^c = z_B{}^{\wedge}$;
iii. $(z_B{}^{\vee} \bigvee w_B{}^{\vee})^c = z_B{}^{\vee^c} \bigwedge w_B{}^{\vee^c}$;
iv. $(z_B{}^{\wedge} \bigvee w_B{}^{\wedge})^c = z_B{}^{\wedge^c} \bigwedge w_B{}^{\wedge^c}$;
v. $(z_B{}^{\vee} \bigwedge w_B{}^{\vee})^c = z_B{}^{\vee^c} \bigvee w_B{}^{\vee^c}$;
vi. $(z_B{}^{\wedge} \bigwedge w_B{}^{\wedge})^c = z_B{}^{\wedge^c} \bigvee w_B{}^{\wedge^c}$;
vii. $z_B{}^{\vee} \leq w_B{}^{\vee} \iff w_B{}^{\vee^c} \leq z_B{}^{\vee^c}$;
viii. $z_B{}^{\wedge} \leq w_B{}^{\wedge} \iff w_B{}^{\wedge^c} \leq z_B{}^{\wedge^c}$.

Proof. These conditions can be proved by definitions of union, intersection, and complement of bipolar soft rough approximations over Ξ. □

Definition 4.12. Let $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ. Let $\rho = (\Lambda(\Xi), (\Upsilon, \Omega)_\varsigma)$ be a bipolar soft approximation space ($BSA - space$). If $\forall \varrho_i, \varrho_j \in \varsigma$, $\Upsilon(\varrho_i) \wedge \Omega(\neg\varrho_j) = 0$, then $(\Upsilon, \Omega)_\varsigma$ is called semi-intersection bipolar soft set, denoted by $(\Upsilon, \Omega)_\wedge$.

Proposition 4.13. *Let* $(\Upsilon, \Omega)_\wedge$ *be a semi-intersection bipolar soft set over* Ξ*,* $z \in \Xi$*, and let* $z_B{}^{\wedge}$ *and* $z_B{}^{\vee}$ *be bipolar soft rough approximations of* z*. Then*

i. $z^{\vee^+} \wedge z^{\vee^-} = 0$;
ii. $z^{\wedge^+} \wedge z^{\wedge^-} = 0$.

Proof. Obvious. □

Proposition 4.14. *Let* $(\Upsilon, \Omega)_\varsigma$ *be a bipolar soft set over* Ξ*. Let* $\rho = (\Lambda(\Xi), (\Upsilon, \Omega)_\varsigma)$ *be a* $(BSA - space)$ *,* $z \in \Xi$ *if*

$z \leq \Lambda(\Xi) \setminus (\bigvee_{\varrho\in\varsigma} \Upsilon(\varrho) \vee \bigvee_{\neg\varrho\in\neg\varsigma} \Omega(\neg\varrho))$.
Then z *is a bipolar soft* ρ*- definable set, where* $z_B{}^{\wedge} = z_B{}^{\vee}$

Proof. We know that $z^{\vee^+} \leq z^{\wedge^+}$ from [14], and $z^{\wedge^-} \leq z^{\vee^-}$ from Proposition 4.4. Then we prove that $z^{\wedge^+} \leq z^{\vee^+}$ *and* $z^{\vee^-} \leq z^{\wedge^-}$. Let $z \leq \Lambda(\Xi) \setminus (\bigvee_{\varrho\in\varsigma} \Upsilon(\varrho) \vee \bigvee_{\neg\varrho\in\neg\varsigma} \Omega(\neg\varrho))$. Then $z^{\vee^+} = 0$, and so $z^{\wedge^+} \leq z^{\vee^+}$. Let $b \leq z^{\vee^-}$, where $\exists\ b \in \Lambda(\Xi)$. Then $b \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$. From hypothesis $\Omega(\neg\varrho) \wedge z = 0$ for all$\neg\varrho \in \neg\varsigma$ and so $b \leq \Omega(\neg\varrho) \leq z^{\backprime}$. Then $z^{\vee^-} \leq z^{\wedge^-}$. □

Lemma 4.15. *Let* $\Xi = (\Xi, \leq)$ *be a complete atomic Boolean lattice. Then for all* $\{z_i : i \in I\} \subseteq \Xi$

$$\bigvee\{\cap\{z_i : i \in I\}\} \leq \bigwedge \bigvee\{z_i : i \in I\}.$$

Proof. $\forall i \in I$, $\cap\{z_i : i \in I\} \leq z_i$. So $\bigvee\{\cap z_i : i \in I\} \leq \bigvee z_i\ \ \forall i \in I$. This $\bigvee\{\cap z_i : i \in I\}$ is lower bound of $\{\bigvee z_i : i \in I\}$. Hence $\bigvee\{\cap z_i : i \in I\} \leq \bigwedge \bigvee\{z_i : i \in I\}$. □

For all $M \subseteq \Xi$, we denote $M^{\vee^-} = \{z^{\vee^-} : z \in M\}$ and $M^{\wedge^-} = \{z^{\wedge^-} : z \in M\}$.

Definition 4.16. Let $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ. For all $M \subseteq \Xi$.

$$\bigvee_B M^{\wedge} = (\bigvee M^{\wedge^+}\ ,\ \bigwedge M^{\wedge^-})\ ,\ \bigwedge_B M^{\vee} = (\bigwedge M^{\vee^+}\ ,\ \bigvee M^{\vee^-})$$
$$(\bigvee_B M)^{\wedge} = ((\bigvee M)^{\wedge^+}\ ,\ (\bigvee M)^{\wedge^-})\ ,\ (\bigwedge_B M)^{\vee} = ((\bigwedge M)^{\vee^+}\ ,\ (\bigwedge M)^{\vee^-})$$

Proposition 4.17. *Let* $(\Upsilon, \Omega)_\varsigma$ *be a bipolar soft set over* Ξ*. For all* $M \subseteq \Xi$*. The following properties hold.*

i. $(\bigvee_B M)^{\wedge} = \bigvee_B M^{\wedge}$;
ii. $(\bigwedge_B M)^{\vee} = \bigwedge_B M^{\vee}$.

Proof. (i) The first part $(\bigvee M)^{\wedge^{+}} = \bigvee M^{\wedge^{+}}$ was proved in [14]. We prove the other part $(\bigvee M)^{\wedge^{-}} = \bigwedge M^{\wedge^{-}}$.

$(\Rightarrow)$ We prove that $(\bigvee M)^{\wedge^{-}} \leq \bigwedge M^{\wedge^{-}}$. Let $\lambda \in \Lambda(\Xi)$ s.t $\lambda \leq (\bigvee M)^{\wedge^{-}}$ then $\exists\ \neg\varrho \in \neg\varsigma\ s.t\ \lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \leq (\bigvee M)^{\backprime} = \bigwedge M^{\backprime} = \bigwedge\{z^{\backprime} : z \in M\}$. So $\forall z \in M,\ \Omega(\neg\varrho) \leq z^{\backprime}$, hence $\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma\ ,\ \lambda \leq \Omega(\neg\varrho)\ and\ \Omega(\neg\varrho) \leq (\bigvee M)^{\backprime}\} \leq \cap_{z\in M}\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma, \lambda \leq \Omega(\neg\varrho)\ and\ \Omega(\neg\varrho) \leq z^{\backprime}\}$. Therefore $(\bigvee M)^{\wedge^{-}} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma\ s.t\ \lambda \leq \Omega(\neg\varrho)\ and\ \Omega(\neg\varrho) \leq (\bigvee M)^{\backprime}\} \leq \bigvee\{\cap_{z\in M}\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma\ s.t\ \lambda \leq \Omega(\neg\varrho)\ and\ \Omega(\neg\varrho) \leq z^{\backprime}\} \leq \bigwedge_{z\in M} \bigvee\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\varsigma\ s.t\ \lambda \leq \Omega(\neg\varrho)\ and\ \Omega(\neg\varrho) \leq z^{\backprime}\} = \bigwedge M^{\wedge^{-}}$.

$(\Leftarrow)$ Let $M \subseteq \Xi$, the map $\wedge^{-} : \Xi \to \Xi$ is order preserving. which implies that $\bigwedge M^{\wedge^{-}} \leq (\bigvee M)^{\wedge^{-}}$.

(ii) The first part $(\bigwedge M)^{\vee^{+}} = \bigwedge M^{\vee^{+}}$ was proved in [14]. We prove the other part $(\bigwedge M)^{\vee^{-}} = \bigvee M^{\vee^{-}}$. Let $M \subseteq \Xi$, the map $\vee^{-} : \Xi \to \Xi$ is order preserving. which implies that $\bigwedge M^{\vee^{-}} \leq (\bigwedge M)^{\vee^{-}} \leq \bigvee M^{\vee^{-}}$. On the other hand, let $\lambda \in \Lambda(\Xi)\ s.t\ \lambda \leq \bigvee M^{\vee^{-}} = \bigvee\{z^{\vee^{-}} : z \in M\}$, $\exists\ z \in M\ s.t\ \lambda \leq z^{\vee^{-}}$, so $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)\ and\ \Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$. So $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$, then $\Omega(\neg\varrho) \wedge \bigvee\{z^{\backprime} : z \in M\} \neq 0$. So $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$, then $\Omega(\neg\varrho) \wedge (\bigwedge M)^{\backprime} \neq 0$, where $(\bigwedge M)^{\backprime} = \bigvee M^{\backprime} = \bigvee\{z^{\backprime} : z \in M\}$. Then $\lambda \leq (\bigwedge M)^{\vee^{-}}$. □

Proposition 4.18. *Let $(\Upsilon, \Omega)_{\varsigma}$ be a bipolar soft set over Ξ. If $(\Upsilon, \Omega)_{\varsigma}$ is a partition. Then $(\Xi^{\vee_B}, \geq) \cong (\Xi^{\wedge_B}, \leq)$.*

Proof. The first part $(\Xi^{\vee^{+}}, \geq) \cong (\Xi^{\wedge^{+}}, \leq)$ was proved in [14]. We prove the other part $(\Xi^{\vee^{-}}, \leq) \cong (\Xi^{\wedge^{-}}, \geq)$.

We prove that $z^{\vee^{-}} \to (z^{\backprime})^{\wedge^{-}}$ is dual order isomorphism. It is obvious that $z^{\vee^{-}} \to (z^{\backprime})^{\wedge^{-}}$ is onto $(\Xi^{\vee^{-}}, \leq)$, then we prove that $z^{\vee^{-}} \to (z^{\backprime})^{\wedge^{-}}$ is order embedding. Assume that $z^{\vee^{-}} \leq w^{\vee^{-}}$ then $\forall\ \lambda \in \Lambda(\Xi), \lambda \leq z^{\vee^{-}}$ this implies that $\lambda \leq w^{\vee^{-}}$. So $\lambda \in \Lambda(\Xi)$ s.t $\exists\ \neg\varrho_1 \in \neg\varsigma, \lambda \leq \Omega(\neg\varrho_1)$ and $\Omega(\neg\varrho_1) \wedge z^{\backprime} \neq 0$ then $\exists\ \neg\varrho_2 \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho_2)$ and $\Omega(\neg\varrho_2) \wedge w^{\backprime} \neq 0$. Since $(\Upsilon, \Omega)_{\varsigma}$ is a partition and $\lambda \leq \Omega(\neg\varrho_1) \wedge \Omega(\neg\varrho_2)$ then $\Omega(\neg\varrho_1) = \Omega(\neg\varrho_2)$. Hence if $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$ then $\Omega(\neg\varrho) \wedge w^{\backprime} \neq 0$. Assume that $(w^{\backprime})^{\wedge^{-}} \nleq (z^{\backprime})^{\wedge^{-}}$, so $\exists\ \lambda \in \Lambda(\Xi)$ s.t $\lambda \leq (w^{\backprime})^{\wedge^{-}}$ and $\lambda \nleq (z^{\backprime})^{\wedge^{-}}$. Since $\lambda \leq (w^{\backprime})^{\wedge^{-}}$ then $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \leq w$, since $\lambda \leq \Omega(\neg\varrho)$ and $\lambda \nleq (z^{\backprime})^{\wedge^{-}}$ then $\Omega(\neg\varrho) \leq z$ equivalent to $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$ then by hypothesis $\Omega(\neg\varrho) \wedge w^{\backprime} \neq 0$ this implies that $\Omega(\neg\varrho) \nleq w$ a contradiction. Hence $(w^{\backprime})^{\wedge^{-}} \leq (z^{\backprime})^{\wedge^{-}}$. On the other hand, assume that $(w^{\backprime})^{\wedge^{-}} \leq (z^{\backprime})^{\wedge^{-}}$. Since $(\Upsilon, \Omega)_{\varsigma}$ is a partition, then $\lambda \in \Lambda(\Xi)$ s.t $\exists\ \neg\varrho \in \neg\varsigma, \lambda \leq \Omega(\neg\varrho)$ and $\Omega(\neg\varrho) \leq w$ this implies that $\Omega(\neg\varrho) \leq z$. Suppose that $z^{\vee^{-}} \nleq w^{\vee^{-}}$, so $\exists\ \lambda \in \Lambda(\Xi)$s.t $\lambda \leq z^{\vee^{-}}$ and $\lambda \nleq w^{\vee^{-}}$, so $\exists\ \neg\varrho \in \neg\varsigma, \lambda \leq \Omega(\neg\varrho), \Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$ and $\Omega(\neg\varrho) \wedge w^{\backprime} = 0$. But this implies that $\Omega(\neg\varrho) \leq w$. Since $z^{\vee^{-}} \nleq w^{\vee^{-}}$ then $\Omega(\neg\varrho) \leq z$. This equivalent to $\Omega(\neg\varrho) \wedge z^{\backprime} = 0$, a contradiction. □

Theorem 4.19. *Let $(\Upsilon, \Omega)_{\varsigma}$ is a partition bipolar soft set. Then $(\Xi^{\wedge_B}, \leq)$ is linearly ordered iff $(\Upsilon(\varsigma), \leq)$ and $(\Omega(\neg\varsigma), \leq)$ are linearly ordered.*

Proof. We prove the first part $(\Xi^{\wedge^{+}}, \leq)$ is linearly ordered iff $(\Upsilon(\varsigma), \leq)$ is linearly ordered. Assume that $(\Upsilon(\varsigma), \leq)$ is linearly ordered. Let $w, z \in \Xi^{\wedge^{+}}$ then $z = h^{\wedge^{+}}$, $w = k^{\wedge^{+}}$, for some $h, k \in \Xi$.

Suppose that w and z are not comparable in $\Xi^{\wedge^{+}}$, then $\exists\ \lambda, b \in \Lambda(\Xi)$ s.t $\lambda \leq z = h^{\wedge^{+}}$ and $\lambda \nleq w = k^{\wedge^{+}}$, $b \nleq z = h^{\wedge^{+}}$ and $b \leq w = k^{\wedge^{+}}$. So $\exists\ \varrho_1, \varrho_2 \in \varsigma$ s.t $\lambda \leq \Upsilon(\varrho_1)$, $\Upsilon(\varrho_1) \wedge h \neq 0$ and $\Upsilon(\varrho_1) \wedge k = 0$, $\Upsilon(\varrho_2) \wedge h = 0$ and $\Upsilon(\varrho_2) \wedge k \neq 0$. Since $(\Upsilon(\varsigma), \leq)$ is linearly ordered. Then we have $\Upsilon(\varrho_1) \leq \Upsilon(\varrho_2)$ or $\Upsilon(\varrho_2) \leq \Upsilon(\varrho_1)$.

If $\Upsilon(\varrho_1) \leq \Upsilon(\varrho_2)$, then $\Upsilon(\varrho_1) \wedge h \neq 0$ implies $\Upsilon(\varrho_2) \wedge h \neq 0$. Which is a contradiction to the hypothesis, a similar contradiction occurs when $\Upsilon(\varrho_2) \leq \Upsilon(\varrho_1)$. Therefore $(\Xi^{\wedge^{+}}, \leq)$ is linearly ordered.

Conversely, assume that $(\Xi^{\wedge^{+}}, \leq)$ is linearly ordered, suppose that $(\Upsilon(\varsigma), \leq)$ is not linearly ordered, then $\exists\ \varrho_1, \varrho_2 \in \varsigma$ s.t neither $\Upsilon(\varrho_1) \leq \Upsilon(\varrho_2)$ nor $\Upsilon(\varrho_2) \leq \Upsilon(\varrho_1)$. Then $\exists\ c, d \in \Lambda(\Xi)$ s.t $c \leq \Upsilon(\varrho_1)$ and $c \nleq \Upsilon(\varrho_2)$, $d \leq \Upsilon(\varrho_2)$ and $d \nleq \Upsilon(\varrho_1)$. Then $c \leq \Upsilon(\varrho_1)$ and $c \nleq \Upsilon(\varrho_2)$ implies $\Upsilon(\varrho_1) \wedge c \neq 0$ and $\Upsilon(\varrho_2) \wedge c = 0$. Also $d \leq \Upsilon(\varrho_2)$ and $d \nleq \Upsilon(\varrho_1)$ implies $\Upsilon(\varrho_2) \wedge d \neq 0$ and $\Upsilon(\varrho_1) \wedge d = 0$. So $c \leq c^{\wedge^{+}}$ and $d \leq d^{\wedge^{+}}$. Assume that $\exists\ \varrho_3 \in \varsigma$, $c \leq \Upsilon(\varrho_3)$. Then $c \leq \Upsilon(\varrho_1) \wedge \Upsilon(\varrho_3)$. Since $(\Upsilon, \Omega)_{\varsigma}$ is a partition, then $\Upsilon(\varrho_1) = \Upsilon(\varrho_3)$, and therefore $\Upsilon(\varrho_3) \wedge d = 0$. Consequently $c \nleq d^{\wedge^{+}}$. Similarly we can show that $d \nleq c^{\wedge^{+}}$, so we prove that $c \leq c^{\wedge^{+}}$ and $c \nleq d^{\wedge^{+}}$, $d \leq d^{\wedge^{+}}$ and $d \nleq c^{\wedge^{+}}$. This implies that $\exists\ c^{\wedge^{+}}$, $d^{\wedge^{+}} \in \Xi^{\wedge^{+}}$ s.t $c^{\wedge^{+}} \nleq d^{\wedge^{+}}$ and $d^{\wedge^{+}} \nleq c^{\wedge^{+}}$, a contradiction. Hence $(\Upsilon(\varsigma), \leq)$ is linearly ordered. The second part $(\Xi^{\wedge^{-}}, \leq)$ is linearly ordered iff $(\Omega(\neg\varsigma), \leq)$ is linearly ordered can prove similarly. □

Proposition 4.20. *Let $(\Upsilon, \Omega)_{\varsigma}$ be a bipolar soft set over Ξ. Then, the following properties hold.*

i. *If $(\Upsilon, \Omega)_{\varsigma}$ is infimum-preserving, then*
   (a) $\forall z \in \Xi, \exists\ \neg\varrho \in \neg\varsigma\ s.t\ z^{\vee^{-}} = \Omega(\neg\varrho)$;
   (b) $\forall z \in \Xi, \exists\ \neg\varrho \in \neg\varsigma\ s.t\ z^{\wedge^{-}} = \Omega(\neg\varrho)$.

ii. *If* $(\Upsilon,\Omega)_\varsigma$ *is full and infimum-preserving, then* $z^{\vee^-}=1$ *for every* $z\in\Xi$ *and* $z\neq 1$.

Proof. (i)-(a) Since $z^{\vee^-}=\bigvee\{\Omega(\neg\varrho):\neg\varrho\in\neg\varsigma \text{ and } \Omega(\neg\varrho)\wedge z^{\backprime}\neq 0\}$, and $(\Upsilon,\Omega)_\varsigma$ is infimum-preserving. Then for any $\neg\varrho_1,\neg\varrho_2\in\neg\varsigma$ there exists $\Omega(\neg\varrho_1)\vee\Omega(\neg\varrho_2)=\Omega(\neg\varrho_3)$, where $\neg\varrho_3\in\neg\varsigma$. Then $z^{\vee^-}=\Omega(\neg\varrho)$.
(i)-(b) Proof easily from (i)-(a) .
(ii) Let $z\in\Xi$. It generally holds that $z^{\backprime}\leq 1$. Given that $(\Upsilon,\Omega)_\varsigma$ is both full and infimum-preserving, then $\exists\ \neg\varrho^*\in\neg\varsigma$ s.t $\bigvee_{\neg\varrho\in\neg\varsigma}\Omega(\neg\varrho)=\Omega(\neg\varrho^*)=1$. So $b\leq\Omega(\neg\varrho^*)=1$ and $1=\Omega(\neg\varrho^*)\wedge z^{\backprime}\neq 0$, consequently $b\leq z^{\vee^-}$. So $z^{\vee^-}=1$, $z\neq 1$. □

Theorem 4.21. *Let* $(\Upsilon,\Omega)_\varsigma$ *be a bipolar soft set over* Ξ, $\rho=(\Lambda(\Xi),(\Upsilon,\Omega)_\varsigma)$ *be a* $(BSA-space)$*. Then the following conditions are equivalent:*

i. *If* $(\Upsilon,\Omega)_\varsigma$ *is full, then*
ii. $1_B{}^{\vee}=(1,0)$;
iii. $1_B{}^{\wedge}=(0,1)$;
iv. $z_B^{\vee}\leq z_B\leq z_B^{\wedge}$.

Proof. (ii) Since $(\Upsilon,\Omega)_\varsigma$ is full, then the first part $1^{\vee^+}=1$ was proved in [14], and the other part is $1^{\vee^-}=0$ by Proposition 4.9. (iii) It follows by Proposition 4.9 and Proposition 3.8 in [14].
(iv) Since $(\Upsilon,\Omega)_\varsigma$ is full, then the first part $z^{\vee^+}\leq z\leq z^{\wedge^+}$ was proved in [14]. Then we prove the other part $z^{\wedge^-}\leq z^{\backprime}\leq z^{\vee^-}$. Since $z^{\wedge^-}\leq z^{\backprime}$ by Proposition 4.8. Suppose that $\lambda\in\Lambda(\Xi)$ and $\lambda\leq z^{\backprime}$, since $(\Upsilon,\Omega)_\varsigma$ is full, then $\exists\ \neg\varrho\in\neg\varsigma$ s.t $\lambda\leq\Omega(\neg\varrho)$. So $\lambda\leq\Omega(\neg\varrho)\wedge z^{\backprime}$, and therefore $\Omega(\neg\varrho)\wedge z^{\backprime}\neq 0$. Consequently $\lambda\leq z^{\vee^-}$ then $z^{\backprime}\leq z^{\vee^-}$. □

Lemma 4.22. *Let* $(\Upsilon,\Omega)_\varsigma$ *be a bipolar soft set over* Ξ*. If* $(\Upsilon,\Omega)_\varsigma$ *is full. Then the maps* $\wedge^-:\Xi\to\Xi$ *and* $\vee^-:\Xi\to\Xi$ *are mutually dual.*

Proof. (a) We show that $(z^{\wedge^-})^{\backprime}=(z^{\backprime})^{\vee^-}$. Let $\lambda\in\Lambda(\Xi)$, then $\lambda\leq(z^{\wedge^-})^{\backprime}$ iff $\lambda\nleq z^{\wedge^-}$ $\forall\ \neg\varrho\in\neg\varsigma$ s.t $\lambda\leq\Omega(\neg\varrho)$, $\Omega(\neg\varrho)\nleq z^{\backprime}$. Since $(\Upsilon,\Omega)_\varsigma$ is full, and $\lambda\in\Lambda(\Xi),\exists\ \neg\varrho\in\neg\varsigma$ s.t $\lambda\leq\Omega(\neg\varrho)$ and $\Omega(\neg\varrho)\nleq z^{\backprime}$ iff $\exists\ \neg\varrho\in\neg\varsigma$, $\lambda\leq\Omega(\neg\varrho)$ and $\Omega(\neg\varrho)\wedge z\neq 0$, then $\lambda\leq(z^{\backprime})^{\vee^-}$. Consequently $(z^{\wedge^-})^{\backprime}=(z^{\backprime})^{\vee^-}$.
(b) We show that $(z^{\vee^-})^{\backprime}=(z^{\backprime})^{\wedge^-}$. Let $\lambda\in\Lambda(\Xi)$, then $\lambda\leq(z^{\vee^-})^{\backprime}$ iff $\lambda\nleq z^{\vee^-}$, $\forall\ \neg\varrho\in\neg\varsigma$ s.t $\lambda\leq\Omega(\neg\varrho)$, $\Omega(\neg\varrho)\wedge z^{\backprime}=0$. Since $(\Upsilon,\Omega)_\varsigma$ is full, and $\lambda\in\Lambda(\Xi),\exists\ \neg\varrho\in\neg\varsigma$ s.t $\lambda\leq\Omega(\neg\varrho)$, so by assume that $\Omega(\neg\varrho)\wedge z^{\backprime}=0$ iff $\Omega(\neg\varrho)\leq z$, then $\lambda\leq(z^{\backprime})^{\wedge^-}$. Consequently $(z^{\vee^-})^{\backprime}=(z^{\backprime})^{\wedge^-}$. □

Proposition 4.23. *Let* $(\Upsilon,\Omega)_\varsigma$ *be a bipolar soft set over* Ξ*. If* $(\Upsilon,\Omega)_\varsigma$ *is a partition. Then*

i. $(z^{\backprime})^{\wedge^-}=z^{\vee^-\wedge^-}$;
ii. $z^{\wedge^-\wedge^-}\leq(z^{\wedge^-})^{\backprime}$;
iii. $z^{\vee^-\vee^-}=(z^{\vee^-})^{\backprime}$.

Proof. (i) Since $(\Upsilon,\Omega)_\varsigma$ is full, from (iv) in Theorem 4.21. Then $(z^{\backprime})^{\wedge^-}\leq z^{\vee^-\wedge^-}$. Conversely, we prove that $z^{\vee^-\wedge^-}\leq(z^{\backprime})^{\wedge^-}$. Assume that $b\leq z^{\vee^-\wedge^-}$, then $\exists\ \neg\varrho\in\neg\varsigma$ s.t $b\leq\Omega(\neg\varrho)$ and $\Omega(\neg\varrho)\leq(z^{\vee^-})^{\backprime}$ then $\Omega(\neg\varrho)\wedge z^{\vee^-}=0$, since $b\leq\Omega(\neg\varrho)$. So $b\nleq z^{\vee^-}$. Hence $\forall\ \neg\varrho_1\in\neg\varsigma$ s.t $b\leq\Omega(\neg\varrho_1)$, $\Omega(\neg\varrho_1)\wedge z^{\backprime}=0$, since $b\leq\Omega(\neg\varrho)$. Then $\Omega(\neg\varrho)\wedge z^{\backprime}=0$, and therefore $\Omega(\neg\varrho)\leq z$. Consequently $b\leq(z^{\backprime})^{\wedge^-}$, then $z^{\vee^-\wedge^-}=(z^{\backprime})^{\wedge^-}$.
(ii) Let $b\in\Lambda(\Xi)$ and $b\leq z^{\wedge^-\wedge^-}$, then $\exists\ \neg\varrho\in\neg\varsigma$ s.t $b\leq\Omega(\neg\varrho)$, $\Omega(\neg\varrho)\leq(z^{\wedge^-})^{\backprime}$. Since $(z^{\wedge^-})^{\backprime}=(z^{\backprime})^{\vee^-}$, then $b\leq\Omega(\neg\varrho)$, $\Omega(\neg\varrho)\leq(z^{\backprime})^{\vee^-}$. $\exists\ \lambda\in\Lambda(\Xi)$ s.t $\lambda\leq\Omega(\neg\varrho)$ and $\lambda\leq(z^{\backprime})^{\vee^-}$. But $\lambda\leq(z^{\backprime})^{\vee^-}$, then $\exists\ \neg\varrho_1\in\neg\varsigma$ s.t $\lambda\leq\Omega(\neg\varrho_1)$ and $\Omega(\neg\varrho_1)\wedge z\neq 0$. Since $(\Upsilon,\Omega)_\varsigma$ is a partition and $\lambda\leq\Omega(\neg\varrho)\wedge\Omega(\neg\varrho_1)$, then $\Omega(\neg\varrho)=\Omega(\neg\varrho_1)$. So $\exists\ \neg\varrho\in\neg\varsigma$ s.t $b\leq\Omega(\neg\varrho)$, $\Omega(\neg\varrho)\wedge z\neq 0$, then $b\leq(z^{\backprime})^{\vee^-}=(z^{\wedge^-})^{\backprime}$.
(iii) Since $(\Upsilon,\Omega)_\varsigma$ is full, from(iv) in Theorem 4.21. Then $(z^{\vee^-})^{\backprime}\leq z^{\vee^-\vee^-}$. Conversely, we show that $z^{\vee^-\vee^-}\leq(z^{\vee^-})^{\backprime}$. Let $b\in\Lambda(\Xi)$ and $b\leq z^{\vee^-\vee^-}$, so $\exists\ \neg\varrho\in\neg\varsigma$ s.t $b\leq\Omega(\neg\varrho)$ and $\Omega(\neg\varrho)\wedge(z^{\vee^-})^{\backprime}=\Omega(\neg\varrho)\wedge(z^{\backprime})^{\wedge^-}\neq 0$. Hence $\exists\ \lambda\in\Lambda(\Xi)$ s.t $\lambda\leq\Omega(\neg\varrho)$ and $\lambda\leq(z^{\backprime})^{\wedge^-}$. But $\lambda\leq(z^{\backprime})^{\wedge^-}$ this implies that $\exists\ \neg\varrho_1\in\neg\varsigma$, $\lambda\leq\Omega(\neg\varrho_1)$, $\Omega(\neg\varrho_1)\leq z$. Since $(\Upsilon,\Omega)_\varsigma$ is a partition and $\lambda\leq\Omega(\neg\varrho)\wedge\Omega(\neg\varrho_1)$. Then $\Omega(\neg\varrho)=\Omega(\neg\varrho_1)$. Hence $\exists\ \neg\varrho\in\neg\varsigma$, $b\leq\Omega(\neg\varrho)$, $\Omega(\neg\varrho)\leq z$, and therefore $b\leq(z^{\backprime})^{\wedge^-}=(z^{\vee^-})^{\backprime}$ by using Lemma 4.22. Consequently $z^{\vee^-\vee^-}\leq(z^{\vee^-})^{\backprime}$, then $z^{\vee^-\vee^-}=(z^{\vee^-})^{\backprime}$. □

Remark 4.24. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice and $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ. The map $\wedge^- : \Xi \to \Xi$ is not a closure operator.

Now, the following example explains the above remark.

Example 4.25. Let $\Xi = \{0, a, b, c, d, e, f, 1\}$ and the order $\leq$ be defined as in Figure (1). Let $\varsigma = \{\varrho_1, \varrho_2, \varrho_3\}$ and $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ defined as follows:
$\Upsilon(\varrho_1) = b \quad \Upsilon(\varrho_2) = a \quad \Upsilon(\varrho_3) = b$
$\Omega(\neg\varrho_1) = a \quad \Omega(\neg\varrho_2) = c \quad \Omega(\neg\varrho_3) = c.$
Let $z = c,\ w = f$. Then $z^{\wedge^-} = c^{\wedge^-} = a$, implies $c \nleq a \Rightarrow z \nleq z^{\wedge^-}$. Then the map $\wedge^-$ is not extensive .
Since $z \leq w \Rightarrow c \leq f \Rightarrow z^{\wedge^-} \leq w^{\wedge^-} \Rightarrow a \leq e$, then the map $\wedge^-$ is order- preserving. Hence $\wedge^-$ is not a closure operator.

5. New mpping induced by bpolar soft rough approximation on a complete atomic Boolean lattice

In this section, we introduce a new mapping induced by bipolar soft rough approximation over a complete atomic Boolean lattice(*BSRA*) and compare it with bipolar soft rough approximation in section (4). Also, give some properties.

Definition 5.1. Let $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ. Define a mapping $\sigma_\Upsilon : \Lambda(\Xi) \to \Xi$ *and* $\psi_\Omega : \Lambda(\Xi) \to \Xi$ for every $\lambda, b \in \Lambda(\Xi)$.

i. $\lambda \leq \sigma_\Upsilon(b) \iff \exists\ \varrho \in \varsigma$ s.t $\lambda \leq \Upsilon(\varrho),\ b \leq \Upsilon(\varrho)$;
ii. $\lambda \leq \psi_\Omega(b) \iff \exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho),\ b \leq \Omega(\neg\varrho)$.

The σ_Υ *and* ψ_Ω are referred to as the mapping induced by $(\Upsilon, \Omega)_\varsigma$ on Ξ .

Proposition 5.2. *Let $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ. Let $\sigma_\Upsilon : \Lambda(\Xi) \to \Xi$ and $\psi_\Omega : \Lambda(\Xi) \to \Xi$ be mapping induced by $(\Upsilon, \Omega)_\varsigma$ on Ξ. The following properties are satisfied:*

i. *$\sigma_\Upsilon, \psi_\Omega$ are symmetric ;*
ii. *If $(\Upsilon, \Omega)_\varsigma$ is full, then $\sigma_\Upsilon, \psi_\Omega$ are extensive;*
iii. *If $(\Upsilon, \Omega)_\varsigma$ is a partition , then $\sigma_\Upsilon, \psi_\Omega$ are extensive, symmetric and closed.*

Proof. (i) We only show that Proposition about map ψ_Ω, where map σ_Υ was proved in Proposition 4.1 in [14]. We prove that ψ_Ω is symmetric. Let $\lambda, b \in \Lambda(\Xi)$, suppose that $\lambda \leq \psi_\Omega(b)$, where $\psi_\Omega : \Lambda(\Xi) \to \Xi$ is mapping induced by $(\Upsilon, \Omega)_\varsigma$ on Ξ. $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho), b \leq \Omega(\neg\varrho)$, this implies that $b \leq \psi_\Omega(\lambda)$. Then ψ_Ω is symmetric.
(ii) Let $\lambda \in \Lambda(\Xi)$. Since $(\Upsilon, \Omega)_\varsigma$ is full, then $\exists\ \neg\varrho \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho)$. Hence $\lambda \leq \psi_\Omega(\lambda)$. Then ψ_Ω is extensive .
(iii) If $(\Upsilon, \Omega)_\varsigma$ is a partition, then $(\Upsilon, \Omega)_\varsigma$ is full. Also ψ_Ω is extensive, since ψ_Ω is symmetric. We only show that ψ_Ω is closed. Let $\lambda, b \in \Lambda(\Xi)$ s.t $\lambda \leq \psi_\Omega(b)$. We prove that $\psi_\Omega(\lambda) \leq \psi_\Omega(b)$, since $\lambda \leq \psi_\Omega(b)$. Then $\exists\ \neg\varrho_1 \in \neg\varsigma$ s.t $\lambda \leq \Omega(\neg\varrho_1)$ and $b \leq \Omega(\neg\varrho_1)$. Assume that $\psi_\Omega(\lambda) \nleq \psi_\Omega(b)$. So $\exists\ a \in \Lambda(\Xi)$ s.t $a \leq \psi_\Omega(\lambda)$ and $a \nleq \psi_\Omega(b)$. But $a \nleq \psi_\Omega(b)$ for every $\neg\varrho \in \neg\varsigma$ either $a \nleq \Omega(\neg\varrho)$ or $b \nleq \Omega(\neg\varrho)$. Since $a \leq \psi_\Omega(\lambda)$, then $\exists\ \neg\varrho_2 \in \neg\varsigma$ s.t $a \leq \Omega(\neg\varrho_2)$ and $\lambda \leq \Omega(\neg\varrho_2)$. Since $(\Upsilon, \Omega)_\varsigma$ is a partition , and $\lambda \leq \Omega(\neg\varrho_1) \wedge \Omega(\neg\varrho_2)$, then $\Omega(\neg\varrho_1) = \Omega(\neg\varrho_2)$. Hence $\exists\ \neg\varrho_1 \in \neg\varsigma$ s.t $a \leq \Omega(\neg\varrho_1)$ and $b \leq \Omega(\neg\varrho_1)$, a contradiction. Then $\psi_\Omega(\lambda) \leq \psi_\Omega(b)$, and ψ_Ω is closed. □

Proposition 5.3. *Let $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ. Let $\psi_\Omega : \Lambda(\Xi) \to \Xi$ be the mapping induced by $(\Upsilon, \Omega)_\varsigma$ on Ξ. The following properties are satisfied:*

i. *If $\lambda \leq \Omega(\neg\varrho)$ for every $\neg\varrho \in \neg\varsigma$ and $\lambda \in \Lambda(\Xi)$. Then $\Omega(\neg\varrho) \leq \psi_\Omega(\lambda)$;*
ii. *If $(\Upsilon, \Omega)_\varsigma$ is a partition, and $\lambda \leq \Omega(\neg\varrho)$ for every $\neg\varrho \in \neg\varsigma$ and $\lambda \in \Lambda(\Xi)$. Then $\Omega(\neg\varrho) = \psi_\Omega(\lambda)$;*
iii. *If $(\Upsilon, \Omega)_\varsigma$ is infimum-preserving. Then $\forall \lambda \in \Lambda(\Xi)$, $\neg\varrho \in \neg\varsigma$ s.t $\psi_\Omega(\lambda) = \Omega(\neg\varrho)$.*

Proof. (i) Let $b \in \Lambda(\Xi)$ *s.t* $b \leq \Omega(\neg\varrho)$. Since $\lambda \leq \Omega(\neg\varrho)$, then $b \leq \psi_{\Omega}(\lambda)$. Hence $\Omega(\neg\varrho) \leq \psi_{\Omega}(\lambda)$.
(ii) Suppose that $(\Upsilon, \Omega)_{\varsigma}$ is a partition, and assume that $\lambda \leq \Omega(\neg\varrho)$ *for every* $\neg\varrho \in \neg\varsigma$ *and* $\lambda \in \Lambda(\Xi)$. Since $\Omega(\neg\varrho) \leq \psi_{\Omega}(\lambda)$ from (i). Conversely, let $b \in \Lambda(\Xi)$ s.t $b \leq \psi_{\Omega}(\lambda)$, then $\exists\ \neg\varrho_1 \in \neg\varsigma$ s.t $b \leq \Omega(\neg\varrho_1)$ and $\lambda \leq \Omega(\neg\varrho_1)$, so $\lambda \leq \Omega(\neg\varrho) \wedge \Omega(\neg\varrho_1)$. Since $(\Upsilon, \Omega)_{\varsigma}$ is a partition, then $\Omega(\neg\varrho) = \Omega(\neg\varrho_1)$. Hence $b \leq \Omega(\neg\varrho)$, and therefore $\psi_{\Omega}(\lambda) \leq \Omega(\neg\varrho)$. Consequently $\Omega(\neg\varrho) = \psi_{\Omega}(\lambda)$.
(iii) Suppose that $(\Upsilon, \Omega)_{\varsigma}$ is infimum-preserving , and $\lambda \in \Lambda(\Xi)$. Let $b \in \Lambda(\Xi)$ s.t $b \leq \psi_{\Omega}(\lambda)$, then $\exists\ \neg\varrho \in \neg\varsigma$ s.t $b \leq \Omega(\neg\varrho)$ *and* $\lambda \leq \Omega(\neg\varrho)$. Since $\Omega(\neg\varrho) \leq \psi_{\Omega}(\lambda)$ from (i). Hence $\psi_{\Omega}(\lambda) = \bigvee_{b \in \Lambda(\Xi)}\{\Omega(\neg\varrho) : b \leq \psi_{\Omega}(\lambda)\} = \Omega(\neg\varrho)$ for every $\neg\varrho \in \neg\varsigma$. Therefore $\psi_{\Omega}(\lambda) = \Omega(\neg\varrho)$. □

Definition 5.4. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice, and let $(\Upsilon, \Omega)_{\varsigma}$ be a bipolar soft set over Ξ. Assume that $\Lambda(\Xi)$ denotes the set of all atoms of Ξ. Let $\sigma_{\Upsilon} : \Lambda(\Xi) \to \Xi$ and $\psi_{\Omega} : \Lambda(\Xi) \to \Xi$ be the mappings induced by $(\Upsilon, \Omega)_{\varsigma}$ on Ξ. We define the following bipolar soft rough approximation operators: $\triangledown_{\Upsilon},\ \vartriangle_{\Upsilon},\ \triangledown_{\Omega},\ \vartriangle_{\Omega} : \Xi \to \Xi$ for any $z \in \Xi$ as follows:

$z^{\triangledown_{\Upsilon}} = \bigvee\{\lambda \in \Lambda(\Xi) \mid \sigma_{\Upsilon}(\lambda) \leq z\}$;
$z^{\triangledown_{\Omega}} = \bigvee\{\lambda \in \Lambda(\Xi) \mid \psi_{\Omega}(\lambda) \wedge z^{\backprime} \neq 0\}$;
$z^{\vartriangle_{\Upsilon}} = \bigvee\{\lambda \in \Lambda(\Xi) \mid \sigma_{\Upsilon}(\lambda) \wedge z \neq 0\}$;
$z^{\vartriangle_{\Omega}} = \bigvee\{\lambda \in \Lambda(\Xi) \mid \psi_{\Omega}(\lambda) \leq z^{\backprime}\}$.
Here, $z^{\triangledown_{\Upsilon}}$ and $z^{\vartriangle_{\Upsilon}}$ are called the *lower* and *upper* approximations of z with respect to the mapping σ_{Υ}, respectively, while $z^{\triangledown_{\Omega}}$ and $z^{\vartriangle_{\Omega}}$ are called the *lower* and *upper* approximations of z with respect to the mapping ψ_{Ω}, respectively. The pairs $(z^{\triangledown_B}, z^{\vartriangle_B})$ are referred to as the *bipolar soft rough approximations* of z induced by $(\Upsilon, \Omega)_{\varsigma}$.

Proposition 5.5. *Let $(\Upsilon, \Omega)_{\varsigma}$ be a bipolar soft set over Ξ, and let $\sigma_{\Upsilon} : \Lambda(\Xi) \to \Xi$ and $\psi_{\Omega} : \Lambda(\Xi) \to \Xi$ be the mappings induced by $(\Upsilon, \Omega)_{\varsigma}$ on Ξ. Then, for all $\lambda \in \Lambda(\Xi)$ and $z, w \in \Xi$, the following statements hold:*

i. $\lambda \leq z^{\triangledown_{\Upsilon}}$ *and* $\lambda \leq z^{\triangledown_{\Omega}} \iff \sigma_{\Upsilon}(\lambda) \leq z$ *and* $\psi_{\Omega}(\lambda) \wedge z^{\backprime} \neq 0$.
ii. $\lambda \leq z^{\vartriangle_{\Upsilon}}$ *and* $\lambda \leq z^{\vartriangle_{\Omega}} \iff \sigma_{\Upsilon}(\lambda) \wedge z \neq 0$ *and* $\psi_{\Omega}(\lambda) \leq z^{\backprime}$.
iii. *If $(\Upsilon, \Omega)_{\varsigma}$ is full, then $z^{\triangledown_B} \leq z \leq z^{\vartriangle_B}$.*
iv. *If $(\Upsilon, \Omega)_{\varsigma}$ is full, then $0^{\triangledown_B} = 0^{\vartriangle_B} = (0, 1)$ and $1^{\triangledown_B} = 1^{\vartriangle_B} = (1, 0)$.*
v. *If $z \leq w$, then $z^{\triangledown_B} \leq w^{\triangledown_B}$ and $z^{\vartriangle_B} \leq w^{\vartriangle_B}$.*
vi. *The operators $\triangledown_B : \Xi \to \Xi$ and $\vartriangle_B : \Xi \to \Xi$ are mutually dual.*
vii. *For any subset $M \subseteq \Xi$,*
$\bigvee M^{\vartriangle_B} = \left(\bigvee M\right)^{\vartriangle_B}$.
viii. *For any subset $M \subseteq \Xi$,*
$\bigwedge M^{\triangledown_B} = \left(\bigwedge M\right)^{\triangledown_B}$.
ix. *$(\Xi^{\vartriangle_B}, \leq)$ is a complete lattice, whose least element is $(0, 1)$ and whose greatest element is $1^{\vartriangle_B}$.*
x. *The pair $(\triangledown_B, \vartriangle_B)$ forms a dual Galois connection on Ξ.*
xi. $(\Xi^{\triangledown_B}, \geq) \cong (\Xi^{\vartriangle_B}, \leq)$.

Proof. It follows by Proposition 5.2 and Proposition 4.1, 4.4 in [14]. □

In the following, we examine the relation between the above pairs of bipolar soft rough approximation operators given in Definitions 4.1 and 5.4.

Proposition 5.6. *Let $(\Upsilon, \Omega)_{\varsigma}$ be a bipolar soft set over Ξ. Let $\sigma_{\Upsilon} : \Lambda(\Xi) \to \Xi$ and $\psi_{\Omega} : \Lambda(\Xi) \to \Xi$ be the mappings induced by $(\Upsilon, \Omega)_{\varsigma}$ on Ξ. Then the following relations hold:*

i. *If $(\Upsilon, \Omega)_{\varsigma}$ is a partition, supremum-preserving, and infimum-preserving, then $z^{\vartriangle_B} \leq z_{B}^{\wedge}$.*
ii. *If $(\Upsilon, \Omega)_{\varsigma}$ is a partition and infimum-preserving, then $z^{\triangledown_B} \leq z_{B}^{\vee}$.*
iii. *If $(\Upsilon, \Omega)_{\varsigma}$ is a partition and infimum-preserving, thenz$^{\triangledown_B} = z_{B}^{\vee}$, $z^{\vartriangle_B} = z_{B}^{\wedge}$.*

Proof. (i) The inequality $z^{\Delta_\Upsilon} \leq z^{\wedge^+}$ is proved in [14]. We show that $z^{\wedge^-} \leq z^{\Delta_\Omega}$. Let $b \in \Lambda(\Xi)$ such that $b \leq z^{\wedge^-}$. Then, there exists $\neg\varrho \in \neg\varsigma$ such that $b \leq \Omega(\neg\varrho) \leq z^{\backprime}$. Since $(\Upsilon,\Omega)_\varsigma$ is a partition, Proposition 5.3(ii) implies $\Omega(\neg\varrho) = \psi_\Omega(b)$. Hence $b \leq \psi_\Omega(b) \leq z^{\backprime} \quad \Longrightarrow \quad b \leq z^{\Delta_\Omega}$. Therefore, $z^{\wedge^-} \leq z^{\Delta_\Omega}$.
(ii) The inequality $z^{\nabla_\Upsilon} \leq z^{\vee^+}$ is proved in [14]. We show that $z^{\vee^-} \leq z^{\nabla_\Omega}$. Let $b \in \Lambda(\Xi)$ such that $b \leq z^{\vee^-}$. Then there exists $\neg\varrho \in \neg\varsigma$ such that $\Omega(\neg\varrho) \wedge z^{\backprime} \neq 0$. Using the partition property, $\Omega(\neg\varrho) = \psi_\Omega(b)$, hence $b \leq \psi_\Omega(b), \quad \psi_\Omega(b) \wedge z^{\backprime} \neq 0 \quad \Longrightarrow \quad b \leq z^{\nabla_\Omega}$. Therefore, $z^{\vee^-} \leq z^{\nabla_\Omega}$.
(iii) For the equality $z^{\nabla_B} = z^{\vee}_B$: ($\Rightarrow$) by infimum-preserving property, $z^{\nabla_B} \leq z^{\vee}_B$.
($\Leftarrow$) The first part $z^{\vee^+} \leq z^{\nabla_\Upsilon}$ is proved in [14]. For the second part, if $z = 1$, then $z^{\vee^-} = 1^{\vee^-} = 0 = 1^{\nabla_\Omega} = z^{\nabla_\Omega}$. If $z \neq 1$, by infimum-preserving, $z^{\vee^-} = 1$. Hence $z^{\nabla_\Omega} \leq z^{\vee^-}$, and consequently $z^{\nabla_B} = z^{\vee}_B$. For the equality $z^{\Delta_B} = z^{\wedge}_B$: ($\Rightarrow$) By supremum- and infimum-preserving, $z^{\Delta_B} \leq z^{\wedge}_B$.
($\Leftarrow$) First, $z^{\wedge^+} \leq z^{\Delta_\Upsilon}$ is proved in [14]. Let $b \in \Lambda(\Xi)$ such that $b \leq z^{\Delta_\Omega}$. Since $(\Upsilon,\Omega)_\varsigma$ is full, there exists $\neg\varrho \in \neg\varsigma$ such that $b \leq \Omega(\neg\varrho) \leq \psi_\Omega(b) \leq z^{\backprime}$. Hence $b \leq z^{\wedge^-}$, which implies $z^{\Delta_\Omega} \leq z^{\wedge^-}$. Therefore, $z^{\Delta_B} = z^{\wedge}_B$. □

In the following example, we give the opposite of the Proposition 5.5(v).

Example 5.7. Let $\Xi = \{0, a, b, c, d, e, f, 1\}$ and the order $\leq$ be defined as in Figure(1). Let $\varsigma = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$ and $(\Upsilon,\Omega)_\varsigma$ be a bipolar soft set over Ξ defined as follows:
$\Upsilon(\varrho_1) = b \quad \Upsilon(\varrho_2) = c \quad \Upsilon(\varrho_3) = f \quad \Upsilon(\varrho_4) = 0$
$\Omega(\neg\varrho_1) = a \quad \Omega(\neg\varrho_2) = b \quad \Omega(\neg\varrho_3) = 0 \quad \Omega(\neg\varrho_4) = d$
Obviously, $(\Upsilon,\Omega)_\varsigma$ is not full. Since $\psi_\Omega(a) = \bigvee\{b \in \Lambda(\Xi) : b \leq \psi_\Omega(a)\} = a \vee b = d$, $\psi_\Omega(b) = a \vee b = d$, $\psi_\Omega(c) = 0$.
Let $z = b$, $w = a$, so $z^{\nabla_\Omega} = \bigvee\{\lambda \in \Lambda(\Xi) : \psi_\Omega(\lambda) \wedge z^{\backprime} \neq 0\} = a \vee b = d$, and $z^{\Delta_\Omega} = \bigvee\{\lambda \in \Lambda(\Xi) : \psi_\Omega(\lambda) \leq z^{\backprime}\} = c$, where $z^{\backprime} = e$. On the other hand $w^{\nabla_\Omega} = 0$ and $w^{\Delta_\Omega} = c$. Hence $w^{\nabla_\Omega} \leq z^{\nabla_\Omega}$ and $w^{\Delta_\Omega} \leq z^{\Delta_\Omega}$, but $z \nleq w$.

Example 5.8. Let $\Xi = \{0, a, b, c, d, e, f, 1\}$ and the order $\leq$ be defined as in Figure (1). Let $\varsigma = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$ and $(\Upsilon_1,\Omega_1)_\varsigma$ be a bipolar soft set over Ξ defined as follows:
$\Upsilon_1(\varrho_1) = d \quad \Upsilon_1(\varrho_2) = f \quad \Upsilon_1(\varrho_3) = a \quad \Upsilon_1(\varrho_4) = c$
$\Omega_1(\neg\varrho_1) = f \quad \Omega_1(\neg\varrho_2) = a \quad \Omega_1(\neg\varrho_3) = c \quad \Omega(\neg\varrho_4) = d$
Obviously, $(\Upsilon_1,\Omega_1)_\varsigma$ is full. Also $\psi_\Omega(a) = a \vee b \vee c = 1$, $\psi_\Omega(b) = 1$, *and* $\psi_\Omega(c) = 1$.
Let $z = f$, $z^{\backprime} = d$, then $z^{\nabla_\Omega} = a \vee b \vee c = 1$, and $z^{\Delta_\Omega} = 0$. Hence $z^{\Delta_\Omega} \leq z^{\backprime} \leq z^{\nabla_\Omega}$.

In the following, we give a relation between the mapping induced approximations and bipolar soft rough set on a complete atomic Boolean lattice in section(4).

Definition 5.9. Let $\Xi = (\Xi, \leq)$ represent a complete atomic Boolean lattice. Let $\sigma : \Lambda(\Xi) \to \Xi$ and $\psi : \Lambda(\Xi) \to \Xi$ be extensive, symmetric, and closed mapping. Define a mapping $\Upsilon_\sigma : \Lambda \to \Xi$ by $\Upsilon_\sigma(\varrho) = \sigma(\varrho)\ \forall\varrho \in \Lambda$ and $\Omega_\psi : \Lambda \to \Xi$ by $\Omega_\psi(\neg\varrho) = \psi(\varrho)\ \ \forall\varrho \in \Lambda$, where $\Lambda = \Lambda(\Xi)$. Then $(\Upsilon_\sigma,\Omega_\psi)_\Lambda$ is called a bipolar soft set induced by σ, ψ on Ξ.

Proposition 5.10. *Let $\Xi = (\Xi, \leq)$ represent a complete atomic Boolean lattice, and $(\Upsilon_\sigma,\Omega_\psi)_A$ be a bipolar soft set induced by σ, ψ on Ξ. Then for every $z \in \Xi$. $z^{\vee_\sigma} = z^{\nabla_\Upsilon}$, $z^{\wedge_\sigma} = z^{\Delta_\Upsilon}$, and $z^{\vee_\psi} = z^{\nabla_\Omega}$, $z^{\wedge_\psi} = z^{\Delta_\Omega}$, where*

$z^{\vee_\sigma} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists\ \varrho \in \Lambda\ s.t\ \lambda \leq \Upsilon_\sigma(\varrho),\ \Upsilon_\sigma(\varrho) \leq z\}$, $z^{\nabla_\Upsilon} = \{\lambda \in \Lambda(\Xi) : \sigma(\lambda) \leq z\}$;
$z^{\vee_\psi} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\Lambda\ s.t\ \lambda \leq \Omega_\psi(\neg\varrho),\ \Omega_\psi(\neg\varrho) \wedge z^{\backprime} \neq 0\}$, $z^{\nabla_\Omega} = \{\lambda \in \Lambda(\Xi) : \psi(\lambda) \wedge z^{\backprime} \neq 0\}$;
$z^{\wedge_\sigma} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists\ \varrho \in \Lambda\ s.t\ \lambda \leq \Upsilon_\sigma(\varrho),\ \Upsilon_\sigma(\varrho) \wedge z \neq 0\}$, $z^{\Delta_\Upsilon} = \{\lambda \in \Lambda(\Xi) : \sigma(\lambda) \wedge z \neq 0\}$;
$z^{\wedge_\psi} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists\ \neg\varrho \in \neg\Lambda\ s.t\ \lambda \leq \Omega_\psi(\neg\varrho),\ \Omega_\psi(\neg\varrho) \leq z^{\backprime}\}$, $z^{\Delta_\Omega} = \{\lambda \in \Lambda(\Xi) : \psi(\lambda) \leq z^{\backprime}\}$

Proof. Obvious. □

Definition 5.11. Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice. Let $\sigma_\Upsilon : \Lambda(\Xi) \to \Xi$ and $\psi_\Omega : \Lambda(\Xi) \to \Xi$ be extensive, symmetric, and closed mappings. Define the mappings $\Upsilon_\sigma : \varsigma \to \Xi$, $\Upsilon_\sigma(\varrho) = \sigma_\Upsilon(\varrho)$, $\forall\varrho \in \varsigma$, $\Omega_\psi : \neg\varsigma \to \Xi$, $\Omega_\psi(\neg\varrho) = \psi_\Omega(\neg\varrho)$, $\forall\neg\varrho \in \varsigma$, where $\varsigma = \Lambda(\Xi)$. Then $(\Upsilon_\sigma,\Omega_\psi)_\varsigma$ is called the *bipolar soft set induced by σ_Υ and ψ_Ω on Ξ.*

Theorem 5.12. *Let $\Xi = (\Xi, \leq)$ be a complete atomic Boolean lattice, and let $(\Upsilon_\sigma, \Omega_\psi)_\varsigma$ be a bipolar soft set induced by σ_Υ and ψ_Ω on Ξ. Then, for every $z \in \Xi$, the following equalities hold: $z^{\vee_\sigma} = z^{\nabla_\Upsilon}$, $z^{\wedge_\sigma} = z^{\Delta_\Upsilon}$, $z^{\vee_\psi} = z^{\nabla_\Omega}$, $z^{\wedge_\psi} = z^{\Delta_\Omega}$, where:*

i. $z^{\vee_\sigma} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists \varrho \in \varsigma \ s.t. \ \lambda \leq \Upsilon_\sigma(\varrho), \ \Upsilon_\sigma(\varrho) \leq z\}$, $z^{\nabla_\Upsilon} = \bigvee\{\lambda \in \Lambda(\Xi) : \sigma(\lambda) \leq z\}$,
ii. $z^{\vee_\psi} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists \neg\varrho \in \neg\varsigma \ s.t. \ \lambda \leq \Omega_\psi(\neg\varrho), \ \Omega_\psi(\neg\varrho) \wedge z' \neq 0\}$, $z^{\nabla_\Omega} = \bigvee\{\lambda \in \Lambda(\Xi) : \psi(\lambda) \wedge z' \neq 0\}$,
iii. $z^{\wedge_\sigma} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists \varrho \in \varsigma \ s.t. \ \lambda \leq \Upsilon_\sigma(\varrho), \ \Upsilon_\sigma(\varrho) \wedge z \neq 0\}$, $z^{\Delta_\Upsilon} = \bigvee\{\lambda \in \Lambda(\Xi) : \sigma(\lambda) \wedge z \neq 0\}$,
iv. $z^{\wedge_\psi} = \bigvee\{\lambda \in \Lambda(\Xi) : \exists \neg\varrho \in \neg\varsigma \ s.t. \ \lambda \leq \Omega_\psi(\neg\varrho), \ \Omega_\psi(\neg\varrho) \leq z'\}$, $z^{\Delta_\Omega} = \bigvee\{\lambda \in \Lambda(\Xi) : \psi(\lambda) \leq z'\}$.

Proof. The proof follows directly from the definitions of the mappings σ_Υ and ψ_Ω and the construction of the induced bipolar soft set. The correspondences between the operators are immediate by definition. □

6. Application of bipolar soft rough sets on Ξ in a decision-making problem

Let $\Xi = \{0, a, b, c, d, e, f, 1\}$ represent eight employees, where 0 represents an employee who has no skills, a represents an employee who has the highest IQ, b represents an employee who has leadership skills and emotional intelligence, c represents an employee who has work experience, d represents an employee who has the highest IQ and leadership skills, f represents an employee who has leadership skills and work experience, e represents an employee who has the highest IQ and work experience, and 1 represents an employee who has all skills. The order $\leq$ is illustrated in Figure(1). Let $\varsigma_1 = \{\varrho_1, \varrho_2, \varrho_3\}$, where ϱ_1 represents experience, ϱ_2 represents leadership and leadership management skills, ϱ_3 represents IQ score. $\neg\varsigma_1 = \{\neg\varrho_1, \neg\varrho_2, \neg\varrho_3\}$, where $\neg\varrho_1$ represents not experience, $\neg\varrho_2$ represents not leadership skills and emotional intelligence, $\neg\varrho_3$ represents not IQ score (low IQ), and $(\Upsilon, \Omega)_\varsigma$ be a bipolar soft set over Ξ represents the employee, who will get promoted as a project manager.

Algorithm

Let Ξ be a complete atomic Boolean lattice $\Xi = (\Xi, \leq)$ and $\Lambda(\Xi) = \{a, b, c\}$ be set of atoms, $\varsigma_1 = \{\varrho_1, \varrho_2, \varrho_3\}$, $\neg\varsigma_1 = \{\neg\varrho_1, \neg\varrho_2, \neg\varrho_3\}$ be a set of parameters , and $(\Upsilon, \Omega)_{\varsigma_1}$ be a bipolar soft set over Ξ. Assume that $H = \{P_1, P_2, P_3\}$, where $d, e, f \in \Xi$ denote results of primary evaluations of experts P_1, P_2, P_3 *and* $T_1, T_2, T_3 \in BSs$.

Step1: Take primary evaluations d, e, f of experts $P_1, P_2, ..., P_k$.
Step2: Form $T_1, T_2, ..., T_r$ bipolar soft sets by using real results.
Step3: Count $z_B^{\vee}$ *and* $z_B^{\wedge}$ for each $q = 1, 2, ..., r$ and $j = 1, 2, ..., k$, where $z_B^{\vee} = (z^{\vee^+}, z^{\vee^-})$, $z_B^{\wedge} = (z^{\wedge^+}, z^{\wedge^-})$.
Step4: Form bipolar soft matrices for the lower and upper approximations. $\underline{\upsilon} = \upsilon^{\vee}, \overline{\upsilon} = \upsilon^{\wedge}$.
Step 5: Count $\varepsilon^{\vee}$ and $\varepsilon^{\wedge}$.
Step 6: Count $\varepsilon^{\vee} \bigoplus \varepsilon^{\wedge}$.
Step 7: Find $max_{i \in I_n} \varepsilon_i$.

Illustrative example

Step1: Primary evaluations of $P_1, P_2, and\ P_3$ are $d, e, f \in \Xi$.
Step2: Real results in different three periods are expressed as bipolar soft set over Ξ, defined as follows:
$\Upsilon_1(\varrho_1) = b \quad \Upsilon_1(\varrho_2) = a \quad \Upsilon_1(\varrho_3) = b$
$\Omega_1(\neg\varrho_1) = a \quad \Omega_1(\neg\varrho_2) = c \quad \Omega_1(\neg\varrho_3) = c$
$T_1 = \{(\{b\}, \{a\}, \varrho_1), (\{a\}, \{c\}, \varrho_2), (\{b\}, \{c\}, \varrho_3)\}$

$\Upsilon_2(\varrho_1) = e \quad \Upsilon_2(\varrho_2) = c \quad \Upsilon_2(\varrho_3) = b$
$\Omega_2(\neg\varrho_1) = b \quad \Omega_2(\neg\varrho_2) = 0 \quad \Omega_2(\neg\varrho_3) = e$
$T_2 = \{(\{e\}, \{b\}, \varrho_1), (\{c\}, \{0\}, \varrho_2), (\{b\}, \{e\}, \varrho_3)\}$

$\Upsilon_3(\varrho_1) = d \quad \Upsilon_3(\varrho_2) = c \quad \Upsilon_3(\varrho_3) = f$
$\Omega_3(\neg\varrho_1) = f \quad \Omega_3(\neg\varrho_2) = d \quad \Omega_3(\neg\varrho_3) = a$
$T_3 = \{(\{d\}, \{f\}, \varrho_1), (\{c\}, \{d\}, \varrho_2), (\{f\}, \{a\}, \varrho_3)\}$

Step3:
$(z_B^{\vee})_1 = (d_B^{\vee}) = (d,c)$, $(z_B^{\wedge})_1 = (d_B^{\wedge}) = (d,1)$;
$(e_B^{\vee})_1 = (a,0)$, $(e_B^{\wedge})_1 = (a,0)$;
$(f_B^{\vee})_1 = (b,a)$, $(f_B^{\wedge})_1 = (b,a)$.

$(z_B^{\vee})_2 = (d_B^{\vee}) = (b,c)$, $(z_B^{\wedge})_2 = (d_B^{\wedge}) = (d,0)$;
$(e_B^{\vee})_2 = (e,b)$, $(e_B^{\wedge})_2 = (e,b)$;
$(f_B^{\vee})_2 = (f,a)$, $(f_B^{\wedge})_2 = (1,0)$.

$(d_B^{\vee})_3 = (d,f)$, $(d_B^{\wedge})_3 = (d,0)$;
$(e_B^{\vee})_3 = (c,1)$, $(e_B^{\wedge})_3 = (e,0)$;
$(f_B^{\vee})_3 = (f,d)$, $(f_B^{\wedge})_3 = (f,a)$.

Step4:

$$\underline{\upsilon} = \upsilon^{\vee} = \left\{\begin{matrix} \{\varepsilon_1^{1\vee^+}, \varepsilon_1^{1\vee^-}\} & \{\varepsilon_2^{1\vee^+}, \varepsilon_2^{1\vee^-}\} & \{\varepsilon_3^{1\vee^+}, \varepsilon_3^{1\vee^-}\} \\ \{\varepsilon_1^{2\vee^+}, \varepsilon_1^{2\vee^-}\} & \{\varepsilon_2^{2\vee^+}, \varepsilon_2^{2\vee^-}\} & \{\varepsilon_3^{2\vee^+}, \varepsilon_3^{2\vee^-}\} \\ \{\varepsilon_1^{3\vee^+}, \varepsilon_1^{3\vee^-}\} & \{\varepsilon_2^{3\vee^+}, \varepsilon_2^{3\vee^-}\} & \{\varepsilon_3^{3\vee^+}, \varepsilon_3^{3\vee^-}\} \end{matrix}\right\} \tag{1}$$

$$\overline{\upsilon} = \upsilon^{\wedge} = \left\{\begin{matrix} \{\varepsilon_1^{1\wedge^+}, \varepsilon_1^{1\wedge^-}\} & \{\varepsilon_2^{1\wedge^+}, \varepsilon_2^{1\wedge^-}\} & \{\varepsilon_3^{1\wedge^+}, \varepsilon_3^{1\wedge^-}\} \\ \{\varepsilon_1^{2\wedge^+}, \varepsilon_1^{2\wedge^-}\} & \{\varepsilon_2^{2\wedge^+}, \varepsilon_2^{2\wedge^-}\} & \{\varepsilon_3^{2\wedge^+}, \varepsilon_3^{2\wedge^-}\} \\ \{\varepsilon_1^{3\wedge^+}, \varepsilon_1^{3\wedge^-}\} & \{\varepsilon_2^{3\wedge^+}, \varepsilon_2^{3\wedge^-}\} & \{\varepsilon_3^{3\wedge^+}, \varepsilon_3^{3\wedge^-}\} \end{matrix}\right\} \tag{2}$$

These are referred to as bipolar soft lower and upper approximation matrices, respectively, and denoted by $\upsilon^{\vee}$ *and* $\upsilon^{\wedge}$. Here

$$\varepsilon_j^{q\vee^+} = \left(z_{1j}^{q\vee^+}, z_{2j}^{q\vee^+}, ..., z_{nj}^{q\vee^+}\right) \tag{3}$$

$$\varepsilon_j^{q\vee^-} = \left(z_{1j}^{q\vee^-}, z_{2j}^{q\vee^-}, ..., z_{nj}^{q\vee^-}\right) \tag{4}$$

$$\varepsilon_j^{q\wedge^+} = \left(z_{1j}^{q\wedge^+}, z_{2j}^{q\wedge^+}, ..., z_{nj}^{q\wedge^+}\right) \tag{5}$$

$$\varepsilon_j^{q\wedge^-} = \left(z_{1j}^{q\wedge^-}, z_{2j}^{q\wedge^-}, ..., z_{nj}^{q\wedge^-}\right) \tag{6}$$

Where

$$z_{ij}^{q\vee^+} = \begin{cases} 1 & z_i \leq z^{\vee^+} \\ 0 & z_i \nleq z^{\vee^+} \end{cases}$$

$$z_{ij}^{q\vee^-} = \begin{cases} -\frac{1}{2} & z_i \leq z^{\vee^-} \\ 0 & z_i \nleq z^{\vee^-} \end{cases}$$

$$z_{ij}^{q\wedge^+} = \begin{cases} \frac{1}{2} & z_i \leq z^{\wedge^+} \\ 0 & z_i \nleq z^{\wedge^+} \end{cases}$$

$$z_{ij}^{q\wedge^-} = \begin{cases} -1 & z_i \leq z^{\wedge^-} \\ 0 & z_i \nleq z^{\wedge^-} \end{cases}$$

Then

$$v^{\vee} = \left\{ \begin{array}{ccc} \{(1,1,1,1,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0)\} & \{(1,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)\} & \{(0,1,0,0,0,0,0,0),(-\frac{1}{2},0,0,0,0,0,0,0)\} \\ \{(0,1,1,0,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0)\} & \{(1,1,0,0,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0)\} & \{(0,1,1,0,0,0,0,0),(-\frac{1}{2},0,0,0,0,0,0,0)\} \\ \{(1,1,1,1,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0)\} & \{(0,0,1,1,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},0,0,0,0)\} & \{(0,1,1,0,0,0,0,0),(-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0)\} \end{array} \right\}$$

$$v^{\wedge} = \left\{ \begin{array}{ccc} \{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0),(-1,-1,-1,-1,0,0,0,0)\} & \{(\frac{1}{2},0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0)\} & \{(0,\frac{1}{2},0,0,0,0,0,0),(-1,0,0,0,0,0,0,0)\} \\ \{(0,\frac{1}{2},\frac{1}{2},0,0,0,0,0),(-1,-1,0,0,0,0,0,0)\} & \{(\frac{1}{2},\frac{1}{2},0,0,0,0,0,0),(-1,-1,0,0,0,0,0,0)\} & \{(0,\frac{1}{2},\frac{1}{2},0,0,0,0,0),(-1,0,0,0,0,0,0,0)\} \\ \{(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,0,0),(-1,-1,-1,-1,0,0,0,0)\} & \{(0,0,\frac{1}{2},\frac{1}{2},0,0,0,0),(-1,-1,-1,-1,0,0,0,0)\} & \{(0,\frac{1}{2},\frac{1}{2},0,0,0,0,0),(-1,-1,0,0,0,0,0,0)\} \end{array} \right\}$$

Step5: By using equations

$$\varepsilon^{\vee} = \bigoplus_{j=1}^{k} \bigoplus_{q=1}^{r} (\varepsilon_j^{q\vee^+} \oplus \varepsilon_j^{q\vee^-})$$

$$\varepsilon^{\wedge} = \bigoplus_{j=1}^{k} \bigoplus_{q=1}^{r} (\varepsilon_j^{q\wedge^+} \oplus \varepsilon_j^{q\wedge^-})$$

The bipolar soft lower and upper approximation vectors are derived as follows:

$$\varepsilon^{\vee} = \left(\frac{9}{2}, 5, \frac{11}{2}, \frac{11}{2}, \frac{11}{2}, \frac{7}{2}, 4, 5\right), \; \varepsilon^{\wedge} = \left(-\frac{9}{2}, -\frac{9}{2}, -2, -2, -2, -\frac{3}{2}, -\frac{3}{2}, -\frac{3}{2}\right).$$

Step 6: The decision vector is obtained as

$$\varepsilon^{\vee} \bigoplus \varepsilon^{\wedge} = \left(0, \frac{1}{2}, \frac{7}{2}, \frac{7}{2}, \frac{7}{2}, \frac{1}{2}, \frac{5}{2}, \frac{7}{2}\right).$$

Step 7: Since

$$0 < a < e < f < b = c = d = \frac{7}{2}.$$

The optimal element is 1, which corresponds to the employee possessing all required skills. Hence, the bipolar soft rough decision-making model selects the employee represented by 1 as the most suitable candidate for promotion to the position of project manager.

7. Conclusion

In this paper, we present a novel concept of bipolar soft sets on a complete atomic Boolean lattice, extending the traditional notion of bipolar soft sets. We elucidate the lattice structure inherent in these expanded bipolar soft sets and examine both positive and negative bipolar soft rough approximation operators on a complete atomic Boolean lattice, detailing their respective properties. Furthermore, we demonstrate that the approximations induced by mappings can be interpreted as a specific instance of our broader bipolar soft rough approximations. The applicability of bipolar soft rough sets on a complete atomic Boolean lattice is showcased through its utilization in decision-making problems. In future work, the present framework may be extended to develop modified bipolar soft rough set models on complete atomic Boolean lattices by integrating the bipolar structure with modified soft rough approximation techniques, as initiated in [15]. Such an extension is expected to provide more flexible approximation mechanisms and enhance the expressive power of lattice-based bipolar uncertainty models, particularly in advanced decision-making applications.

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