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Projectively coresolved Gorenstein flat modules over semi-trivial ring extensions

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Abstract. We establish necessary and sufficient conditions for modules over a semi-trivial ring extension $R \ltimes_{\Phi} M$ to be projectively coresolved Gorenstein flat. Some applications are given in the Morita context ring $\begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$.

1. Introduction and preliminaries

The origin of Gorenstein homological algebra may date back to 1960s when Auslander and Bridger introduced the concept of G-dimension for finitely generated modules over a two-sided Noetherian ring [1]. In 1990s, Enochs, Jenda and Torrecillas extended the ideas of Auslander and Bridger and introduced the concepts of Gorenstein projective, injective and flat modules over arbitrary rings, and then developed Gorenstein homological algebra [6, 7], which has significant applications in representation theory of algebras, algebraic geometry and other fields. It is well known that projective modules are always flat. An open problem of the area is whether every Gorenstein projective module is Gorenstein flat. In [21], Šaroch and Šťovíček introduced the notion of projectively coresolved Gorenstein flat modules (*PGF* modules, for short). They are the cycles of exact complexes of projective modules that remain exact when tensored with any injective module. The *PGF* modules played a crucial role in Šaroch and Šťovíček's proof that the cotorsion pair generated by the Gorenstein flat modules is complete. These modules are simultaneously Gorenstein projective and Gorenstein flat. Thus *PGF* modules in Gorenstein homological algebra may be viewed as the role of projective modules in classical homological algebra and have been studied by many authors [3, 4, 10, 21, 23].

On the other hand, the notion of semi-trivial extensions of rings is an important extension of rings and has played a crucial role in ring theory and homological algebra. Let R be an associative ring and M an R-R-bimodule, $\Phi : M \otimes_R M \to R$ an R-R-bimodule homomorphism. Define a multiplication in the direct sum $R \oplus M$ of Abelian groups R and M by

$$(r_1, m_1)(r_2, m_2) = (r_1r_2 + \Phi(m_1 \otimes m_2), r_1m_2 + m_1r_2).$$

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This multiplication is associative if and only if $\Phi(m_1 \otimes m_2)m_3 = m_1\Phi(m_2 \otimes m_3)$ for all $m_1, m_2, m_3 \in M$. In such a case, $R \oplus M$ becomes a ring. This ring is called the *semi-trivial extension* of the ring R by M and Φ [16] or Φ-*trivial extension* of the ring R by M [18], and denoted by $R \ltimes_\Phi M$. Clearly, $\Phi = 0$ corresponds to the classical trivial extension $R \ltimes M$ [8]. One important example of the semi-trivial extension is the Morita context ring $\begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$. Note that the ring R is a subring of $R \ltimes_\Phi M$ but in general not a quotient ring. In fact, $R/\mathrm{im}(\Phi)$ is a quotient ring of $R \ltimes_\Phi M$. Semi-trivial ring extensions have been investigated by many scholars. For example, Palmér developed results on the global homological dimensions of semi-trivial ring extensions [16]. Sakano exhibited some ring theoretic properties of semi-trivial ring extensions [18]-[20]. Valtonen gave some homological properties of commutative semi-trivial ring extensions [22]. The author characterized when the semi-trivial extension of a ring is a left self-injective ring (right coherent ring, left V-ring, etc.) [13] and described Gorenstein projective, injective and flat modules over semi-trivial ring extensions [14].

In the present paper, we will describe *PGF* modules over semi-trivial ring extensions. In Section 2, we first exhibit necessary conditions of *PGF* modules over a semi-trivial ring extension $R \ltimes_\Phi M$. Write $I = \mathrm{im}(\Phi)$. Since R/I is a quotient ring of $R \ltimes_\Phi M$, R/I and M/MI may be viewed $R \ltimes_\Phi M$-$R \ltimes_\Phi M$-bimodules. It is proven that, if $fd({}_{R\ltimes_\Phi M}R/I) < \infty$, $fd(R/I_{R\ltimes_\Phi M})$ or $id(R/I_{R\ltimes_\Phi M}) < \infty$, $fd(M/MI_{R\ltimes_\Phi M})$ or $id(M/MI_{R\ltimes_\Phi M}) < \infty$, (X, f) is a *PGF* left $R \ltimes_\Phi M$-module, then the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact and $\mathrm{coker}(f)$ is a *PGF* left R/I-module (see Theorem 2.4). Next we provide sufficient conditions of *PGF* modules over $R \ltimes_\Phi M$. It is proven that, if $\mathrm{Tor}_1^R(R/I, R/I \oplus M) = 0$, $M \otimes_R I = 0$, $fd({}_{R/I}I) < \infty$, $fd({}_{R/I}M) < \infty$, $fd(M_{R/I})$ or $id(M_{R/I}) < \infty$, (X, f) is a left $R \ltimes_\Phi M$-module such that the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact, $\mathrm{coker}(f)$ and IX are *PGF* left R/I-modules, then (X, f) is a *PGF* left $R \ltimes_\Phi M$-module (see Theorem 2.6). In Section 3, some applications are given in Morita context rings. We describe *PGF* modules over Morita context rings.

Throughout this paper, all rings are nonzero associative rings with identity and all modules are unitary. For a ring R, we write R-Mod (resp. Mod-R) for the category of left (resp. right) R-modules. ${}_RX$ (resp. X_R) denotes a left (resp. right) R-module. The character module $\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Q}/\mathbb{Z})$ of X is denoted by X^+. $\mathrm{Add}(X)$ denotes the class of R-modules isomorphic to direct summands of direct sums of copies of X, $pd(X)$, $id(X)$ and $fd(X)$ denote the projective, injective and flat dimensions of X, respectively. $R \ltimes_\Phi M$ stands for the semi-trivial extension of a ring R by M and Φ. We always write $I = \mathrm{im}(\Phi)$, $l_Y(I) = \{y \in Y : yI = 0\}$ for a right R-module Y.

We next briefly recall some known results about semi-trivial extensions of rings [16]. It is well known that the category $R \ltimes_\Phi M$-Mod is isomorphic to the category Ξ: whose objects are couples (X, f) with $X \in R$-Mod and $f : M \otimes_R X \to X$ a left R-homomorphism such that the following diagram commutes.

$$\begin{array}{ccc} M \otimes_R M \otimes_R X & \xrightarrow{M\otimes f} & M \otimes_R X \\ {\scriptstyle \Phi\otimes X}\downarrow & & \downarrow{\scriptstyle f} \\ R \otimes_R X & \xrightarrow{\cong} & X \end{array}$$

A morphism $\alpha : (X, f) \to (Y, g)$ in Ξ is a morphism $\alpha : X \to Y$ in R-Mod such that the following diagram

$$\begin{array}{ccc} M \otimes_R X & \xrightarrow{M\otimes \alpha} & M \otimes_R Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

commutes. Precisely, if $(X, f) \in \Xi$, then X obtains a left $R \ltimes_\Phi M$-module structure by $(r, m)x = rx + f(m \otimes x)$ for $r \in R, m \in M$ and $x \in X$.

For $(X, f) \in \Xi$, $\ker(f)$ and $\mathrm{coker}(f)$ are annihilated by the image I of Φ and so they are left R/I-modules. In particular, $(X, 0) \in \Xi$ if and only if X is a left R/I-module. A sequence $(X_1, f_1) \xrightarrow{\alpha_1} (X_2, f_2) \xrightarrow{\alpha_2} (X_3, f_3)$ in Ξ is exact if and only if the sequence $X_1 \xrightarrow{\alpha_1} X_2 \xrightarrow{\alpha_2} X_3$ in R-Mod is exact.

In view of the well-known adjointness relation, the category Mod-$R \ltimes_\Phi M$ is isomorphic to the category Υ: whose objects are couples $[Y, g]$ with $Y \in$ Mod-R and $g : Y \to \mathrm{Hom}_R(M, Y)$ a right R-homomorphism such that the following diagram commutes.

$$\begin{array}{ccccc} Y & \xrightarrow{\;g\;} & \mathrm{Hom}_R(M,Y) & \xrightarrow{\;g_*\;} & \mathrm{Hom}_R(M,\mathrm{Hom}_R(M,Y)) \\ \cong\downarrow & & & & \downarrow\cong \\ \mathrm{Hom}_R(R,Y) & & \xrightarrow{\qquad\Phi^*\qquad} & & \mathrm{Hom}_R(M\otimes_R M, Y) \end{array}$$

A morphism $\beta : [X, f] \to [Y, g]$ in Υ is a morphism $\beta : X \to Y$ in Mod-R such that the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\;\beta\;} & Y \\ f\downarrow & & \downarrow g \\ \mathrm{Hom}_R(M,X) & \xrightarrow{\;\beta_*\;} & \mathrm{Hom}_R(M,Y) \end{array}$$

commutes. A sequence $[Y_1, g_1] \xrightarrow{\beta_1} [Y_2, g_2] \xrightarrow{\beta_2} [Y_3, g_3]$ in Υ is exact if and only if the sequence $Y_1 \xrightarrow{\beta_1} Y_2 \xrightarrow{\beta_2} Y_3$ in Mod-R is exact.

Similar to trivial extensions of rings [8], we define some functors for semi-trivial extensions of rings:

The functor $\mathbf{T} : R$-Mod $\to \Xi$ is given, for every object $X \in R$-Mod, by $\mathbf{T}(X) = (X \oplus (M \otimes_R X), \mu)$ with $\mu = \begin{pmatrix} 0 & \Phi_X \\ 1 & 0 \end{pmatrix} : (M \otimes_R X) \oplus (M \otimes_R M \otimes_R X) \to X \oplus (M \otimes_R X)$, where $\Phi_X : M \otimes_R M \otimes_R X \to X$ is the composition $M \otimes_R M \otimes_R X \xrightarrow{\Phi\otimes X} R \otimes_R X \xrightarrow{\cong} X$ and for morphisms by $\mathbf{T}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & M\otimes\alpha \end{pmatrix}$.

The functor $\mathbf{U} : \Xi \to R$-Mod is given, for every object $(X, f) \in \Xi$, by $\mathbf{U}(X, f) = X$ and for morphisms by $\mathbf{U}(\alpha) = \alpha$.

The functor $\mathbf{Z} : R/I$-Mod $\to \Xi$ is given, for every object $X \in R/I$-Mod, by $\mathbf{Z}(X) = (X, 0)$ and for morphisms by $\mathbf{Z}(\alpha) = \alpha$.

The functor $\mathbf{C} : \Xi \to R/I$-Mod is given, for every object $(X, f) \in \Xi$, by $\mathbf{C}(X, f) = \mathrm{coker}(f)$ and for morphisms by $\mathbf{C}(\alpha) =$ the induced morphism.

It is easy to verify that $(\mathbf{T}, \mathbf{U})$ and $(\mathbf{C}, \mathbf{Z})$ are adjoint pairs (see [17, Lemma 11.59] and [16, Lemma 5]). Then we get the following diagram:

$$R\text{–Mod} \underset{\mathbf{U}}{\overset{\mathbf{T}}{\rightleftarrows}} \Xi \underset{\mathbf{Z}}{\overset{\mathbf{C}}{\rightleftarrows}} R/I\text{–Mod}.$$

It is clear that $\mathbf{T}$ and $\mathbf{C}$ are right exact, $\mathbf{U}$ and $\mathbf{Z}$ are exact, $\mathbf{UZ}(X) = X$ and $\mathbf{CT}(X) \cong X/IX \cong R/I \otimes_R X$.

There are analogous functors for right modules.

In the rest of the paper, we always identify $R \ltimes_\Phi M$-Mod with Ξ and identify Mod-$R \ltimes_\Phi M$ with Υ.

2. *PGF* modules over semi-trivial ring extensions

Lemma 2.1. *[14, Lemma 2.1] Let (X, f) be a left $R \ltimes_\Phi M$-module and $[Y, g]$ a right $R \ltimes_\Phi M$-module, $\rho : X \to \mathrm{coker}(f)$ and $\pi : X \to X/IX$ the canonical epimorphisms, $\lambda : \ker(g) \to Y$ and $\chi : l_Y(I) \to Y$ the inclusions.*

1. *There is the exact sequence $M \otimes_R \mathrm{coker}(f) \xrightarrow{\kappa} X/IX \xrightarrow{\sigma} \mathrm{coker}(f) \to 0$ such that $\kappa(M \otimes \rho) = \pi f$. Moreover, the sequence $0 \to M \otimes_R \mathrm{coker}(f) \xrightarrow{\kappa} X/IX \xrightarrow{\sigma} \mathrm{coker}(f) \to 0$ is exact if and only if the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact, where $\widetilde{f}$ is the composition $M \otimes_R X \xrightarrow{f} X \xrightarrow{\pi} X/IX$.*

2. *There is the exact sequence* $0 \to \ker(g) \xrightarrow{\iota} l_Y(I) \xrightarrow{\varsigma} \mathrm{Hom}_R(M, \ker(g))$ *such that* $g\chi = \lambda_*\varsigma$. *Moreover, the sequence* $0 \to \ker(g) \xrightarrow{\iota} l_Y(I) \xrightarrow{\varsigma} \mathrm{Hom}_R(M, \ker(g)) \to 0$ *is exact if and only if the sequence* $l_Y(I) \xrightarrow{\widehat{g}} \mathrm{Hom}_R(M, Y) \xrightarrow{g_*} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, Y))$ *is exact, where* $\widehat{g}$ *is the composition* $l_Y(I) \xrightarrow{\chi} Y \xrightarrow{g} \mathrm{Hom}_R(M, Y)$.
3. *There are the exact sequences* $0 \to \mathbf{Z}(\mathrm{im}(f)/IX) \to (X/IX, \overline{f}) \to \mathbf{Z}(\mathrm{coker}(f)) \to 0$ *and* $0 \to (IX, \zeta) \to (X, f) \longrightarrow (X/IX, \overline{f}) \to 0$, *where* $\overline{f}$ *and* ζ *are the homomorphisms induced by* f.
4. *There are the exact sequences* $0 \to \mathbf{Z}(\ker(g)) \to [l_Y(I), \underline{g}] \to \mathbf{Z}(l_Y(I)/\ker(g)) \to 0$ *and* $0 \to [l_Y(I), \underline{g}] \to [Y, g] \to [Y/l_Y(I), \varepsilon] \to 0$, *where* $\underline{g}$ *and* ε *are the homomorphisms induced by* g.

Lemma 2.2. *[14, Lemma 2.2] Let* (X, f) *be a left* $R \ltimes_\Phi M$*-module and* $[Y, g]$ *a right* $R \ltimes_\Phi M$*-module.*

1. *If* (X, f) *is a projective left* $R \ltimes_\Phi M$*-module, then the sequence* $M \otimes_R M \otimes_R X \xrightarrow{M \otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ *is exact and* $\mathrm{coker}(f)$ *is a projective left* R/I*-module. The converse holds if* I *is nilpotent and* $\mathrm{Tor}_1^R(R/I, X) = 0$.
2. *If* $[Y, g]$ *is an injective right* $R \ltimes_\Phi M$*-module, then the sequence* $l_Y(I) \xrightarrow{\widehat{g}} \mathrm{Hom}_R(M, Y) \xrightarrow{g_*} \mathrm{Hom}_R(M, \mathrm{Hom}_R(M, Y))$ *is exact and* $\ker(g)$ *is an injective right* R/I*-module. The converse holds if* I *is nilpotent and* $\mathrm{Ext}_R^1(R/I, Y) = 0$.

Lemma 2.3. *[14, Lemma 2.4] Let* (X, f) *be a left* $R \ltimes_\Phi M$*-module,* N *a right* R*-module,* W *a left* R*-module,* G *a right* R/I*-module. Then*

1. $\boldsymbol{T}(N) \otimes_{R \ltimes_\Phi M} (X, f) \cong N \otimes_R X$.
2. $\mathbf{Z}(G) \otimes_{R \ltimes_\Phi M} (X, f) \cong G \otimes_{R/I} \mathrm{coker}(f)$.
3. $W^+/l_{W^+}(I) \cong (IW)^+$.

Recall that a left R-module X is a *projectively coresolved Gorenstein flat module* (*PGF module*, for short) [21] if there is an exact sequence

$$\cdots \to P^{-1} \to P^0 \xrightarrow{h^0} P^1 \to P^2 \to \cdots$$

of projective left R-modules with $X \cong \ker(h^0)$ and $E \otimes_R -$ leaves the sequence exact for any injective right R-module E.

Theorem 2.4. *Let* (X, f) *be a PGF left* $R \ltimes_\Phi M$*-module.*

1. *If* $fd({}_{R \ltimes_\Phi M}\mathbf{Z}(R/I)) < \infty$, $fd(\mathbf{Z}(R/I)_{R \ltimes_\Phi M})$ *or* $id(\mathbf{Z}(R/I)_{R \ltimes_\Phi M}) < \infty$, *then* $\mathrm{coker}(f)$ *is a PGF left* R/I*-module.*
2. *If* $fd(\mathbf{Z}(M/MI)_{R \ltimes_\Phi M})$ *or* $id(\mathbf{Z}(M/MI)_{R \ltimes_\Phi M}) < \infty$, *then the sequence* $M \otimes_R M \otimes_R X \xrightarrow{M \otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ *is exact.*

Proof. There is an exact sequence of projective left $R \ltimes_\Phi M$-modules

$$\Delta : \cdots \longrightarrow (A^{-1}, g^{-1}) \longrightarrow (A^0, g^0) \xrightarrow{\gamma^0} (A^1, g^1) \longrightarrow (A^2, g^2) \longrightarrow \cdots$$

with $(X, f) \cong \ker(\gamma^0)$ and the sequence $Q \otimes_{R \ltimes_\Phi M} \Delta$ is exact for any injective right $R \ltimes_\Phi M$-module Q.

(1) Since $fd(\mathbf{Z}(R/I)_{R \ltimes_\Phi M})$ or $id(\mathbf{Z}(R/I)_{R \ltimes_\Phi M}) < \infty$, we have $\mathbf{Z}(R/I) \otimes_{R \ltimes_\Phi M} \Delta$ is exact by [5, Lemma 2.3] and [23, Lemma 2.1]. Thus $\mathbf{C}(\Delta) \cong (R/I) \otimes_{R/I} \mathbf{C}(\Delta) \cong \mathbf{Z}(R/I) \otimes_{R \ltimes_\Phi M} \Delta$ is exact by Lemma 2.3(2). Also $\mathrm{coker}(g^i)$ is projective by Lemma 2.2(1). So we get the exact sequence of projective left R/I-modules

$$\mathbf{C}(\Delta) : \cdots \longrightarrow \mathrm{coker}(g^{-1}) \longrightarrow \mathrm{coker}(g^0) \xrightarrow{\mathbf{C}(\gamma^0)} \mathrm{coker}(g^1) \longrightarrow \mathrm{coker}(g^2) \longrightarrow \cdots$$

with $\mathrm{coker}(f) \cong \ker(\mathbf{C}(\gamma^0))$.

Let G be an injective right R/I-module. Then there is a split monomorphism $G \to \prod (R/I)^+$, which induces the split monomorphism

$$\mathbf{Z}(G) \to \mathbf{Z}(\prod (R/I)^+) \cong \prod \mathbf{Z}((R/I)^+).$$

Since $fd({}_{R\ltimes_\Phi M}\mathbf{Z}(R/I)) < \infty$, we have $id(\mathbf{Z}(G)_{R\ltimes_\Phi M}) < \infty$. Thus $G \otimes_{R/I} \mathbf{C}(\Delta) \cong \mathbf{Z}(G) \otimes_{R\ltimes_\Phi M} \Delta$ is exact by Lemma 2.3(2). So coker(f) is a *PGF* left R/I-module.

(2) Let ϵ : coker$(f) \to$ coker(g^0) be the inclusion. By Lemma 2.1(1), we get the following commutative diagram:

$$\begin{array}{ccc} M \otimes_R \mathrm{coker}(f) & \xrightarrow{M\otimes\epsilon} & M \otimes_R \mathrm{coker}(g^0) \\ \downarrow{\scriptstyle\kappa} & & \downarrow{\scriptstyle\tau} \\ X/IX & \longrightarrow & A^0/IA^0. \end{array}$$

Since $fd(\mathbf{Z}(M/MI)_{R\ltimes_\Phi M})$ or $id(\mathbf{Z}(M/MI)_{R\ltimes_\Phi M}) < \infty$, we have $\mathbf{Z}(M/MI) \otimes_{R\ltimes_\Phi M} \Delta$ is exact. Note that

$$M \otimes_R \mathbf{C}(\Delta) \cong M \otimes_R (R/I) \otimes_{R/I} \mathbf{C}(\Delta) \cong (M/MI) \otimes_{R/I} \mathbf{C}(\Delta) \cong \mathbf{Z}(M/MI) \otimes_{R\ltimes_\Phi M} \Delta.$$

Thus the complex $M \otimes_R \mathbf{C}(\Delta)$ is exact. Hence $M \otimes \epsilon$ is a monomorphism. Since τ is a monomorphism by Lemmas 2.1(1) and 2.2(1), κ is a monomorphism. So the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact by Lemma 2.1(1). □

Corollary 2.5. *Let $R \ltimes_\Phi M$ be a semi-trivial ring extension and X a left R-module.*

1. *If $fd({}_{R\ltimes_\Phi M}\mathbf{Z}(R/I)) < \infty$, $fd(\mathbf{Z}(R/I)_{R\ltimes_\Phi M})$ or $id(\mathbf{Z}(R/I)_{R\ltimes_\Phi M}) < \infty$, $\mathbf{T}(X)$ is a PGF left $R \ltimes_\Phi M$-module, then X/IX is a PGF left R/I-module.*
2. *If $fd({}_RM) < \infty$, $fd(M_R)$ or $id(M_R) < \infty$, X is a PGF left R-module, then $\mathbf{T}(X)$ is a PGF left $R \ltimes_\Phi M$-module.*

Proof. (1) is an immediate consequence of Theorem 2.4(1).

(2) There is an exact sequence of projective left R-modules

$$\Psi : \cdots \to G^{-2} \to G^{-1} \to G^0 \xrightarrow{g^0} G^1 \to \cdots$$

such that $X \cong \ker(g^0)$ and the sequence $E \otimes_R \Psi$ is exact for any injective right R-module E. Since $fd(M_R)$ or $id(M_R) < \infty$, we get the exact sequence of projective left $R \ltimes_\Phi M$-modules

$$\mathbf{T}(\Psi) : \cdots \longrightarrow \mathbf{T}(G^{-2}) \longrightarrow \mathbf{T}(G^{-1}) \longrightarrow \mathbf{T}(G^0) \xrightarrow{\mathbf{T}(g^0)} \mathbf{T}(G^1) \longrightarrow \cdots$$

with $\mathbf{T}(X) \cong \ker(\mathbf{T}(g^0))$. For any injective right $R\ltimes_\Phi M$-module $[N, \xi]$, we have $id(N_R) < \infty$ since $fd({}_RM) < \infty$. So $[N, \xi] \otimes_{R\ltimes_\Phi M} \mathbf{T}(\Psi) \cong N \otimes_R \Psi$ is exact by Lemma 2.3(1). Therefore $\mathbf{T}(X)$ is a *PGF* left $R \ltimes_\Phi M$-module. □

The following theorem extends [23, Theorem 2.7] not only from trivial ring extensions to semi-trivial ring extensions but also remove the unnecessary condition that $\mathrm{Ext}^1_R(G, \mathrm{Add}(M)) = 0$ for every *PGF* left R-module G.

Suppose that $M \otimes_R I = 0$ (for example, $M \otimes_R M \otimes_R M = 0$). It is easy to see that $IM = MI = 0$ and so $I^2 = 0$. In this case, both M and I may be viewed as R/I-R/I-bimodules.

Theorem 2.6. *Let $R \ltimes_\Phi M$ be a semi-trivial ring extension such that $\mathrm{Tor}^R_1(R/I, R/I \oplus M) = 0$, $M \otimes_R I = 0$, $fd({}_{R/I}I) < \infty$, $fd({}_{R/I}M) < \infty$, $fd(M_{R/I})$ or $id(M_{R/I}) < \infty$. If (X, f) is a left $R \ltimes_\Phi M$-module such that the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact,* coker(f) *and IX are PGF left R/I-modules, then (X, f) is a PGF left $R \ltimes_\Phi M$-module.*

Proof. There is an exact sequence of projective left R/I-modules

$$\Psi : \cdots \longrightarrow P^{-1} \xrightarrow{h^{-1}} P^0 \xrightarrow{h^0} P^1 \xrightarrow{h^1} P^2 \longrightarrow \cdots$$

such that $\mathrm{coker}(f) \cong \ker(h^0)$ and the sequence $E \otimes_{R/I} \Psi$ is exact for any injective right R/I-module E. Since $fd(M_{R/I})$ or $id(M_{R/I}) < \infty$, we get the exact sequence of left R/I-modules

$$M \otimes_{R/I} \Psi : \cdots \longrightarrow M \otimes_{R/I} P^{-1} \overset{M\otimes h^{-1}}{\longrightarrow} M \otimes_{R/I} P^0 \overset{M\otimes h^{0}}{\longrightarrow} M \otimes_{R/I} P^1 \overset{M\otimes h^{1}}{\longrightarrow} M \otimes_{R/I} P^2 \longrightarrow \cdots$$

with $M \otimes_{R/I} \mathrm{coker}(f) \cong \ker(M \otimes h^0)$.

Let $\rho : X \to \mathrm{coker}(f)$ and $\pi : X \to X/IX$ be the canonical epimorphisms. Since the sequence $M \otimes_R M \otimes_R X \overset{M\otimes f}{\longrightarrow} M \otimes_R X \overset{\widetilde{f}}{\longrightarrow} X/IX$ is exact, we obtain the exact sequence of left R-modules

$$0 \to M \otimes_R \mathrm{coker}(f) \overset{\kappa}{\to} X/IX \overset{\sigma}{\to} \mathrm{coker}(f) \to 0$$

such that $\kappa(M \otimes \rho) = \pi f$ by Lemma 2.1(1). Note that $M \otimes_R \mathrm{coker}(f) \cong M \otimes_{R/I} \mathrm{coker}(f)$. So we obtain the exact sequence of left R/I-modules

$$0 \to M \otimes_{R/I} \mathrm{coker}(f) \overset{\kappa}{\to} X/IX \overset{\sigma}{\to} \mathrm{coker}(f) \to 0.$$

Since $M \otimes_{R/I} P^i \in \mathrm{Add}({}_{R/I}M)$ and $fd({}_{R/I}M) < \infty$, we have $fd({}_{R/I}M \otimes_{R/I} P^i) < \infty$. By [10, Corollary 1] and [12, Lemma 3.1], $\mathrm{Ext}^1_{R/I}(\ker(h^i), M \otimes_{R/I} P^i) = 0$.

Let $\iota : \mathrm{coker}(f) \longrightarrow P^0$ be the inclusion and $\tau : P^{-1} \longrightarrow \mathrm{coker}(f)$ the obvious epimorphism such that $\iota\tau = h^{-1}$. Then there is a left R/I-module homomorphism $\omega : X/IX \longrightarrow M \otimes_{R/I} P^0$ such that $\omega\kappa = M \otimes \iota$ and there is a left R/I-module homomorphism $\theta : P^{-1} \longrightarrow X/IX$ such that $\sigma\theta = \tau$.

Define $\lambda : X/IX \longrightarrow P^0 \oplus (M \otimes_{R/I} P^0)$ by

$$\lambda(\overline{x}) = (\iota\sigma(\overline{x}), \omega(\overline{x})), x \in X$$

and define $\xi : P^{-1} \oplus (M \otimes_{R/I} P^{-1}) \longrightarrow X/IX$ by

$$\xi(x, y) = \theta(x) + \kappa(M \otimes \tau)(y), x \in P^{-1}, y \in M \otimes_{R/I} P^{-1}.$$

It is easy to check that λ is a monomorphism and ξ is an epimorphism. By the generalized Horseshoe Lemma (see [24, Lemma 1.6]), we get the following commutative diagrams:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M \otimes_{R/I} \mathrm{coker}(f) & \overset{\kappa}{\longrightarrow} & X/IX & \overset{\sigma}{\longrightarrow} & \mathrm{coker}(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle M\otimes\iota} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & M \otimes_{R/I} P^0 & \longrightarrow & P^0 \oplus (M \otimes_{R/I} P^0) & \longrightarrow & P^0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle M\otimes h^0} & & \vdots{\scriptstyle g^0} & & \downarrow{\scriptstyle h^0} & & \\
0 & \longrightarrow & M \otimes_{R/I} P^1 & \longrightarrow & P^1 \oplus (M \otimes_{R/I} P^1) & \longrightarrow & P^1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle M\otimes h^1} & & \vdots{\scriptstyle g^1} & & \downarrow{\scriptstyle h^1} & & \\
0 & \longrightarrow & M \otimes_{R/I} P^2 & \longrightarrow & P^2 \oplus (M \otimes_{R/I} P^2) & \longrightarrow & P^2 & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

and

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow{\scriptstyle g^{-3}} & & \downarrow & & \\
0 & \longrightarrow & M\otimes_{R/I} P^{-2} & \longrightarrow & P^{-2}\oplus(M\otimes_{R/I} P^{-2}) & \longrightarrow & P^{-2} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle M\otimes h^{-2}} & & \downarrow{\scriptstyle g^{-2}} & & \downarrow{\scriptstyle h^{-2}} & & \\
0 & \longrightarrow & M\otimes_{R/I} P^{-1} & \longrightarrow & P^{-1}\oplus(M\otimes_{R/I} P^{-1}) & \longrightarrow & P^{-1} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle M\otimes\tau} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \tau} & & \\
0 & \longrightarrow & M\otimes_{R/I}\mathrm{coker}(f) & \xrightarrow{\kappa} & X/IX & \xrightarrow{\sigma} & \mathrm{coker}(f) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns and $g^i = \begin{pmatrix} h^i & 0 \\ \delta^i & M\otimes h^i \end{pmatrix}$.

Let $g^{-1} = \lambda\xi$. Then we get the exact sequence of left R/I-modules

$$\cdots \to P^{-1}\oplus(M\otimes_{R/I} P^{-1}) \xrightarrow{g^{-1}} P^{0}\oplus(M\otimes_{R/I} P^{0}) \xrightarrow{g^{0}} P^{1}\oplus(M\otimes_{R/I} P^{1}) \xrightarrow{g^{1}} P^{2}\oplus(M\otimes_{R/I} P^{2}) \to \cdots$$

with $X/IX \cong \ker(g^0)$, which also may be viewed the exact sequence of left R-modules.

Define $\overline{f} : M\otimes_R (X/IX) \to X/IX$ by $\overline{f}(m\otimes\overline{x}) = \overline{f(m\otimes x)}$ for $m \in M$ and $x \in X$. Then the following commutative diagram

$$
\begin{array}{ccccccc}
 & M\otimes_R (X/IX) & \xrightarrow{M\otimes\lambda} & M\otimes_R (P^0\oplus(M\otimes_R P^0)) & \xrightarrow{M\otimes g^0} & M\otimes_R (P^1\oplus(M\otimes_R P^1)) & \longrightarrow \cdots\\
 & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu_1} & \\
0 \longrightarrow & X/IX & \xrightarrow{\lambda} & P^0\oplus(M\otimes_R P^0) & \xrightarrow{g^0} & P^1\oplus(M\otimes_R P^1) & \longrightarrow \cdots
\end{array}
$$

implies that the sequence $0 \longrightarrow (X/IX, \overline{f}) \xrightarrow{\lambda} \mathbf{T}(P^0) \xrightarrow{g^0} \mathbf{T}(P^1) \longrightarrow \cdots$ is exact.

The following commutative diagram

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & M\otimes_R (P^{-2}\oplus(M\otimes_R P^{-2})) & \xrightarrow{M\otimes g^{-2}} & M\otimes_R (P^{-1}\oplus(M\otimes_R P^{-1})) & \xrightarrow{M\otimes\xi} & M\otimes_R (X/IX) & \\
 & \downarrow{\scriptstyle \mu_{-2}} & & \downarrow{\scriptstyle \mu_{-1}} & & \downarrow{\scriptstyle \overline{f}} & \\
\cdots \longrightarrow & P^{-2}\oplus(M\otimes_R P^{-2}) & \xrightarrow{g^{-2}} & P^{-1}\oplus(M\otimes_R P^{-1}) & \xrightarrow{\xi} & X/IX & \longrightarrow 0
\end{array}
$$

implies that the sequence $\cdots \longrightarrow \mathbf{T}(P^{-2}) \xrightarrow{g^{-2}} \mathbf{T}(P^{-1}) \xrightarrow{\xi} (X/IX, \overline{f}) \longrightarrow 0$ is exact. Since $\mathrm{Tor}_1^R(R/I, R/I\oplus M) = 0$ and $I^2 = 0$, each $\mathbf{T}(P^i)$ is a projective left $R \ltimes_\Phi M$-module by Lemma 2.2(1). So there is the exact sequence of projective left $R \ltimes_\Phi M$-modules

$$\Delta : \cdots \longrightarrow \mathbf{T}(P^{-1}) \xrightarrow{g^{-1}} \mathbf{T}(P^0) \xrightarrow{g^0} \mathbf{T}(P^1) \xrightarrow{g^1} \mathbf{T}(P^2) \longrightarrow \cdots$$

with $(X/IX, \overline{f}) \cong \ker(g^0)$.

Next, we show that the sequence Δ remains exact when tensored with any injective right $R \ltimes_\Phi M$-module $[Q, \gamma]$. By Lemmas 2.1(2) and 2.2(2), $\ker(\gamma)$ is an injective right R/I-module and there is the exact sequence of right R/I-modules

$$0 \to \ker(\gamma) \to l_Q(I) \to \mathrm{Hom}_{R/I}(M, \ker(\gamma)) \to 0.$$

Define $\underline{\gamma} : l_Q(I) \to \mathrm{Hom}_R(M, l_Q(I))$ by $\underline{\gamma}(y)(m) = \gamma(y)(m)$ for $m \in M$ and $y \in l_Q(I)$. Then $\underline{\gamma}$ is well defined and $[l_Q(I), \underline{\gamma}] \in$ Mod-$R \ltimes_\Phi M$. By Lemma 2.1(4), we get the two exact sequences

$$0 \to \mathbf{Z}(\ker(\gamma)) \to [l_Q(I), \underline{\gamma}] \to \mathbf{Z}(\mathrm{Hom}_{R/I}(M, \ker(\gamma))) \to 0,$$

$$0 \to [l_Q(I), \underline{\gamma}] \to [Q, \gamma] \to \mathbf{Z}(Q/l_Q(I)) \to 0.$$

So we get the two exact sequences of complexes

$$0 \to \mathbf{Z}(\ker(\gamma)) \otimes_{R\ltimes_\Phi M} \Delta \to [l_Q(I), \underline{\gamma}] \otimes_{R\ltimes_\Phi M} \Delta \to \mathbf{Z}(\mathrm{Hom}_{R/I}(M, \ker(\gamma))) \otimes_{R\ltimes_\Phi M} \Delta \to 0,$$

$$0 \to [l_Q(I), \underline{\gamma}] \otimes_{R\ltimes_\Phi M} \Delta \to [Q, \gamma] \otimes_{R\ltimes_\Phi M} \Delta \to \mathbf{Z}(Q/l_Q(I)) \otimes_{R\ltimes_\Phi M} \Delta \to 0.$$

Since $fd({}_{R/I}M) < \infty$ and $\ker(\gamma)$ is an injective right R/I-module, $id(\mathrm{Hom}_{R/I}(M, \ker(\gamma))_{R/I}) < \infty$. Thus

$$\mathbf{Z}(\mathrm{Hom}_{R/I}(M, \ker(\gamma))) \otimes_{R\ltimes_\Phi M} \Delta \cong \mathrm{Hom}_{R/I}(M, \ker(\gamma)) \otimes_{R/I} \Psi$$

is exact. Also $\mathbf{Z}(\ker(\gamma) \otimes_{R\ltimes_\Phi M} \Delta \cong \ker(\gamma) \otimes_{R/I} \Psi$ is exact. Therefore $[l_Q(I), \underline{\gamma}] \otimes_{R\ltimes_\Phi M} \Delta$ is exact.

Since $[Q, \gamma]$ is an injective right $R \ltimes_\Phi M$-module, there is the split monomorphism $[Q, \gamma] \to (\coprod \mathbf{T}(R))^+$. So there is the split monomorphism $Q \to (\coprod (R \oplus M))^+$, which induces the split monomorphism

$$Q/l_Q(I) \to (\coprod (R \oplus M))^+/l_{(\coprod(R\oplus M))^+}(I).$$

By Lemma 2.3(3), we have

$$(\coprod (R \oplus M))^+/l_{(\coprod(R\oplus M))^+}(I) \cong (I \coprod (R \oplus M))^+ \cong \prod I^+.$$

Thus we get the split monomorphism $Q/l_Q(I) \to \prod I^+$. Since $fd({}_{R/I}I) < \infty$, we have $id(Q/l_Q(I)_{R/I}) < \infty$. So $\mathbf{Z}(Q/l_Q(I)) \otimes_{R\ltimes_\Phi M} \Delta \cong Q/l_Q(I) \otimes_{R/I} \Psi$ is exact. Thus the complex $[Q, \gamma] \otimes_{R\ltimes_\Phi M} \Delta$ is exact. Hence $(X/IX, \overline{f})$ is a *PGF* left $R \ltimes_\Phi M$-module.

Since $M \otimes_R I = 0$, one gets $M \otimes_R IX = 0$. By Lemma 2.1(3), there is the exact sequence in $R \ltimes_\Phi M$-Mod

$$0 \to \mathbf{Z}(IX) \to (X, f) \to (X/IX, \overline{f}) \to 0.$$

Since the sequence $M \otimes_R M \otimes_R IX \xrightarrow{0} M \otimes_R IX \xrightarrow{0} IX/I^2X$ is exact and IX is a *PGF* left R/I-module, we have $\mathbf{Z}(IX)$ is a *PGF* left $R \ltimes_\Phi M$-module by the above proof. Thus (X, f) is a *PGF* left $R \ltimes_\Phi M$-module by [21, Theorem 4.9]. □

Corollary 2.7. *Let $R \ltimes_\Phi M$ be a semi-trivial ring extension such that $\mathrm{Tor}_1^R(R/I, R/I \oplus M) = 0$, $M \otimes_R I = 0$, $fd({}_{R/I}I) < \infty$, $fd({}_{R/I}M) < \infty$, $fd(M_{R/I})$ or $id(M_{R/I}) < \infty$, $fd({}_{R\ltimes_\Phi M}\mathbf{Z}(R/I)) < \infty$, $fd(\mathbf{Z}(R/I)_{R\ltimes_\Phi M})$ or $id(\mathbf{Z}(R/I)_{R\ltimes_\Phi M})$ $< \infty$, $fd(\mathbf{Z}(M/MI)_{R\ltimes_\Phi M})$ or $id(\mathbf{Z}(M/MI)_{R\ltimes_\Phi M}) < \infty$. Then (X, f) is a PGF left $R \ltimes_\Phi M$-module if and only if the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact,* coker(f) *and IX are PGF left R/I-modules.*

Proof. " ⇒ " By Theorem 2.4, the sequence $M \otimes_R M \otimes_R X \xrightarrow{M\otimes f} M \otimes_R X \xrightarrow{\widetilde{f}} X/IX$ is exact and coker(f) is a *PGF* left R/I-module.

By Lemma 2.1(3), there is the exact sequence in $R \ltimes_\Phi M$-Mod

$$0 \to \mathbf{Z}(IX) \to (X, f) \to (X/IX, \overline{f}) \to 0.$$

By the proof of Theorem 2.6, $(X/IX, \overline{f})$ is also a *PGF* left $R \ltimes_\Phi M$-module. So $\mathbf{Z}(IX)$ is a *PGF* left $R \ltimes_\Phi M$-module by [21, Theorem 4.9]. Thus IX is a *PGF* left R/I-module by Theorem 2.4.

" ⇐ " follows from Theorem 2.6. □

3. Applications to Morita context rings

Morita context rings, originated from equivalences of module categories [15] and formulated by Bass [2], have been studied explicitly in a large variety of literature [2, 8, 9, 11, 15]. In this section, we apply the foregoing results to arbitrary modules over arbitrary Morita context rings (each bimodule homomorphism may be nonzero) since this kind of rings is one special case of semi-trivial ring extensions.

Let $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$, where A and B are rings, ${}_AV_B$ and ${}_BU_A$ are bimodules, $\varphi : V \otimes_B U \to A$ and $\psi : U \otimes_A V \to B$ are bimodule homomorphisms such that $\varphi(v \otimes u)v' = v\psi(u \otimes v'), \psi(u \otimes v)u' = u\varphi(v \otimes u')$ for $u, u' \in U$ and $v, v' \in V$. Λ is called a *Morita context ring* or *formal matrix ring* [11, 15], where the addition of elements of Λ is componentwise and multiplication is given by

$$\begin{pmatrix} a_1 & v_1 \\ u_1 & b_1 \end{pmatrix}\begin{pmatrix} a_2 & v_2 \\ u_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + \varphi(v_1 \otimes u_2) & a_1v_2 + v_1b_2 \\ u_1a_2 + b_1u_2 & \psi(u_1 \otimes v_2) + b_1b_2 \end{pmatrix}.$$

Green [9] proved that the category Λ-Mod is equivalent to the category Ω: whose objects are tuples (X, Y, f, g), where $X \in A$-Mod, $Y \in B$-Mod, $f \in \mathrm{Hom}_B(U \otimes_A X, Y)$ and $g \in \mathrm{Hom}_A(V \otimes_B Y, X)$ such that the following diagrams

$$\begin{array}{ccc} U \otimes_A V \otimes_B Y & \xrightarrow{U\otimes g} & U \otimes_A X \\ \downarrow{\scriptstyle \psi\otimes Y} & & \downarrow{\scriptstyle f} \\ B \otimes_B Y & \xrightarrow{\cong} & Y, \end{array} \qquad \begin{array}{ccc} V \otimes_B U \otimes_A X & \xrightarrow{V\otimes f} & V \otimes_B Y \\ \downarrow{\scriptstyle \varphi\otimes X} & & \downarrow{\scriptstyle g} \\ A \otimes_A X & \xrightarrow{\cong} & X \end{array}$$

are commutative and whose morphisms from (X_1, Y_1, f_1, g_1) to (X_2, Y_2, f_2, g_2) are pairs (α, β) such that $\alpha \in \mathrm{Hom}_A(X_1, X_2), \beta \in \mathrm{Hom}_B(Y_1, Y_2)$ and the following diagrams commute.

$$\begin{array}{ccc} U \otimes_A X_1 & \xrightarrow{U\otimes\alpha} & U \otimes_A X_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow{\beta} & Y_2, \end{array} \qquad \begin{array}{ccc} V \otimes_B Y_1 & \xrightarrow{V\otimes\beta} & V \otimes_B Y_2 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ X_1 & \xrightarrow{\alpha} & X_2. \end{array}$$

In what follows, we will identify Λ-Mod with Ω.

Consider $U \oplus V$ as a natural $A\times B$-$A\times B$-bimodule and define $\Phi = (\varphi, \psi) : (U \oplus V) \otimes_{A\times B} (U \oplus V) \to A \times B$ by $\Phi((u_1, v_1)\otimes(u_2, v_2)) = (\varphi(v_1\otimes u_2), \psi(u_1\otimes v_2))$. Then Φ is an $A\times B$-$A\times B$-bimodule homomorphism and we get the semi-trivial extension $(A\times B) \ltimes_\Phi (U\oplus V)$. The rings $(A\times B) \ltimes_\Phi (U\oplus V)$ and $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$ are isomorphic under the correspondence: $((a, b), (u, v)) \to \begin{pmatrix} a & v \\ u & b \end{pmatrix}$. Therefore Λ-Mod is isomorphic to $(A\times B) \ltimes_\Phi (U\oplus V)$-Mod by the functor $\Theta : \Lambda\text{-Mod} \to (A \times B) \ltimes_\Phi (U \oplus V)\text{-Mod}$ given by $\Theta(X, Y, f, g) = ((X, Y), (g, f))$.

Let X be a left A-module. Then we obtain a left Λ-module $(X, U \otimes_A X, 1, \varphi_X)$, where $\varphi_X : V \otimes_B U \otimes_A X \to X$ is the composition $V \otimes_B U \otimes_A X \xrightarrow{\varphi\otimes X} A \otimes_A X \xrightarrow{\cong} X$.

Let Y be a left B-module. Then we obtain a left Λ-module $(V \otimes_B Y, Y, \psi_Y, 1)$, where $\psi_Y : U \otimes_A V \otimes_B Y \to Y$ is the composition $U \otimes_A V \otimes_B Y \xrightarrow{\psi\otimes Y} B \otimes_B Y \xrightarrow{\cong} Y$.

In this section, we always write $J = \mathrm{im}(\varphi)$ and $L = \mathrm{im}(\psi)$. Clearly, (J, L) is an ideal of the ring $A \times B$.

Let $(X, Y, f, g) \in \Lambda$-Mod, then there exist the two sequences

$$V \otimes_B U \otimes_A X \xrightarrow{V\otimes f} V \otimes_B Y \xrightarrow{\widetilde{g}} X/JX,$$

$$U \otimes_A V \otimes_B Y \xrightarrow{U\otimes g} U \otimes_A X \xrightarrow{\widetilde{f}} Y/LY$$

such that $\widetilde{g}(V \otimes f) = 0$ and $\widetilde{f}(U \otimes g) = 0$, where $\widetilde{f}$ is the composition $U \otimes_A X \xrightarrow{f} Y \to Y/LY$ and $\widetilde{g}$ is the composition $V \otimes_B Y \xrightarrow{g} X \to X/JX$.

Theorem 3.1. *Suppose that $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$ is a Morita context ring and (X, Y, f, g) is a PGF left Λ-module.*

1. *If $fd({}_\Lambda(A/J, B/L, 0, 0)) < \infty$, $fd((A/J, B/L, 0, 0)_\Lambda)$ or $id((A/J, B/L, 0, 0)_\Lambda) < \infty$, then* coker$(g)$ *is a PGF left A/J-module and* coker(f) *is a PGF left B/L-module.*
2. *If $fd((U/UJ, V/VL, 0, 0)_\Lambda)$ or $id((U/UJ, V/VL, 0, 0)_\Lambda) < \infty$, then the sequences $V \otimes_B U \otimes_A X \xrightarrow{V\otimes f} V \otimes_B Y \xrightarrow{\widetilde{g}} X/JX$ and $U \otimes_A V \otimes_B Y \xrightarrow{U\otimes g} U \otimes_A X \xrightarrow{\widetilde{f}} Y/LY$ are exact.*

Proof. Define $\Phi = (\varphi, \psi) : (U \oplus V) \otimes_{A\times B} (U \oplus V) \to A \times B$ by $\Phi((u_1, v_1) \otimes (u_2, v_2)) = (\varphi(v_1 \otimes u_2), \psi(u_1 \otimes v_2))$. Then $(A \times B) \ltimes_\Phi (U \oplus V) \cong \Lambda$. Since (X, Y, f, g) is a *PGF* left Λ-module, $((X, Y), (g, f))$ is a *PGF* left $(A \times B) \ltimes_\Phi (U \oplus V)$-module.

(1) By hypothesis, we have $fd({}_{(A\times B)\ltimes_\Phi(U\oplus V)}\mathbf{Z}((A\times B)/(J, L))) < \infty$, $fd(\mathbf{Z}((A\times B)/(J, L))_{(A\times B)\ltimes_\Phi(U\oplus V)})$ or $id(\mathbf{Z}((A\times B)/(J, L))_{(A\times B)\ltimes_\Phi(U\oplus V)}) < \infty$. By Theorem 2.4(1), coker$(g, f) \cong$ (coker(g), coker(f)) is a *PGF* left $(A \times B)/(J, L)$-module. So coker(g) is a *PGF* left A/J-module and coker(f) is a *PGF* left B/L-module.

(2) By hypothesis, $fd(\mathbf{Z}((U\oplus V)/(U\oplus V)(J, L))_{(A\times B)\ltimes_\Phi(U\oplus V)})$ or $id(\mathbf{Z}((U\oplus V)/(U\oplus V)(J, L))_{(A\times B)\ltimes_\Phi(U\oplus V)}) < \infty$. By Theorem 2.4(2), the sequence $(U\oplus V)\otimes_{A\times B}(U\oplus V)\otimes_{A\times B}(X, Y) \xrightarrow{(U\oplus V)\otimes(g,f)} (U\oplus V)\otimes_{A\times B}(X, Y) \xrightarrow{\widetilde{(g,f)}} (X, Y)/(J, L)(X, Y)$ is exact and so the two sequences $V \otimes_B U \otimes_A X \xrightarrow{V\otimes f} V \otimes_B Y \xrightarrow{\widetilde{g}} X/JX$ and $U \otimes_A V \otimes_B Y \xrightarrow{U\otimes g} U \otimes_A X \xrightarrow{\widetilde{f}} Y/LY$ are exact. □

Theorem 3.2. *Suppose that $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$ is a Morita context ring, X is a left A-module and Y is a left B-module.*

1. *If $fd({}_\Lambda(A/J, B/L, 0, 0)) < \infty$, $fd((A/J, B/L, 0, 0)_\Lambda)$ or $id((A/J, B/L, 0, 0)_\Lambda) < \infty$, $(X, U \otimes_A X, 1, \varphi_X) \oplus (V \otimes_B Y, Y, \psi_Y, 1)$ is a PGF left Λ-module, then X/JX is a PGF left A/J-module and Y/LY is a PGF left B/L-module.*
2. *If $fd({}_{A\times B}(U \oplus V)) < \infty$, $fd((U \oplus V)_{A\times B})$ or $id((U \oplus V)_{A\times B}) < \infty$, X is a PGF left A-module and Y is a PGF left B-module, then $(X, U \otimes_A X, 1, \varphi_X) \oplus (V \otimes_B Y, Y, \psi_Y, 1)$ is a PGF left Λ-module.*

Proof. (1) By hypothesis, we have $fd({}_{(A\times B)\ltimes_\Phi(U\oplus V)}\mathbf{Z}((A \times B)/(J, L))) < \infty$, $fd(\mathbf{Z}((A \times B)/(J, L))_{(A\times B)\ltimes_\Phi(U\oplus V)})$ or $id(\mathbf{Z}((A \times B)/(J, L))_{(A\times B)\ltimes_\Phi(U\oplus V)}) < \infty$. Since $\mathbf{T}(X, Y)$ is a *PGF* left $(A \times B) \ltimes_\Phi (U \oplus V)$-module, $(X, Y)/(J, L)(X, Y)$ is a *PGF* left $(A \times B)/(J, L)$-module by Corollary 2.5(1). So X/JX is a *PGF* left A/J-module and Y/LY is a *PGF* left B/L-module.

(2) Since (X, Y) is a *PGF* left $A \times B$-module, we have $(X, U \otimes_A X, 1, \varphi_X) \oplus (V \otimes_B Y, Y, \psi_Y, 1)$ is a *PGF* left Λ-module by Corollary 2.5(2). □

Theorem 3.3. *Suppose that $\Lambda = \begin{pmatrix} A & {}_AV_B \\ {}_BU_A & B \end{pmatrix}_{(\varphi,\psi)}$ is a Morita context ring such that* $\mathrm{Tor}_1^A(A/J, A/J \oplus V) = 0 = \mathrm{Tor}_1^B(B/L, B/L\oplus U)$, *$U\otimes_A J = 0 = V\otimes_B L$, $fd({}_{(A\times B)/(J,L)}(J, L)) < \infty$, $fd({}_{(A\times B)/(J,L)}(U\oplus V)) < \infty$, $fd((U\oplus V)_{(A\times B)/(J,L)})$ or $id((U\oplus V)_{(A\times B)/(J,L)}) < \infty$. If (X, Y, f, g) is a left Λ-module such that the sequences $V\otimes_B U\otimes_A X \xrightarrow{V\otimes f} V\otimes_B Y \xrightarrow{\widetilde{g}} X/JX$ and $U \otimes_A V \otimes_B Y \xrightarrow{U\otimes g} U \otimes_A X \xrightarrow{\widetilde{f}} Y/LY$ are exact,* coker(g) *and JX are PGF left A/J-modules,* coker(f) *and LY are PGF left B/L-modules, then (X, Y, f, g) is a PGF left Λ-module.*

Proof. By hypothesis, we have $\mathrm{Tor}_1^{A\times B}((A \times B)/(J, L), (A \times B)/(J, L) \oplus (U \oplus V)) = 0$, $(U \oplus V) \otimes_{A\times B} (J, L) = 0$. Since the sequence $(U \oplus V) \otimes_{A\times B} (U \oplus V) \otimes_{A\times B} (X, Y) \xrightarrow{(U\oplus V)\otimes(g,f)} (U \oplus V) \otimes_{A\times B} (X, Y) \xrightarrow{\widetilde{(g,f)}} (X, Y)/(J, L)(X, Y)$ is exact, and coker(g, f) and $(J, L)(X, Y)$ are *PGF* left $(A \times B)/(J, L)$-modules, we have $((X, Y), (g, f))$ is a *PGF* left $(A \times B) \ltimes_\Phi (U \oplus V)$-module by Theorem 2.6. So (X, Y, f, g) is a *PGF* left Λ-module. □

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